# Geometrical Structures of Space-Time in General Relativity<sup>1</sup>

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**Abstract.** Space-Time in general relativity is a dynamical entity because it is subject to the Einstein field equations. The space-time metric provides different geometrical structures: conformal, volume, projective and linear connection. A deep understanding of them has consequences on the dynamical role played by geometry. We present a unified description of those geometrical structures, with a standard criterion of naturalness, and then we establish relationships among them and try to clarify the meaning of associated geometric magnitudes.

**Keywords:** Volume of space-time, linear connection, Lorentzian conformal structure, projective differential geometry.

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# INTRODUCTION

The space-time of general relativity (GR), from the point of view of differential geometry, is a 4-dimensional manifold M, with a  $C^{\infty}$  atlas  $\mathscr{A}$ . The atlas is the *differential structure* of our space-time.

The *principle of general covariance* of GR establishes the invariance by diffeomorphisms. This leads us to think that a *physical event* is not a point, but a geometrical structure on a neighborhood. The *fundamental geometrical structures* that we can consider defined in the space-time are:

- Volume (4-form)
- Conformal structure (Lorentzian)
- Metric (Lorentzian)
- Linear connection (symmetric)
- Projective structure

They are defined in terms of the *most primitive* differential structure, via the concept of *G*-structure. Volume, conformal structure and metric are *first order G-structures*. But linear connection and projective structure are *second order G-structures*.

For certain G's, *classified in* [8], every first order G-structure lead to a unique second order structure, named its *prolongation*. This is the case for the volume, metric and conformal structures.

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#### **FRAME BUNDLES**

The *r*-th order frame bundle  $\mathscr{F}^r M$  is a quotient space of a subset of  $\mathscr{A}$  [13, p. 38]. An *r*-frame,  $j^r \varphi \in \mathscr{F}^r M$  is an *r*-jet at 0, where  $x = \varphi^{-1}$  is a chart with 0 as a target.

We restrict our interest to first and second order. The first order frame bundle  $\mathscr{F}^1M$  is usually identified with the *linear frame bundle LM*. For a better understanding of the second order frame bundle consider *LLM*, the linear bundle of *LM*. There is a *canonical inclusion*  $\mathscr{F}^2M \hookrightarrow LLM$ ,  $j^2\varphi \mapsto j^1\tilde{\varphi}$ , where  $\tilde{\varphi}$  is the diffeomorphism induced by  $\varphi$ , between neighborhoods of  $0 \in \mathbb{R}^{n+n^2}$  and  $j^1\varphi \in LM$  [10, p. 139].

Let  $J^1LM$  be the bundle of 1-*jets of (local) sections* of LM and s be a section of LM. Each  $j_p^1 s$  is characterized by the *transversal n-subspace*  $H_l = s_*(T_pM) \subset T_lLM$  [6]. Then, there is also a *canonical inclusion*  $J^1LM \hookrightarrow LLM$ ,  $j_p^1 s \mapsto z$ , where z is the basis of  $T_lLM$ , whose first n vectors span  $H_l$  and correspond to the usual basis of  $\mathbb{R}^n$ , via the *canonical form of* LM, and the last  $n^2$  vectors are the fundamental vectors corresponding to the standard basis of  $\mathfrak{gl}(n, \mathbb{R})$  [9].

By the previous canonical maps, it happens that  $\mathscr{F}^2M$  is mapped one to one into the subset of  $J^1LM$ , corresponding with the *torsion-free transversal n-subspaces* in *TLM*.

**Theorem 1** We have the canonical embeddings: [13, p. 54]

$$\mathscr{F}^2M \hookrightarrow J^1LM \hookrightarrow LLM$$

The *r*-th order frame bundles are principal bundles. They are fundamental because every *natural bundle*, in the categorial approach, can be described as an associated bundle to some  $\mathcal{F}^r M$  [12] and the so-called *geometrical objects* can be identified with sections of those associated bundles.

#### STRUCTURAL GROUPS

The structural group of the principal bundle  $\mathscr{F}^r M$  is the group  $G_n^r$  of *r*-jets at 0 of diffeomorphisms of  $\mathbb{R}^n$ ,  $j_0^r \phi$ , with  $\phi(0) = 0$ .

The group  $G_n^1$  is identified with  $GL(n, \mathbf{R})$ . Then there is a canonical inclusion of  $G_n^1$  into  $G_n^r$ , if we take the *r*-jet at 0 of every linear map of  $\mathbf{R}^n$ . Furthermore,  $G_n^r$  is the *semidirect product* of  $G_n^1$  with a nilpotent normal subgroup [16]. Let us see this decomposition for  $G_n^2$ . We consider the underlying additive group of the vector space  $S_n^2$  of symmetric bilinear maps of  $\mathbf{R}^n \times \mathbf{R}^n$  into  $\mathbf{R}^n$ . Then there is a monomorphism  $\iota: S_n^2 \to G_n^2$  defined by  $\iota(s) = j_0^2 \phi$ , with  $s = (s_{jk}^i)$  and  $\phi(u^i) := (u^i + \frac{1}{2}s_{jk}^i u^j u^k)$ .

**Theorem 2** We obtain the split exact sequence of groups:

$$0 \rightarrow S_n^2 \xrightarrow{\iota} G_n^2 \underset{\scriptstyle \supset}{\rightleftharpoons} G_n^1 \rightarrow 1$$

It makes  $G_n^2$  isomorphic to the semidirect product  $G_n^1 \rtimes S_n^2$ , whose multiplication rule is  $(a,s)(b,t) := (ab, b^{-1}s(b,b) + t)$ . The isomorphism is given by  $j_0^2 \phi \mapsto (D\phi|_0, D\phi|_0^{-1}D^2\phi|_0)$ .

#### **G-STRUCTURES**

We define an *r*-th order *G*-structure on *M* as a reduction of  $\mathscr{F}^r M$  to a subgroup  $G \subset G_n^r$  [10]. This idea of geometrical structure on *M* concerns the classification of charts in  $\mathscr{A}$ , when the meaningful classes are chosen guided by an structural group.

We exemplify the concept of a *G*-structure studying a volume on a manifold, which rarely is treated this way [3]. Let us define a *volume* on *M* as a first order *G*-structure *V*, with  $G = SL_n^{\pm} := \{a \in GL(n, \mathbb{R}) : |\det a| = 1\}$ . For an orientable *M*, *V* has two components for two  $SL(n, \mathbb{R})$ -structures, for two equal, except sign, *volume n-forms*. For a general *M*, volume corresponds to *odd type n-form*, as in [4, pp. 21-27].

From *principal bundle theory* [9],  $SL_n^{\pm}$ -structures are the sections of the *bundle asso-ciated* with *LM* and the left action of  $G_n^1$  on  $G_n^1/SL_n^{\pm}$ . This is the *volume bundle*,  $\mathcal{V}M$ . Furthermore, the sections of  $\mathcal{V}M$  correspond to  $G_n^1$ -equivariant functions f of *LM* to  $G_n^1/SL_n^{\pm}$ . The equivariance condition is  $f(la) = |\det a|^{-1/n} I_n \cdot f(l), \forall a \in G_n^1$ .

We have the bijections:

Volumes on 
$$M \iff \operatorname{Sec} \mathscr{V}M \iff C^{\infty}_{\operatorname{eq}}(LM, \operatorname{G}^{1}_{n}/\operatorname{SL}^{\pm}_{n})$$

The isomorphisms  $G_n^1/SL_n^{\pm} \simeq H_n$ , with  $H_n := \{kI_n : k > 0\}$  and  $H_n \simeq \mathbb{R}^+$ , the multiplicative group of positive numbers, allow to represent a volume as an *(odd) scalar density* on *M*.

#### SECOND ORDER STRUCTURES

We can view a symmetric linear connection (SLC) on M as a  $G_n^1$ -structure of second order. A SLC is also the image of an *injective homomorphism* of LM to  $\mathscr{F}^2M$  [10].

From the *principal bundle theory* [9], SLC's on M are sections of the *SLC bundle*,  $\mathscr{D}M$ , associated with  $\mathscr{F}^2M$  and the action of  $G_n^2$  on  $G_n^2/G_n^1 \simeq S_n^2$ . Furthermore, each SLC,  $\nabla$ , corresponds to a  $G_n^2$ -equivariant function  $f^{\nabla} : \mathscr{F}^2M \to S_n^2$ , verifying  $f^{\nabla}(z(a,s)) = a^{-1}f^{\nabla}(z)(a,a) + s$ .

We have the bijections:

SLC's on 
$$M \quad \longleftrightarrow \quad \operatorname{Sec} \mathscr{D}M \quad \longleftrightarrow \quad C^{\infty}_{\operatorname{eq}}(\mathscr{F}^2M, \mathbf{S}^2_n)$$

Given two SLC's,  $\nabla$  and  $\widehat{\nabla}$ , the difference function  $f^{\nabla} - f^{\widehat{\nabla}} \colon \mathscr{F}^2 M \to S_n^2$  verifies  $z(a,s) \mapsto a^{-1}(f^{\nabla}(z) - f^{\widehat{\nabla}}(z))(a,a)$ . Then, it is projectable to a function  $f \colon LM \to S_n^2$  verifying  $f(la) = a^{-1}f(l)(a,a)$ , which corresponds to a tensor  $\rho = (\rho_{jk}^i)$  on M.

A projective structure (PS) is an equivalence class of SLC's which have the same family of pregeodesics. This is the cornerstone to understand the freely falling bodies in GR [5]. We can define a PS on M as a second order  $G_n^1 \rtimes \mathfrak{p}$ -structure, Q, with  $\mathfrak{p} := \{s \in S_n^2 : s_{jk}^i = \delta_j^i \mu_k + \mu_j \delta_k^i, \ \mu = (\mu_i) \in \mathbf{R}^{n*}\}.$ 

Now, for two SLC *included* in the same PS (i.e. literally  $\nabla, \widehat{\nabla} \subset Q$ ) the tensor  $\rho$ , expressing *their difference*, is determined by the contraction  $C(\rho) = (\rho_{si}^s)$ , which is *an 1-form* on *M*.

#### PROLONGATIONS

Let *B* be a *first order G-structure*. A connection in *B* is a distribution *H* of transversal *n*-subspaces,  $H_l \subset T_l B$ . If the subspaces are *free-torsion*, these determine a *second order G-structure*, whose  $G_n^1$ -extension [7, p. 206] is a SLC on *M*. Then, we say that *B admits a SLC*. Let us give two examples:

- A SLC and a parallel volume is an *equiaffine structure* on M [11]; hence, it is a second order  $SL_n^{\pm}$ -structure.
- A SLC compatible with a conformal structure is a *Weyl structure*; hence, it is a second order CO(*n*)-structure [2].

For a linear group *G*, let  $\mathfrak{g}$  denote the Lie algebra of *G*. The *first prolongation of*  $\mathfrak{g}$  is defined by  $\mathfrak{g}_1 := S_n^2 \cap L(\mathbb{R}^n, \mathfrak{g})$ . We obtain that  $G \rtimes \mathfrak{g}_1$  is a subgroup of  $G_n^1 \rtimes S_n^2$ , and hence, a subgroup of  $G_n^2$  (see more details in [1]).

**Theorem 3** Let  $B \subset LM$  be a *G*-structure, admitting a SLC. Then, the set of 2-frames, corresponding with torsion-free transversal n-subspaces which are included in TB, is a reduction of  $\mathscr{F}^2M$  to  $G \rtimes \mathfrak{g}_1$ . It is named the prolongation of *B* and denoted by  $B^2$  (for a proof, see [13, pp. 150-155]).

Let us give a well known example: if *B* is an O(n)-structure,  $B^2$  is isomorphic to *B* on account of  $\mathfrak{o}(n)_1 = \{0\}$ ; this explain the uniqueness of Levi-Civita connection.

There is an important theorem [8] *classifying the groups G* such that *every G-structure admits a SLC*: only the groups of *volume, metric and conformal* structures, and a class of groups preserving an 1-dimensional distribution, have this property.

### **CONCLUDING REMARKS**

We have done a unified description of the geometrical structures that have been used by GR to define intrinsic properties of the space-time. The unifying criterion, we used for it, not only is natural in the sense that geometric objects are sections of bundles associated with the  $\mathscr{F}^rM$  frame bundles [16], but also in the sense that the objects themselves are reductions of  $\mathscr{F}^rM$ . Therefore, we have not considered a linear connection with torsion because it is a section of an associated bundle of  $\mathscr{F}^2M$ , but not a reduction.

We have tried to clarify the relationships between the structures involved. Only simple relations, such as intersection, inclusion, reduction and extension, have been used for it, on account of the previous *prolongation* of *G*-structures admitting SLC. For instance, it follows readily from the last section that the classical *equiaffine* or *Weyl structures* can be defined as the intersection of a SLC with the prolongation of a volume or a conformal structure, respectively.

Recently, some of my research [14] have been taken into consideration for one of the lines of thought about quantum gravity [15]. This contribution is a set of my latest reflections and conclusions about geometrical structures with an eye on the applications to physics.

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