AIP Journal of Mathematical Physics

Jet bundles and path structures

R. A. Coleman and H. Korte

Citation: Journal of Mathematical Physics **21**, 1340 (1980); doi: 10.1063/1.524598
View online: http://dx.doi.org/10.1063/1.524598
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/21/6?ver=pdfcov
Published by the AIP Publishing

Jet bundles and path structures[a)]

R. A. Coleman

Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7 and Mathematics and Physics Department, Ryerson Polytechnical Institute, 50 Gould St., Toronto, Ontario, Canada M5B 1E8

H. Korte

Department of Philosophy, University of Western Ontario, London, Ontario, Canada and Department of Philosophy, University of Regina, Regina, Saskatchewan, Canada

(Received 27 March 1979; accepted for publication 25 June 1979)

The analysis of path structures is formulated in terms of jet bundles with particular emphasis on the transformation laws and symmetry properties of geodesic path structures. The role played by geodesic path structures in the constructive axioms of Ehlers, Pirani, and Schild for GRT is discussed and it is shown that these axioms are decidable.

1. INTRODUCTION

Ehlers, Pirani, and Schild[1] (EPS) proposed a set of constructive axioms for general relativity theory based on the local behavior of arbitrary massive particles, freely falling massive particles, and light propagation. The analysis of the aspect of spacetime structure revealed by the paths followed by freely falling massive particles leads to the study of path structures on manifolds. Ehlers and Köhler[2] have presented an analysis of path structures and their symmetries using the standard formalism of the first and second order tangent bundles, $T(M)$ and $T(T(M))$, of the spacetime manifold M. However, the simplest and most natural description of structures of higher order contact is in terms of the jets and jet bundles of Ehresmann.[3]

In the present paper, the analysis of curve and path structures is developed using jets. A great simplification, both conceptual and technical, results. Conceptually, the elements of the second order jet bundle $J^2(\mathbb{R}_0, M)$ have a direct interpretation as second degree Taylor approximations to curves through a given point of M. The derivation of the coordinate, parameter, and active transformation laws is a straightforward exercise in the application of the chain law. In contrast, the elements of $T(T(M))$ which in jet language is $J^1(\mathbb{R}_0, J^1(\mathbb{R}_0, M))$, are more complicated. The desired elements of the sub-bundle $J^2(\mathbb{R}_0, M)$ of $J^1(\mathbb{R}_0, J^1(\mathbb{R}_0, M))$, must in the standard approach be selected by imposing the spray condition on the elements of $J^1(\mathbb{R}_0, J^1(\mathbb{R}_0, M))$. Of course, the interpretation of the elements of the subbundle and the discussion of their coordinate, parameter, and active transformation laws is considerably obscured by the use of this indirect approach. The definition and analysis of sprays, called acceleration fields in this paper, are also much simplified by using jets. Moreover, the relationship between these fields and second order differential equations becomes transparent. Finally, the discussion of the corresponding structures for paths is very difficult if the standard approach is used. The discussion in terms of jets is easy in comparison. In the case of geodesic acceleration fields and the projective analog, geodesic directing fields, the description in terms of jets is so much simpler that such fields are readily obtained as cross sections of appropriate fiber bundles. The bundle of geodesic directing fields, $\mathscr{G}\Xi(M)$, (4.22), provides an elegant coordinate free formulation[4] of the second projective axiom of EPS; namely, the directing field which governs the motion of freely falling particles is a cross section of $\mathscr{G}\Xi(M)$.

The definitions and notations for jets and jet bundles are established in Sec. 2. In Sec. 3, curve structures, acceleration fields, and the one to one relationship between them are discussed. Also, geodesic acceleration fields are defined and it is shown how these may be obtained as cross sections of a fiber bundle. The analogous discussion for path structures and directing fields is presented in Sec. 4.

The definitions of active transformations and symmetries of curve and path structures are given in Sec. 5. The discussion is presented for the three customary levels of analysis, global, local, and micro (infinitesimal neighborhood of a point p of M). The formulas for the microtransformations and microsymmetry conditions are particularly relevant for this paper and are presented in detail. These results are then used in Sec. 6 to prove some theorems concerning geodesic curve and path structures and their microsymmetry groups. Theorem 4 states that a curve structure is geodesic if and only if its microsymmetry group is isomorphic to $\mathrm{GL}^1(n)$. The maximal microsymmetry group of a geodesic path structure is derived in Theorem 5. In comparison with the standard treatment of this projective group, the jet bundle language offers a marked improvement in conceptual clarity. Theorems 6 and 7 correspond to Theorems 2 and 3 of Ehlers and Köhler.[2] The first of these theorems states that a path structure which admits a microsymmetry transformation at every point whose first order part is a dilatation other than the identity is geodesic. The proof given by Ehlers and Köhler is reproduced for completeness. The second theorem states that a path structure which is maximally isotropic to first order in the sense that it admits, at every point of the manifold, a microsymmetry group whose first order part acts transitively on the space of one-directions $\mathbb{D}^1_p(M)$ is geodesic and conversely. Ehlers and Köhler present the proof of this theorem only for analytic path structures and for manifold dimension $n = 2$. The proof presented below does not require analyticity (only C^6), and the organization of the proof is sufficiently improved so that it can be written down

a)Supported in part by the Natural Sciences and Engineering Research Council Canada.

in reasonably concise form for the case of arbitrary manifold dimension n.

The geodesic method of EPS has recently been criticized.[5-8] It has been argued that the geodesic method is beset with logical and derivatively with epistemological circularity. Specifically, criteria that determine which bodies are suitable as freely falling test bodies and permit their identification presuppose metrical considerations, thereby leading to circularity. A particle which has a gravitational multipole structure will not in general travel along a timelike geodesic even if no forces act on it. Without already knowing the spacetime structure, how are we to know which particles are gravitational monopoles and which are not?

In Sec. 7, these criticisms are briefly analyzed. It is shown that they rest on a serious misunderstanding of the nature of inertial laws and the geodesic method.

Morever, using radar coordinates and the concept of a directing field, it is shown that the criticisms are without any substance; that is, it is shown that the truth of the projective axioms concerning free fall motion is epistemically decidable in a noncircular way.

2. JETS AND JET BUNDLES

Let M and N be C^∞ differentiable manifolds of dimensions m and n, respectively. Let $(U,x)_p$ and $(V,y)_q$ be charts for neighborhoods $p\in M$ and $q\in N$. The k-jet $j_p^k(f)$ of a C^k map $f:M\to N$ with source $p\in M$ and target $q=f(p)\in N$ is the equivalence class of such maps which agree at the point $p\in M$ and for which the derivatives of the maps $y\circ f\circ x^{-1}$ agree at $x(p)$ up to and including order k. That the equivalence is not dependent on the choice of coordinate charts follows from the chain rule. The set of such k-jets is denoted by $J^k(M_p,N_q)$. If the source, target or both are unrestricted, the sets of k-jets are denoted by $J^k(M,N_q)$, $J^k(M_p,N)$, and $J^k(M,N)$, respectively. These four sets of k-jets are differentiable manifolds, and the coefficients of the k th order Taylor expansion of $y\circ f\circ x^{-1}$ may be used as local coordinates of the k-jet $j_p^k(f)$. Moreover, the source and target maps σ: $J^k(M,N)\to M$ and $\tau: J^k(M,N)\to N$ defined by

$$\sigma(j_p^k(f)) = p,\qquad \tau(j_p^k(f)) = f(p) \tag{2.1}$$

are differentiable.

If $m=n$, denote by $D(M_p,N_q)$ the set of diffeomorphisms $f:M\to N$ such that $f(p)=q$, and by $J^kD(M_p,N_q)$ the set of k-jets $j_p^k(f)$. The Lie group $\mathrm{GL}^k(n)$ is defined to be the set of k-jets $J^kD(R_0^n,R_0^n)$ with the group product defined by k-jet composition

$$j_0^k(f_1)\circ j_0^k(f_2) = j_0^k(f_1\circ f_2). \tag{2.2}$$

This group acts on $J^k(\mathbb{R}_0,\mathbb{R}_0^n)$, the set of k-jets of curves through $\mathbf{0}\in\mathbb{R}^n$, according to

$$j_0^k(f)\circ j_0^k(\gamma) = j_0^k(f\circ\gamma), \tag{2.3}$$

where $\gamma:\mathbb{R}\to\mathbb{R}^n$ and $\gamma(0)=\mathbf{0}$.

A local grid for $p\in M$ is a diffeomorphism $\zeta:\mathbb{R}^n \to \zeta_{\vdash}(\mathbb{R}^n)\subset M$ such that $\zeta(0)=p$. A k-grid is a k-jet $j_0^k(\zeta)$ of a grid. Let $Z^k(M)$ denote the set of k-grids for all $p\in M$. $Z^k(M)$ is the total space of a principal fiber bundle (PFB)

$$\mathscr{Z}^k(M) = \langle Z^k(M),\pi_{Zk},M,\mathrm{GL}^k(n)\rangle, \tag{2.4}$$

where the differentiable projection map $\pi_{Zk}:Z^k(M)\to M$ is defined by

$$\pi_{Zk}(j_0^k(\zeta)) = \zeta(0). \tag{2.5}$$

In view of the action (2.3) of $\mathrm{GL}^k(n)$ on $J^k(\mathbb{R}_0,\mathbb{R}_0^n)$, one may construct the associated fiber bundle (AFB)[9] of k-arcs on M,

$$\mathscr{A}^k(M) = \langle J^k(\mathbb{R}_0,M),\pi_k,M,J^k(\mathbb{R}_0,\mathbb{R}_0^n),\mathscr{Z}^k(M)\rangle, \tag{2.6}$$

with typical fiber $J^k(\mathbb{R}_0,\mathbb{R}_0^n)$. As the notation indicates, the elements of the total space $J^k(\mathbb{R}_0,M)$ may be more directly obtained as the k-jets of curves $\gamma:\mathbb{R}\to M$ in M. The projection map $\pi_k: J^k(\mathbb{R}_0,M)\to M$ is defined by

$$\pi_k(j_0^k(\gamma)) = \gamma(0). \tag{2.7}$$

There is a sequence of natural, differentiable projection maps $\pi_k^l: J^k(\mathbb{R}_0,M)\to J^l(\mathbb{R}_0,M)$ for $1\leqslant l<k$ defined by

$$\pi_k^l(j_0^k(\gamma)) = j_0^l(\gamma). \tag{2.8}$$

In many cases of physical interest, the parameter of a curve is either arbitrary or not specified in advance; for example, in general relativity the world line of a freely falling test particle is determined by a point on it and its direction (a nonzero multiple of its tangent vector) at that point. Since the tangent vectors of physical particles are everywhere nonzero, curves such as $\gamma:\mathbb{R}\to\mathbb{R}^2$ with

$$\gamma(t) = (t^2,t^3) \tag{2.9}$$

need not be considered in the definition of a path ("parameter free curve") for the purposes of this paper.

A parameter transformation is an element of $D(\mathbb{R},\mathbb{R})$. The k-jets $j_0^k(\mu)\in J^kD(\mathbb{R}_0,\mathbb{R}_0)$ for $\mu\in D(\mathbb{R}_0,\mathbb{R}_0)$ form a group P^k where the group product is k-jet composition.

Define an equivalence relation in the set of curves with nowhere vanishing tangent vectors by $\hat\gamma\sim\gamma$ iff $\exists\mu\in D(\mathbb{R}_0,\mathbb{R}_0)$, $\hat\gamma=\gamma\circ\mu$. Then a path is an equivalence class of such curves.

There is an action of the group P^k on $J^k(\mathbb{R}_0,\mathbb{R}_0^n)$ and on $J^k(\mathbb{R}_0,M)$ which will be denoted by R_k in both cases. It is given by

$$R_k(j_0^k(\gamma),j_0^k(\mu)) = j_0^k(\gamma)\circ j_0^k(\mu). \tag{2.10}$$

This right action is compatible with the structure of the bundle $\mathscr{A}^k(M)$; that is, $R_k\circ j^k(f) = j^k(f)\circ R_k$ for $f:M\to M$ and $\pi_k\circ R_k=\pi_k$ and $\pi_k^l\circ R_k = R_l\circ\pi_k^l$. Denote by $\mathbb{D}^k$ and $\mathbb{D}^k(M)$ the sets of equivalence classes of elements of $J^k(\mathbb{R}_0,\mathbb{R}_0^n)$ and $J^k(\mathbb{R}_0,M)$ defined by R_k. These equivalence classes will be called k-directions (or simply directions for $k=1$). Note that 2-directions are called special directions in Ehlers and Köhler.[2]

For $k>1$, the manifold of k-directions $\mathbb{D}^k(M)$ is the total space of an AFB with typical fiber $\mathbb{D}^k$ and PFB $\mathscr{Z}^k(M)$

$$\mathscr{D}^k(M) = \langle\mathbb{D}^k(M),\pi_k,M,\mathbb{D}^k,\mathscr{Z}^k(M)\rangle. \tag{2.11}$$

For $k=1$, the structure group of the bundle is $\mathrm{PG}(n)$, the projective group in n dimensions. $\mathrm{PG}(n)$ is the factor group of $\mathrm{GL}(n)$ with respect to the invariant subgroup of elements of the form $(\lambda\delta^i_j)$ with $\lambda\neq 0$ called dilatations. The appropriate PFB is the bundle of projective 1-grids.

$$\mathscr{P}\mathscr{Z}^1(M) = \langle PZ^1(M),\pi_{Z1},M,\mathrm{PG}(n)\rangle, \tag{2.12}$$

where the elements of the fiber $PZ^1_p(M)$ at $p\in M$ are equivalence classes of 1-grids in $Z^1_p(M)$ related by a dilatation. The AFB of 1-directions is then

$$\mathscr{D}^1(M) = \langle \mathrm{D}^1(M),\pi_1,M,\mathrm{D}^1,\mathscr{P}\mathscr{Z}^1(M)\rangle. \tag{2.13}$$

The above considerations do not require C^∞ manifolds. A differentiability class C^r for some finite r would be sufficient. In the following, it is assumed that mappings are sufficiently differentiable that any derivative maps which occur are at least C^1. It is also assumed that the base manifold M has dimension $n\geqslant 2$.

3. CURVE STRUCTURES

Following Ehlers and Köhler,[2] we restrict the concept of a curve in M, $\gamma:I\to M$, where I is an open interval of $\mathbb{R}$ by requiring: For every $s_1,s_2\in I$ such that $\gamma(s_1)=\gamma(s_2)$ and $\dot\gamma(s_1)=\dot\gamma(s_2)$, there exist open intervals $I_1\ni s_1$ and $I_2\ni s_2$ and a smooth invertible map $\mu:I_1\to I_2$ such that $\mu(s_1)Is_2$ and $\gamma|I_1=(\gamma\circ\mu)|I_1$.

A curve which retraces itself periodically such as $\gamma:\mathbb{R}\to\mathbb{R}^2$

$$\gamma(s) = (\cos s, \sin s) \tag{3.1}$$

is not excluded, nor is a curve which touches itself or retraces a portion of its track in the opposite sense $[\dot\gamma(s_1) = -\dot\gamma(s_2)]$. However, a curve which touches itself or retraces part of its track in the same sense $[\dot\gamma(s_1)=\dot\gamma(s_2)]$ is excluded for the condition is not satisfied at the point (s) where the curve bifurcates. For such a curve, at the point of bifurcation, information of higher order [say $\ddot\gamma(s)$] would be required to determine which branch to follow; consequently, the curve could not satisfy everywhere a differential equation of second order. Note that the condition excludes bifurcation for both increasing and decreasing values of the parameter.

One may also consider curves $\gamma^1:I\to J^1(\mathbb{R}_0,M)$ and $\gamma^2:I\to J^2(\mathbb{R}_0,M)$ given in terms of local coordinates by

$$\begin{aligned}\gamma^1(s) &= (\gamma^{1i}(s),\gamma_1^{1i}(s)),\\ \gamma^2(s) &= (\gamma^{2i}(s),\gamma_1^{2i}(s),\gamma_2^{2i}(s)).\end{aligned} \tag{3.2}$$

For those special curves for which

$$\begin{aligned}\gamma_1^{1i}(s) &= \dot\gamma^{1i}(s),\\ \gamma_1^{2i}(s) &= \dot\gamma^{2i}(s),\\ \gamma_2^{2i}(s) &= \dot\gamma_1^{2i}(s),\end{aligned} \tag{3.3}$$

the γ^1 and γ^2 are called the first and second lifts of the curves $\pi_1\circ\gamma^1:\mathbb{R}\to M$ and $\pi_2\circ\gamma^2:\mathbb{R}\to M$, respectively. If $\gamma=\pi_1\circ\gamma^1$: and $\gamma=\pi_2\circ\gamma^2$:, then one writes

$$\begin{aligned}j^1(\gamma) &= \gamma^1,\\ j^2(\gamma) &= \gamma^2.\end{aligned} \tag{3.4}$$

The relations (3.3) do not hold in general since the coordinates γ_1^i,γ_2^i are defined as derivatives only at a point.

Definition: A curve structure (CS) $\mathscr{C}$, on M is a set of curves in M such that for every element $\gamma^1\in J^1(\mathbb{R}_0,M)$, there exists exactly one maximal curve $\gamma\in\mathscr{C}$ such that $j^1(\gamma)$ passes through γ^1.

Definition: An acceleration field on M is a map $A:J^1(\mathbb{R}_0,M)\to J^2(\mathbb{R}_0,M)$ such that $\pi_2^1\circ A=\mathrm{id}$.

Lemma: Every curve structure on M defines a unique acceleration field on M and conversely.

Given a curve structure $\mathscr{C}$ and $\gamma^1\in J^1(\mathbb{R}_0,M)$, let $\gamma\in\mathscr{C}$ be the unique curve such that $j^1(\gamma)$ passes through γ^1. Then define

$$A(\gamma^1) = j^2_s(\gamma), \tag{3.5}$$

where $s\in I$ is uniquely defined by $j^1(\gamma)(s)=\gamma^1$. Because of the restriction on curves in M stated above, $j^1(\gamma)$ does not self-intersect.

Conversely, in a given coordinate system an acceleration field A is given by

$$\begin{aligned}A(\gamma^{1i},\gamma_1^{1i}) &= (A^i(\gamma^{1i},\gamma_1^{1i}),A^i_1(\gamma^{1i},\gamma_1^{1i}),A^i_2(\gamma^{1i},\gamma_1^{1i}))\\ &= ((\gamma^{1i},\gamma_1^{1i},A^i_2(\gamma^{1i},\gamma_1^{1i})).\end{aligned} \tag{3.6}$$

Then, the initial conditions $\gamma^i(0)=\gamma^{1i}$ and $\dot\gamma^i(0)=\gamma_1^{1i}$ and the differential equation

$$\ddot\gamma^i(s) = A^i_2(\gamma^i(s),\dot\gamma^i(s)) \tag{3.7}$$

determine a unique curve γ up to a translation in parameter space such that $j^1(\gamma)$ passes through $\gamma^1\in J^1(\mathbb{R}_0,M)$.

Unless required for clarity, the superscript denoting the order of the jet and the coordinates of the base point will be suppressed. Let $(U,x)_p$ and $(\bar U,\bar x)_p$ be charts of $p\in M$. Set $X=x\circ\bar x^{-1}$ and $\bar X=\bar x\circ x^{-1}$. Then $X\circ\bar X=\mathrm{id}$ and $\bar X\circ X=\mathrm{id}$ with suitable domain restrictions. The coordinates of a 2-jet with respect to these charts, (γ_1^i,γ_2^i) and $(\bar\gamma_1^i,\bar\gamma_2^i)$, are related by

$$\begin{aligned}\bar\gamma_1^i &= \bar X^i_j\gamma_1^j,\\ \bar\gamma_2^i &= \bar X^i_j\gamma_2^j + \bar X^i_{jk}\gamma_1^j\gamma_1^k,\end{aligned} \tag{3.8}$$

where $(\bar X^i_j,\bar X^i_{jk})\in\mathrm{GL}^2(n)$ is the 2-jet of $\bar X$ at $x(p)$. The coordinates of a 1-jet, (γ_1^i) and $\bar\gamma_1^i$, are related by the first of equations (3.8).

Definition: A geodesic acceleration field Γ: $J^1(\mathbb{R}_0,M)\to J^2(\mathbb{R}_0,M)$ is an acceleration field for which, at each $p\in M$, there is a chart, say $(\bar U,\bar x)_p$, such that

$$\bar\Gamma(\bar\gamma_1^i) = (\bar\gamma_1^i,0). \tag{3.9}$$

This definition is a modern formulation of Weyl's definition of a symmetric linear connection. (See Ref. 14b, Sec. 15, p. 114.)

Theorem 1: An acceleration field is geodesic iff relative to any given chart $(U,x)_p$

$$\Gamma^i_2(\gamma_1^j) = -\Gamma^i_{jk}\gamma_1^j\gamma_1^k, \tag{3.10}$$

where the Γ^i_{jk} are functions only of $p\in M$.

If an acceleration field is geodesic, then relative to some chart $(\bar U,\bar x)_p$, it is given by Eq. (3.9). Then relative to $(U,x)_p$

$$\begin{aligned}\Gamma(\gamma_1^i) &= (\bar X^{-1})(\bar X)\Gamma(\bar X^{-1})(\bar X)(\gamma_1^i)\\ &= (X)\bar\Gamma(\bar\gamma_1^i) = (X)(\bar\gamma_1^i,0)\\ &= (\gamma_1^i,X^i_{lm}\bar X^l_j\bar X^m_k\gamma_1^j\gamma_1^k).\end{aligned} \tag{3.11}$$

Thus (3.10) holds with $\Gamma^i_{jk}=-X^i_{lm}\bar X^l_j\bar X^m_k$ which are functions only of $p\in M$. Conversely, if an acceleration field is given by (3.10), then relative to $(\bar{U},\bar{x})_p$

$$\bar{\Gamma}^i_2(\bar{\gamma}^j_1) = (-\bar{X}^i_l\Gamma^l_{jk}X^j_pX^k_q + \bar{X}^i_{jk}X^j_pX^k_q)\bar{\gamma}^p_1\bar{\gamma}^q_1 \,. \quad (3.12)$$

Consequently, there exist charts in which $\bar{\Gamma}^i_2(\bar{\gamma}^j_2)$ vanishes; namely, those for which

$$\bar{X}^i_{jk} = \bar{X}^i_l\Gamma^l_{jk} \,. \quad (3.13)$$

Geodesic acclaration fields can all be obtained as cross sections of an AFB with PFB $\mathscr{L}^2(M)$. Consider the space of maps $\Gamma: J^1(\mathbb{R}_0,\mathbb{R}^n_0) \to J^2(\mathbb{R}_0,\mathbb{R}^n_0)$ such that $\pi^1_2 \circ \Gamma = \text{id}$ and

$$\Gamma(\gamma^i_1) = (\gamma^i_1, -\Gamma^i_{jk}\gamma^j_1\gamma^k_1) \,, \quad (3.14)$$

where the Γ^i_{jk} are just numbers. This function space is a manifold of dimension $n^2(n+1)/2$ with the global coordinates Γ^i_{jk}. The group $\mathrm{GL}^2(n)$ acts on this space according to

$$[(a)\Gamma(a^{-1})]^i_{jk} = (a^i_l\Gamma^l_{pq} - a^i_{pq})a_j^{-1p}a_k^{-1q} \,. \quad (3.15)$$

Note that this equation has nothing at all to do with the manifold M whereas (3.12) refers to a particular $p \in M$ and the Γ^i_{jk} in (3.12) are functions of p.

Denoted the space of maps defined by (3.14) by GA. Then using the $\mathrm{GL}^2(n)$ action on GA given by (3.15) construct the AFB

$$\mathscr{G}\mathscr{A}(M) = \langle GA(M), \pi_{GA}, M, GA, \mathscr{L}^2(M)\rangle \,. \quad (3.16)$$

Then every geodesic acceleration field on M is given by a cross section $\Gamma: M \to GA(M)$ of $GA(M)$.

4. PATH STRUCTURES

A path in M will be denoted by ξ. That a curve γ is a member of the equivalence class defining ξ will be denoted by $\gamma \in \xi$, or $\xi = [\gamma]$. The k-lift of a curve $\gamma: I \to M$ is the curve $j^k(\gamma): I \to J^k(\mathbb{R}_0, M)$ which defines a curve $j^k_{R_k}(\gamma)$: $I \to \mathbb{D}^k(M)$ by means of the right action R_k of P^k on $J^k(\mathbb{R}_0, M)$. The k-lift of the path ξ in M is the path $j^k(\xi) \equiv [j^k_{R_k}(\gamma)]$ in $\mathbb{D}^k(M)$. (Note that it is not appropriate to define $j^k(\xi)$ to be the path $[j^k(\gamma)]$ since the set of parameter transformations allowed for $[j^k(\gamma)]$ is in general the subset of those for $[\gamma]$ such that $j^k_s(\mu)$ is the identity of P^k.) A general element of $\mathbb{D}^k(M)$ will be denoted by ξ^k. General curves and paths in $\mathbb{D}^k(M)$ may be defined but will not be needed for the purposes of this paper.

Relative to a coordinate chart $(U,x)_p$ for $p \in M$, $\gamma^1 \in J^1(\mathbb{R}_0, M_p)$ and $\gamma^2 \in J^2(\mathbb{R}_0, M_p)$ are determined by the coordinates

$$\gamma^1 = (x^i(p), \gamma^{1i}_1) \,,$$
$$\gamma^2 = (x^i(p), \gamma^{2i}_1, \gamma^{2i}_2) \,. \quad (4.1)$$

In terms of these coordinates, the right actions R_1 and R_2 defined by (2.10) are given by

$$R_1(\gamma^1, j^1_0(\mu)) = (x^i(p), (D\mu)\gamma^{1i}_1) \,,$$
$$R_2(\gamma^2, j^2_0(\mu)) = (x^i(p), (D\mu)\gamma^{2i}_1, (D\mu)^2\gamma^{2i}_2 + (D^2\mu)\gamma^{2i}_1) \,, \quad (4.2)$$

where $D\mu$ and $D^2\mu$ are the first and second derivatives of the parameter transformation at $t = 0$. From (4.2), it is evident that the portion of $\mathbb{D}^1(M)$ over U is covered by the n coordinate charts defined by taking $D\mu = 1/\gamma^{1b}_1$ for $b = 1,...,n$. Similarly, the portion of $\mathbb{D}^2(M)$ over U is covered by the n coordinate charts defined by taking $D\mu = 1/\gamma^{2b}_1$ and

$$D^2\mu = -\gamma^{2b}_2/(\gamma^{2b}_1)^3 \quad (4.3)$$

for $b = 1,...,n$. In general, equations will only be written for the case $b = n$, a case which is particularly apt for discussing timelike paths when $n = 4$. In this case the parameter transformed coordinates are given by

$$\xi^{1i}_1 = \gamma^{1i}_1/\gamma^{1n}_1 \,,$$
$$\xi^{2i}_1 = \gamma^{2i}_1/\gamma^{2n}_1 \quad \xi^{2i}_2 = (\gamma^{2n}_1\gamma^{2i}_2 - \gamma^{2i}_1\gamma^{2n}_2)/(\gamma^{2n}_1)^3 \quad (4.4)$$

and satisfy $\xi^{1n}_1 = 1$, $\xi^{2n}_1 = 1$, $\xi^{2n}_2 = 0$. In terms of local coordinates, elements $\xi^1 \in \mathbb{D}^1_p(M)$ and $\xi^2 \in \mathbb{D}^2_p(M)$ are given by

$$\xi^1 = (x^i(p), \xi^{1\alpha}_1) \,,$$
$$\xi^2 = (x^i(p), \xi^{2\alpha}_1, \xi^{2\alpha}_2) \,, \quad (4.5)$$

where $\alpha = 1,...,n-1$. For convenience, the superscript denoting the order of the element and the coordinates of $p \in M$ will in general be suppressed.

Let $\gamma: I \to M$ be a curve in M. Then the lifted curves in $\mathbb{D}^1(M)$ and $\mathbb{D}^2(M)$ are given by

$$j^1_{R_1}(\gamma)(s) = (x^i \circ \gamma(s), \xi^\alpha_1(s)) \,,$$
$$j^2_{R_2}(\gamma)(s) = (x^i \circ \gamma(s), \xi^\alpha_1(s), \xi^\alpha_2(s)) \,, \quad (4.6)$$

where

$$\xi^\alpha_1(s) = \dot{\gamma}^\alpha(s)/\dot{\gamma}^n(s) \,,$$
$$\xi^\alpha_2(s) = [\dot{\gamma}^n(s)\ddot{\gamma}^\alpha(s) - \dot{\gamma}^\alpha(s)\ddot{\gamma}^n(s)]/[\dot{\gamma}^n(s)]^3 \,. \quad (4.7)$$

Writing $x^i(s) = x^i \circ \gamma(s) = \gamma^i(s)$, one readily obtains

$$\xi^\alpha_1(s) = \frac{dx^\alpha}{dx^n} \,,$$
$$\xi^\alpha_2(s) = \frac{d^2x^\alpha}{(dx^n)^2} \,. \quad (4.8)$$

For the case $n = 4$, $\xi^\alpha_1(s)$ and $\xi^\alpha_2(s)$ are the 3-velocity and 3-acceleration, respectively.

Definition: A path structure (PS), $\mathscr{P}$, on M is set of paths in M such that for every element $\xi^1 \in \mathbb{D}^1(M)$, there exists exactly one maximal path $\xi \in \mathscr{P}$ such that ξ^1 is on $j^1(\xi)$.

Definition: A directing field on M is a map Ξ: $\mathbb{D}^1(M) \to \mathbb{D}^2(M)$ such that $\pi^1_2 \circ \Xi = \text{id}$.

In terms of local coordinates, a directing field Ξ is given by

$$\begin{aligned}\Xi(x^i(p), \xi^\alpha_1) &= (\Xi^i(x^i(p),\xi^\alpha_1), \Xi^\alpha_1(x^i(p),\xi^\alpha_1), \Xi^\alpha_2(x^i(p),\xi^\alpha_1)) \\ &= (x^i(p), \xi^\alpha_1, \Xi^\alpha_2(x^i(p),\xi^\alpha_1)) \,. \end{aligned} \quad (4.9)$$

Lemma: Every PS on M defines a unique directing field on M and conversely.

Given a PS, choose $\xi^1 \in \mathbb{D}^1(M)$ and let $\xi = [\gamma]$ be the unique path determined by $\xi^1 = (x^i(p), \xi^\alpha_1)$. Let $j^2_{R_2}(\gamma)(s) = (x^i \circ \gamma(s), \xi^\alpha_1(s), \xi^\alpha_2(s))$ and let ξ^α_2 be the value of $\xi^\alpha_2(s)$ at p. Then Ξ is defined at ξ^1 by

$$\Xi^\alpha_2(x^i(p), \xi^\alpha_1) = \xi^\alpha_2 \,. \quad (4.10)$$

Conversely, a directing field Ξ determines a PS by means of the differential equation

$$\xi^\alpha_2(s) = \Xi^\alpha_2(x^i(p), \xi^\alpha_1(s)), \quad (4.11)$$

which by means of (4.8) may be reexpressed as

$$\frac{d^2x^\alpha}{(dx^n)^2}=\Xi^\alpha_2\left(x^i,\frac{dx^\alpha}{dx^n}\right). \tag{4.12}$$

The coordinate transformation formulas (3.8) together with (4.4) yield the transformation formulas

$$\bar\xi^\alpha_1=\frac{\bar X^\alpha_n+\bar X^\alpha_\beta\xi^\beta_1}{\bar X^n_n+\bar X^n_\gamma\xi^\gamma_1},$$
$$\bar\xi^\alpha_2=\frac{\bar X^\alpha_\beta\xi^\beta_2+\bar X^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\bar X^\alpha_{n\rho}\xi^\rho_1+\bar X^\alpha_{nn}}{(\bar X^n_n+\bar X^n_\gamma\xi^\gamma_1)^2}-\frac{\bar X^n_\beta\xi^\beta_2+\bar X^n_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\bar X^n_{n\rho}\xi^\rho_1+\bar X^n_{nn}}{(\bar X^n_n+\bar X^n_\gamma\xi^\gamma_1)^2}\bar\xi^\alpha_1. \tag{4.13}$$

Definition: A geodesic directing field $\Pi:\mathbb{D}^1(M)\to\mathbb{D}^2(M)$ is a directing field for which, at each $p\in M$, there is a chart, say $(\bar U,\bar x)_p$, such that

$$\bar\Pi(\bar\xi^\alpha_1)=(\bar\xi^\alpha_1,0). \tag{4.14}$$

Note that every geodesic directing field corresponds to a class of symmetric linear connections which are projectively equivalent. (See Ref. 12, Sec. 22, p. 56.)

Theorem 2: A directing field is geodesic iff relative to any given chart $(U,x)_p$

$$\Pi^\alpha_2(\xi^\beta_1)=\xi^\alpha_1(\Pi^n_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\Pi^n_{n\rho}\xi^\rho_1+\Pi^n_{nn})-(\Pi^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\Pi^\alpha_{n\rho}\xi^\rho_1+\Pi^\alpha_{nn}), \tag{4.15}$$

where the Π^i_{jk} are functions only of $p\in M$ and $\Pi^i_{ji}=0$ [so that $\Pi^n_{n\rho}$ and Π^n_{nn} can be eliminated from (4.15)].

Let Π be a geodesic directing field satisfying (4.14). Then relative to the chart $(U,x)_p$

$$\Pi(\xi^\alpha_1)=(X)(\bar X)\Pi(X)(\bar X)(\xi^\alpha_1)=(X)\bar\Pi(\bar\xi^\alpha_1)=(X)(\bar\xi^\alpha_1,0). \tag{4.16}$$

Using the inverse of (4.13), one obtains

$$\Pi^\alpha_2(\xi^\alpha_1)=\frac{X^\alpha_{\rho\sigma}\bar\xi^\rho_1\bar\xi^\sigma_1+2X^\alpha_{n\rho}\bar\xi^\rho_1+X^\alpha_{nn}}{(X^n_n+X^n_\gamma\bar\xi^\gamma_1)^2}-\frac{X^n_{\rho\sigma}\bar\xi^\rho_1\bar\xi^\sigma_1+2X^n_{n\rho}\bar\xi^\rho_1+X^n_{nn}}{(X^n_n+X^n_\gamma\bar\xi^\gamma_1)^2}\xi^\alpha_1. \tag{4.17}$$

Substitution for $\bar\xi^\alpha_1$ in terms of ξ^α_1 gives

$$\Pi^\alpha_2(\xi^\alpha_1)=-\xi^\alpha_1(X^n_{ij}\bar X^i_\rho\bar X^j_\sigma\xi^\rho_1\xi^\sigma_1+2X^n_{ij}\bar X^i_n\bar X^j_\rho\xi^\rho_1+X^n_{ij}\bar X^i_n\bar X^j_n)+(X^\alpha_{ij}\bar X^i_\rho\bar X^j_\sigma\xi^\rho_1\xi^\sigma_1+2X^\alpha_{ij}\bar X^i_n\bar X^j_\rho\xi^\rho_1+X^\alpha_{ij}\bar X^i_n\bar X^j_n). \tag{4.18}$$

Since $\Gamma^i_{ji}=-X^i_{pq}\bar X^p_j\bar X^q_l$ by (3.11), (4.18) is the same as (4.15) with the Π^i_{jk} replaced by Γ^i_{jk}. However, one can define $\Gamma_i\equiv\Gamma^k_{ki}$ and

$$\Pi^i_{jk}=\Gamma^i_{jk}-[1/(n+1)](\delta^i_j\Gamma_k+\delta^i_k\Gamma_j), \tag{4.19}$$

where $\Pi^i_{ij}=0$. The terms in (4.18) involving Γ_i cancel, giving (4.15).

Conversely, suppose a directing field is given by (4.15). Then apply (4.13) (for simplicity choose $\bar X^\alpha_\beta=\delta^\alpha_\beta$) to obtain

$$\bar\Pi^\alpha_2(\bar\xi^\beta_1)=\Pi^\alpha_2(\xi^\beta_1)+\bar X^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\bar X^\alpha_{n\rho}\xi^\rho_1+\bar X^\alpha_{nn}-\xi^\alpha_1(\Pi^n_2(\xi^\beta_1)+\bar X^n_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+\bar X^n_{n\rho}\xi^\rho_1+\bar X^n_{nn}). \tag{4.20}$$

The right-hand side vanishes for the choice $\bar X^i_{jk}=\Pi^i_{jk}$; so that, a coordinate chart exists in which (4.14) holds.

Geodesic directing fields can all be obtained as cross sections of an AFB with PFB $\mathscr{L}^2(M)$. With apologies for the multiple use of the same symbols, consider the space of maps $\Pi:\mathbb{D}^1(\mathbb{R}^n_0)\to\mathbb{D}^2(\mathbb{R}^n_0)$ such that $\pi^1_2\circ\Pi=\mathrm{id}$ and with $\Pi^\alpha_2(\xi_1{}^\beta)$ given by the expression (4.15) with the understanding that ξ^α_1 denotes an element of $\mathbb{D}^1(\mathbb{R}^n_0)$ [not of $\mathbb{D}^1(M_p)$] and that the Π^i_{jk} are just numbers (not functions of $p\in M$). This function space, denoted by $G\Xi$, is a manifold of dimension $n^2(n+1)/2-4$ (since $\Pi^i_{ij}=0$). Again, there are n coordinate charts. Corresponding to the chart in which $\xi^b_1=1$ and $\xi^b_2=0$, one may choose to eliminate Π^b_{bb} and $\Pi^b_{b\rho}$. An element $(a)\in GL^2(n)$ acts on $G\Xi$ according to

$$\Pi\to(a)\Pi(a)^{-1}. \tag{4.21}$$

The effect of this transformation of the Π^i_{jk} can be found by successive application of (4.13) with the $(\bar X^i_j,\ \bar X^i_{jk})$ replaced by $(a^i_j,\ a^i_{jk})$. Thus one can construct the AFB

$$\mathscr{G}\Xi(M)=\langle G\Xi(M),\pi_{G\Xi},M,G\Xi,\mathscr{L}^2(M)\rangle \tag{4.22}$$

and every geodesic directing field on M is given by a cross section $\Pi:M\to G\Xi(M)$ of $G\Xi(M)$.

Finally, it is clear from Theorem 2 that if Π is a geodesic directing field, then $\Pi^\alpha_2(\xi^\beta_1)$ is a cubic polynomial in $\xi_1{}^\beta$ in every coordinate chart $(U,x)_p$. The converse is also true.

Theorem 3: If with respect to every coordinate chart $(U,x)_p$, the corresponding function $\Xi^\alpha_2(\xi^\beta_1)$ which determines the directing field Ξ is cubic, that is, if

$$\Xi^\alpha_2(\xi^\beta_1)=A^\alpha+B^\alpha_\rho\xi^\rho_1+C^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+D^\alpha_{\rho\sigma\tau}\xi^\rho_1\xi^\sigma_1\xi^\tau_1, \tag{4.23}$$

where the coefficients A, B, C, D are functions only of $p\in M$, then Ξ is geodesic.

Under a coordinate transformation, a directing field tranforms according to

$$\bar\Xi=(\bar X)\Xi(X). \tag{4.24}$$

In terms of the function $\Xi^\alpha_2(\xi^\beta_1)$, this law becomes

$$\bar\Xi^\alpha_2(\bar\xi^\beta_1)=\overline{\Xi^\alpha_2(\xi^\beta_1)}=\frac{\bar X^\alpha_\rho\Xi^\rho_2(\xi^\beta_1)+\bar X^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\bar X^\alpha_{n\rho}\xi^\rho_1+\bar X^\alpha_{nn}}{(\bar X^n_n+\bar X^n_\gamma\xi^\gamma_1)^2}-\frac{\bar X^n_\rho\Xi^\rho_2(\xi^\beta_1)+\bar X^n_{\rho\sigma}\xi^\rho_1\xi^\sigma_1+2\bar X^n_{n\rho}\xi^\rho_1+\bar X^n_{nn}}{(\bar X^n_n+\bar X^n_\gamma\xi^\gamma_1)^2}\bar\xi^\alpha_1, \tag{4.25}$$

where $\overline{\Xi^\alpha(\xi^\beta_1)}$ is given by the second equation of (4.13). The expression for $\bar\Xi^\alpha_2(\bar\xi^\beta_1)$ is obtained by substituting (4.23) into (4.25) and by expressing ξ^α_1 in terms of $\bar\xi^\alpha_1$ using the inverse of the first of Eqs. (4.13). The result is not in general a polynomial unless the coefficients $D^\alpha_{\rho\sigma\tau}$ have the form

$$D^\alpha_{\rho\sigma\tau}=\tfrac{1}{3}(\delta^\alpha_\rho D_{\sigma\tau}+\delta^\alpha_\sigma D_{\rho\tau}+\delta^\alpha_\tau D_{\rho\sigma}). \tag{4.26}$$

However, if this condition is satisfied, then (4.23) may be put into the form (4.15) by redefining the coefficients in the following way. Set

$$B^{\alpha}_{\rho} = 2\tilde{B}^{\alpha}_{\rho} + \delta^{\alpha}_{\rho} B\,, \qquad B = [1/(n+1)] B^{\alpha}_{\alpha}\,, \tag{4.27}$$

and

$$C^{\alpha}_{\rho\sigma} = \tilde{C}^{\alpha}_{\rho\sigma} + \delta^{\alpha}_{\rho} C_{\sigma} + \delta^{\alpha}_{\sigma} C_{\rho}\,, \qquad C_{\rho} = [1/(n+1)] C^{\alpha}_{\alpha\rho}\,. \tag{4.28}$$

Then it is only necessary to make the identifications

$$D_{\rho\sigma} = \Pi^{n}_{\rho\sigma}, \quad \tilde{C}^{\alpha}_{\rho\sigma} = -\Pi^{\alpha}_{\rho\sigma}\,, \qquad \tilde{B}^{\alpha}_{\rho} = -\Pi^{\alpha}_{n\rho}, \quad A^{\alpha} = -\Pi^{\alpha}_{nn}\,, \tag{4.29}$$

from which follow (recall $\Pi^{i}_{ij} = 0$)

$$B = \tilde{B}^{\alpha}_{\alpha} = -\Pi^{\alpha}_{\alpha n} = \Pi^{n}_{nn}\,, \qquad C_{\rho} = \tilde{C}^{\alpha}_{\alpha\rho} = -\Pi^{\alpha}_{\alpha\rho} = \Pi^{n}_{n\rho}\,. \tag{4.30}$$

5. SYMMETRIES OF CURVE AND PATH STRUCTURES

For the examination of differentiable manifolds and for the discussion of the symmetries of geometric objects defined on them, there are three qualitatively different scales to consider; namely, global, local, and micro. In each case, a symmetry is an invertible, active transformation of the manifold which preserves the geometric object when attention is restricted to the appropriate scale. For a given scale, the set of transformations which preserve a given geometric object form a group or pseudogroup called its global, local, or microsymmetry group, respectively. Note that the use of the term "infinitesimal symmetry group" instead of "microsymmetry group" would *incorrectly* suggest that the Lie algebra of some finite group was under consideration. The symmetry groups will be defined for the cases of curve and path structures and for the corresponding acceleration and directing fields, the geometric objects of central interest in this paper; however, similar definitions would apply to any geometric object.[10]

First, consider global symmetries of a CS $\mathscr{C}$. Let $f{:}M \to M$ be a diffeomorphism. Then for every $\gamma \in \mathscr{C}$, $\gamma^f = f \circ \gamma$ is a curve in M and $\mathscr{C}^f = \{\gamma^f | \gamma \in \mathscr{C}\}$ is a CS for M. If $\mathscr{C}^f = \mathscr{C}$, then f is a symmetry of $\mathscr{C}$ and the set of all diffeomorphisms $f{:}M \to M$ such that $\mathscr{C}^f = \mathscr{C}$ is the global symmetry group of $\mathscr{C}$.

Moreover, if $\mathscr{P}$ is a PS on M and $\xi = [\gamma]$ is a path, $\xi \in \mathscr{P}$, then $\xi^f = [f \circ \gamma]$ is a path on M and $\mathscr{P}^f = \{\xi^f | \xi \in \mathscr{P}\}$ is a PS for M. If $\mathscr{P}^f = \mathscr{P}$, then f is a symmetry of $\mathscr{P}$ and the set of all diffeomorphisms f such that $\mathscr{P}^f = \mathscr{P}$, is the global symmetry group of $\mathscr{P}$.

Because of the bijective correspondence between CS's and acceleration fields and between PS's and directing fields, the above definitions may be reformulated in terms of these fields. Let $A{:}J^1(\mathbb{R}_0, M) \to J^2(\mathbb{R}_0, M)$ be the acceleration field corresponding to the CS $\mathscr{C}$. Then the acceleration field A^f corresponding to the CS, $\mathscr{C}^f$ is given by

$$A^f = j^2(f) \circ A \circ j^1(f)^{-1}\,, \tag{5.1}$$

where $j^k(f){:}J^k(\mathbb{R}_0, M) \to J^k(\mathbb{R}_0, M)$ is the k-prolongation of $f{:}M \to M$. The condition that the CS remain invariant under f is

$$A^f = A\,. \tag{5.2}$$

If $\gamma{:}I \to M$ is a curve on M and μ is a parameter transformation, then since

$$j^k(f) \circ j^k_0(\gamma \circ \mu) = j^k(f \circ \gamma) \circ j^k_0(\mu)\,, \tag{5.3}$$

This action of the $j^k(f)$ can be factored by the projective transformations $j^k(\mu)$ to define the action on $\mathbb{D}^k(M)$

$$j^k(f) \circ j^k_{R_k}(\gamma) = j^k_{R_k}(f \circ \gamma)\,. \tag{5.4}$$

Consequently, if $\Xi{:}\mathbb{D}^1(M) \to \mathbb{D}^2(M)$ is the directing field corresponding to the PS, $\mathscr{P}$ then the directing field Ξ^f corresponding to the PS $\mathscr{P}^f$ is given by

$$\Xi^f = j^2(f) \circ \Xi \circ j^1(f)^{-1} \tag{5.5}$$

and the condition for invariance of the PS, $\mathscr{P}$, becomes

$$\Xi^f = \Xi\,. \tag{5.6}$$

If the global diffeomorphism is replaced by a local diffeomorphism $f{:}U \to V$ in the above considerations and if the invariance conditions are applied to the restrictions of curves and paths to U and V, then one refers to the local diffeomorphism f as a local symmetry and the set of such local symmetries forms a local symmetry pseudogroup. If, in addition, the local diffeomorphisms are required to leave some point $p \in M$ fixed, the terms p-local symmetry and p-local symmetry pseudogroup will be used. In this case, the invariance conditions are applied only to those curves and paths which pass through the point p.

The set $J^k D(M_p, M_p)$ of k-jets $j^k_p(f)$ of diffeomorphisms $f{:}M \to M$ which leave $p \in M$ fixed form a finite dimensional Lie group GL^k_p. The group product is k-jet composition. The group GL^k_p is isomorphic to the group $\mathrm{GL}^k(n)$. For $l < k$, there is a natural projection from GL^k_p to GL^l_p which maps $j^k_p(f)$ into $j^l_p(f)$. The group GL^k_p acts on $J^k(\mathbb{R}_0, M_p)$ according to

$$j^k_p(f) \circ j^k_0(\gamma) = j^k_0(f \circ \gamma)\,. \tag{5.7}$$

Again, parameter transformations commute with the action (5.7) so that the group GL^k_p also acts on $\mathbb{D}^k_p(M)$ according to

$$j^k_p(f) \circ j^k_{R_k}(\gamma) = j^k_{R_k}(f \circ \gamma)\,. \tag{5.8}$$

[See (4.6) and (4.7).]

As noted above, a diffemorphism f induces transformations (5.1) and (5.5) of acceleration and directing fields, respectively. If $f(p) = p$, then one may restrict these transformations to the point $p \in M$ to obtain

$$A^f_p = j^2_p(f) \circ A_p \circ j^1_p(f)^{-1}\,, \qquad \Xi^f_p = j^2_p(f) \circ \Xi_p \circ j^1_p(f)^{-1}, \tag{5.9}$$

called the microtransformations at p of the curve and path structures.

Definition: A microsymmetry of CS,$\mathscr{C}$ (or a PS,$\mathscr{P}$) at a point $p\in M$ is an element of GL^2_p which leaves the corresponding acceleration field A (or directing field Ξ) invariant at p. The set of such microsymmetries forms a group which is a Lie subgroup of GL^2_p called the microsymmetry group at p.

The invariance conditions are

$$A^f_p = A_p, \quad \Xi^f_p = \Xi_p. \tag{5.10}$$

Relative to a chart $(U,x)_p$, the microtransformation $j^2_p(f)$ is represented by

$$j^2_{x(p)}(x\circ f\circ x^{-1}) = (f^i{}_j, f^i{}_{jk}), \tag{5.11}$$

where $f^i{}_j$ is the Jacobian and $f^i{}_{jk}$ is the Hessian at $x(p)$. For $(\gamma^i_1)\in J^1(\mathbb{R}_0,M_p)$,

$$\begin{aligned} A^f(\gamma^i_1) &= (f)A(f)^{-1}(\gamma^i_1) \\ &= (f)A(f_j^{-1i}\gamma^j_1) = (f)(f_j^{-1i}\gamma^j_1, A^i_2(f_j^{-1i}\gamma^j_1)) \\ &= (\gamma^i_1, f^i{}_j A^j_2(f_j^{-1i}\gamma^j_1) + f^i{}_{jk} f_l^{-1j} f_m^{-1k}\gamma^l_1\gamma^m_1). \end{aligned} \tag{5.12}$$

Consequently, the transformation law is

$$A^{fi}_2(\gamma^i_1) = f^i{}_j A^j_2(f_j^{-1i}\gamma^j_1) + f^i{}_{jk} f_l^{-1j} f_m^{-1k}\gamma^l_1\gamma^m_1. \tag{5.13}$$

Replacement of γ^i_1 by $f^i{}_j\gamma^j_1$ gives for the invariance condition

$$A^i_2(f^i{}_j\gamma^j_1) = f^i{}_j A^j_2(\gamma^i_1) + f^i{}_{jk}\gamma^j_1\gamma^k_1. \tag{5.14}$$

For an infinitesimal microtransformation

$$(f) = (\delta^i_j + \epsilon F^i{}_j, \epsilon F^i{}_{jk}), \tag{5.15}$$

where ϵ is infinitesimal; consequently, the infinitesimal version of (5.14) is

$$F^k{}_j\gamma^j(\partial/\partial\gamma^k_1)A^i_2(\gamma^i_1) = F^i{}_j A^j_2(\gamma^i_1) + f^i_{jk}\gamma^j_1\gamma^k_1. \tag{5.16}$$

The corresponding formulas for directing fields are obtained as follows. Choose one of the n coordinate charts for $\mathbb{D}^1_p(M)$ and $\mathbb{D}^2_p(M)$ corresponding to $(U,x)_p$, say the nth. Then apply (5.9) in the form

$$\Xi^f(f) = (f)\Xi. \tag{5.17}$$

Using (4.13), one obtains for $\xi_1\in\mathbb{D}^1_p(M)$

$$\begin{aligned} \Xi^f(f)(\xi^\alpha_1) &= \Xi^f\left(\frac{f^\alpha_n + f^\alpha_\beta\xi^\beta_1}{f^n_n + f^n_\gamma\xi^\gamma_1}\right) \\ &= \left(\frac{f^\alpha_n + f^\alpha_\beta\xi^\beta_1}{f^n_n + f^n_\gamma\xi^\gamma_1}, \Xi^{f\alpha}\left(\frac{f^\alpha_n + f^\alpha_\beta\xi^\beta_1}{f^n_n + f^n_\gamma\xi^\gamma_1}\right)\right) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} &(f)\Xi(\xi^\alpha_1) \\ &= (f)(\xi^\alpha_1, \Xi^\alpha_2(\xi^\alpha_1)) = \left(\frac{f^\alpha_n + f^\alpha_\beta\xi^\beta_1}{f^n_n + f^n_\gamma\xi^\gamma_1},\right. \\ &\quad \frac{f^\alpha_\beta\Xi^\beta_2(\xi^\alpha_1) + f^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1 + 2f^\alpha_n\xi^\rho_1 + f^\alpha_{nn}}{(f^n_n + f^n_\gamma\xi^\gamma_1)^2} \\ &\quad - \frac{f^n_\beta\Xi^\beta_2(\xi^\alpha_1) + f^n_{\rho\sigma}\xi^\rho_1\xi^\sigma_1 + 2f^n_{n\rho}\xi^\rho_1 + f^n_{nn}}{(f^n_n + f^n_\gamma\xi^\gamma_1)^2} \\ &\quad \times\left.\left(\frac{f^\alpha_n + f^\alpha_\beta\xi^\beta_1}{f^n_n + f^n_\gamma\xi^\gamma_1}\right)\right). \end{aligned} \tag{5.19}$$

Using the convention $\xi^n_1 = 1$, the result may be expressed more compactly as

$$\begin{aligned} &\Xi^{f\alpha}_2\left(\frac{f^\alpha_i\xi^i_1}{f^n_i\xi^i_1}\right) \\ &= \frac{f^\alpha_\beta\Xi^\beta_2(\xi^\alpha_1)f^n_i\xi^i_1 - f^n_\beta\Xi^\beta_2(\xi^\alpha_1)f^\alpha_i\xi^i_1}{(f^n_i\xi^i_1)^3} \\ &\quad + \frac{f^\alpha_{jk}\xi^j_1\xi^k_1 f^n_i\xi^i_1 - f^n_{jk}\xi^j_1\xi^k_1 f^\alpha_i\xi^i_1}{(f^n_i\xi^i_1)^3}. \end{aligned} \tag{5.20}$$

The invariance condition corresponding to (5.14) is obtained by replacing Ξ^f by Ξ in (5.20). Finally, using (5.15), one obtains for the infinitesimal version of the invariance condition

$$\begin{aligned} &\frac{\partial\Xi^\alpha_2}{\partial\xi^\beta_1}(\xi^\alpha_1)[F^\beta_\gamma\xi^\gamma_1 + F^\beta_n - \xi^\beta_1(F^n_\gamma\xi^\gamma_1 + F^n_n)] \\ &\quad + 2\Xi^\alpha_2(\xi^\alpha_1)[F^n_\gamma\xi^\gamma_1 + F^n_n] \\ &\quad + \Xi^\beta_2(\xi^\alpha_1)[F^n_\beta\xi^\alpha_1 - F^\alpha_\beta] \\ &\quad\quad = F^\alpha_{\rho\sigma}\xi^\rho_1\xi^\sigma_1 + 2F^\alpha_{n\rho}\xi^\rho_1 + F^\alpha_{nn} \\ &\quad\quad\quad - \xi^\alpha_1[F^n_{\rho\sigma}\xi^\rho_1\xi^\sigma_1 + 2F^n_{n\rho}\xi^\rho_1 + F^n_{nn}]. \end{aligned} \tag{5.21}$$

6. SYMMETRIES OF GEODESIC CURVE AND PATH STRUCTURES

In this section, a number of theorems are stated and proved which serve to characterize geodesic curve and path structures geometrically in terms of their microsymmetry groups.

Theorem 4: A curve structure $\mathscr{C}$ is geodesic if and only if its microsymmetry group for every $p\in M$ is a subgroup of GL^2_p isomorphic to $GL^1(n)$.

Let A be the acceleration field corresponding to a geodesic CS,$\mathscr{C}$. Then with respect to any chart $(U,x)_p$

$$A^i_2(\gamma_1) = -\Gamma^i{}_{jk}\gamma^j_1\gamma^k_1. \tag{6.1}$$

Substitution of (6.1) into the invariance condition (5.14) gives

$$f^i{}_{jk} = f^i_l\Gamma^l{}_{jk} - \Gamma^i{}_{lm}f^l{}_j f^m_k. \tag{6.2}$$

Thus the microsymmetry group is the subgroup of GL^2_p of elements of the form

$$(f^i{}_j, f^i_l\Gamma^l{}_{jk} - \Gamma^i{}_{lm}f^l{}_j f^m_k). \tag{6.3}$$

It is straightforward to verify that this subgroup of GL^2_p is isomorphic to $GL^1(n)$.

Conversely, assume that the microsymmetry group is isomorphic to $GL^1(n)$. An infinitesimal element of GL^2_p has the form (5.15). For any element in the microsymmetry group, the $F^i{}_{jk}$ are determined by the $F^i{}_j$. The product of two such elements is

$$(\delta^i_j + \epsilon(F^i{}_j + G^i{}_j), \epsilon(\alpha^i{}_{jk}(F^r_s) + \alpha^i{}_{jk}(G^r_s)). \tag{6.4}$$

Closure requires linearity

$$\alpha^i_{jk}(F^r_s + G^r_s) = \alpha^i_{jk}(F^r_s) + \alpha^i_{jk}(G^r_s) \tag{6.5}$$

and $\alpha^i{}_{jk}(F^r{}_s)$ vanish when $F^r{}_s$ vanish since the identity element is $(\delta^i{}_j,0)$. Thus

$$\alpha^i{}_{jk}(F^r_s) = \alpha^i{}_{jkr}{}^s F^r{}_s , \tag{6.6}$$

where the $\alpha^i{}_{jkr}{}^s$ depend only on the point $p\in M$.

Now assume that $A^i{}_2(\gamma^i_1)$ is at least C^4 and set

$$A^i{}_2(\gamma^i_1) = A^i + B^i{}_j\gamma^j_1 + C^i{}_{jk}\gamma^j_1\gamma^k_1 + w^i(\gamma^i_1), \tag{6.7}$$

where A^i,$B^i{}_j$, and $C^i{}_{jk}$ depend only on $p\in M$ and $w^i(\gamma^i_1)$ is of order $(\gamma^i_1)^3$. Substitute (6.7) and (6.6) into (5.1) and note that the $F^i{}_j$ are arbitrary. Set the coefficients of $F^i{}_j$ equal to zero to obtain

$$\begin{aligned}\gamma^s[B^i{}_r + 2C^i{}_{jr}\gamma^j_1 + w^i{}_{,r}(\gamma^i_1)]\\ = \delta^i_r[A^s + B^s{}_j\gamma^j_1 + C^s{}_{jk}\gamma^j_1\gamma^k_1 + w^s(\gamma^i_1)]\\ + \alpha^i{}_{jkr}{}^s\gamma^j_1\gamma^k_1 .\end{aligned} \tag{6.8}$$

Equating the coefficients of terms of corresponding order, one obtains

$$\begin{aligned}&A^s = 0 ,\\ &\delta^s_j B^i{}_r = -\delta^i_r B^s{}_j ,\\ &\alpha^i{}_{jkr}{}^s = -\delta^i_r C^s{}_{jk} + \delta^s_j C^i{}_{rk} + \delta^s_k C^i{}_{jr} ,\\ &\gamma^s_1 w^i{}_{,r}(\gamma^i_1) = \delta^i_r w^s(\gamma^i_1) .\end{aligned} \tag{6.9}$$

From the last equation of (6.9), $w^i{}_{,r} = 0$ for $i\neq r$; so that, $\forall i$ the ith component of w depends only on the ith component of γ_1. But then choosing $r = i$ and $s\neq i$, w^s would have to be of first order in γ^s_1 contrary to assumption; consequently, $w^i(\gamma_1) = 0$. From the second equation of (6.9) by contracting on s and j

$$B^i{}_r = \delta^i_r(1/n)B^s{}_s = \delta^i_r B . \tag{6.10}$$

The third equation of (6.9) shows that the $\alpha^i{}_{jkr}{}^s$ have the form required in order that $(\delta^i{}_j + \epsilon F^i{}_j, \epsilon\alpha^i{}_{jkr}{}^s F^r{}_s)$ is a microsymmetry group element which is the infinitesimal version of (6.3) where

$$\alpha^i{}_{jk} = C^i{}_{jk} = -\Gamma^i{}_{jk} . \tag{6.11}$$

Using these results in (6.9), one obtains

$$A^i{}_2(\gamma^i_1) = B\gamma^i_1 - \Gamma^i{}_{jk}\gamma^j_1\gamma^k_1 . \tag{6.12}$$

The CS defined by (6.12) is geodesic since the term containing B can be eliminated by a suitable choice of parameter.

The fact that the microsymmetry group of a geodesic CS is isomorphic to $GL^1(n)$ is closely related to the existence of affine normal coordinates[11] and the fact that such coordinates are unique up to a $GL^1(n)$ transformation.

The next theorem characterizes the maximal microsymmetry group of a geodesic path structure $\mathscr{P}$ with corresponding directing field Ξ.

Theorem 5: If a path structure $\mathscr{P}$ is geodesic then its microsymmetry group for every $p\in M$ is a subgroup of GL^2_p isomorphic to the subgroup of $GL^2(n)$ with elements of the form

$$(a^i{}_j, a^i{}_j a_k + a^i_k a_j) . \tag{6.13}$$

The proof of this theorem is tedious but straightforward. Consider an arbitrary infinitesimal element (5.15) of GL^2_p. To be an element of the microsymmetry group $\mathscr{P}$, the parameters $F^i{}_j$ and $F^i{}_{jk}$ must satisfy the invariance condition (5.21) for arbitrary ξ^α_1 where the $\Xi^\alpha_2(\xi^\alpha_1)$ are given by the expression (4.15) for $\Pi^\alpha_2(\xi^\alpha_1)$ in terms of the $\Pi^i{}_{jk}$ which are known and depend only on $p\in M$. Carrying out the substitutions, one obtains a polynomial in ξ^α_1 which must vanish for arbitrary ξ^α_1. Equating the coefficients of this polynomial to zero yields the various components of the relation

$$F^i{}_{jk} = F^i_l\Pi^l{}_{jk} - F^l{}_j\Pi^i_{lk} - F^l_k\Pi^i{}_{jl} + \delta^i{}_j F_k + \delta^i_k F_j . \tag{6.14}$$

Consequently, the $F^i{}_{jk}$ are determined in terms of the parameters $F^i{}_j$ and F_i which may be chosen arbitrarily. To organize the computation for (6.14), it is useful to define

$$\begin{aligned}&F_i = [1/(n+1)]F^l_{li} ,\\ &\hat F^i{}_{jk} = F^i{}_{jk} - (\delta^i{}_j F_k + \delta^i_k F_j),\end{aligned} \tag{6.15}$$

so that $\hat F^l_{li} = 0$. After substituting for $F^i{}_{jk}$ in terms of $\hat F^i{}_{jk}$ and F_i, the terms containing F_i drop out and the $\hat F^i{}_{jk}$ are determined by the first part of (6.14) involving the $\Pi^i{}_{jk}$. It is also useful to recall that $\Pi^l_{li} = 0$.

The finite form of the microtransformation (5.15) with $F^i{}_{jk}$ given by (6.14) is

$$(f^i{}_j, f^i_l\Pi^l{}_{jk} - \Pi^i_{lm}f^l{}_j f^m_k + f^i{}_j f_k + f^i{}_k f_j) . \tag{6.16}$$

It is a straightforward matter to verify that the subgroup of GL^2_p of such elements is isomorphic to the projective subgroup of $GL^2(n)$ with elements given by (6.13).

Corresponding to the normal coordinates of a space with a geodesic curve structure, there are for a space with a geodesic path structure special projective normal coordinates[12] determined up to a projective transformation.

Consider the action of $GL^1(n)$ on a flat n-dimensional affine space. Straight lines through the origin are mapped into straight lines through the origin. Moreover, the dilatation subgroup of $GL^1(n)$ of elements $(e^s\delta^i{}_j)$ for $s\in\mathbb{R}$ (one might also include reflections) maps each straight line through the origin into itself. The following theorem states that if the paths of a path structure are straight to second order at every point $p\in M$, then the path structure is geodesic.

Theorem 6: If a PS, $\mathscr{P}$, admits at every $p\in M$ a microsymmetry $j^2_p(f)\in GL^2_p$ with $j^1_p(f) = (\lambda\delta^i{}_j)$ and $\lambda\neq 1$, then $\mathscr{P}$ is geodesic and conversely.

The converse follows from Theorem 5. Let $j^2_p(f)$ be a microsymmetry of $\mathscr{P}$ and let the corresponding directing field be Ξ. The invariance condition is given by (5.20) with $\Xi^{f\alpha}_2 = \Xi^\alpha_2$. Since $f^i{}_j = \lambda\delta^i{}_j$ with $\lambda\neq 0$ and $\lambda\neq 1$,

$$\Xi^\alpha_2(\xi^\alpha_1) = [1/(\lambda^2 - \lambda)][f^\alpha_{jk}\xi^j_1\xi^k_1 - \xi^\alpha_1 f^n_{jk}\xi^j_1\xi^k_1], \tag{6.17}$$

which is of the required form (4.15).

The following theorem states that if a path structure $\mathscr{P}$ is microisotropic to first order, then it is geodesic.

Theorem 7: If a PS, $\mathscr{P}$, admits at every $p\in M$ a microsymmetry group $G_p(\mathscr{P})$, a subgroup of GL^2_p, which induces a transitive action on $\mathbb{D}^1_p(M)$, then $\mathscr{P}$ is geodesic and conversely.

Again, the converse follows easily from Theorem 5. Suppose, then, that $G_p(\mathscr{P})$ induces a transitive action on

$\mathbb{D}_p^1(M)$. An arbitrary infinitesimal element of GL_p^2 is given by (5.15). For every such element that is an element of the microsymmetry group $G_p(\mathscr{P})$, the directing field Ξ of $\mathscr{P}$ satisfies the constraints (5.21). Since, $G_p(\mathscr{P})$ acts transitively on $\mathbb{D}_p^1(M)$, an n-dimensional projective space, and since dilatations do not affect points of $\mathbb{D}_p^1(M)$, the $F^i{}_j$ in (5.21) may be chosen arbitrarily up to a dilatation. In particular, if

$$F^i{}_j = \chi^i{}_j + (1/n)\delta^i_j\,\chi\,, \qquad \chi^i_i = 0, \quad \chi = F^i{}_i. \tag{6.18}$$

Then the $\chi^n_\alpha, \chi^\alpha_n$ and χ^α_β may be chosen arbitrarily. Now, assume that $\Xi^\alpha_2(\xi_1)$ is at least C^6 and expand in a Taylor series about $\xi^\alpha_1 = 0$,

$$\Xi^\alpha_2(\xi_1) = A^\alpha + B^\alpha{}_\rho\xi^\rho_1 + C^\alpha{}_{\rho\pi}\xi^\rho_1\xi^\pi_1 + D^\alpha{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 + E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 + w^\alpha(\xi_1)\,, \tag{6.19}$$

where it is assumed that $w^\alpha(\xi_1)$ is of order five in the variables $\xi^\alpha_1(\alpha = 1,...,n-1)$. Substitute (6.19) into (5.21) and pick out the terms of order at least four in ξ^α_1. Expressed in terms of the $\chi^n_\alpha, \chi^\alpha_n\chi^\alpha_\beta$, and χ, the result, which does not depend on the $F^i{}_{jk}$, is

$$\begin{aligned}
&\chi^\beta_n[w^\alpha_{,\beta}(\xi_1)] + \chi^n_\beta[-\xi^\beta_1\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) + 2\xi^\beta_1 w^\alpha(\xi_1) \\
&+ \xi^\alpha_1 w^\beta(\xi_1) - 2\xi^\beta_1 E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 + \xi^\alpha_1 E^\beta{}_{\rho\pi\sigma\tau} \\
&\times\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 + \xi^\alpha_1 D^\beta{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 - \xi^\beta_1 D^\alpha{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1] \\
&+ \chi^\beta_\gamma[-\xi^\gamma_1 w^\alpha_{,\beta}(\xi_1) + \delta^\gamma_\beta\xi^\eta_1 w^\alpha_{,\eta}(\xi_1) \\
&- \delta^\alpha_\beta w^\gamma(\xi_1) - \delta^\gamma_\beta w^\alpha(\xi_1) + 4\xi^\gamma_1 E^\alpha{}_{\rho\pi\sigma\beta}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 \\
&- \delta^\alpha_\beta E^\gamma{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 + 2\delta^\gamma_\beta E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1] \\
&+ (1/n)\chi[-2\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) + E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1] = 0\,.
\end{aligned} \tag{6.20}$$

Since $\chi^\beta_n, \chi^n_\beta$, and χ^β_γ may be chosen arbitrarily, for arbitrary parameters λ_β, μ^β, and ν^γ_β which depend only on $p\in M$, one obtains the relations

$$w^\alpha_{,\beta}(\xi_1) = \lambda_\beta[-2\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) + E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1]\,, \tag{6.21}$$

$$\begin{aligned}
&-\xi^\beta_1\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) + 2\xi^\beta_1 w^\alpha(\xi_1) + \xi^\alpha_1 w^\beta(\xi_1) \\
&-2\xi^\beta_1 E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 + \xi^\alpha_1 E^\beta{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 \\
&+ \xi^\alpha_1 D^\beta{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 - \xi^\beta_1 D^\alpha{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 \\
&\quad = \mu^\beta[-2\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) + E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1]\,,
\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}
&-\xi^\gamma_1 w^\alpha_{,\beta}(\xi_1) + \delta^\gamma_\beta\xi^\eta_1 w^\alpha_{,\eta}(\xi_1) - \delta^\alpha_\beta w^\gamma(\xi_1) - \delta^\gamma_\beta w^\alpha(\xi_1) \\
&\quad + 4\xi^\gamma_1 E^\alpha{}_{\rho\pi\sigma\beta}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 - \delta^\alpha_\beta E^\gamma{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 \\
&\quad + 2\delta^\gamma_\beta E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 \\
&\quad = \nu^\gamma_\beta[-2\xi^\eta_1 w^\alpha_{,\eta}(\xi_1) + E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1]\,.
\end{aligned} \tag{6.23}$$

The terms of order four of (6.22) give

$$\xi^\alpha_1 D^\beta{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 - \xi^\beta_1 D^\alpha{}_{\rho\pi\sigma}\xi^\rho_1\xi^\pi_1\xi^\sigma_1 = \mu^\beta E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1\,. \tag{6.24}$$

Thus

$$\mu^\alpha E^\beta{}_{\rho\pi\sigma\tau} + \mu^\beta E^\alpha{}_{\rho\pi\sigma\tau} = 0\,. \tag{6.25}$$

Suppose $\exists\alpha\; E^\alpha{}_{\rho\pi\sigma\tau}\neq 0$. Then (6.25) for $\beta = \alpha$ gives $\mu^\alpha = 0$ and then (6.25) for $\beta\neq\alpha$ gives $\mu^\beta = 0$. On the other hand, suppose $\exists\alpha\;\mu^\alpha\neq 0$. Then (6.25) for $\beta = \alpha$ gives $E^\alpha{}_{\rho\pi\sigma\tau} = 0$ and (6.25) for $\beta\neq\alpha$ gives $E^\beta{}_{\rho\pi\sigma\tau} = 0$.

Consequently, the right side of (6.24) must vanish, and it follows that

$$D^\alpha{}_{\rho\pi\sigma} = \tfrac{1}{3}[\delta^\alpha_\rho D_{\pi\sigma} + \delta^\alpha_\pi D_{\rho\sigma} + \delta^\alpha_\sigma D_{\rho\pi}]\,. \tag{6.26}$$

Moreover, one must have $\forall_\alpha E^\alpha{}_{\rho\pi\sigma\tau} = 0$, for if $\exists\delta$ $E^\delta{}_{\rho\pi\sigma\tau}\neq 0$ then $\forall_\alpha\;\mu^\alpha = 0$ and the right side of (6.22) vanishes. The terms of order five of (6.22) then give for any α, β

$$-2\xi^\beta_1 E^\alpha{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 + \xi^\alpha_1 E^\beta{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 = 0 \tag{6.27}$$

and for $\alpha = \beta = \delta$ this gives

$$\xi^\delta_1 E^\delta{}_{\rho\pi\sigma\tau}\xi^\rho_1\xi^\pi_1\xi^\sigma_1\xi^\tau_1 = 0 \tag{6.28}$$

whence $E^\delta{}_{\rho\pi\sigma\tau} = 0$ which contradicts the assumption. Thus $\forall_\alpha E^\alpha{}_{\rho\pi\sigma\tau} = 0$.

Next (6.21) gives

$$w^\alpha_{,\beta}(\xi_1) = -2\lambda_\beta\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1)\,. \tag{6.29}$$

Contraction with ξ^β_1 gives

$$(1 + 2\xi^\beta_1\lambda_\beta)\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) = 0\,. \tag{6.30}$$

Consequently, since $w^\alpha_{,\beta}(\xi_1)$ is C^1,

$$\xi^\gamma_1 w^\alpha_{,\gamma}(\xi_1) = 0\,, \quad w^\alpha_{,\beta}(\xi_1) = 0\,. \tag{6.31}$$

Finally, (6.23) gives

$$\delta^\alpha_\beta w^\gamma(\xi_1) + \delta^\gamma_\beta w^\alpha(\xi_1) = 0\,, \tag{6.32}$$

and by contraction of γ and β

$$w^\alpha(\xi_1) = 0\,. \tag{6.33}$$

Since it has been shown that $\Xi^\alpha_2(\xi_1)$ is a polynomial of degree at most three, the proof may be completed by appealing to Theorem 3 above in Sec. 4. Alternatively, since (6.26) is just (4.26), the redefinition argument following (4.26) may be applied directly.

7. DECIDABILITY OF THE CONSTRUCTIVE AXIOMS OF GRT

Recent criticisms of the geodesic method of EPS were outlined in the Introduction. Before proving their invalidity, we shall briefly analyze their philosophical basis and contrast the latter with the conceptual motivation, significance, and aim of the constructive axiomatics of EPS. This will clarify to what extent the work of EPS constitutes a solution to the controversy between realism and geometric conventionalism in favor of realism.

Einstein suggested the distinction between *principle* theories and *constructive* theories.[13] The aim of a constructive theory is to reduce a wide class of diverse complex physical processes to simpler ones. Our understanding of the former is constructed out of hypotheses concerning the latter; for example, the kinetic theory of gases constructs mechanical, thermal, and diffusional processes from the hypothesis of molecular motion. On the other hand, a principal theory postulates abstract structural constraints which events are held to satisfy. Einstein's example is the classical theory of thermodynamics.

The special and general theories of relativity are principle theories of spacetime structure. The four dimensional pseudo-Riemannian manifold is the mathematical model of the physical spacetime of the theory of general relativity. It was Weyl who first distinguished between two more primitive structures of the model: the *conformal* structure, and the *projective* structure of paths defined by the set of all unparametrized geodesics.[14]

Weyl suggested that the conformal structure represents the *causal* structure and may be identified with the propagation of light, and that the projective structure represents the *inertial* structure of spacetime that is revealed by the *path* structure of free fall motions of suitable test particles.

Using these structures and their compatibility relation, Ehlers, Pirani, and Schild[1] have derived a unique Riemannian spacetime metric solely as a consequence of a set of "geometry free" axioms concerning the incidence and differential-topological properties of light propagation and free fall.

The "geometry free" axioms are propositions about a few general qualitative assumptions concerning free fall motion and light propagation that can be verified directly through experience in a way that does not presuppose the full blown edifice of the theory of general relativity. From these axioms, the theoretical basis of the theory is reconstructed step by step. Following Reichenbach,[15] EPS call their approach *constructive axiomatics*.

The aim of a constructive axiomatic approach to a principle theory of space–time is to exhibit the physical basis for the particular structural constraints which the principle theory postulates certain events must satisfy. The structures contained in the mathematical model of a principle theory should all have in principle a link to physical experience. Spacetime models with inherent structures that do not relate to experience (e.g., absolute time) are defective for that reason.[16] Hence, it must be theoretically possible, that is, possible in principle, to relate the various structures to experience in a way that is consistent with the theory.

Hence a constructive axiomatic approach should satisfy the basic requirement of any *proper* and *complete* theory. *Completeness* requires that the reconstruction of the various structures inherent in the mathematical model of a principle theory of spacetime be realizable by means of relatively simple physical systems that are themselves well defined within the specific theory being considered, that is, that can be considered as an interpretation of the inherent structures of the spacetime model and are consistent with the theoretical consequences of the theory which presupposes that model. Einstein was well aware of this problem and considered the use of clocks and rigid rods an undesirable makeshift.[17] Unlike light propagation and freely falling particles, rigid rods and ideal clocks are relativistically ill defined and are thus unsuitable for the determination of the inherent structures of the spacetime of general relativity. The concepts of a theory, its formulation and measuring devices should all lead to a unified, self-sufficient and conceptually coherent world picture. There are essentially two types of conventionalist viewpoints. The less radical type may be called *epistemological conventionalism*. On this view, observationally indistinguishable theories may utilize alternative geometries, but it is in principle not possible to single out that theory whose underlying geometry is the true geometry of the world. Any such decision, whether or not it is guided by criteria of simplicity, is essentially epistemically conventional. Epistemological conventionality permits the existence of a true geometry, but access to it is not possible in a nonconventional manner.

Ontological conventionalism asserts that the continuous spacetime manifold is metrically amorphous. All nontopological structures are *extrinsic* to spacetime and are stipulated by means of the behavior of material entities such as clocks, light rays, and geodesic particles; that is, the metric structure of spacetime is always relative to which class of material entities is chosen as the standard of measurement (which choice is arbitrary). According to this view; metrical relations within spacetime reduce to the relations of the chosen material standards of measurement; that is, the latter are ontologically *constitutive* of the former.

We are now able to see what the criticisms leveled against EPS really amount to. The charge of epistemic circularity is directed against the geodesic method because the latter employs the concept of free fall as a standard of inertial motion. The criticism is thus essentially about the status of the infinitesimal law of inertia. Since, as the argument goes, the inertial law does not by itself furnish *independent* criteria by which one can decide when a test particle is free, it is considered to be conventional in character. But this reasoning rests on a serious misunderstanding of both the law of inertia and the geodesic method which employs it.

First, the essential idea of the geodesic method is to *discover* through the behavior of physical systems various intrinsic, primitive geometrical spacetime structures. It is in spirit analogous to Helmholtz's procedure of deducing the existence and form of the metric of physical space.[18] Helmholtz asked "what must the geometric structure of space be in order that a mechanics of rigid bodies is realizable in that space?" Thus Helmholtz is essentially asking what abstract structural constrainst must a principle theory of mechanics postulate that certain events must satisfy. According to Helmholtz, the structure of space follows from the possibility of congruent transport of rigid bodies; that is, the structure of space constitutes a necessary condition for the possibility of the realizability of certain physical processes and operations within that space; in particular, whether or not space possesses a constant curvature, or whether space is a general Riemannian space depends on whether or not physics allows the introduction of ideal rigid bodies.

The structure of space is, according to Helmholtz, the framework for possible physical laws. Certain types of laws presuppose certain types of spaces. Hence, on this view, the law of inertia presupposes an affine structure and may thus be regarded as a geometrical statement.

The conventionalist view that considers the behavior of material entities as being ontologically constitutive of the metrical structure of spacetime is clearly at variance with the notion of a principle theory. It is clear that the views of Weyl and Helmholtz are directly opposed to those of ontological conventionalism. According to Weyl, "... the behavior of rigid bodies and clocks is almost exclusively determined through the metric structure, as is the pattern of the motion of a force free mass point and the propagation of a light source. And only through these effects on the concrete natural processes can we recognize this structure."[19] Thus according to Weyl we *discover* through the behavior of physical phenomena an *already determined* metrical structure of spacetime; that is, the metrical relations of physical objects are determined by the second rank *physical* metric tensor field which is only revealed by, not defined by, those relations. Although distinct from physical objects in space–time, the metric tensor *explains* the geometric relations between them.

Secondly, Newton's first law and the corresponding infinitesimal version thereof, is physically realized by a suitable class of objects in free motion. These laws are geometrical statements concerning the underlying spacetime structure. The inertial laws serve to define an affine structure on the spacetime manifold. It is the affine structure that plays the essential role in the formulation of all physical laws that are expressed in terms of differential equations. In both Newtonian physics and general relativity, all dynamical laws presuppose that structure. Now, inability to identify or single out a class of suitable test objects in an epistemologically noncircular way whose free motion exhibit the projective structure of spacetime means only that the truth of the axioms concerning free fall is epistemically undecidable. But any argument from the epistemic inaccessibility of free test particles—even if this inaccessibility has a sound logical and physical basis—does not establish that the structures derived from the axioms are ontologically conventional. The most that is entailed is epistemological conventionality.

However, epistemological conventionality permits the assertion of the truth of the axioms and hence the inference from them to a unique metric structure at least in this conditional sense:

> If the geometry-free axioms are true of the world and are hence satisfied by an actual or possible nonempty class of suitable test objects (light rays and symmetric, nonrotating, neutral, freely falling particles), then there exists a unique and intrinsic spacetime metric.

The truth of this conditional claim is *incompatible* with the truth of ontological conventionalism, for if the latter were true, then there could be *no factual* reasons, known or unknown, for preferring one metric over another. But EPS have at least shown that certain facts, *if known*, would single out a unique intrinsic metric. That we may not perhaps avail ourselves of these facts in an epistemically noncircular way supports only epistemological conventionalism.

We shall now show that one does have epistemic access to freely falling particles in a way that does not beset the geodesic method of EPS with either logical or epistemological circularity. First, note that freely falling particles are *not* required to construct the radar coordinate systems. For this purpose, any massive particles may be employed. Then, relative to such a coordinate system the trajectory of any other particle may be determined.

If the motion of a particle is governed by a directing field Ξ, then, by definition, such a particle's spacetime trajectory is determined uniquely by an event on the trajectory and its direction at that event. Assume that there are many particles governed by a given directing field Ξ if there are any at all. Then collections of particles corresponding to various directing fields can be built up by means of the following comparison procedure. Two particles belong to the same directing field class if and only if whenever they are launched from infinitesimally neighboring spacetime events with directions which differ only infinitesimally, their subsequent spacetime trajectories remain infinitesimally near. Here, the notion of near does not require a metric. Only an appeal to the differentiable structure of the manifold is required. The fact that in practice such a differentiable topological concept of nearness would require limiting sequences of experiments[20] would only complicate the matching procedure. Note that requiring the directions to differ only infinitesimally does not presuppose a connection since the infinitesimal transformation has been left arbitrary. This matching procedure permits the separation of particles into classes, each class associated with a distinct directing field. The EPS axiom regarding the existence of freely falling particles asserts the existence of at least one such class.

Particles with higher order gravitational multipole moments can almost be eliminated from consideration at this point. One would expect that their spacetime trajectories would not be uniquely determined solely by an event on the trajectory and the direction at the event but would also depend on the orientation of the multipole moment as is the case for particles with higher electromagnetic multipole moments. The motion of such particles would not be governed by a directing field and the above matching procedure would fail. The analyses of the motion of particles with gravitational multipole moments,[21] both relativistic and nonrelativistic, indicate that the motion of such particles is indeed not governed by directing fields; however, it is not possible to rely on such analyses here because they presuppose a metric. Consequently, the conceivable degenerate case in which only the scalar magnitude of such a particle's multipole moment interacts with the gravitational field must be considered.

For each class of particles, the corresponding directing field Ξ could be measured at any given spacetime event as follows. Take a large number of the particles and launch them from many different directions in such a way that they all pass through an infinitesimal neighborhood of the given spacetime event. Track each of the particles in some radar coordinate system. Then by curve fitting and differentiation (4.8), the one and two directions (ξ_1, ξ_2) for each of the particles may be determined at the given event. These pairs in turn determine the directing field

$$\xi_2^\alpha = \Xi_2^\alpha(\xi_1^\beta) \tag{7.1}$$

at the event in the given coordinate system. By repeating the procedure for many spacetime events the directing *fields* for the given class of particles may be measured.

Having measured the directing field with sufficient accuracy at a large number of spactime points, the analytic criterion (4.15) of Theorem 2 may be used to determine whether or not it is geodesic. Assume a polynomial form for the functions Ξ_2^α in (7.1) of degree greater that three, say five or six. Then use the measured data pairs (ξ_1, ξ_2) at the given spacetime event to determine the coefficients by, for example, the method of least squares. Then if the coefficients of the terms of degree greater than three are essentially zero and if the third degree terms other than $\xi_1^\alpha D_{\rho\sigma} \xi_1^\rho \xi_1^\sigma$ are also essentially zero, and if this turned out to be the case for every spacetime event considered, then one would conclude that the directing field was geodesic. If it turned out that Ξ_2^α were not cubic polynomials of the desired form even at a single spacetime event, then one would conclude that the directing field was not geodesic. This curve fitting technique also serves to determine the projective coefficients Π^i_{jk} ($\Pi^i_{ik} = 0$) as functions of the spacetime event. In turn, these coefficients uniquely determine a geodesic path structure.

The determination and measurement of the conformal tensor density $\mathscr{G}_{ab}$ and the conformal connection coefficients

$$K^i_{jk} = \tfrac{1}{2}\mathscr{G}^{il}(\mathscr{G}_{lj,k} + \mathscr{G}_{lk,j} - \mathscr{G}_{jk,l}) \tag{7.2}$$

is adequately discussed elsewhere in the literature. Ehlers, Pirani, and Schild have shown that the necessary and sufficient condition that a geodesic path structure determined by Π^i_{jk} is compatible with the conformal structure determined by $\mathscr{G}_{ab}$ is that[1]

$$\Delta^i_{jk} \equiv \Pi^i_{jk} - K^i_{jk} = 5\mathscr{G}_{jk}\mathscr{G}^{il}q_l - \delta^i_j q_k - \delta^i_k q_j \,, \tag{7.3}$$

where the coefficients q_i depend only on the spacetime event. The Eqs. (7.3) form a system of $n^2(n+1)/2$ linear equations in the n unknowns q_i. The structures are compatible if and only if a solution exists for every spacetime event. If (7.3) holds then the q_i are given by

$$q_i = \tfrac{1}{18}\mathscr{G}_{il}\mathscr{G}^{pq}\Delta^l_{pq} \tag{7.4}$$

(for four dimensional spacetime); so that, the compatibility conditions that must be satisfied by the Δ^i_{jk} may be obtained by substituting (7.4) into the right-hand side of (7.3). If the structures are compatible, the unique symmetric linear connection which preserves nullity of vectors is given by

$$\begin{aligned}\Gamma^i_{jk} &= K^i_{jk} + 5\mathscr{G}^{il}(\mathscr{G}_{jk}q_l - \mathscr{G}_{lj}q_k - \mathscr{G}_{lk}q_j) \\ &= \Pi^i_{jk} - 4(\delta^i_j q_k + \delta^i_k q_k)\,.\end{aligned} \tag{7.5}$$

It is clear from this relation that it is possible to have any number of distinct projective structures all compatible with the same conformal structure.

If extensive investigation failed to reveal even a single class of particles governed by a geodesic directing field, then the EPS construction would fail to demonstrate the existence of a unique Riemannian metric. Such a structure might still exist, but other means would have to be sought to establish evidence for its existence.

If one or more classes of particles governed by geodesic directing fields were found and if none of the projective structures were compatible with the conformal structure, the construction would fail as before. If two or more projective structures were found which were compatible with the conformal structure, then not even a unique Weyl structure would exist let alone a unique Riemannian structure. There remains the case in which exactly one class of particles governed by a geodesic directing field compatible with the conformal structure is found. Then the projective path structure revealed by these particles and the conformal structure revealed by light propagation together determine a unique Weyl structure. As discussed by EPS, parallel transport along non-null curves is then well defined. Finally, the absence of the second clock effect is then the necessary and sufficient condition for the existence of a unique Riemannian metric.

In conclusion, the truth of the constructive axioms of EPS is epistemically decidable in a noncircular manner, and the metric structure derived from the conformal and projective structures and their compatibility relation is therefore not even epistemologically conventional but constitutes an intrinsic feature of the spacetime manifold that is revealed through light propagation and free fall.

[1] J. Ehlers, F.A.E. Pirani, and A. Schild, "The Geometry of Free Fall and Light Propagation," in *General Relativity, Papers in Honour of J.L. Synge*, edited by L. O'Raifeartaigh (Oxford U.P., New York, 1972), pp. 63–84.

[2] J. Ehlers and E. Köhler, "Path Structures on Manifolds," J. Math. Phys. **18**, 2014 (1977).

[3] C. Ehresmann, "Les Prolongements d'une Variété Différentiable I-V," C. R. Acad. Sci. **233**, 598, 777, 1081 (1951); **234**, 1028, 1424 (1952).

[4] F.A.E. Pirani, Symposia Mathematica, Rome, pp. 67–83 (see p. 74) (1973); J. Ehlers, "Survey of General Relativity Theory," in *Relativity, Astrophysics and Cosmology*, edited by W. Israel (Reidel, Dordrecht, (1973), pp. 1–125, see p. 30.

[5] A. Grübaum, *Philosophical Problems of Space and Time*, (Reidel, Dordrecht, 1973), Chap. 22.

[6] J. Stachel, "Space-Time Problems" (A review of the Synge Festschrift cf. 1) in Science **180**, 292 (1973).

[7] L. Sklar, "Facts, Conventions and Assumptions," in *Foundations of Space-Time Theories*, Minnesota Studies in the Philosophy of Science, VIII.

[8] J.A. Winnie, "The Causal Theory of Space-Time" in *Foundations of Space–Time Theories* (cf., Ref. 7).

[9] S. Kobayashi, and K. Nomizu, *Foundations of Differential Geometry*, (Interscience, New York, 1963), Vol. I.

[10] J.A. Schouten, *Ricci-Calculus* (Springer, Berlin, 1954), p. 67.

[11] Ref. 10, p. 155.

[12] L.P. Eisenhart, "Non-Riemannian Geometry," Am. Math. Soc. Colloq. Publ. **VIII**, 111 (1964).

[13] A. Einstein, "What is the Theory of Relativity?" in *Essays in Science* (Philosophical Library, New York, 1934).

[14] (a) H. Weyl, "Zur Infinitessimal Geometry: Einordnug der Projectiven und Konformen Auffassung," in *Gessammelte Abhandlungen*, edited by K. Chandrasekharan (Springer, New York, 1968), Vol. II, p. 195. (b) H. Weyl, *Raum, Zeit, Materie* (Springer, Berlin, 1970).

[15] H. Reichenbach, *Axiomatization of the Theory of Relativity* (Univeristy of California, Berkely, 1969).

[16] A. Trautman, "Invariance of Lagrangian Systems" in *General Relativity*, edited by L. O'Raifeartaigh, (Oxford U.P., New York, 1972).

[17] A. Einstein, "Reply to Criticisms," in *Albert Einstein Philosopher–Scientist*, edited by P.A. Schilpp (Open Court, LaSalle, Illinois, 1970), Vol. II.

[18] H. von Helmholtz *Über die Tatsachen, Welche der Geometrie Zugrunde Liegen*, Nachr. Ges. Wiss. Göttingen (1868).

[19] H. Weyl, "Geometrie und Physik," in *Gesammelte Abhandlungen*, Vol. III, p. 336 (cf., Ref. 14).

[20] J. Ehlers, and A. Schild, "Geometry in a Manifold with Projective Structure," Comm. Math. Phys. **32**, 119 (1973), see p. 146.

[21] L. Infeld, and A. Schild, Rev. Mod. Phys. **21**, 408 (1949); A. Papapetrou, Proc. R. Soc. London A **209**, 248 (1951); J.A. Wheeler, Rev. Mod. Phys. **33**, 63 (1961).