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Citation: Journal of Mathematical Physics **22**, 2598 (1981); doi: 10.1063/1.524838
View online: http://dx.doi.org/10.1063/1.524838
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/22/11?ver=pdfcov
Published by the AIP Publishing

Spacetime G structures and their prolongations[a)]

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(Received 30 December 1980; accepted for publication 11 May 1981)

An elementary procedure for obtaining the prolongations of arbitrary G structures is described and applied to the affine, projective, Lorentz, conformal, and Weyl structures. Also, the condition that a projective and conformal structure be compatible and thereby define a unique Weyl structure is discussed from the G-structure viewpoint. Throughout, an invariant notation for jets is employed.

PACS numbers: 04.20.Cv, 02.40. + m

1. INTRODUCTION

The geometric structures required for the formulation of the constructive axioms of general relativity theory proposed by Ehlers, *et al.*[1] are the affine, projective, Lorentz, conformal, and Weyl structures.[2,3] In this paper, these geometric[4] structures are all discussed in a uniform manner as group or G structures of first and higher order. This approach has a number of advantages. First, G structures are naturally described by gauge fields in a way that is described below in Sec. 2. Second, the theory of prolongations of G structures reveals how a G structure of a given order determines an infinite sequence of G structures of progressively higher order and leads to the notion of normal coordinates for each type of G structure. Also, the interrelationships of the geometric structures are more apparent because the groups which characterize the geometric structures play a more prominent role in the G-structure approach, a fact which also emphasizes the role of symmetry which is of central importance for spacetime theories.

The standard approach to the theory of G structures[5–7] and their prolongations evolved from the analysis of automorphism groups of G structures. Kobayashi[8] has given a quite readable account of this approach. A very elegant and quite thorough presentation of the standard theory of G structures has been given by P. Molino.[9]

The theory of G structures may also be subsumed under the broader theory of systems of partial differential equations and Lie pseudogroups developed in the work of Spencer,[10] Guillemin and Sternberg.[11] Important new contributions to this theory have been presented by J. F. Pommaret[12] in a recent book. Some of the G structures mentioned above are discussed therein from this more advanced and general but less intuitive perspective.[13]

The presentation given below focuses solely on the description of the above mentioned G structures and the process of prolongation of these G structures to higher order. Although the results obtained are the same as those obtained by the standard approach mentioned above, the conceptual basis for the prolongation procedure is quite different. Moreover, the prolongation procedure used below rests on *entirely elementary considerations* which have a *vividly intuitive interpretation*, and it is simple to apply. The procedure may be described using either frames or coframes, but it is more simply stated using coframes.

Briefly, if $h: U \to \mathbb{R}^n$ is a local diffeomorphism of an open neighborhood U of a manifold M into $\mathbb{R}^n$, then an n^k coframe at a point $p \in M$ is a k-jet $j_p^k[h - h(p)]$ for some h, where $h(p)$ is the constant function on U with image $h(p)$. Roughly, the chart (U,h) provides an image of the neighborhood U of M and the n^k coframe at p is the k th order Taylor polynomial approximation to $[h - h(p)]$ at p. It is clear that the Taylor polynomial approximation of order k at a point p determines at an infinitesimally near point q the Taylor polynomial approximation of order $k-1$ at q to first order in the coordinate difference $x^i(q) - x^i(p)$; that is, an n^k coframe at p determines an n^{k-1} coframe at q. Since a G structure of order $k-1$ specifies *at each* $p \in M$ an equivalence class of n^{k-1} coframes, it is clear that a G structure of order k must satisfy the constraint that it induce the same equivalence classes as the lower order G structure in order to be a prolongation. In fact, this simple constraint generates not only the prolonged G structure but also the prolonged gauge group which characterizes it. This prolongation procedure which rests on the notion of approximation to the manifold is discussed more fully in Sec. 2.

In Secs. 3–7, the affine, projective, Lorentz, conformal, and Weyl structures are treated. The sequence of presentation reflects the increase in algebraic complexity. The affine, projective, and conformal structures are prolonged to 3rd order, beyond which order no novel features appear for these G structures. For the Lorentz and Weyl structures, one may stop at 2nd order because these structures define a unique affine structure and further prolongations may be made by appeal to the results for the affine case.

In Sec. 8, the condition that a conformal and a projective structure be compatible and hence define a unique Weyl structure, a condition originally derived by Ehlers *et al.*[1] is discussed from the G structure viewpoint.

Although the prolongation procedure is applied to only five G structures in this paper, it may be applied with equal facility to any G structure such as symplectic, complex, Hermitian, and Kaehler structures. Also, one of the authors (Coleman) and R. Mann[14] have applied the procedure to subspace G structures of which the Galilean structure is a

a)Supported in part by the Natural Sciences and Engineering Research Council of Canada.

special case.

Indeed, the procedure is applicable in more general circumstances; for example, Rogers[15] has presented a view of supermanifolds in which k jets of mappings are well defined and the k jet of a composition is the composition of the k jets. Since the constructions needed rest on these properties, the theory of super coframe bundles, super G structures and their prolongations could be carried through in a manner directly analogous to that used for the theory of such structures on ordinary manifolds.

Throughout the paper, an invariant notation for jets is used which is explained in Appendix A. The germ of the idea behind the notation may be found in the work of Ehresmann.[16] The notation used for the jet bundles that are employed is described in Appendix B and the notation used for the various groups that appear is described in Appendix C.

2. G STRUCTURES AND PROLONGATIONS

Let $\mathscr{P}(M)$ be a principal fiber bundle with total space $P(M)$, base space M, projection $\pi_P: P(M) \to M$ and structure group G; that is, $\mathscr{P}(M)$ is the structure

$$\mathscr{P}(M) = \langle P(M), \pi_P, M, G\rangle. \tag{2.1}$$

Let H be a closed subgroup of G. An H structure on M is a reduction to the subgroup H of the structure group G of $\mathscr{P}(M)$. Such structures are in bijective correspondence with cross sections of the associated fiber bundle

$$\mathscr{P}(M)/H = \langle P(M)/H, \pi_{P/H}, M, G/H, \mathscr{P}(M)\rangle \tag{2.2}$$

of orbits under the action of H on elements of $P(M)$. In the original context, $\mathscr{P}(M)$ was the principal bundle of linear frames and the closed subgroup of the structure group $G^1_n(\equiv \mathrm{GL(n)})$ was denoted by G rather than H.

An H structure on M given by a cross section of the bundle (2.2) specifies an equivalence class of H related elements of $P(M_p)$ for every $p \in M$. Since equivalence classes are difficult to work with, the cross section $\sigma: M \to P(M)/H$ is represented by a family of local cross sections $\sigma_U: U \to P(U)$ such that

$$\forall\, p \in U, \quad \sigma_U(p) \in \sigma(p). \tag{2.3}$$

Moreover, if σ_{U_1} and σ_{U_2} are any two such local cross sections such that $U_1 \cap U_2 \neq \phi$, then there must exist a cross section $\rho_{12}: U_1 \cap U_2 \to (U_1 \cap U_2) \times H$ such that

$$\forall\, p \in U_1 \cap U_2, \quad \sigma_{U_2}(p) = \sigma_{U_1}(p) \circ \rho_{12}(p). \tag{2.4}$$

The transformation (2.4) is a local gauge transformation. Note that it is an active transformation and not a passive coordinate transformation. For a given computation, it is often convenient to work with a particular local cross section σ_U. Such a representative is usually selected by imposing a gauge-fixing condition which amounts to choosing a particular set of coordinates for the coset space G/H. Such conditions are coordinate dependent because the local trivializing maps depend on the chart (U,x). Suitable examples of such conditions will appear below in the analysis of the various G structures.

The relevant principal bundles for the analysis of spacetime G structures and their prolongations are the bundles of frames or coframes of order k. While the prolongation procedure used in this paper can be stated either in terms of frames or coframes, it is more natural to use coframes. The bundle of n^k coframes is denoted by

$$\mathscr{H}^{k*}(M) = \langle H^{k*}(M), \pi_{H^{k*}}, M, G^k_n\rangle. \tag{2.5}$$

Let $p \in U \subset M$ and let $h: U \to \mathbb{R}^n$ be a local diffeomorphism, and let $h(p)$, the image of p in $\mathbb{R}^n$, also denote the constant map $h(p): U \to \mathbb{R}^n$ such that $\forall q \in U\,[h(p)(q) = h(p)]$. A pair (U,h), called a chart, provides an image of a local neighborhood U in the standard space $\mathbb{R}^n$. If only one such image were allowed, the local region would have all the rigidity of $\mathbb{R}^n$; however, for a manifold with only a differentiable structure, all such images are allowed provided that they are smoothly related and compatible. An element of $H^{k*}(M_p)$, called an n^k coframe at p, is a k jet $j^k_p(h - h(p))$ which is the equivalence class of local diffeomorphisms which agree with $h - h(p)$ at p and have the same Taylor expansion of order k at p with respect to some arbitrarily chosen chart (U,x) for a neighborhood of p. Thus an n^k coframe is essentially the k th order polynomial approximation to a chart at a given point p.

A local n^k coframe field is given by a local cross section $h^k: U \to H^{k*}(U)$ which is represented with respect to a given chart (U,x) by

$$h^k(p) = \partial^k_0 I_n\big(h^i_j d^k_p x^j + (1/2!)\, h^i_{j_1 j_2} d^k_p x^{j_1} d^k_p x^{j_2} + \dots \\ \dots + (1/k!)\, h^i_{j_1 j_2 \dots j_k} d^k_p x^{j_1} d^k_p x^{j_2} \dots d^k_p x^{j_k}\big), \tag{2.6}$$

where the coefficients $(h^i_j, h^i_{j_1 j_2}, \dots, h^i_{j_1 j_2 \dots j_k})$ are functions of p and are symmetric with respect to their lower indices.

A field of n^1 coframes is a rather restrictive geometric structure because at each point p only a single linear image of the neighborhood of p is regarded as giving an appropriate representation. More flexible geometric structures may be defined by specifying at each p a class of distinguished n^1 coframes rather than a unique one; for example, for any closed *Lie* subgroup SG^1_n of G^1_n, a cross section of the associated fiber bundle $SG^1_n \backslash \mathscr{H}^{1*}(M)$ defines a field of equivalence classes of SG^1_n related n^1 coframes, an SG^1_n structure. If $SG^1_n = G^1_n$, then all linear images of the neighborhood of any point p are regarded as equivalent and the manifold M acquires none of the structural features of $\mathbb{R}^n$. At the other extreme, SG^1_n consists of the identity element alone and the manifold M acquires a great deal of the structure of $\mathbb{R}^n$. For other choices of SG^1_n, such as $O^1_{p,q}$ and $C^1_{p,q}$, the infinitesimal neighborhoods of points $p \in M$ acquire structural attributes from $\mathbb{R}^n$ that have the microsymmetry[17] of the corresponding group.

Structures of higher order are defined in a similar way. If SG^k_n is a closed Lie subgroup of G^k_n, then an SG^k_n structure on a manifold M is given by a cross section of the associated fiber bundle $SG^k_n \backslash \mathscr{H}^{k*}(M)$. Such a cross section specifies at each $p \in M$ an equivalence class of SG^k_n related n^k coframes. As in the first-order case, structure is pulled back from $\mathbb{R}^n$ by virtue of the fact that not every k th order approximate image of the infinitesimal neighborhood of a given point p is considered faithful. However, for $k > 1$, there is an additional self-consistency condition which the SG^k_n structure must satisfy, a condition which underlies the prolongation procedure used throughout the paper.

The self-consistency condition follows from the elementary observation that the k th order Taylor expansion of a function at some given point determines the $(k-1)$th order Taylor expansion of that function at an infinitesimally near point accurately to first order in the infinitesimal vector which describes the separation of the two points. Let $w^i = x^i(q) - x^i(p)$, where p and q are infinitesimally near points in M. Then

$$x^i - x^i(p) = x^i - x^i(q) + w^i. \tag{2.7}$$

An n^k coframe at p given by (2.6) determines an n^{k-1} coframe at q by means of the replacement

$$d_p^k x^i \rightarrow d_q^{k-1} x^i + w^i. \tag{2.8}$$

Note that the constant term is discarded because a coframe by definition has zero constant term. The result is

$$\begin{aligned}\partial_0^{k-1} I_n(&(h^i_j + h^i_{j\ell} w^\ell) d_q^{k-1} x^{j_1} \\ &+ (1/2!)(h^i_{j_1 j_2} + h^i_{j_1 j_2 \ell} w^\ell) d_q^{k-1} x^{j_1} d_q^{k-1} x^{j_2} + \cdots \\ &+ (1/(k-1)!)(h^i_{j_1 j_2 \cdots j_{k-1}} + h^i_{j_1 j_2 \cdots j_{k-1} \ell} w^\ell) \\ &\times d_q^{k-1} x^{j_1} d_q^{k-1} x^{j_2} \ldots d_q^{k-1} x^{j_{k-1}}.\end{aligned} \tag{2.9}$$

An n^k-coframe field determines an n^{k-1}-coframe field simply by the discarding of the k th order term. The n^k coframe at q determines an n^{k-1} coframe at q; namely,

$$\begin{aligned}\partial_0^{k-1} I_n(&h^i_j + h^i_{j,\ell} w^\ell d_q^{k-1} x^j \\ &+ (1/2!) h^i_{j_1 j_2} + h^i_{j_1 j_2, \ell} w^\ell) d_q^{k-1} x^{j_1} d_q^{k-1} x^{j_2} \\ &+ \cdots \\ &+ (1/(k-l)!)(h^i_{j_1 j_2 \cdots j_{k-1}} + h^i_{j_1 j_2 \cdots j_{k-1}, \ell} w^\ell) \\ &\times d_q^{k-1} x^{j_1} d_q^{k-1} x^{j_2} \ldots d_q^{k-1} x^{j_{k-1}}),\end{aligned} \tag{2.10}$$

where the coefficients describing the n^{k-1} coframe at q have been reexpressed in terms of the field coefficients and their derivatives at p means of a 1st order Taylor expansion. If the n^k coframe field is coherent, then each of the n^k coframes should be the k th order Taylor approximation at the appropriate point in M of one and the same chart. Thus self-consistency requires that the n^{k-1} coframes (2.9) and (2.10) be identical. However, there are two additional complications to consider.

First, one is usually not concerned with a simple n^k-coframe field, but rather with an SG_n^k structure which specifies at each $p \in M$ not a unique n^k coframe but an equivalence class of SG_n^k related coframes. Thus the self-consistency condition is weakened to the requirement that the n^{k-1}-coframes (2.9) and (2.10) are the same up to an SG_n^{k-1} transformation where SG_n^{k-1} is the group obtained by discarding the k th order terms of the elements of SG_n^k. This complication may be efficiently handled by transforming (2.9) and (2.10) to standard representatives before making the identification.

Second, it is clear that the coefficients of w^ℓ in (2.9) are necessarily symmetric with respect to all of their lower indices since they arise as the coordinates of a k jet. However, the corresponding coefficients of w^ℓ in (2.10) need not be totally symmetric in their lower indices since they arise by differentiation of the coefficient functions describing a $(k-1)$-jet field. In general, (2.9) and (2.10) differ by a term of the form $B^i_{j_1 j_2 \cdots j_r \ell} w^\ell$ for each $r \in \{1, 2 \ldots, k-1\}$, where $B^i_{j_1 j_2 \cdots j_r \ell}$ has the symmetry with respect to the lower indices obtained by first antisymmetrizing with respect to j_r and ℓ and then symmetrizing with respect to $j_1, j_2, \ldots j_r$.

The prolongation procedure may now be described as follows: assume that a self-consistent SG_n^{k-1} structure is given; for any given point $p \in M$, assume for an arbitrary n^{k-1}-coframe at p an extension to an n^k coframe; compute the n^{k-1}-coframes at q corresponding to (2.9) and (2.10) and reduce these to standard form using SG_n^{k-1} transformations; equate the results allowing for the terms of the form $B^i_{j_1 j_2 \cdots j_r \ell} w^\ell$. This procedure results in a system of linear equations for the k th order coefficient of the n^k coframe. These equations determine both the typical element of the prolonged group SG_n^k and the standard representative of the equivalence class of n^k coframes belonging to the SG_n^k structure at p with respect to the chosen coordinate system. It may happen that the system of equations is underdetermined in which case the group SG_n^k has new parameters in addition to those of SG_n^{k-1}. Such new parameters arise in the prolongation of the conformal or $C^1_{p,q}$ structure to a $C^2_{p,q}$ structure.

For every case, the process must begin at some lowest order. Not every case begins in the first order. The affine and projective structures begin at second order, and although the conformal and Weyl structures are indistinguishable in first order, they are distinguished at second order by the imposition of additional structure in the Weyl case. In all of the cases discussed below, no new group parameters arise beyond second order; however, in the case of subspace structures which will be discussed elsewhere,[14] new group parameters arise at each successive order.

We conclude this section with a formal characterization of the notion of prolongation. For any bundle $\mathscr{E}(M)$, denote by $J^r\mathscr{E}(M)$ the bundle of r jets of local cross sections of $\mathscr{E}(M)$. An SG_n^k structure is determined by a cross section $\sigma^k: M \rightarrow SG_n^k \backslash H^{k*}(M)$, which in turn may be described by a family of local cross sections $h^k: U \rightarrow H^{k*}(U)$ which satisfy $h^k(p) \in \sigma^k(p)$ for $p \in U$. Such a local cross section is given explicitly by (2.6). The natural projection maps $G_n^k \rightarrow G_n^{k-1}$ and $H^{k*}(M) \rightarrow H^{k-1*}(M)$ defined by the discarding of the k th order term induce projection maps $SG_n^k \rightarrow SG_n^{k-1}$ and $SG_n^k \backslash H^{k*}(M) \rightarrow SG_n^{k-1} \backslash H^{k-1*}(M)$. Thus the compositions of the maps σ^k and h^k with the appropriate projection maps yield maps $\sigma^{k-1}: M \rightarrow SG_n^{k-1} \backslash H^{k-1*}(M)$ and $h^{k-1}: U \rightarrow H^{k-1*}(U)$. In this way, every SG_n^k structure uniquely determines an SG_n^{k-1} structure by truncation.

In an infinitesimal neighborhood of $p \in U \subset M$, h^{k-1} is determined to first order by $j_p^1 h^{k-1} \in J^1 H^{k-1*}(M_p)$. Explicitly,

$$\begin{aligned} j_p^1 h_{k-1} = \partial_0^{k-1} I_n(&(h^i_j + h^i_{j,\ell} d_p^1 x^\ell) j_p^1 d^{k-1} x^j + \ldots \\ &+ \frac{1}{(k-1)!} (h^i_{j_1 \cdots j_{k-1}} + h^i_{j_1 \cdots j_{k-1}, \ell} d_p^1 x^\ell) \\ &\times j_p^1 d^{k-1} x^{j_1} \ldots j_p^1 d^{k-1} x^{j_{k-1}}).\end{aligned} \tag{2.11}$$

The expression (2.10) is obtained by composing the 1-jet aspect of (2.11) with the 1-jet $\partial_p^1 x(w^i d_0^1 I)$.

Define a mapping $e: H^{k*}(M) \rightarrow J^1 H^{k-1*}(M)$ by means of the substitution

$$d_p^k x \to j_p^1 d^{k-1} x + d_p^1 x \tag{2.12}$$

with the understanding that the constant term of the (k − 1) jet is discarded. One obtains

$$\begin{aligned} e[h^k(p)] = \partial_0^{k-1} I_n((h^i_j + h^i_{j\ell} d_p^1 x^\ell) j_p^1 d^{k-1} x^j + \dots \\ + (1/(k-1)!)(h^i_{j_1 \cdots j_{k-1}} \\ + h^i_{j_1 \cdots j_{k-1}\ell} d_p^1 x^\ell) j_p^1 d^{k-1} x^{j_1} \dots j_p^1 d^{k-1} x^{j_{k-1}}). \end{aligned} \tag{2.13}$$

The expression (2.9) is related to (2.13) in the same way that (2.10) is related to (2.11).

Both $j_p^1 h^{k-1}$ and $e(h^k(p))$ may be regarded as generalized elements of $H^{k-1*}(M)$ with 1-jet coefficients. The inverse $e(h^k(p))^{-1}$ is a generalized element of $H^{k-1}(M)$. The compostion $j_p^1 h^{k-1} \circ e(h^k(p))^{-1}$ [regarded as generalized $(k-1)$ jets] has the form

$$\begin{aligned} \partial_0^{k-1} I_n [(\delta^i_j + E^i_{j\ell} d_p^1 x^\ell) d_0^{k-1} I^j_n + \dots \\ + \frac{1}{(k-1)!} E^i_{j_1 \cdots j_{k-1}\ell} d_p^1 x^\ell d_0^{k-1} I^{j_1}_n \dots d_0^{k-1} I^{j_{k-1}}_n]. \end{aligned} \tag{2.14}$$

This is an infinitesimal element of G_n^{k-1}, where the sense of infinitesimal is made precise by the use of the 1 jet $d_p^l x$.

The concept of a self-consistent SG_n^k structure may now be formulated as follows.

Definition: An SG_n^k structure is *self-consistent* iff $j_p^1 h^{k-1} \circ e(h^k(p))^{-1}$ is the composition of an infinitesimal element of SG_n^{k-1} and an infinitesimal element of G_n^{k-1} of the form (2.14) with E replaced by $\hat{B}$ such that the $\hat{B}^i_{j_1 \cdots j_r q} h^{-1q}_{\ell}$ have the same type of symmetry with respect to the lower indices as $B^i_{j_1 \cdots j_r \ell}$ discussed above.

Finally, the definition of "prolongation" may be stated as follows.

Definition: An SG_n^k structure is the *prolongation* of an SG_n^{k-1} structure iff the SG_n^{k-1} structure is determined by the SG_n^k structure by truncation and the SG_n^k structure is self-consistent.

Note that the self-consistency of the SG_n^k structure entails the self-consistency of the induced SG_n^{k-1} structure.

3. THE AFFINE STRUCTURE

It is customary to describe the affine structure of space-time as a connection on the principal bundle of linear frames or n[1] frames $\mathscr{H}^1(M)$. In this section, however, the affine structure will be presented as a G structure defined by a reduction to the affine subgroup Γ_n^2 of the structure group G_n^2 of the principal bundle of n^2 coframes

$$\mathscr{H}^{2*}(M) = \langle H^{2*}(M), \pi_{H^{2*}}, M, G_n^2 \rangle. \tag{3.1}$$

This approach has the advantages that other related structures can be described in a similar way, that the structure is described explicitly by local Γ_n^2 gauge fields, and that the concept of approximating the manifold by second-order polynomials is emphasized. Moreover, the relationship of the affine structure to affine curves and parallel transport and to the higher-order prolongations of these structures is quite direct and intuitive from the viewpoint of higher-order approximation of the manifold.

The group Γ_n^2 is the subgroup of G_n^2 of elements of the form

$$\partial_0^2 I_n (a^i_j d_0^2 I^j_n). \tag{3.2}$$

The restriction of the left, free action of G_n^2 on $H^{2*}(M)$ to the subgroup Γ_n^2 may be used to construct the associated bundle of equivalence classes of Γ_n^2 related n^2 coframes

$$\begin{aligned} \Gamma_n^2 \backslash \mathscr{H}^{2*}(M) \\ = \langle \Gamma_n^2 \backslash H^{2*}(M), \pi_{\Gamma_n^2 \backslash H^{2*}}, M, \Gamma_n^2 \backslash G_n^2, \mathscr{H}^{2*}(M) \rangle, \end{aligned} \tag{3.3}$$

where the typical fiber $\Gamma_n^2 \backslash G_n^2$ is the set of left cosets. The affine structure Γ is a cross section $\Gamma: M \to \Gamma_n^2 \backslash H^{2*}(M)$. Since for each $p \in M$, $\Gamma(p)$ is an equivalence class of n^2 coframes, the affine structure may be represented locally by cross sections $h: U \to H^{2*}(U)$,

$$h(p) = \partial_0^2 I_n (h^i_j d_p^2 x^j + (1/2!) \mathrm{h}^i_{jk} d_p^2 x^j d_p^2 x^k), \tag{3.4}$$

which are determined up to a Γ_n^2 left-acting gauge transformation

$$\begin{aligned} &L: U \to U \times \Gamma_n^2, \\ &L(p) = \partial_0^2 I_n (a^i_j d_0^2 I^j_n). \end{aligned} \tag{3.5}$$

Thus under a gauge transformation,

$$\begin{aligned} \tilde{h}(p) = L(p) \circ h(p) \\ \times \partial_0^2 I_n (a^i_\ell h^\ell_j d_p^2 x^j + \frac{1}{2!} a^i_\ell h^\ell_{jk} d_p^2 x^j d_p^2 x^k). \end{aligned} \tag{3.6}$$

Whenever desired, the gauge may be chosen by the coordinate dependent gauge fixing condition

$$a^i_\ell h^\ell_j = \delta^i_j. \tag{3.7}$$

Then, the only nontrivial coefficient in (3.6) is

$$\Gamma^i_{jk} = h_r^{-1i} h^r_{jk}. \tag{3.8}$$

The bundle of second order curve elements is

$$\mathscr{L}_1^2(M) = \langle L_1^2(M), \pi_{L_1^2}, M, L^2_{1,n}, \mathscr{H}^2(M) \rangle. \tag{3.9}$$

If $\gamma: \mathbb{R} \to M$ and $\gamma(0) = p$, then an element $j_0^2 \gamma \in L_1^2(M_p)$ is given by

$$j_0^2 \gamma = \partial_p^2 x(\gamma_1^i d_0^2 I + (1/2!) \gamma_2^i (d_0^2 I)^2). \tag{3.10}$$

The n^2 coframe (3.4) may be used to give an image of this curve element in $\mathbb{R}^n$.

$$\begin{aligned} h(p) \circ j_0^2 \gamma = \partial_0^2 I_n [h^i_j \gamma^j_1 d_0^2 I \\ + (1/2!) (h^i_r \gamma^r_2 + h^i_{rs} \gamma^r_1 \gamma^s_1)(d_0^2 I)^2]. \end{aligned} \tag{3.11}$$

Thus the image line element is linear to 2nd order iff

$$\gamma^i_2 + \Gamma^i_{jk} \gamma^j_1 \gamma^k_1 = 0, \tag{3.12}$$

where (3.8) has been used.

The idea that n^k coframes are k th order Taylor approximations to local diffeomorphisms can be used to explicate the relationship of the affine structure to the affine connection and the prolongation of these structures to higher order. Let $f: U \to V \subset \mathbb{R}^n$ be a local diffeomorphism such that $f(p) = \mathbf{0}$. Let q be infinitesimally near p and set $\hat{f} = f - f(q)$. Then $j_q^{k-1} \hat{f}$ is determined by $j_p^k f$ to first order in the coordinate difference $w = x(q) - x(p)$, where (U,x) is a chart. Set $F = f \circ x^{-1}$ and $\hat{F} = \hat{f} \circ x^{-1}$. Then to third order

$$F^i(x) = F^i_j x^j_p + \frac{1}{2!} F^i_{jk} x^j_p x^k_p + \frac{1}{3!} F^i_{jk\ell} x^j_p x^k_p x^\ell_p, \tag{3.13}$$

where $x_p^i = x^i - x^i(p)$. Since

$$x_p^i = x_q^i + w^i \quad \text{and} \quad f^i(q) = F^i_j w^j,$$

$$\begin{aligned}\widehat{F}^i(x) &= F^i(x) - F^i_j w^j \\ &= F^i_j x_q^j + \frac{1}{2!} F^i_{jk} x_q^j x_q^k + \frac{1}{3!} F^i_{jk\ell} x_q^j x_q^k x_q^\ell + F^i_{js} x_q^j w^s \\ &\quad + \frac{1}{2!} F^i_{jks} x_q^j x_q^k w^s,\end{aligned} \tag{3.14}$$

where only terms up to first order in w^i have been kept. An affine structure defines an equivalence class of second-order coframes; namely,

$$\partial_0^2 I_n (a^i_j d^2_p x^j + \frac{1}{2!} a^i_\ell \Gamma^\ell_{jk} d^2_p x^j d^2_p x^k) \tag{3.15}$$

for $(a^i_j) \in G^1_n$. Each such second-order coframe at p determines a first-order coframe at p

$$\partial^1_0 I_n (a^i_j d^1_p x^j) \tag{3.16}$$

by projection and a first-order coframe at an infinitesimally near point q

$$\partial^1_0 I_n ((a^i_j + a^i_\ell \Gamma^\ell_{js} w^s) d^1_q x^j) \tag{3.17}$$

by the substitution (3.14). The *parallel transport* of coframes is defined by mapping (3.16) into (3.17).

Now consider the problem of prolonging the affine structure to the next order. An extension of an arbitrary n^2 coframe (3.15) to a n^3 coframe has the form

$$\begin{aligned}\partial^3_0 I_n \Big[& a^i_j d^3_p x^j + \frac{1}{2!} a^i_s \Gamma^s_{jk} d^3_p x^j d^3_p x^k \\ & + \frac{1}{3!} h^i_{jk\ell} d^3_p x^j d^3_p x^k d^3_p x^\ell \Big],\end{aligned} \tag{3.18}$$

which by the substitution (3.14) defines a n^2-coframe equivalence class at q near p which contains

$$\begin{aligned}\partial^2_0 \mathrm{I}_n (&(a^i_j + a^i_s \Gamma^s_{jr} w^r) d^2_q x^j \\ & + \frac{1}{2!} (a^i_s \Gamma^s_{jk} + h^i_{jks} w^s) d^2_q x^j d^2_q x^k).\end{aligned} \tag{3.19}$$

However, the affine structure also defines another equivalence class of n^2 coframes at q, namely, the equivalence class containing

$$\partial^2_0 I_n (a^i_j d^2_q x^j + \tfrac{1}{2} a^i_s (\Gamma^s_{jk} + \Gamma^s_{jk,r} w^r) d^2_q x^j d^2_q x^k). \tag{3.20}$$

It is natural to require that the two equivalence classes defined by (3.19) and (3.20) be the same and to impose this condition by reducing each of the n^2 coframes to standard form and equating the results; however, it is necessary to allow for a term of the form $B^i_{jkr} w^r$, where B^i_{jkr} has the symmetry in the lower indices obtained by first antisymmetrizing in the indices k and r and then symmetrizing in the indices j and k. Such a term arises because the coefficients describing the 1 jet of the local 2-jet field need not be symmetric in the lower indices. Similar terms arise in higher orders. One obtains

$$\begin{aligned}\Gamma^i_{jk} &+ \Gamma^i_{jk,r} w^r + B^i_{jkr} w^r \\ &= \Gamma^i_{jk} + a^{-1}{}^i_s h^s_{jkr} w^r - \Gamma^i_{sr} \Gamma^s_{jk} w^r.\end{aligned} \tag{3.21}$$

Set $\Gamma^i_{jk\ell} = a^{-1}{}^i_s h^s_{jk\ell}$. Then, the relation (3.21) gives, after appropriate symmetrization, the result

$$\begin{aligned}\Gamma^i_{jk\ell} = \tfrac{1}{3}(&\Gamma^i_{jk,\ell} + \Gamma^i_{k\ell,j} + \Gamma^i_{\ell j,k} \\ & + \Gamma^i_{sj} \Gamma^s_{k\ell} + \Gamma^i_{sk} \Gamma^s_{\ell j} + \Gamma^i_{s\ell} \Gamma^s_{jk}),\end{aligned} \tag{3.22}$$

which upon substitution into (3.18) gives the affine frame of third order corresponding to the affine frame of second order (3.15). Clearly, this procedure could be iterated order-by-order to define a reduction of the structure group G^k_n of $\mathscr{H}^{k*}(M)$ to the affine subgroup Γ^k_n. Moreover, such a reduction defines a connection on the principal bundle $\mathscr{H}^{k-1*}(M)$; for example, suppose affine n^3 coframes have been defined by (3.18) with $h^i_{jk\ell} = a^i_s \Gamma^s_{jk\ell}$, where $\Gamma^i_{jk\ell}$ is given by (3.22). Then by projection one obtains the frames (3.15) and by substitution the frames (3.19). Taking the composition of (3.19) with the inverse of (3.15) [applied on the left of (3.19)], one obtains an invertible map from $H^{2*}(M_p)$ to $H^{2*}(M_q)$; namely,

$$\begin{aligned}\partial^2_p x \Big[& (\delta^i_j + \Gamma^i_{jr} w^r) d^2_q x^j + \frac{1}{2!} (\Gamma^i_{jkr} w^r \\ & - \Gamma^i_{sj} \Gamma^s_{kr} w^r - \Gamma^i_{sk} \Gamma^s_{jr} w^r) d^2_q x^j d^2_q x^k \Big].\end{aligned} \tag{3.23}$$

To parallel transport the second-order curve element $j^2_0 \gamma$ given by (3.10) from p to q, the inverse of (3.23) is applied on the left of $j^2_0 \gamma$. Also, note that the image of the third order curve element

$$j^3_0 \gamma = \partial^3_p x \Big[\gamma^i_1 d^3_0 I + \frac{1}{2!} \gamma^i_2 (d^3_0 I)^2 + \frac{1}{3!} \gamma^i_3 (d^3_0 I)^3 \Big] \tag{3.24}$$

under an affine n^3 coframe is

$$\begin{aligned}\partial^3_0 I_n \Big[& a^i_s \gamma^s_1 d^3_0 I + \frac{1}{2!} (a^i_s \gamma^s_2 + a^i_s \Gamma^s_{jk} \gamma^j_1 \gamma^k_1)(d^3_0 I)^2 \\ & + \frac{1}{3!} (a^i_s \gamma^s_3 + 3 a^i_s \Gamma^s_{jk} \gamma^j_2 \gamma^k_1 + a^i_s \Gamma^s_{jk\ell} \gamma^j_1 \gamma^k_1 \gamma^\ell_1)(d^3_0 I)^3 \Big].\end{aligned} \tag{3.25}$$

This image curve is linear to third order provided that (3.12) holds and

$$\gamma^i_3 + 3 \Gamma^i_{jk} \gamma^j_2 \gamma^k_1 + \Gamma^i_{jk\ell} \gamma^j_1 \gamma^k_1 \gamma^\ell_1 = 0 \tag{3.26}$$

holds. The result agrees with the usual result obtained by differentiating the equation for affine curves. In addition, the prolongation to order k gives the k jet of the coordinate transformation that relates the given coordinate system to the normal coordinate system for a given point p. In the normal coordinate system all of the functions $\Gamma^i_{jk}, \Gamma^i_{jk\ell}, \ldots$ vanish at p.

4. THE PROJECTIVE STRUCTURE

The projective structure of space-time is a second-order G structure defined by a reduction of the group G^2_n of $\mathscr{H}^{2*}(M)$ to the subgroup P^2_n consisting of elements of the form

$$\partial^2_0 I_n (a^i_j d^2_0 I^j + \frac{1}{2!} (a^i_j a_k + a^i_k a_j) d^2_0 I^j_n d^2_0 I^k_n). \tag{4.1}$$

As in the affine case, the restriction of the action of G^2_n on $H^{2*}(M)$ to the subgroup P^2_n may be used to construct the associated bundle of equivalence classes of P^2_n related n^2 coframes

$P^2_n \backslash \mathscr{H}^{2*}(M)$

$$= \langle P_n^2 \backslash H^{2*}(M), \pi_{P_n^2\backslash H^{2*}}, M, P_n^2 \backslash G_n^2, \mathscr{H}^{2*}(M) \rangle. \tag{4.2}$$

A projective structure on space time is a cross section $\Pi{:}M \to P_n^2 \backslash H^{2*}(M)$. Such a cross section may be represented locally by a family of cross sections $h{:}U \to H^{2*}(U)$ [see (3.4)], which are determined up to a P_n^2 gauge transformation $L{:}U \to U \times P_n^2$, where $L(p)$ is given by (4.1) and where the a_j^i and a_k depend smoothly on $p \in U$. The cross sections h satisfy

$$\forall\, p \in U, \;\; h(p) \in \Pi(p) \tag{4.3}$$

and any two such are related by a gauge transformation of the form

$$\begin{aligned} \bar{h}(p) &= L(p) \cdot h(p) \\ &= \partial_0^2 I_n \big[(a_r^i h_j^r d_p^2 x^j \\ &\quad + \frac{1}{2!}(a_r^i h_{jk}^r + a_r^i h_j^r a_m h_k^m + a_r^i h_k^r a_m h_j^m) d_p^2 x^j \\ &\quad \times d_p^2 x^k \big]. \end{aligned} \tag{4.4}$$

The gauge fixing condition (3.7) may also be used in this case. Then the coefficient for the second-degree term in (4.4) becomes

$$h^{-1i}{}_r h_{jk}^r + \delta_j^i a_m h_k^m + \delta_k^i a_m h_j^m. \tag{4.5}$$

Since a_m is still undetermined, the additional gauge-fixing condition

$$h^{-1i}{}_r h_{ik}^r + (n+1) a_m h_k^m = 0 \tag{4.6}$$

may be imposed. The coefficient of the second-degree term is then

$$\Pi_{jk}^i = h^{-1i}{}_r h_{jk}^r - \frac{1}{n+1}(\delta_j^i h^{-1m}{}_r h_{mk}^r + \delta_k^i h^{-1m}{}_r h_{mj}^r). \tag{4.7}$$

The Π_{jk}^i which satisfy $\Pi_{ij}^i = 0$ are called the *projective coefficients* and relative to a given coordinate system uniquely determine and are uniquely determined by the projective structure. Clearly, the process of choosing a particular gauge is just the selection of standard representatives for the cosets $P_n^2 \backslash G_n^2$. For a n^2 coframe $h(p) \in \Pi(p)$,

$$h(p) = \partial_0^2 I_n (a_j^i d_p^2 x^j + \frac{1}{2!}(a_r^i \Pi_{jk}^r + a_j^i a_k + a_k^i a_j) d_p^2 x^j d_p^2 x^k), \tag{4.8}$$

where $a_j^i = h_j^i$ and $a_k = \dfrac{1}{n+1} h^{-1m}{}_r h_{mk}^r$.

The n^2 coframe (4.8) may be composed with the second-order curve element (3.10) to give an image of the curve element in $\mathbb{R}^n$.

$$\begin{aligned} h(p) \circ j_0^2 \gamma &= \partial_0^2 I_n (a_j^i \gamma_1^j d_0^2 I \\ &\quad + \frac{1}{2!}(a_r^i \gamma_2^r + a_r^i \Pi_{jk}^r \gamma_1^j \gamma_1^k + 2 a_j^i \gamma_1^j a_k \gamma_1^k)(d_0^2 I)^2). \end{aligned} \tag{4.9}$$

The condition that this image curve element is linear to second order is just

$$\gamma_2^i + \Pi_{jk}^i \gamma_1^j \gamma_1^k + 2 a_k \gamma_1^k \gamma_1^i = 0. \tag{4.10}$$

Since a_k is arbitrary at every p, the factor $a_k \gamma_1^k$ may be chosen freely in a smooth way along any integral curve. It is readily shown that this freedom allows an arbitrary choice of parameter for the integral curves; consequently, (4.10) describes a path element rather than a curve element.

The projective structure determines only a family of connections on the manifold. By projection and substitution, the coframe (4.8) determines the two coframes

$$\partial_0^1 I_n (a_j^i d_p^1 x^j)$$

and

$$\partial_0^1 I_n ((a_j^i + a_r^i \Pi_{js}^r w^s + a_s^i w^s a_j + a_j^i a_s w^s) d_q^1 x^j). \tag{4.11}$$

The map from $H^{1*}(M_p)$ to $H^{1*}(M_q)$ is then

$$\partial_p^1 x ((\delta_j^i + \Pi_{js}^i w^s + w^i a_j + \delta_j^i a_s w^s) d_q^1 x^j) \tag{4.12}$$

which depends on the parameters a_i. Consider a 1^1 frame $j_0^1 \gamma \in H_1^1(M_p)$,

$$j_0^1 \gamma = \partial_p^1 x (\gamma_1^i d_0^1 I). \tag{4.13}$$

Under parallel transport by (4.12) for any a_i, the 1^1 frame becomes

$$\partial_q^1 x ((\gamma_1^i - \Pi_{jk}^i w^k \gamma_1^j - w^i a_j \gamma_1^j - a_j w^j \gamma_1^i) d_0^1 I). \tag{4.14}$$

For autoparallel transport, $w^i = \lambda \gamma_1^i$ and (4.14) becomes

$$\partial_q^1 x ((\gamma_1^i - \lambda \Pi_{jk}^i \gamma_1^j \gamma_1^k - 2\lambda a_j \gamma_1^j \gamma_1^i) d_0^1 I). \tag{4.15}$$

It is clear that the autoparallel transport of projective 1^1 frames (directions) is uniquely defined by the projective structure because the term (4.15) which depends on a_i is proportional to γ_1^i and consequently may be removed by a parameter, G_1^1, transformation.

The above discussion may be readily extended to 2^1 frames and projective 2^1 frames (a set of all 2^1 frames for a given two-dimensional subspace of the tangent space). Let $j_0^1 f \in H_2^1(M_p)$, where $f{:}\mathbb{R}^2 \to M$ such that $f(0) = p$. Then

$$j_0^1 f = \partial_p^1 x (f_\alpha^i d_0^1 I_2^\alpha). \tag{4.16}$$

A projective 2^1 frame is an equivalence class of 2^1-frames under the action of the group G_2^2 of parameter transformations. Now, choose any vector which belongs to the subspace of $L_1^1(M_p)$ corresponding to a given projective 2^1 frame, say $w^i = \lambda_1 f_1^i + \lambda_2 f_2^i$. Next, parallel transport (4.16) along w^i using (4.12). The result is

$$\begin{aligned} \partial_q^1 x \big[\{ & f_\alpha^i - \Pi_{jk}^i f_\alpha^i (\lambda_1 f_1^k + \lambda_2 f_2^k) \\ & - (\lambda_1 f_1^i + \lambda_2 f_2^i) a_j f_\alpha^j + a_j (\lambda_1 f_1^j + \lambda_2 f_2^j) f^i \} d_0^1 I_2^\alpha \big]. \end{aligned} \tag{4.17}$$

Again, the terms which depend on the a_i may be removed by a G_2^2 parameter transformation; consequently, a projective structure determines the parallel transport of projective 2^1 frames. Also, the autoparallel transport of the initial direction lies in the subspace determined by the new projective 2^1 frame; therefore, the procedure may be iterated to define a parallel field of tangential projective 2^1 frames along a geodesic. This procedure corresponds to the *strip-forming* construction discussed by Ehlers and Schild.[18] It would probably be useful to reformulate the rest of their results in the language of jets.

The problem of determining the prolongations of the projective structure can be solved by the same procedure used in the affine case; however, the algebra is slightly more complicated. Consider an extension of the projective n^2 coframe (4.8),

$$\partial_0^3 I_n\Big[a^i_j d^3_p x^j + \frac{1}{2!}(a^i_s \Pi^s_{jk} + a^i_j a_k + a^i_k a_j) d^3_p x^j d^3_p x^k + \frac{1}{3!} a^i_s \Gamma^s_{jk\ell} d^3_p x^j d^3_p x^k d^3_p x^\ell\Big]. \tag{4.18}$$

Such a frame defines an equivalence class of n^2 coframes at q near p containing

$$\partial_0^2 I_n\Big[(a^i_j + a^i_s \Pi^s_{jr} w^r + a^i_r a_j w^r + a^i_j a_r w^r) d^2_q x^j + \frac{1}{2!}(a^i_s \Pi^s_{jk} + a^i_j a_k + a^i_k a_j + a^i_s \Gamma^s_{jk\ell} w^\ell) d^2_q x^j d^2_q x^k\Big]. \tag{4.19}$$

This n^2 coframe should define the same P^2_n equivalence class at q as the n^2 coframe

$$\partial_0^2 I_n\Big[a^i_j d^2_q x^j + \frac{1}{2!}(a^i_s \Pi^s_{jk} + a^i_s \Pi^s_{jk,r} w^r + a^i_j a_k + a^i_k a_j) d^2_q x^j d^2_q x^k\Big]. \tag{4.20}$$

The standard representative corresponding to the n^2 coframes given by (4.20) is

$$\partial_0^2 I_n\Big(d^2_q x^i + \frac{1}{2!}(\Pi^i_{jk} + \Pi^i_{jk,\ell} w^\ell) d^2_q x^j d^2_q x^k\Big). \tag{4.21}$$

The reduction of (4.19) to standard form is somewhat more complicated. If the first-order coefficient is reduced to δ^i_j, the second-order coefficient becomes

$$\Pi^i_{jk} + \Gamma^i_{jk\ell} w^\ell - \Pi^i_{r\ell} \Pi^r_{jk} w^\ell - w^i a_r \Pi^r_{jk} - a_\ell w^\ell \Pi^i_{jk} - a_k \Pi^i_{j\ell} w^\ell - a_j \Pi^i_{k\ell} w^\ell - 2 w^i a_j a_k. \tag{4.22}$$

The next step is to remove the trace from the expression (4.22). The resulting expression is equated with the coefficient of the second-order part of (4.21) allowing for a term $B^i_{jk\ell}$ with the same symmetries as in the affine case and also traceless, $B^r_{rk\ell} = 0$. The compatibility condition so obtained is

$$\begin{aligned}\Gamma^i_{jk\ell} - \frac{1}{n+1}(\delta^i_j \Gamma^r_{rk\ell} + \delta^i_k \Gamma^r_{rj\ell}) &= B^i_{jk\ell} + \Pi^i_{jk,\ell} + \Pi^i_{r\ell} \Pi^r_{jk} \\ &\quad - \frac{1}{n+1}(\delta^i_j \Pi^r_{sk} \Pi^s_{r\ell} + \delta^i_k \Pi^r_{sj} \Pi^s_{r\ell}) \\ &\quad + \delta^i_\ell a_r \Pi^r_{jk} - \frac{2}{n+1}(\delta^i_j a_r \Pi^r_{k\ell} + \delta^i_k a_r \Pi^r_{j\ell}) \\ &\quad + a_j \Pi^i_{k\ell} + a_k \Pi^i_{\ell j} + a_\ell \Pi^i_{jk} \\ &\quad + 2\delta^i_\ell a_j a_k - \frac{2}{n+1}(\delta^i_j a_\ell a_k + \delta^i_k a_\ell a_j).\end{aligned} \tag{4.23}$$

Take the totally symmetric part of this equation. The term $B^i_{jk\ell}$ drops out and one obtains

$$\begin{aligned}\Gamma^i_{jk\ell} - \frac{2}{3(n+1)}(\delta^i_j \Gamma^r_{rk\ell} + \delta^i_k \Gamma^r_{r\ell j} + \delta^i_\ell \Gamma^r_{rjk}) &= \frac{1}{3}(\Pi^i_{jk,\ell} + \Pi^i_{k\ell,j} + \Pi^i_{\ell j,k} + \Pi^i_{rj} \Pi^r_{k\ell} \\ &\quad + \Pi^i_{rk} \Pi^r_{\ell j} + \Pi^i_{r\ell} \Pi^r_{jk}) \\ &\quad - \frac{2}{3(n+1)}(\delta^i_j \Pi^r_{sk} \Pi^s_{r\ell} + \delta^i_k \Pi^r_{s\ell} \Pi^s_{rj} + \delta^i_\ell \Pi^r_{sj} \Pi^s_{rk}) \\ &\quad + \frac{n-3}{2(n+1)}(\delta^i_j a_r \Pi^r_{k\ell} + \delta^i_k a_r \Pi^r_{\ell j} + \delta^i_\ell a_r \Pi^r_{jk}) \\ &\quad + a_j \Pi^i_{k\ell} + a_k \Pi^i_{\ell j} + a_\ell \Pi^i_{jk} \\ &\quad + \frac{2}{3}\frac{(n-1)}{(n+1)}(\delta^i_j a_k a_\ell + \delta^i_k a_\ell a_j + \delta^i_\ell a_j a_k).\end{aligned} \tag{4.24}$$

The trace of this equation gives

$$\begin{aligned}\frac{1}{3}\frac{(n-1)}{(n+1)} \Gamma^r_{rk\ell} &= \frac{1}{3} \Pi^r_{k\ell,r} - \frac{2}{3}\frac{1}{(n+1)} \Pi^r_{sk} \Pi^s_{r\ell} \\ &\quad + \frac{(n-1)(n+3)}{3(n+1)} a_r \Pi^r_{k\ell} \\ &\quad + \frac{2}{3}\frac{(n-1)(n+2)}{(n+1)} a_k a_\ell.\end{aligned} \tag{4.25}$$

Then (4.24) gives the final result

$$\begin{aligned}\Gamma^i_{jk\ell} &= \frac{1}{3}(\Pi^i_{jk,\ell} + \Pi^i_{k\ell,j} + \Pi^i_{\ell j,k}) \\ &\quad + \frac{2}{3(n-1)}(\delta^i_j \Pi^r_{k\ell,r} + \delta^i_k \Pi^r_{\ell j,r} + \delta^i_\ell \Pi^r_{jk,r}) \\ &\quad + \frac{1}{3}(\Pi^i_{rj} \Pi^r_{k\ell} + \Pi^i_{rk} \Pi^r_{\ell j} + \Pi^i_{r\ell} \Pi^r_{jk}) \\ &\quad - \frac{2}{3(n-1)}(\delta^i_j \Pi^r_{sk} \Pi^s_{r\ell} + \delta^i_k \Pi^r_{s\ell} \Pi^s_{rj} + \delta^i_\ell \Pi^r_{sj} \Pi^s_{rk}) \\ &\quad + a_j \Pi^i_{k\ell} + a_k \Pi^i_{\ell j} + a_\ell \Pi^i_{jk} \\ &\quad + \delta^i_j a_r \Pi^r_{k\ell} + \delta^i_k a_r \Pi^r_{\ell j} + \delta^i_\ell a_r \Pi^r_{jk} \\ &\quad + 2(\delta^i_j a_k a_\ell + \delta^i_k a_\ell a_j + \delta^i_\ell a_j a_k).\end{aligned} \tag{4.26}$$

Define $\Pi^i_{jk\ell}$ to be the expression given by (4.26) for $a_j = 0$. Then, the general n^3-coframe belonging to the P^3_n structure may be expressed as the composition of

$$\partial_0^3 I_n\Big[a^i_j d^3_0 I^j_n + \frac{1}{2!}(a^i_j a_k + a^i_k a_j) d^3_0 I^j_n d^3_0 I^k_n + \frac{2}{3!}(a^i_j a_k a_\ell + a^i_k a_\ell a_j + a^i_\ell a_j a_k) d^3_0 I^j_n d^3_0 I^k_n d^3_0 I^\ell_n\Big] \tag{4.27}$$

and

$$\partial_0^3 I_n\Big(d^3_p x^i + \frac{1}{2!}\Pi^i_{jk} d^3_p x^j d^3_p x^k + \frac{1}{3!}\Pi^i_{jk\ell} d^3_p x^j d^3_p x^k d^3_p x^\ell\Big). \tag{4.28}$$

The 3 jet (4.27) is the typical element of the prolonged group P^3_n.

5. RIEMANN AND LORENTZ STRUCTURES

The local Lorentz gauge invariant vierbein formulation of Einstein's general relativity theory is well known. It was first developed by Weyl[19] in 1929. Much later in 1950, he discussed the coupling of the vierbein fields to the spin density of matter fields and showed that an equivalent metric theory could be formulated.[20] Thereafter, the theory was analyzed further by Utiyama,[21] Sciama[22] and Kibble.[23] The natural setting for this formulation of general relativity is the theory of an $O^1_{1,n-1}$ structure on spacetime. In the following presentation, the bundles and formulas needed will be given for the case of a Lorentz structure; however, the corresponding bundles and formulas for the case of a Riemann structure may be obtained simply by substituting the group O^1_n for the group $O^1_{1,n-1}$. Both the first-order $O^1_{1,n-1}$ structure and its prolongation to the second-order $O^2_{1,n-1}$ structure are discussed.

The restriction of the action of G^1_n on $H^{1*}(M)$ to the subgroup $O^1_{1,n-1}$ may be used to construct the bundle of equivalence classes of $O^1_{1,n-1}$ related n^1 coframes

$$O^1_{1,n-1}\backslash\mathscr{H}^{1*}(M) = \langle O^1_{1,n-1}\backslash H^{1*}(M), \pi_{O^1_{1,n-1}\backslash H^{1*}}, M, O^1_{1,n-1}\backslash G_n, \mathscr{H}^{1*}(M)\rangle. \tag{5.1}$$

An $O^1_{1,n-1}$ structure on M is a (global) cross section of $O^1_{1,n-1}\backslash\mathscr{H}^{1*}(M)$. Such a cross section may be locally represented by a family of local cross sections of $\mathscr{H}^{1*}(M)$. Any two such local cross sections $h:U\to H^{1*}(U)$ and $\tilde{h}:U\to H^{1*}(U)$ are related by a local $O^1_{1,n-1}$ gauge transformation defined by a cross section $\Lambda:U\to U\times O^1_{1,n-1}$

$$\begin{aligned} h(p) &= \partial^1_0 I_n [h^i_j(p) d^1_p x^j], \\ \tilde{h}(p) &= \Lambda(p)\circ h(p) \\ &= \partial^1_0 I_n [\Lambda^i_r(p) h^r_j(p) d^1_p x^j], \end{aligned} \tag{5.2}$$

where

$$\eta_{ij}\Lambda^i_\ell(p)\Lambda^j_m(p) = \eta_{\ell m}. \tag{5.3}$$

For both the Lorentz and Riemann cases, a suitable gauge-fixing condition is

$$h^i_j(p) = 0;\ i>j. \tag{5.4}$$

For such a choice of gauge, the matrix $[h^i_j(p)]$ is upper triangular.

The local cross sections h are usually called *n-bein* fields. The Lorentz metric tensor defined by the $O^1_{1,n-1}$ structure is locally given by

$$g = \eta_{ij} h^i_k h^j_\ell dx^k \otimes dx^\ell. \tag{5.5}$$

Since these local metric tensor fields are $O^1_{1,n-1}$ gauge independent, they can be patched together to define a global metric tensor field.

Now, consider the problem of prolonging an $O^1_{1,n-1}$ structure to an $O^2_{1,n-1}$ structure. At $p\in M$, a general element of the equivalent class of frames defined by the $O^1_{1,n-1}$ structure is

$$\partial^1_0 I_n(\Lambda^i_r h^r_j d^1_p x^j). \tag{5.6}$$

A general n^2 coframe having the same first-order part and belonging to some $O^2_{1,n-1}$ structure is

$$\partial^2_0 I_n\Big(\Lambda^i_r h^r_j d^2_p x^j + \frac{1}{2!}\Lambda^i_r h^r_s \Gamma^s_{jk} d^2_p x^j d^2_p x^k\Big). \tag{5.7}$$

Note that factoring out h^r_s in the second-order part is a mere convenience. If $q\in M$ is near $p\in M$ and $w^i = x^i(q) - x^i(p)$, then (5.7) defines at q an equivalence class of n^1 coframes containing the element

$$\partial^1_0 I_n((\Lambda^i_r h^r_j + \Lambda^i_r h^r_s \Gamma^s_{j\ell} w^\ell) d^1_q x^j). \tag{5.8}$$

However, the $O^1_{1,n-1}$ structure or Lorentz structure already determines at q an equivalence class of n^1 coframes which contains the standard element [some gauge fixing condition such as (5.4) is assumed]

$$\partial^1_0 I_n((h^i_j + h^i_{j,\ell} w^\ell) d^1_q x^j). \tag{5.9}$$

The condition that the $O^2_{1,n-1}$ structure is compatible with the $O^1_{1,n-1}$ structure is that the two equivalence classes of $O^1_{1,n-1}$ related n^1 coframes determined by (5.8) and (5.9) are the same. In order to make the desired comparison, it is necessary to transform the n^1coframe to standard form. Now, the matrix

$$h^i_j + h^i_r \Gamma^r_{j\ell} w^\ell \tag{5.10}$$

differs only infinitesimally from standard form. Hence, for some infinitesimal Lorentz transformation, say

$$\begin{aligned} \Lambda^i_j &= \delta^i_j + \lambda^i_{j\ell} w^\ell, \\ \eta_{ir}\lambda^r_{j\ell} &= \lambda_{ij\ell} = -\lambda_{ji\ell} \end{aligned} \tag{5.11}$$

the matrix

$$\Lambda^i_s(h^s_j + h^s_r \Gamma^r_{j\ell} w^\ell) = h^i_j + h^i_r \Gamma^r_{j\ell} w^\ell + h^r_j \lambda^i_{r\ell} w^\ell \tag{5.12}$$

is in standard form. If the particular gauge-fixing condition (5.4) is used, then

$$h^i_r \Gamma^r_{j\ell} + h^r_j \lambda^i_{r\ell} = 0;\ i>j. \tag{5.13}$$

Since $h^r_j = 0$ unless $j\geqslant r$, Eqs. (5.13) contain $\lambda^i_{r\ell}$ for $i>r$ and may be solved successively for $\lambda^i_{j\ell} (i>j)$ in terms of h^i_j and $\Gamma^i_{j\ell}$. The values for $i<j$ follow from the skew symmetry of $\lambda_{ij\ell}$ in i and j. Although Eq. (5.13) are easy to solve, it is somewhat tedious to write the solution out explicitly. Fortunately, in this case, it is not necessary to do so; nevertheless, the expression (5.12) should be regarded as a function of h^i_j, $\Gamma^i_{j\ell}$, and w^ℓ. If this expression is equated with the matrix of the standard form (5.9), one obtains the condition

$$h^i_{j,\ell} = h^i_r \Gamma^r_{j\ell} + h^r_j \lambda^i_{r\ell}. \tag{5.14}$$

Since $\Gamma^i_{j\ell} = \Gamma^i_{\ell j}$, the skew part of (5.14) gives

$$h^i_{j,\ell} - h^i_{\ell,j} = h^r_j \lambda^i_{r\ell} - h^r_\ell \lambda^i_{rj}. \tag{5.15}$$

Define

$$\hat{\lambda}_{ijk} = \eta_{is}\lambda^s_{jr} h^{-1r}{}_k. \tag{5.16}$$

The (5.15) may be rewritten as

$$\eta_{i\ell}(h^\ell_{r,s} - h^\ell_{s,r}) h^{-1r}{}_j h^{-1s}{}_k = \hat{\lambda}_{ijk} - \hat{\lambda}_{ikj}. \tag{5.17}$$

Cyclic permutation of the indices and use of the skew symmetry (5.11) yields the solution given by De Witt[24]; namely,

$$\begin{aligned} 2\hat{\lambda}_{ijk} &= \eta_{i\ell}(h^\ell_{r,s} - h^\ell_{s,r}) h^{-1r}{}_j h^{-1s}{}_k \\ &\quad + \eta_{j\ell}(h^\ell_{r,s} - h^\ell_{s,r}) h^{-1r}{}_k h^{-1s}{}_i \\ &\quad - \eta_{k\ell}(h^\ell_{r,s} - h^\ell_{s,r}) h^{-1r}{}_i h^{-1s}{}_j. \end{aligned} \tag{5.18}$$

This result may then be used in (5.14) to give the usual result for Γ^i_{jk} which may be written as

$$\Gamma^i_{jk} = h^{-1i}{}_r h^r_{j,k} - \eta^{rs} h^{-1i}{}_r \hat{\lambda}_{sab} h^a_j h^b_k. \tag{5.19}$$

Although the standard gauge defined by (5.4) was assumed in order to make the discussion more concrete, the result (5.18) is gauge-independent provided that the quantities $\lambda^i_{j\ell}$ transfrom in the manner appropriate to an O(1,3) gauge connection under a local gauge transformation.

It is useful to discuss an alternate approach to the Lorentz structure of spacetime. Consider the bundle of 1^2 cospeeds

$$\mathscr{L}^{2*}_1(M) = \langle L^{2*}_1(M), \pi_{L^{2*}_1}, M, L^2_{n,1}, \mathscr{H}^{2*}(M)\rangle. \tag{5.20}$$

If $f:M\to\mathbb{R}$ such that f(p) = 0, then $j^2_p f\in L^{2*}_1(M_p)$ and

$$j^2_p f = \partial^2_0 I\Big(f_{,i} d^2_p x^i + \frac{1}{2!} f_{ij} d^2_p x^i d^2_p x^j\Big). \tag{5.21}$$

If only those functions f are considered which are singular but Morse with signature $2-n$ at p, then a subbundle $\hat{\mathscr{L}}_1^{2*}(M)$ of $\mathscr{L}_1^{2*}(M)$ is obtained consisting of elements of the form

$$j_p^2 g = \partial_0^2 I\left(\frac{1}{2!} g_{ij} d_p^2 x^i d_p^2 x^j\right), \tag{5.22}$$

where the matrix (g_{ij}) assumes in some coordinate system the standard Minkowski form. Clearly, a Lorentz structure may also be defined on M by a global cross section of the $\hat{\mathscr{L}}^{2*}(M)$ as is evident from a comparison of (5.5) and (5.22).

The principal bundle associated with $\hat{\mathscr{L}}^{2*}(M)$ is $\mathscr{H}^{1*}(M)$ rather than $\mathscr{H}^{2*}(M)$; consequently, if (5.22) is parallel transported using (3.23), only the first-order part will have any effect. The element (5.22) is mapped into

$$\partial_0^2 I\left[\frac{1}{2!}(g_{ij} + g_{rj}\Gamma^r_{i\ell}w^\ell + g_{ir}\Gamma^r_{j\ell}w^\ell) d_q^2 x^i d_q^2 x^j\right]. \tag{5.23}$$

If the Lorentz structure defined by the cross section of $\hat{\mathscr{L}}_1^{2*}(M)$ is horizontal under the connection induced by the second-order structure, the element (5.23) must equal the element

$$\partial_0^2 I\left[\frac{1}{2!}(g_{ij} + g_{ij,\ell}w^\ell) d_q^2 x^i d_q^2 x^j\right]. \tag{5.24}$$

Hence

$$g_{ij,k} = g_{ir}\Gamma^r_{jk} + g_{rj}\Gamma^r_{ik}. \tag{5.25}$$

This relation may also be derived using (5.5) and (5.14). The well-known solution is

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{ir}(g_{rj,k} + g_{rk,j} - g_{jk,r}), \tag{5.26}$$

where g^{ij} is the inverse of g_{ij}.

The compatibility conditions (5.14) and (5.25) are not the most general possible. Just as in the affine and projective cases, allowance should be made for a possible term B^i_{jk} with $B^i_{jk} = -B^i_{kj}$. Thus (5.14) becomes

$$h^i_{j,\ell} + h^i_r B^r_{j\ell} = h^i_r \Gamma^r_{j\ell} + h^r_j \lambda^i_{r\ell}, \tag{5.27}$$

and (5.25) becomes

$$g_{ij,k} + g_{ir}B^r_{jk} + g_{rj}B^r_{ik} = g_{ir}\Gamma^r_{jk} + g_{rj}\Gamma^r_{ik}. \tag{5.28}$$

This equation leads to the formula given by Schrödinger.[25]

$$\begin{aligned}\Gamma^i_{jk} &= \tfrac{1}{2}g^{ir}(g_{rj,k} + g_{rk,j} - g_{jk,r}) \\ &\quad + g_{js}g^{ir}B^s_{rk} + g_{ks}g^{ir}B^s_{rj}.\end{aligned} \tag{5.29}$$

Finally, since the affine connection induced by the $O^2_{1,n-1}$ structure is completely defined, further prolongations are determined by the formulas that apply in the affine case. Also under infinitesimal parallel transport, the connection induces infinitesimal $O^1_{1,n-1}$ transformations.

6. THE CONFORMAL STRUCTURE

The analysis of a first-order conformal or $C^1_{1,n-1}$ structure on M and of its prolongation to a second-order conformal or $C^2_{1,n-1}$ structure on M is similar to the corresponding analysis of the Lorentz structure. The further prolongation of the $C^2_{1,n-1}$ structure to a $C^3_{1,n-1}$ structure may be derived by an analysis entirely analogous to that given for the corresponding prolongation in the affine and projective cases; however, the algebra required is rather tedious although in principle straight forward. Thus, the explicit solution for the $C^3_{1,n-1}$ structure will not be given although the problem will be set up and a solution shown to exist.

A first-order conformal or $C^1_{1,n-1}$ structure on M is a reduction of the structure group G^1_n of the bundle of n^1 coframes $\mathscr{H}^{1*}(M)$ to the subgroup $C^1_{1,n-1}$; that is, a $C^1_{1,n-1}$ structure is defined by a cross section of the bundle of equivalence classes of $C^1_{1,n-1}$ related n^1 coframes

$$\begin{aligned}&C^1_{1,n-1}\backslash\mathscr{H}^{1*}(M)\\ &\quad = \langle C^1_{1,n-1}\backslash H^{1*}(M), \pi_{C^1_{1,n-1}\backslash H^{1*}}, M,\\ &\qquad\qquad C^1_{1,n-1}\backslash G^1_n, \mathscr{H}^{1*}(M)\rangle.\end{aligned} \tag{6.1}$$

Such a cross section is locally represented by a family of local cross sections of $\mathscr{H}^{1*}(M)$. Any two local cross sections $h: U \to H^{1*}(U)$ and $\tilde{h}: U \to H^{1*}(U)$ are related by a local $C^1_{1,n-1}$ gauge transformation defined by a cross section $c: U \to U \times C^1_{1,n-1}$. Thus

$$\begin{aligned}h(p) &= \partial_0^1 I_n(h^i_j(p) d^1_p x^j),\\ \tilde{h}(p) &= c(p)\circ h(p) = \partial_0^1 I_n(c^i_r(p) h^r_j(p) d^1_p x^j),\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}&c^i_j(p) = e^{\lambda(p)}\Lambda^i_j(p),\\ &\eta_{ij}\Lambda^i_k(p)\Lambda^j_\ell(p) = \eta_{k\ell}.\end{aligned} \tag{6.3}$$

As for the Riemann and Lorentz cases, many gauge-fixing conditions are possible. One possible choice is

$$\begin{aligned}&h^i_j = 0;\ i>j\\ &h^0_0 = 1.\end{aligned} \tag{6.4}$$

The metric tensor fields g and $\tilde{g}$ defined by h and $\tilde{h}$ by (5.5) are related by

$$\tilde{g}(p) = e^{2\lambda(p)} g(p). \tag{6.5}$$

To prolong the $C^1_{1,n-1}$ structure to a $C^2_{1,n-1}$ structure consider a general n^2 coframe belonging to a $C^1_{1,n-1}$ equivalence class of n^2 coframes and having a first-order part belonging to the $C^1_{1,n-1}$ structure; namely

$$\partial_0^2 I_n\left(c^i_r h^r_j d^2_p x^j + \frac{1}{2!} c^i_r h^r_s \Gamma^s_{jk} d^2_p x^j d^2_p x^k\right). \tag{6.6}$$

This n^2-coframe at p defines a n^1 coframe at a neighboring point q; namely,

$$\partial_0^1 I_n((c^i_r h^r_j + c^i_r h^r_s \Gamma^s_{j\ell} w^\ell) d^1_q x^j), \tag{6.7}$$

where $w^i = x^i(q) - x^i(p)$. However, the $C^1_{1,n-1}$ structure already determines at q an equivalence class of n^1 coframes at q determined by the standard n^1-coframe

$$\partial_0^1 I_n((h^i_j + h^i_{j,\ell} w^\ell) d^1_q x^j). \tag{6.8}$$

For compatibility, the equivalence classes of n^1 coframes defined by (6.7) and (6.8) must be the same. To reduce the n^1 coframe to standard form, one must determine the infinitesimal conformal transformation

$$\begin{aligned}&c^i_j = \delta^i_j + \lambda_\ell w^\ell \delta^i_j + \lambda^i_{j\ell} w^\ell,\\ &\eta_{ir}\lambda^r_{j\ell} = \lambda_{ij\ell} = -\lambda_{ji\ell},\end{aligned} \tag{6.9}$$

such that the matrix

$$c^i_r(h^r_j + h^r_s\Gamma^s_{j\prime}w^\prime) = h^i_j + h^i_r\Gamma^r_{j\prime}w^\prime + h^r_j\lambda^i_{r\prime}w^\prime + \lambda_\prime w^\prime\delta^i_j \tag{6.10}$$

satisfies the conditions (6.4); that is,

$$h^i_r\Gamma^r_{j\prime} + h^r_j\lambda^i_{r\prime} = 0;\ i>j$$
$$h^0_r\Gamma^r_{0\prime} + \lambda_\prime = 0. \tag{6.11}$$

Note that $h^r_0\lambda^0_{r\prime} = 0$ since $h^r_0 = 0$ unless $r=0$ and $\lambda_{ij\prime}$ is skew in i and j. The discussion given for the Lorentz case applies equally to Eqs. (6.11). The solutions for $\lambda^i_{j\prime}$ and $\lambda_\prime$ are substituted into (6.10) which is then in standard form. Equating the matrix of (6.8) with the matrix (6.10) gives the compatibility condition

$$h^i_{j,\prime} = h^i_r\Gamma^r_{j\prime} + h^r_j\lambda^i_{r\prime} + \lambda_\prime h^i_j. \tag{6.12}$$

As for the Lorentz case, the possibility of a torsion term is for simplicity not considered.

The algebraic difficulties are considerably reduced if the problem is reformulated in terms of metric tensor fields. For this purpose, it is necessary to introduce the bundles $G^2_1\backslash\mathscr{L}^{2}_1{}^*(M)$ and $G^2_1\backslash\hat{\mathscr{L}}^{2}_1{}^*(M)$ of equivalence classes of elements of $\mathscr{L}^{2}_1{}^*(M)$ and $\hat{\mathscr{L}}^{2}_1{}^*(M)$ under the action of the parameter group G^2_1. If $\mu:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism with $\mu(0)=0$, then $j^2_0\mu\in G^2_1$ and

$$j^2_0\mu = \partial^2_0\, I\,(\mu_1 d^2_0\, I + \frac{1}{2!}\mu_2(d^2_0\, I)^2), \tag{6.13}$$

where μ_1 and μ_2 are the first and second derivatives of μ at $0\in\mathbb{R}$. Also, if $j^2_p f\in L^{2}_1{}^*(M_p)$ is given by (5.21) and $\tilde{f}=\mu\circ f$, the action of G^2_1 is given by

$$j^2_p\tilde{f} = j^2_0\mu\circ j^2_p f. \tag{6.14}$$

In component form, this action is given by

$$\tilde{f}_i = \mu_1 f_i,$$
$$\tilde{f}_{ij} = \mu_1 f_{ij} + \mu_2 f_i f_j. \tag{6.15}$$

For elements of the subbundle $\hat{\mathscr{L}}^{2}_1{}^*(M)$ given by (5.22), this action simplifies to

$$\tilde{g}_{ij} = \mu_1\, g_{ij}. \tag{6.16}$$

A $C^1_{1,n-1}$ structure on M may also be defined by a (global) cross section of $G^2_1\backslash\hat{\mathscr{L}}^{2}_1{}^*(M)$ which may be represented by local cross sections of $\mathscr{L}^{2}_1{}^*(M)$, any two of which are related by a gauge transformation (6.16). Numerous gauge-fixing conditions are possible. A simple choice [which is not consistent with (6.4)] is

$$g_{00} = 1. \tag{6.17}$$

Again, under the parallel transport induced by the prolonged structure, the standard representative (5.22) is transformed into (5.23). For some infinitesimal gauge transformation ($\mu_1 = 2\lambda_\prime w^\prime$) the element (5.23) is transformed into standard form

$$g_{ij} + g_{rj}\Gamma^r_{i\prime}w^\prime + g_{ir}\Gamma^r_{j\prime}w^\prime + 2g_{ij}\lambda_\prime w^\prime. \tag{6.18}$$

If the gauge-fixing condition (6.17) is used, then

$$\lambda_\prime = -g_{0r}\Gamma^r_{0\prime}. \tag{6.19}$$

Define the conformal connection coefficients K^i_{jk} by

$$\Gamma^i_{jk} = K^i_{jk} + \frac{1}{n}(\delta^i_j\Gamma_k + \delta^i_k\Gamma_j - g_{jk}g^{ir}\Gamma_r), \tag{6.20}$$

$$K^i_{ik} = 0, \quad \Gamma_k = \Gamma^i_{ik}.$$

Substitute (6.20) into (6.18). The terms involving Γ_k cancel. Thus the matrix of the standard metric representative at q is

$$g_{ij} + g_{rj}K^r_{i\prime}w^\prime + g_{ir}K^r_{j\prime}w^\prime - 2g_{ij}g_{0r}K^r_{0\prime}w^\prime, \tag{6.21}$$

which must equal the matrix of the standard element (5.24); consequently,

$$g_{ij,k} = g_{rj}K^r_{ik} + g_{ir}K^r_{jk} - 2g_{ij}g_{0r}K^r_{0k}. \tag{6.22}$$

Then

$$g^{ij}g_{ij,k} = -2ng_{0r}K^r_{0k} \tag{6.23}$$

and

$$g_{ir}K^r_{jk} + g_{rj}K^r_{ik} = g_{ij,k} - \frac{1}{n}g_{ij}g^{rs}g_{rs,k}. \tag{6.24}$$

This equation may be solved to give

$$K^i_{jk} = \tfrac{1}{2}g^{ir}(g_{rj,k} + g_{rk,j} - g_{jk,r})$$
$$-\frac{1}{2n}(\delta^i_j\, g^{rs}\, g_{rs,k} + \delta^i_k\, g^{rs}\, g_{rs,j} - g_{jk}\, g^{it}\, g^{rs}\, g_{rs,t}). \tag{6.25}$$

Although a particular gauge was used for the computation, the result (6.25) is in fact gauge independent. Note that the trace Γ_k of Γ^i_{jk} is completely undetermined. If these results are now substituted into (6.6), it is readily shown that the typical n^2 coframe belonging to the prolonged structure may be expressed as the composition of

$$\partial^2_0 I_n\left[c^i_j d^2_0 I^j_n + \frac{1}{2!}(c^i_j c_k + c^i_k c_j - \eta_{jk}\eta^{ir}c_r)d^2_0 I^j_n d^2_0 I^k_n\right] \tag{6.26}$$

and

$$\partial^2_0 I_n(h^i_j d^2_p x^j + \frac{1}{2!}h^i_r K^r_{jk} d^2_p x^j d^2_p x^k), \tag{6.27}$$

where

$$\Gamma_k = c_r h^r_k. \tag{6.28}$$

The 2 jet (6.26) is a general element of the prolonged group $C^2_{1,n-1}$. The general n^2 coframe belonging to the $C^2_{1,n-1}$ structure is

$$\partial^2_0 I_n\left[a^i_j d^2_p x^j + \frac{1}{2!}a^i_r(K^r_{jk} + \delta^r_j a_k + \delta^r_k a_j - g_{jk}g^{rs}a_s)d^2_p x^j d^2_p x^k\right], \tag{6.29}$$

where

$$a^i_j = c^i_r h^r_j \qquad a_k = c_r h^r_k. \tag{6.30}$$

The prolongation of the $C^2_{1,n-1}$ structure to a $C^3_{1,n-1}$ structure follows the same pattern. Consider the most general n^3 coframe which projects to the n^2 coframe (6.29).

$$\partial^3_0 I_n\left[a^i_j d^3_p x^j + \frac{1}{2!}\, a^i_r(K^r_{jk} + \delta^r_j a_k + \delta^r_k a_j - g_{jk}g^{rs}a_s)d^3_p x^j d^3_p x^k + \frac{1}{3!}a^i_r\Gamma^r_{jk\prime}d^3_p x^j d^3_p x^k d^3_p x^\prime\right]. \tag{6.31}$$

The n^2-coframe at q determined by (6.31) by the substitution process is

$$\partial_0^2 I_n \, [a^i_r(\delta^r_j + K^r_{js} w^s + \delta^r_j a_s w^s + w^r a_j - g^{rt} a_t g_{js} w^s) d^{\,2}_{q} x^j + \frac{1}{2!} a^i_r (K^r_{jk} + \delta^r_j a_k + \delta^r_k a_j - g_{jk} g^{rt} a_t + \Gamma^r_{jks} w^s) d^{\,2}_{q} x^j d^{\,2}_{q} x^k \,]. \tag{6.32}$$

The next task is to bring the first-order part of (6.32) to a form satisfying the standard gauge conditions (6.4). The problem is essentially the same as that discussed above following (6.9) in connection with the prolongation from a $C^{\,1}_{1,n-1}$ structure to a $C^{\,2}_{1,n-1}$ structure; that is, the form of the linear equations for the parameters of the infinitessimal $C^{\,1}_{1,n-1}$ transformation is the same. However, the number of terms involved is even greater than it was before. Moreover, it does not seem possible to artfully dodge the problem as before because it is necessary to know the additional terms that will appear in the second-order part of (6.32) when the first-order part is brought to standard form. Once the first-order part is in standard form, it is easy to bring the second-order part to standard form by applying (6.26) with $c^i_j = \delta^i_j$ and c_j chosen so that the coefficient of $h^{\,i}_{\,r}$ in the second-order term will have zero trace.

The standard n^2 coframe at q obtained by the above procedure must agree with the appropriate n^2 coframe at q determined by the $C^{\,2}_{1,n-1}$ structure. The n^2 coframe corresponding to (6.27) evaluated at q is

$$\partial_0^2 I_n \, [(h^{\,i}_{\,j} + h^{\,i}_{\,j,s} w^s) d^{\,2}_{q} x^j + \frac{1}{2!} h^{\,i}_{\,r} (K^r_{jk} + K^r_{jk,s} w^s + h^{-1\,r}_{\;\;a} h^{\,a}_{\,b,s} K^b_{jk} w^s) d^{\,2}_{q} x^j d^{\,2}_{q} x^k \,]. \tag{6.33}$$

However, the coefficient of $h^{\,i}_{\,r} = h^{\,i}_{\,r}(p)$ in (6.33) does not have zero trace. This trace must be removed before the comparison with the standard form of (6.32) described above is made.

The first-order terms must necessarily agree because the $K^{\,i}_{jk}$ were defined by that condition at the previous stage. The demand that the second-order terms agree up to a term of the form $h^{\,i}_{\,r} B^{\,r}_{jks} w^s$, where $B^{\,i}_{jks}$ is traceless and has the same symmetries as in the affine and projective cases, gives an equation for the $\Gamma^{\,i}_{jkl}$ which may be solved in a manner similar to that used in the projective case. The solution gives $\Gamma^{\,i}_{jkl}$ in terms of $h^{\,i}_{\,j}$, $K^{\,i}_{jk}$, and a_j. The general n^3 coframe belonging to the $C^{\,3}_{1,n-1}$ structure may be obtained by substituting this solution into (6.31) which may in turn be factored into a $C^{\,3}_{1,n-1}$ gauge group element and the standard n^3 coframe for the $C^{\,3}_{1,n-1}$ structure.

7. THE WEYL STRUCTURE

A Weyl structure on M is a reduction of the structure group $G^{\,2}_{n}$ of $\mathscr{H}^{2*}(M)$ to the Weyl subgroup $W^{\,2}_{1,n-1}$ (see Appendix C). Since $W^{\,2}_{1,n-1}$ is a Lie subgroup of $C^{\,2}_{1,n-1}$ the reduction may be accomplished by first defining a $C^{\,1}_{1,n-1}$ structure and prolonging this structure to a $C^{\,2}_{1,n-1}$ structure and then reducing this structure to a $W^{\,2}_{1,n-1}$ structure.

The general element of an equivalence class belonging to a $C^{\,2}_{1,n-1}$ structure is given by (6.6) together with (6.20) and (6.25). The result is

$$\partial_0^2 I_n \left[c^i_r h^{\,r}_{\,j} d^{\,2}_{p} x^j + \frac{1}{2!} c^i_r h^{\,r}_{\,s} (K^{\,s}_{jk} + \frac{1}{n} (\delta^s_j \Gamma_k + \delta^s_k \Gamma_j - g_{jk} g^{st} \Gamma_t) d^{\,2}_{p} x^j d^{\,2}_{p} x^k \right], \tag{7.1}$$

where $\Gamma_k = c_r h^{\,r}_{\,k}$. Clearly, the reduction to $W^{\,2}_{1,n-1}$ is determined locally by specifying the Γ_k (or c_k) as functions of $p \in U \subset M$. The connection determined by the $\Gamma^{\,i}_{jk}$ given by (6.20) is then fully specified; consequently, the formulas for the higher-order prolongations of an affine structure may be used to determine the higher-order prolongations of a Weyl structure. The Weyl connection induces infinitesimal conformal transformations under infinitesimal parallel transport.

8. THE CONFORMAL–PROJECTIVE COMPATIBILITY CONDITION

In this section, the algebraic condition that expresses the compatibility between a conformal or $C^{\,2}_{1,n-1}$ structure and a projective or $P^{\,2}_{n}$ structure is derived using the analysis of Secs. 4 and 6. The result was originally derived by Ehlers, *et al.*[1], from constructive axioms for general relativity theory. The derivation given here is not a replacement for their argument, but is rather a supplement which clarifies the geometric interpretation of the condition as the necessary and sufficient condition that a $C^{\,2}_{1,n-1}$ structure and a $P^{\,2}_{n}$ structure are compatible and uniquely define a Weyl or $W^{\,2}_{1,n-1}$ structure.[26]

A general element belonging to an equivalence class of $P^{\,2}_{n}$ related n^2 coframes defined by a projective structure is given by (4.8)

$$\partial_0^2 I_n \left[a^i_j d^{\,2}_{p} x^j + \frac{1}{2!} a^i_r (\Pi^{\,r}_{jk} + \delta^i_j a_k) + \delta^i_k a_j \right] d^{\,2}_{p} x^j d^{\,2}_{p} x^k . \tag{8.1}$$

A general element belonging to an equivalence class of $C^{\,2}_{1,n-1}$ related n^2 coframes is given by (7.1)

$$\partial_0^2 I_n \left[c^i_r h^{\,r}_{\,j} d^{\,2}_{p} x^j + \frac{1}{2!} c^i_r h^{\,r}_{\,s} (K^{\,s}_{jk} + \frac{1}{n} (\delta^s_j \Gamma_k + \delta^s_k \Gamma_j - g_{jk} g^{sa} \Gamma_a)) d^{\,2}_{p} x^j d^{\,2}_{p} x^k \right]. \tag{8.2}$$

The two structures are compatible iff the equivalent classes defined by (8.1) and (8.2) have a nonempty intersection. Consideration of the first-order parts shows that a projective coframe is a conformal coframe only if

$$a^i_j = c^i_r h^{\,r}_{\,j}. \tag{8.3}$$

Comparison of the second-order parts then yields the relation

$$\Pi^{\,i}_{jk} + \delta^i_j a_k + \delta^i_k a_j = K^{\,i}_{jk} + \frac{1}{n} (\delta^i_j \Gamma_k + \delta^i_k \Gamma_j - g_{jk} g^{ir} \Gamma_r). \tag{8.4}$$

Define

$$\Delta^{\,i}_{jk} = \Pi^{\,i}_{jk} - K^{\,i}_{jk}. \tag{8.5}$$

Then take the trace of (8.4) to obtain

$$(n+1)a_k = \Gamma_k. \tag{8.6}$$

Then

$$\Delta^{\,i}_{jk} = \frac{1}{n(n+1)}(\delta^i_j \Gamma_k + \delta^i_k \Gamma_j) - \frac{1}{n} g_{jk}\, g^{ir}\Gamma_r \tag{8.7}$$

and

$$g^{rs}\Delta^{\,i}_{rs} = -\frac{(n-1)(n+2)}{n(n+1)} g^{ir}\Gamma_r \tag{8.8}$$

or

$$\Gamma_k = -\frac{n(n+1)}{(n-1)(n+2)} g_{kt} g^{rs}\Delta^{\,t}_{rs}. \tag{8.9}$$

Substitution of (8.9) into (8.7) yields a system of linear equations for $\Delta^{\,i}_{jk}$ which are satisfied iff the projective and conformal structures are compatible; moreover, if they are compatible, Γ_k given by (8.9) and hence $\Gamma^{\,i}_{jk}$ are well defined and the conformal and projective structures define a unique Weyl structure.

APPENDIX A: OF GERMS AND JETS

Let M and N be C^∞ differentiable manifolds. Denote by $C^\infty(M,N)$ the set of C^∞ differentiable maps $f:M\to N$. Any two maps $f_1, f_2 \in C^\infty(M,N)$ are said to be germ equivalent at $p\in M$ iff there exists an open neighborhood U of p on which f_1 and f_2 agree. The *germ* $j_p f$ of a map $f\in C^\infty(M,N)$ is the equivalence class to which f belongs. The set of such equivalence classes will be denoted by $J(M_p,N)$. In certain applications, maps defined only on some open submanifold U of M are considered. The set of germs of such maps is then denoted by $J(U_p,N)$. The map $j_p: C^\infty(M,N)\to J(M_p,N)$ is the canonical projection.

Any two maps $f_1, f_2\in C^\infty(M,N)$ are said to be k-jet equivalent iff they agree *at p*

$$f_1(p) = f_2(p) = q, \tag{A1}$$

and for any charts $(U,x)_p$ and $(V,y)_q$ where $p\in U$ and $q\in V$, the maps $y\circ f_1\circ x^{-1}$ and $y\circ f_2\circ x^{-1}$ have the same partial derivatives of all types at $x(p)$ up to and including order k. That this equivalence relation is independent of the choice of coordinate charts follows by applying the chain rule to the relation

$$\bar{y}\circ f\circ \bar{x}^{-1} = (\bar{y}\circ y^{-1})\circ(y\circ f\circ x^{-1})\circ(x\circ\bar{x}^{-1}). \tag{A2}$$

The k jet $j^k_p f$ of a map $f\in C^\infty(M,N)$ is the equivalence class to which f belongs. The set of k jets with *source* $p\in M$ and *target* $q\in N$ is denoted by $J^k(M_p,N_q)$, and the corresponding sets with unrestricted source, target, or both are denoted by $J^k(M,N_q)$, $J^k(M_p,N)$, and $J^k(M,N)$, respectively. Again, for some applications it may be convenient to consider the set of maps $C^\infty(U,V)$ where U and V are open submanifolds of M and N. Then the corresponding sets of k jets are denoted by $J^k(U_p,V_q)$ and so forth. The map $j^k_p: C^\infty(M,N)\to J^k(M_p,N)$ is the canonical projection. Since $j_p f\supset j^k_p f$ one may also use j^k_p to denote the projection $j^k_p: J(M_p,N)\to J^k(M_p,N)$ and write

$$j^k_p(j_p f) = j^k_p f. \tag{A3}$$

If M, L and N are C^∞ manifolds, then the composition map $\circ: C^\infty(L,N)\times C^\infty(M,L)\to C^\infty(M,N)$ defined by $g\circ f(p) = g(f(p))$ induces a number of compositions between germs and jets; namely,

$$\begin{aligned} j_p(g\circ f) &= j_{f(p)}\, g\circ j_p f \\ j^k_p(g\circ f) &= j^k_{f(p)}\, g\circ j_p f \\ &= j_{f(p)} g\circ j^k_p f \\ &= j^k_{f(p)} g\circ j^k_p f. \end{aligned} \tag{A4}$$

An algebraic structure on the manifold N will in general induce an algebraic structure on $J(M_p,N)$ and $J^k(M_p,N)$; for example, these sets inherit algebra structures if N is $\mathbb{R}$. Define $\forall j^k_p f, j^k_p f_1, j^k_p f_2\in J^k(M_p,\mathbb{R})$ and $\forall \lambda_1,\lambda_2\in\mathbb{R}$

$$\begin{aligned} &\lambda_1 j^k_p f_1 + \lambda_2 j^k_p f_2 = j^k_p(\lambda_1 f_1 + \lambda_2 f_2), \\ &j^k_p f_1 j^k_p f_2 = j^k_p(f_1 f_2), \end{aligned} \tag{A5}$$

where the functions $\lambda_1 f_1 + \lambda_2 f_2$ and $f_1 f_2$ are defined by pointwise scalar multiplication, addition, and multiplication. The definitions of the algebra operations for $J(M_p,\mathbb{R})$ are similar. Also, if N is a vector space, then so are $J(M_p,N)$ and $J^k(M_p N)$, and if N is finite dimensional, then so is $J^k(M_p,N)$.

Any element $j^k_p f\in J^k(M_p,N_q)$ may be invariantly represented with respect to any coordinate charts $(U,x)_p$ and $(V,y)_q$. Denote by $x(p)$ the constant map on U which maps every point of U into $x(p)$, and set $x_p = x - x(p)$. Similarly, define the map $y_q = y - y(q)$. Then

$$\begin{aligned} j^k_p f &= j^k_p(y_q^{-1}\circ y_q\circ f\circ x_p^{-1}\circ x_p) \\ &= j^k_0\, y_q^{-1}\circ j^k_p(F\circ x_p), \end{aligned} \tag{A6}$$

where $F = y_q\circ f\circ x_p^{-1}$. Note that $F(\mathbf{0}) = \mathbf{0}$. Since $F\circ x_p$ may be expanded in powers of the functions x^i_p on U and since j^k_p projects products of the functions x^i_p with more than k factors onto the zero equivalence class, the k jet (A6) may be expanded in the form

$$\begin{aligned} j^k_p f = j^k_0\, y_q^{-1}(&F_{i_1} j^k_p x^{i_1}_p \\ &+ \dots + \frac{1}{k!}F_{i_1\dots i_k} j^k_p x^{i_1}_p \dots j^k_p x^{i_k}_p), \end{aligned} \tag{A7}$$

where the coefficients $F_{i_1\dots i_r}$ are the partial derivatives of the function F at $\mathbf{0}$. The transformation law for these coefficients may readily be determined with the aid of the formulas

$$\begin{aligned} j^k_p x_p &= j^k_p(x_p\circ\bar{x}_p^{-1}\circ\bar{x}_p) = j^k_0 X_p\circ j^k_p\bar{x}_p, \\ j^k_0\, y_q^{-1} &= j^k_0(\bar{y}_q^{-1}\circ\bar{y}_q\circ y_q^{-1}) = j^k_0\,\bar{y}_q^{-1}\circ j^k_0\,\bar{Y}_q, \end{aligned} \tag{A8}$$

where $X_p = x_p\cdot\bar{x}_p^{-1}$ and $\bar{Y}_q = \bar{y}_q\circ y_q^{-1}$.

Both simplicity and flexibility are greatly enhanced by the following notational convention. Define

$$\begin{aligned} d^{\,k}_p x &= j^k_p x_p, \\ \partial^k_p x &= j^k_0 x_p^{-1}. \end{aligned} \tag{A9}$$

Then

$$\begin{aligned} d^{\,k}_p x\circ\partial^k_p x &= j^k_0 I_n, \\ \partial^k_p x\circ d^{\,k}_p x &= j^k_p I_M. \end{aligned} \tag{A10}$$

The expansion (A7) may be written

$$\begin{aligned} j^k_p f = \partial^k_p y(&F_{i_1} d^{\,k}_p x^{i_1} \\ &+ \dots + \frac{1}{k!} f_{i_1\dots i_k} d^{\,k}_p x^{i_1}\dots d^{\,k}_p x^{i_k}). \end{aligned} \tag{A11}$$

For a change of variables (A8), the following formulas are useful.

$$d_p^k\bar{x}\circ\partial_p^k x = j_0^k\bar{X}_p,$$
$$d_p^k x\circ\partial_p^k\bar{x} = j_0^k X_p. \qquad \text{(A12)}$$

For mappings from or into $\mathbb{R}^n$, the standard chart $(\mathbb{R}^n, I_n)$ is used. Thus for a curve $\gamma:\mathbb{R}\to M$ such that $\gamma(0) = p$,

$$j_0^k\gamma = \partial_p^k x(\gamma_1 d_0^k I + \frac{1}{2!}\gamma_2(d_0^k I)^2 + \dots + \frac{1}{k!}\gamma_k(d_0^k I)^k), \qquad \text{(A13)}$$

where

$$\gamma_\alpha^i = \frac{d^\alpha}{dt^\alpha}(x_p^i\circ\gamma)\Big|_{t=0} \qquad \alpha = 1,2,\dots,k. \qquad \text{(A14)}$$

Note that the argument of $\partial_p^k x$ in (A13) is an element of $\mathbb{R}^n$; so that each γ_α is an n-tuple. A similar remark applies to the argument of $\partial_p^k y$ in (A11).

Finally, note that $\partial_p^k x$ and $d_p^k x$ denote k jets at a given point, and that $\partial^k x$ and $d^k x$ denote k jet fields.

APPENDIX B: NOTATION FOR JET BUNDLES

It is useful to have special notations for particular sets of jets and for particular jet bundle structures. A principal bundle structure will be denoted by

$$\mathscr{P}(M) = \langle P(M), \pi_P, M, G\rangle, \qquad \text{(B1)}$$

where $P(M)$ is the *total* space, π_P is the *projection* map, M is the *base* space, and G is the *structure* group of the bundle. A bundle structure associated with a given principal bundle structure will be denoted by

$$\mathscr{E}(M) = \langle E(M), \pi_E, M, F, \mathscr{P}(M)\rangle, \qquad \text{(B2)}$$

where $E(M)$ is the total space and F is the *typical fiber*.

Set

$$L_{m,n}^k = J^k(\mathbb{R}_0^m, \mathbb{R}_0^n),$$
$$A_{m,n}^k = J^k(\mathbb{R}_0^m, \mathbb{R}^n) = \mathbb{R}^n\oplus L_{m,n}^k. \qquad \text{(B3)}$$

Note that $A_{m,n}^k$ may be regarded as the nth power of the algebra $A_{m,1}^k$. Denote by $G_{m,n}^k$ the subset of $L_{m,n}^k$ consisting of jets of maximal rank. For the case $m = n$, set $G_n^k = G_{n,n}^k$. The elements of G_n^k are jets of local diffeomorphisms and form a Lie group under jet composition. The group G_n^1 is customarily denoted by GL(n).

Let M be an n-dimensional C^∞ manifold and let $p\in M$. Set

$$L_m^k(M_p) = J^k(\mathbb{R}_0^m, M_p),$$
$$L_m^{k*}(M_p) = J^k(M_p, \mathbb{R}_0^m), \qquad \text{(B4)}$$
$$A_m^k(M_p) = J^k(M_p, \mathbb{R}^m).$$

The submanifolds of $L_m^k(M_p)$, the m^k speeds, and $L_m^{k*}(M_p)$, the m^k cospeeds, which consist of jets of maximal rank are the m^k frames, $H_m^k(M_p)$, and the m^k coframes, $H_m^{k*}(M_p)$. Normally, $m\leqslant n$, and for $m = n$, one writes simply $H^k(M_p)$ and $H^{k*}(M_p)$ for the sets of n^k frames and n^k coframes, respectively.

The sets obtained by taking the union of the above sets for all $p\in M$ are denoted by $L_m^k(M)$, $A_m^k(M)$, $H_m^k(M)$, and so forth. For an open subset $U\subset M$, the sets obtained by taking the union only for $p\in U$, the *portions* over U, are denoted by $L_m^k(U)$, $A_m^k(U)$, $H_m^k(U)$, and so forth. For a given coordinate chart $(U,x)_p$ for M, the locally trivializing maps may be used to give the sets $L_m^k(M)$, $A_m^k(M)$, $H_m^k(M)$, and so forth their usual topological and differentiable structures; for example, for $L_1^k(M)$, the map $\psi_U: L_1^k(U)\to L_{1,n}^k$ is defined by

$$\psi_U(j_0^k\gamma) = d_p^k x\circ j_0^k\gamma, \qquad \text{(B5)}$$

where $j_0^k\gamma$ is given by (A13) and

$$d_p^k x\circ j_0^k\gamma = \partial_0^k I_n(\gamma_1 d_0^k I + \dots + \frac{1}{k!}\gamma_k(d_0^k I)^k). \qquad \text{(B6)}$$

Then $\pi_{L_1^k}\times\psi_U: L_1^k(U)\to U\times L_{1,n}^k$ is bijective and may be used together with all other such maps corresponding to the charts of an altas for M to pull back the topological and differentiable structures of the spaces $U\times L_{1,n}^k$.

The principal bundle structure for n^k frames is denoted by

$$\mathscr{H}^k(M) = \langle H^k(M), \pi_{H^k}, M, G_n^k\rangle, \qquad \text{(B7)}$$

with a similar notation for the principal bundle of n^k coframes $\mathscr{H}^{k*}(M)$. There are many associated bundle structures that could be written down; for example,

$$\mathscr{L}_m^k(M) = \langle L_m^k(M), \pi_{L_m^k}, M, L_{m,n}^k, \mathscr{H}^k(M)\rangle. \qquad \text{(B8)}$$

There is a natural isomorphism between $\mathscr{L}_1^1(M)$ and the tangent bundle $\mathscr{T}(M)$ and between $\mathscr{H}^1(M)$ and the principal bundle of linear frames $\mathscr{L}(M)$.

APPENDIX C: NOTATION FOR GROUPS

Let $a:\mathbb{R}^n\to\mathbb{R}^n$ be a diffeomorphism such that $a(\mathbf{0}) = \mathbf{0}$. The group G_n^k is the set of k jets $j_0^k a$ together with the group product defined by k-jet composition

$$j_0^k c = j_0^k a\circ j_0^k b. \qquad \text{(C1)}$$

For the case $k = 3$, an element of G_n^3 may be written

$$j_0^3 a = \partial_0^3 I_n(a_j^i d_0^3 I_n^j + \frac{1}{2!}a_{jk}^i d_0^3 I_n^j d_0^3 I_n^k + \frac{1}{3!}a_{jk\ell}^i d_0^3 I_n^j d_0^3 I_n^k d_0^3 I_n^\ell) \qquad \text{(C2)}$$

and the product (C1) is given explicitly by the formulas

$$c_j^i = a_r^i b_j^r,$$
$$c_{jk}^i = a_{rs}^i b_j^r b_k^s + a_r^i b_{jk}^r,$$
$$c_{jk\ell}^i = a_{rst}^i b_j^r b_k^s b_\ell^t + a_{rs}^i(b_{jk}^r b_\ell^s + b_{k\ell}^r b_j^s + b_{\ell j}^r b_k^s) + a_r^i b_{jk\ell}^r. \qquad \text{(C3)}$$

The subgroup of G_n^k which consists of elements of the form

$$\partial_0^k I_n(a_j^i d_0^k I_n^j) \qquad \text{(C4)}$$

will be called the affine group (of order k) and will be denoted by Γ_n^k. Let η_{ij} be the flat metric tensor with signature $s = p - q$, where $n = p + q$. Then $O_{p,q}^k$ is the subgroup of Γ_n^k consisiting of those elements (C4) of Γ_n^k for which

$$\eta_{ij}a_k^i a_\ell^j = \eta_{k\ell}. \qquad \text{(C5)}$$

The orthogonal subgroup $\mathscr{O}_n^k$ of Γ_n^k is the group $\mathscr{O}_{p,q}^k$ for $p = n$ and $q = 0$. The Lorentz subgroup is $O_{1,n-1}^k$. The Weyl subgroup $W_{p,q}^k$ of Γ_n^k consists of those elements (C4) of Γ_n^k which have the form

$$a_j^i = e^b b_j^i, \tag{C6}$$

where $b \in \mathbb{R}$ and b_j^i satisfies (C5).

Let $(\xi^\alpha) \in \mathbb{R}^{n+1}$ and consider the invertible linear transformations

$$\hat{\xi}^\alpha = A^\alpha_\beta \xi^\beta. \tag{C7}$$

Set

$$x^i = \frac{\xi^i}{\xi^{n+1}}, \qquad i = 1,2,...n. \tag{C8}$$

The transformations (C7) induce the projective transformations

$$x^i = \frac{a^i + a^i_j x^j}{1 - a_k x^k}, \tag{C9}$$

where

$$a^i = \frac{A^i_{\ n+1}}{A^{n+1}_{\ n+1}}, \quad a_i = -\frac{A^{n+1}_{\ i}}{A^{n+1}_{\ n+1}}, \quad a^i_j = \frac{A^i_j}{A^{n+1}_{\ n+1}}. \tag{C10}$$

The projective transformations which preserve the origin $(x^i = 0)$ have the form

$$\bar{x}^i = \frac{a^i_j x^j}{1 - a_k x^k}. \tag{C11}$$

The subgroup P^k_n of G^k_n consists of the group of k jets of transformations of the form (C11). In particular, an element of P^2_n has the form

$$\partial^2_0 I_n (a^i_j d^2_0 I^j_n + \frac{1}{2!}(a^i_j a_k + a^i_k a_j) d^2_0 I^j_n d^2_0 I^k_n). \tag{C12}$$

In a slightly more complicated but similar manner, the conformal group in $\mathbb{R}^n$ may be obtained by projecting invertible linear transformations from a higher dimensional space. This group consists of transformations of the form

$$\bar{x}^i = \frac{a^i_j x^j - \frac{1}{2}\eta^{pq} a^i_p a_q \eta_{jk} x^j x^k}{1 - a_l x^l + \frac{1}{4}\eta^{rs} a_r a_s \eta_{lm} x^l x^m}, \tag{C13}$$

where the a^i_j are given by (C6). The subgroup $C^k_{p,q}$ of G^k_n consists of k jets of transformations of the form (C13). In particular, an element of $C^2_{p,q}$ has the form

$$\partial^2_0 I_n (a^i_j d^2_0 I^j_n + \frac{1}{2!}(a^i_j a_k + a^i_k a_j - \eta^{pq} a^i_p a_q \eta_{jk}) d^2_0 I^j_n d^2_0 I^k_n). \tag{C14}$$

The following isomorphisms obtain among the groups introduced above:

$$G^1_n \equiv P^1_n \equiv \Gamma^1_n \cong \Gamma^k_n, \quad C^1_{p,q} \equiv W^1_{p,q} \cong W^k_{p,q},$$
$$O^1_{p,q} \cong O^k_{p,q},$$
$$C^2_{p,q} \cong C^k_{p,q}, \qquad k \geqslant 2$$
$$P^2_n \cong P^\kappa_n, \qquad k \geqslant 2. \tag{C15}$$

In spite of these isomorphisms, it is useful to preserve the distinctions because they permit precise and ready reference to a G structure of a given order in the theory of prolongations of G structures.

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