THE THEORY OF MEASUREMENT IN GENERAL RELATIVITY

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[...]
[p. 14]

0.1 The possibility of Interval Comparisons with Light Rays and Particles.

The elements of a general manifold's geometry are its symmetric affine connection ${\Gamma^i}_{jk}$ and its metric tensor g_{ij}, which is assumed to exist by Russell's axiom of distance. In a Riemann space the two elements are easily related, using any intrinsic coordinate system.

Now in the case of space-time no physical method can exist for determining the metric tensor as a function of any natural coordinates over the whole universe[1] unless we already have a geometrical process of physical measurement, so that in obtaining such a process we must as much as possible avoid the introduction of particular metric or coordinate system.

[...]
[p. 16]

That is, these equation form a contravariant vector. For any separate curves $x^i(t)$ and $x^r(t)$ satisfying these we must have

$$\frac{\frac{d^2x^i}{dt^2} + {\Gamma^i}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}}{\frac{dx^i}{dt}} = \frac{\frac{d^2x^r}{dt^2} + {\Gamma^r}_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}}{\frac{dx^r}{dt}} \tag{2}$$

Suppose now a new connection ${\Lambda^i}_{jk}$ also, satisfying the above equation (2) for the same x^i, x^r. Substraction gives

[1]Note: H. A. Lorentz gave a method whereby an observer may determine the components of the metric tensor numerically for very small regions, over which these components are nearly constant.

The expression for the fundamental metric form in the case of light rays is:

$$ds^2 = g_{ik}dx^i dx^k = 0 \tag{1}$$

The g_{ik} in hour space-time are symmetirc and thus have ten components to be determined. By observations of the progress of nine different light signals in the region, i.e. measurement of nine independent sets of dx^i, we obtain nine independent equations for ds^2 for the g_{ik}.

$$\frac{(\Gamma^i{}_{jk} - \Lambda^i{}_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt}}{\frac{dx^i}{dt}} = \frac{(\Gamma^r{}_{pq} - \Lambda^r{}_{pq})\frac{dx^p}{dt}\frac{dx^q}{dt}}{\frac{dx^r}{dt}} \tag{3}$$

Let :

$$\Gamma^i{}_{jk} - \Lambda^i{}_{jk} = \Phi^i{}_{jk} \tag{4}$$

$$\Phi^i{}_{ik} = (n+1)\Phi_k \tag{5}$$

Then our equations become

$$\Phi^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^r}{dt} = \Phi^r{}_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\frac{dx^i}{dt} \tag{6}$$

$$(\Phi^i{}_{jk}\delta^r_l - \Phi^r{}_{jk}\delta^k_l)\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} = 0 \qquad (j,k \text{ dummy indices}) \tag{7}$$

Now $\frac{dx^i}{dt}$ are arbitrary and thus

$$\Phi^i{}_{jk}\delta^r_l = \Phi^r{}_{jk}\delta^k_l \tag{8}$$

Permuting the indices (i,j,k) and adding:

$$\Phi^i{}_{jk}\delta^r_l + \Phi^i{}_{lj}\delta^r_k + \Phi^i{}_{kl}\delta^r_j = \Phi^r{}_{jk}\delta^i_l + \Phi^r{}_{lj}\delta^i_k + \Phi^r{}_{kl}\delta^i_j \tag{9}$$

Contracting r with l :

$$n\Phi^i{}_{jk} + \Phi^i{}_{jk} = (n+1)\Phi_j\delta^i_k + (n+1)\Phi_k\delta^i_j \tag{10}$$

$$\Gamma^i{}_{jk} - \Lambda^i{}_{jk} = \Phi_j\delta^i_k - \Phi_k\delta^i_j \tag{11}$$

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[p. 46]
determining the plane (see Figure 14), the projection of 2 onto $1CD$ by the light cone of 2 is the intersection of this null cone with the plane. Any point $2'$ along this intersection determines a new standard interval for measurement in the $1CD$ plane. CD is then found in terms of $12'$. But now 12 and $12'$ also lie in a plane. Hence both $12'$ and CD are determinable in terms of 12 by two applications of the planar measuring technique we have just described.

[Figure 14 - Measurement in three dimensions]

In more than three dimensions measurement give no trouble. Four points can always be placed into two intersecting planes, as they were above, and the measurement proceeds in exactly the same fashion.