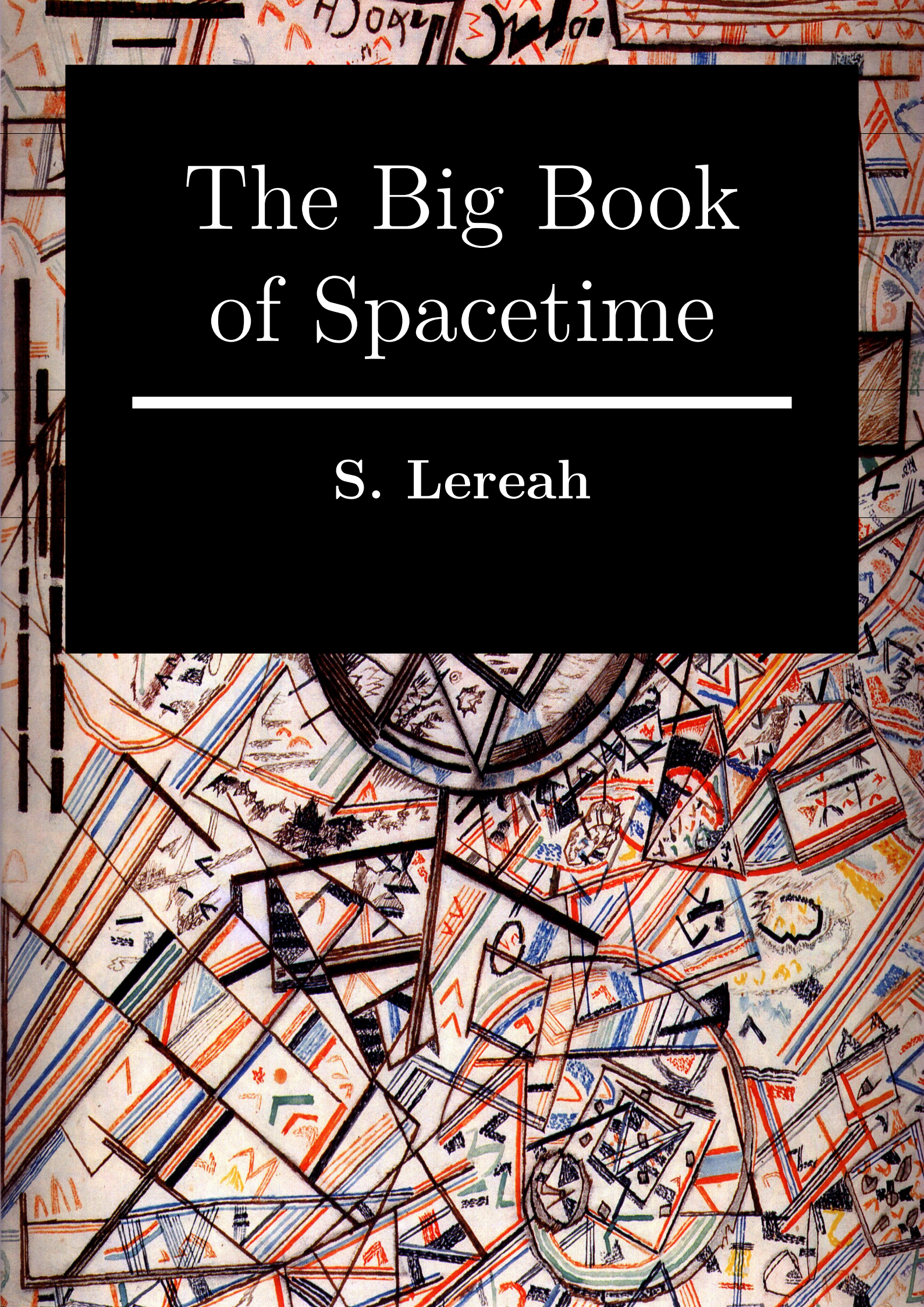




The Big Book of Spacetime

S. Lereah

The big book of spacetime

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Notation and conventions

Conventions on general relativity

As with most general relativity textbooks, we will use the Einstein notation, where summation is implied for identical upper and lower indices. That is, for two quantities X and Y , with components $X^{a_1 b_1 c_2 d_1 \dots p \dots}$ and $Y_{a_2 b_2 c_2 c_2 \dots p \dots}$, with a single common index p , once as an upper index and once as a lower index, we have the equality

$$X^{a_1 b_1 c_2 d_1 \dots p \dots} Y_{a_2 b_2 c_2 c_2 \dots p \dots} = \sum_{p=1}^n X^{a_1 b_1 c_2 d_1 \dots p \dots} Y_{a_2 b_2 c_2 c_2 \dots p \dots} \quad (0.1)$$

where n is the number of dimension implied by the index. Depending on what the typical numbering of indices is done, the sum may be done from 1 to n or from 0 to $n - 1$, but this is purely conventional.

The sign conventions for general relativity are the following :

- The metric signature will be $(- + +)$, unless stated otherwise.
- The Riemann tensor is defined as $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$.
- The Einstein equation is defined as $G_{\mu\nu} = \kappa T_{\mu\nu}$.
- The Levi-Civita tensor's sign is defined by $\varepsilon_{123\dots} = 1$.

In terms of the Misner-Thorne-Wheeler notation, this corresponds to the $(+ + +)$ convention, "mostly plus" or "Westcoast convention".

Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (0.2)$$

0.1 Units and dimensions

The mathematical calculations will all be in natural units, $G = c = \hbar = 1$, while any practical calculations will be in SI units (excluding perhaps the case of a few historical experiments that will mention contemporary units).

The choice of dimensions for the various objects are

$$g : \text{dimensionless} \quad \Gamma : L^{-1} \quad R : L^{-2} \quad \mathcal{L} : J/m^3$$

Nomenclature

∇	Connection.
∂_μ	Coordinate basis of the tangent bundle.
$\approx$	Equivalent with respect to a diffeomorphism.
$\sim$	Equivalence relation.
$\#$	Connected sum of two manifolds $M\#N$.
$\mathcal{A}$	Atlas of a manifold.
$\mathcal{A}_S$	Smooth atlas of a manifold.
α'	Regge slope.
$\mathfrak{A}$	C^* -algebra.
$C\ell$	Clifford algebra.
χ	Euler characteristic.
$\mathbb{C}\ell$	Complexified Clifford algebra.
dx^μ	Coordinate basis of the cotangent bundle.
$\mathcal{D}$	Derivation.
e	Frame field.
F	Electromagnetic tensor. Curvature of the Yang-Mill connection.
ϕ	Coordinate chart. Scalar field.
g	Metric tensor on a manifold, usually Lorentzian, with components $g_{\mu\nu}$.
H	Cauchy horizon.
$\mathcal{H}$	Hilbert space.
$\mathcal{H}$	Hamiltonian density.
I^+	Chronological future.
I^-	Chronological past.

i_X	Interior product by the vector field X .
ι	Inclusion map.
$\mathcal{I}^+$	Future null infinity.
$\mathcal{I}^-$	Past null infinity.
J^+	Causal future.
I^+	Causal past.
K	Killing vector.
$\mathbb{K}$	Klein bottle manifold.
L	Line manifold.
$\mathcal{L}$	Lie derivative.
Λ	Lorentz transformation.
ΛM	Exterior bundle.
$\mathcal{L}$	Lagrangian density.
M	Manifold.
$\mathcal{M}$	Spacetime manifold.
μ	Measure.
$O(p, q)$	Orthogonal group.
p, q, r	Points of a manifold.
R	Ricci scalar.
Riem	Riemann tensor, with components $R^\rho{}_{\mu\nu\sigma}$.
$\mathbb{R}$	Real line.
$\mathbb{R}^4_\Theta$	Exotic $\mathbb{R}^4$.
$SO(p, q)$	Special orthogonal group.
T	String tension.
TM	Tangent bundle to the manifold M .
τ	Proper time.
u	Tangent vector to a curve.
ω	n -form.
ξ	Timelike vector.

Coordinates

There is quite often some debate as to whether objects in general relativity should be written in a given coordinate system, or using coordinate-free notation. I have tried to include both versions in formulas where such things make sense. This can give some difficulties due to the large number of symbols involved. The chosen correspondence is

$$gg_{\mu\nu} \tag{0.3}$$

$$\text{Riem}(X, Y, Z, W)R_{\mu\nu\tau}^{\sigma} \tag{0.4}$$

$$\text{Ric}(X, Y)R_{\mu\nu}X^{\mu}Y^{\nu} \tag{0.5}$$

$$RR \tag{0.6}$$

$$\sqrt{g}d^nx d\mu[g] \tag{0.7}$$

Introduction

This book aims to provide a rather comprehensive set of theorems and informations regarding spacetime, general relativity and related gravitational theories, as well as issues relating to it.

First on a formal level, I have split the two topics of *geometry and kinematics* (what spacetime is and how objects on a given spacetime can be described), and *dynamics* (how the properties of spacetime arise and evolve). While these two issues are fundamentally linked (as is indeed one of the most famous property of general relativity), it remains possible to treat the geometry on its own without any issue, simply by "turning off" the gravity $G \rightarrow 0$ and setting the geometry as a non-dynamical field (dynamical meaning here "depending on the matter content" and not on time).

The section on *matter* describes in more detail how (classical) matter, and specific cases of matter, behave on spacetime, both as free objects, as sources of gravitation, and as fully dynamic objects. This covers a variety of different kinds of forms of matter, from fundamental ones like fields and extended particles to more practical ones like physical objects that we may use in experiments.

Also related to the matter content of the theory, the section on *statistical mechanics*

Cauchy problem

Practical formulas

Alternative theories

Quantum theory

Quantum gravity

Examples

Measurements and philosophy

Experiments and applications

History

I have strived to keep (where applicable) both index-free notation as well as expressions in coordinates.

It can be difficult to find a proper order to introduce all notions in a satisfactory way. Various ways for this are to consider the theory in some historical order, assuming that this will roughly correspond to an intuitive progression towards the theory itself (although usually trimming some of the dead ends or convoluted detours in that process). This is often how many general relativity books are at least for the first few chapters, first considering classical mechanics, special relativity, and then maybe how gravity or accelerated frames interact with special relativity. I have chosen not to go that route, partly due to the abundance of books already following that kind

Another way is to try to consider some kind of logical order, where each following chapter is in some sense a building block for the next. While a tempting way to do it, it runs the risk of exposing a variety of abstract notions that sometimes seem unrelated in between them, and makes it hard to provide concrete examples from objects that have not yet been introduced but which form very intuitive examples of the concepts. If we push the idea to its limit, this may entirely atomize the thematics of each chapter : some elements of algebra will rely on the notion of vector spaces, and some elements of vector spaces will rely on algebras. Short of having a progression that has nearly no structure in terms of mathematical and physical fields, where atomic elements of each are sprinkled throughout the book, this is also not necessarily ideal

I have strived to try and follow some logical order to construct general relativity, but it can be difficult to not have examples at hand that can illustrate those notions, even if those examples themselves depend on notions that have not yet been introduced. In particular, many concepts need to be redefined several times with varying degrees of elaboration to avoid defining them too abstractly from the getgo.

I will however generally assume at least some familiarity with general relativity, mathematics and physics

I don't necessarily recommend to read this book in order, as the order is more focused on the logical construction

Part I

Mathematical background

As part of the goal of this book is to be somewhat self-contained, in the sense that any notion that someone familiar with physics might not be expected to know should be contained, the first part of this book will be a description of various mathematical notions that will be used later on. This is separated from the "geometry" section somewhat arbitrarily, as one could argue that topology, vector spaces, real numbers, etc are part of geometry, but those are a bit more basic and come before that here.

Most of these notions should be fairly well-known, at least in the broad strokes, so that this part may be skipped without too much fear. If a less commonly known theorem is used regarding those notions, I will try my best to refer back to them here.

Chapter 1

Logic

Logic is one of the fundamental basis for most modern mathematics, even if only implicitly in most books. This will not be a particularly involved presentation on the topic of logic but simply some basics to explain the notation that will commonly come up in this book, as well as defining a few logical systems related to our topics later on.

I use basic logic notation quite often in this book. While I try to also write out the propositions fully in words, I will sometimes also include the logical notation for some statements. While I do not write such formulas with any specific logical system in mind, it can be considered to be very generic classical logic as far as propositional logic goes, like the Hilbert system.

Logic is a system about the truth of statements (at least this is true of what we call *classical logic*, which is what we will use here). Mathematically, we have some set of statements, written as variables as $p, q, r, \dots$. Those statements can be either *true* or *false*, what we call their *truth value*.

[example of a statement]

Example 1.1. An example of a statement might be $p = \dots$

To determine this truth value, we use *rules of inference*, which are transformations we can apply to several statements to obtain a new statement,

To apply those rules of inference, we also need a starting set of true statements (the axioms of the theory), and try to figure out whether or not the statement p can be derived by those rules from the initial set of axioms. If the statement p can be so derived, we write it as $\vdash p$. If we wish to be particularly formal, we will also specify under what logical system we operate before the turnstile. For instance if a given statement is true under a theory Theory , we will write

$$\text{Theory} \vdash p \tag{1.1}$$

We say that the theory *entails* p .

From this, a first rule is simply that a proposition p is an axiom of the theory, then the theory implies p :

$$p \text{ axiom of Theory} \Rightarrow \text{Theory} \vdash p \quad (1.2)$$

Formally speaking, if p cannot be derived (or we cannot specifically do it), this does not show the statement to be false in general. To give us the proof of the falsehood of a statement, we introduce the *negation* operator, $\neg$ which is a unary operator on statements : $\neg p$. We then say that a statement is false if we can prove its negation, $\vdash \neg p$.

The basis which forms logical operators can be quite varied, and can even be reduced to a single operator for all of them like the exclusive or $p \oplus q$, but we will not try to reduce our logical system to its minimum and will instead simply use the most common operators for it. In addition to the negation $\neg p$, we also have the four following binary operators :

- The conjunction denoted by the wedge symbol $p \wedge q$ indicates that both p and q must be true
- The disjunction denoted by the vee symbol $p \vee q$ indicates that either p or q or both must be true
- The implication $p \rightarrow q$ indicates that if p is true, then q is also true (but the reverse does not hold : if q is true, p may be true or false)
- The equivalence $p \leftrightarrow q$ indicates that p and q must have the same logical valence.

Those descriptions are only semantics, as there is no notion of truth directly present in propositional logic (although we can define a truth statement $\top$ and a false statement $\perp$ and)

The basic axioms of propositional logic can vary a bit, but are usually something of the form of Hilbert's axioms for logic

- The axiom of identity : if a proposition is true, it is true : $\vdash p \rightarrow p$
- The axiom of simplification : if a proposition is true, then any other proposition will imply it : $\vdash p \rightarrow (q \rightarrow p)$
-

I consider most logical statements to be *theorems* and do not really try to distinguish them from *lemmas*, unless that statement is historically known as a lemma.

1.1 Predicate calculus

Propositional calculus by itself is not a particularly interesting theory as we cannot speak of statements applying to objects, except by making single statements for every object under consideration. The way to generate propositions for objects is to use a *predicate*, which is a symbol accepting objects as arguments to produce propositions. For instance, $p[x]$ [give example of statements]

Interpretation as properties and relations

The operators we consider to generate propositions from predicates without using a specific object are called *quantifiers*, which can take a predicate with n free variables and turn it into a predicate with $n - 1$ variables, which output a proposition in the case where 0 variables are left.

$$Qx. p[x] \Rightarrow q \quad (1.3)$$

The two quantifiers we will commonly use are the *universal quantifier* $\forall x.p[x]$ and the *existential quantification* $\exists x.p[x]$.

$$\exists x. p[x] \leftrightarrow \neg \forall x. \neg p[x] \quad (1.4)$$

the *uniqueness quantification* $\exists! p$

A specific operator applying directly to objects is the *equality* of objects. While it can be defined in other theories, it is also commonly a primitive notion in many systems of predicate logic. As an operator on two objects outputting a proposition, it can simply be considered as a specific type of predicate

$$p[x, y] \leftrightarrow x = y \quad (1.5)$$

[...]

For predicate logic, I use the quantifier symbols of existential quantification

$$\text{There exists an object } x \text{ such that } p \text{ is true} \Leftrightarrow \exists x. p \quad (1.6)$$

and universal quantification

$$\text{For every object } x, p \text{ is true} \Leftrightarrow \forall x. p \quad (1.7)$$

Equality is defined in the sense that if we have two objects A, B , $A = B$ implies that for any provable proposition involving A , $\vdash \phi[A]$ implies $\vdash \phi[B]$: replacing every occurrence of A in the proposition ϕ by B will not change the judgement of its truth value.

For any object x in the domain of discourse of the theory, we say that two objects x, y are equal if, for every proposition p dependent on x $p[x]$, its valuation is equivalent to $p[y]$.

In other words,

$$x = y \vdash p[x] \Rightarrow \vdash p[y] \quad (1.8)$$

1.2 Peano's axioms

As a short example of an entirely self-contained logical system, let's briefly look at Peano's axioms for arithmetics. To keep it in a very abstract logical formalism, I will not try to say that there exists a set of numbers, but instead the predicate that an object n is an integer is its own logical entity : for an object n , we write $N(n)$ to mean " n is an integer".

Peano's theory of arithmetic is based on object

The objects of the theory

Axiom 1.2. • There exists a natural number denoted $0 : \vdash N(0)$

- For any object n which is a natural number, its successor $S(n)$ is a natural number : $\vdash N(n) \rightarrow N(S(n))$
- If the successor of two numbers n, m are equal, they are the same number : $\vdash S(n) = S(m) \rightarrow n = m$
- 0 is not the successor of any natural number : $\vdash \nexists n. S(n) = 0$
- If for some predicate with a single free variable $\phi(n)$, we have that $\vdash \phi(0)$ and $\vdash \phi(n) \rightarrow \phi(S(n))$, then ϕ is true for every number : $\vdash \forall n. N(n) \rightarrow \phi(n)$

Those axioms for natural numbers are pretty universally accepted (except by finitists), and are entirely defined in terms of the theory's own objects and relations, making it a synthetic theory. We will see [X] examples of implementation of Peano's axioms in set theory.

1.3 Type theory

Category theory meshes easily with *type theory*, where we assign types to mathematical objects.

Types

Functions $A \rightarrow B$ define a type $A \rightarrow B$

The natural identification of type theory is that every category C defines a type C to which its objects belong, as well as a type for each of its morphisms (of type $C \rightarrow C$), functors between different categories ($C_1 \rightarrow C_2$), natural transformations $((C_1 \rightarrow C_2) \rightarrow (C_1 \rightarrow C_2))$, and so forth.

Unlike in the case of set theory, this gives us a natural notion

Products and sums

Chapter 2

Sets

The notion of a set is a commonly used basis in mathematics, and it is also usually used to define spaces in physics. While it is a fairly common foundation for a lot of modern mathematics, it is important to remember that formal systems in general do not have to depend on set theory itself and can be defined on their own (or on other types of foundations). This is simply a rather flexible way of defining mathematical objects that became historically popular [ref. to Bourbaki and structuralism?]

As a logical theory, set theory is composed of at least one type of objects (sets), and some binary relation between the objects of that theory and sets. Given any object x of the theory and any set A , we can define a statement

$$x \in A \tag{2.1}$$

If this relationship is true, we say that x is *in* A , or that x *belongs* to A . The relation $\in$ is called the *membership relationship*. It is common to represent sets and membership as a "bag of objects" in diagram form, with the membership being represented by the object being inside the set's boundaries

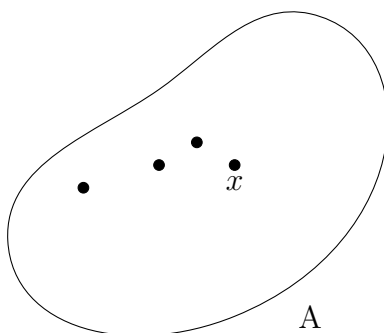


Figure 2.1: Object x belonging to the set A

[diagram]

There is no particular type of object that x should be. If the set theory involved is the entire basis of the mathematics involved here, like for the most common set theory (the Zermelo–Fraenkel set theory with axiom of choice, or ZFC), all will be made of sets, including the objects belonging to them (so-called pure sets). While this can be a useful way to consider it, we will not in general demand that our objects are themselves sets. To denote this, we will write objects as $x, y, z, \dots$ and sets as $A, B, C, \dots$, keeping in mind that objects could also be sets themselves.

Otherwise, we do indeed ask that our sets obey the same axioms as ZFC.

Axiom 2.1. *Axiom of extensionality* : Two sets are equal if they contain the same elements

$$\forall A, B, x, (x \in A \leftrightarrow x \in B) \leftrightarrow A = B \quad (2.2)$$

This means that sets are entirely defined by their elements and by the membership relationship they have with objects. Due to this, sets are very simple objects and do not have a simple way to convey the notion of containing multiple copies of the same element. If we wanted some mathematical structure conveying the notion that it contains two copies of the same object, we could not simply have the set simply containing them both, as this would simply be equivalent to the set containing a single copy (we will see later on how to get around this issue).

Axiom 2.2. *Axiom of pairing* : For any two objects, there exists a set that contains them both.

$$\forall x, y, \exists A, (x \in A) \wedge (y \in A) \quad (2.3)$$

That is to say that we can consider the union of two sets.

[diagram]

The axiom that follow (the axiom of regularity) is written

Axiom 2.3. *Axiom of regularity* : If a set A is non-empty, anyelement

$$\forall A, B, (\forall x, (x \in A \leftrightarrow x \in B) \leftrightarrow A = B) \quad (2.4)$$

Definition 2.4. A subset S of A , denoted by $S \subseteq A$, is a set for which every element is also an element of A :

$$S \subset A \leftrightarrow \forall x. x \in S \rightarrow x \in A \quad (2.5)$$

2.1 Set algebra

A few basic operations that we can define on our sets are the algebra of sets. From the axiom schema of specification, we can define a variety of sets from logical statements. In particular, we will be interested in set builders of the form

$$\{a | a \in A \star a \in B\} \quad (2.6)$$

for two given existing sets A and B and $\star$ some logical operator between those two statements. As there are only 16 possible binary operators, we have a limited number of possible binary set operators in the same way. The two basic such operations are the union and intersection :

$$A \cup B = \{a | a \in A \vee a \in B\} \quad (2.7)$$

$$A \cap B = \{a | a \in A \wedge a \in B\} \quad (2.8)$$

$A \cup B$ is the *union* of A and B , where the elements of $A \cup B$ are all the elements that belong to either A or B , and $A \cap B$ is the *intersection* of A and B , where all of its elements belong to both A and B .

Theorem 2.5. The union of two sets is always defined

Proof. From the axiom of union □

[diagram]

Properties of union and intersection

Proposition 2.6. The union $A \cup B$ and intersection $A \cap B$ are symmetric operations :

$$A \cup B = B \cup A \quad (2.9)$$

$$A \cap B = B \cap A \quad (2.10)$$

Proof. This stems from the symmetry of the logical disjunction and conjunction. □

Proposition 2.7. The union and intersection of a set with the empty set $\emptyset$ are respectively the set itself and the empty set :

$$A \cup \emptyset = A \quad (2.11)$$

$$A \cap \emptyset = \emptyset \quad (2.12)$$

Proof. This stems from the proposition $a \in \emptyset$ being always false, and therefore

$$a \in A \vee a \in \emptyset \leftrightarrow a \in A \quad (2.13)$$

$$a \in A \wedge a \in \emptyset \leftrightarrow \perp \quad (2.14)$$

□

2.2 Numbers

Integers are often defined in terms of set theory in mathematics, where the corresponding number simply corresponds to the cardinality of the set. For instance, zero corresponds to the set with no elements in it, the empty set :

$$0 \leftrightarrow \forall x, x \notin A \quad (2.15)$$

This is a fairly natural correspondence as the empty set is unique in set theories by extensionality, but this is the only case where this uniqueness is obvious. Sets with exactly one element on the other hand, the singletons, can be of a variety . A set with a single object inside can be denoted by $\{\bullet\}$, and obeys the property that for any two different element (two in the theory's meta language), one element belonging to the set implies that the other does not.

$$\forall x, y, (x \in A \wedge x \neq y) \rightarrow y \notin A \quad (2.16)$$

If we have a variety of urelements, the singleton could be any of those, and even in a pure set theory, the set inside that set could be any set.

Then 1 will be put in correspondence with either one such set

There are alternative definitions of integers, for instance by considering levels of inclusion, ie

$$1 = \{\bullet\}, 2 = \{\{\bullet\}\}, \dots \quad (2.17)$$

types of integer definitions

Most definitions of natural numbers stem then from the Peano axioms [X]

Definition 2.8. The *Peano axioms* of arithmetics

Definition 2.9. The *successor function* $s : \mathbb{N} \rightarrow \mathbb{N}$ is the function

$$s(n) = n \cup \{n\} \quad (2.18)$$

$$n = \{0, 1, \dots, n-1\}, s(n) = \{0, 1, \dots, n-1, n\}$$

2.3 Pairing and products

As we mentionned, the basic notions of set theory does not allow us to talk about two identical objects being in the same set, and in addition, there is no notion of ordering

(objects either belong or do not belong, but they cannot "come first" in the set), but notions like that are often useful. To define it, we introduce the notion of ordered pair :

Definition 2.10. An *ordered pair* (a, b) is a set built along a scheme such that we can define the property of an object a coming first in the pair and an object b coming second in the pair.

There are various ways we can define this.

Kuratowski's definition :

$$(a, b) = \{\{a\}, \{a, b\}\} \quad (2.19)$$

What does it mean for an object to be the first element :

$$x = \text{pr}_1(p) \leftrightarrow \forall A \in p, x \in A \quad (2.20)$$

[Get the different definitions of an ordered pair]

This allow us to have both a and b be in the set but have this membership be asymmetric, so that we could both assign a difference in the type of membership, and we could have $a = b$

The ordered pair can be extended to define tuples, which are simply ordered pairs chained together :

$$(a, b, c, \dots) = (((a, b), c), \dots) \quad (2.21)$$

In which case the k -th element of the tuple would be $\text{pr}_1 \circ \dots$

One important construction on sets is the notion of the Cartesian product of two sets, where we consider the set of all ordered pairings

Definition 2.11. The *Cartesian product* of two sets

$$A \times B = \{\{\{a\}, \{a, b\}\} | a \in A \wedge b \in B\} \quad (2.22)$$

This is only one possible expression as a pure set of the Cartesian product, but the main importance of the Cartesian product is its properties. Given a

Projection is a surjection (if no set is empty)

Diagonal map

Definition 2.12. The diagonal map $\Delta : X \rightarrow X \times X$ is the unique map such that $\pi_i \circ \Delta = \text{Id}_X$

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \text{Id} & \downarrow \Delta & \searrow \text{Id} & \\ X & \xleftarrow{\text{pr}_1} & X \times X & \xrightarrow{\text{pr}_2} & X \end{array}$$

Another way of making a set from two old ones is the notion of *disjoint union*.

Definition 2.13. The *disjoint union* $A \sqcup B$ of two sets is

2.4 Relations and orders

As within the context of set theory, the only statements we can really judge about objects is their membership to sets, all relationships between objects that we can define will therefore only depends on their membership to sets (this would be different in say the case of a synthetic axiom system like that of Euclidian geometry where the relationships between elements is a basic notion of the theory). Therefore a relationship in set theory is defined by a set :

Definition 2.14. A *relation* R between objects of a set A and objects of a set B is a subset of the Cartesian product $A \times B$, where we denote for $x \in A$, $y \in B$ that xRy if the pair $(x, y) \in R$.

Example 2.15.

Relation as a graph

Properties of relations

Definition 2.16. A relation R is *reflexive* if for any $x \in A$, xRx

Definition 2.17. A relation R is *symmetric* if for any $x, y \in A$, $xRy \leftrightarrow yRx$. It is *antisymmetric* if this is never true, ie $xRy \leftrightarrow \neg(yRx)$

Definition 2.18. An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

The most common example of an equivalence relation is the equality as defined on sets, where the relation is defined as

$$= \tag{2.23}$$

Definition 2.19. A *partial order* relation $\leq$ is a relation that is reflexive, transitive, and antisymmetric.

[Hasse diagrams]

Definition 2.20. A *total order* relation $\leq$

A basic example of an order relation is that of the set inclusion. For two subsets $A, B \in P(X)$ of a set X ,

2.5 Functions

Functions are meant to associate a unique object to any given object. In the context of set theory, they are therefore a special type of relationship, where the relation is between an object x and its image by a function $f(x)$, such that for any object x , there is exactly one element of the relationship whose first element is x . The relationship we define this way xRy says that y is the image of x by the function that R defines.

[example with something idk, $(1, f(1))$, $(2, f(2))$, etc]

$$f : A \rightarrow B \quad (2.24)$$

$$a \mapsto f(a) \quad (2.25)$$

A function maps an element of A to an element of B , and does so for every element of A

In set theory, functions are typically denoted by some kind of graph, ie. a subset of

Function as a relation $f = \{(a, f(a)) | a \in A\}$, $f \subset A \times B$ such that only one element for each $a \in A$

Inverse of a function : $f^{-1} = \{(b, f^{-1}(b)) | b \in B, \forall a \in A, a \in f^{-1}(b) \leftrightarrow f(a) = b\}$

Two important classes of functions in set theory are

Definition 2.21. For a map $f : A \rightarrow B$, an *injection* or injective function is a function such that for $f(a_1) = f(a_2)$, $a_1 = a_2$. A *surjection* or surjective function is a function for which every element $b \in B$ belongs to the image of f , such that for any $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$. A *bijection* or bijective function is a function that is both surjective and injective, meaning

Theorem 2.22. A function is injective iff it is a monomorphism.

Proof. If a function $f : X \rightarrow Y$ is injective, then right composition by another function $g : Y \rightarrow Z$ implies that for two values z_1, z_2 such that $(f \circ g)(z_1) = (f \circ g)(z_2)$ $\square$

Theorem 2.23. For any bijective function f , there exists an *inverse function* f^{-1}

2.6 Equivalence relations and quotients

If we have some equivalence relation, it is possible to define a new set corresponding to

Definition 2.24. A *quotient* of a set S by an equivalence relation $\sim$

$$S/\sim = \{A \subseteq S \mid \forall x, y \in A, x \sim y\} \quad (2.26)$$

2.7 Subsets, filters and nets

Our axiom of power set guarantees the existence of every possible subset of any given set A .

The power set of any given set X possess an preorder relation defined by the inclusion relation. For the power set $\mathcal{P}(X)$, the inclusion preorder $\subseteq \in \mathcal{P}(X) \times \mathcal{P}(X)$ is defined by

$$\subseteq = \{(S_1, S_2) \mid S_1 \subset S_2\} \quad (2.27)$$

This preorder makes every powerset a poset

[Diagram]

We sometimes wish to define more specific collections of subsets for a given sets

Definition 2.25. A *family of subsets* $\mathcal{B}$ of a given set X is an element of the double power set $\mathcal{P}(\mathcal{P}(X))$.

In other words it is some set of subsets of X , or a subset of $\mathcal{P}(X) : \mathcal{B} \subseteq \mathcal{P}(X)$. As with the power set itself, any family of subsets is itself a poset by inclusion, even if trivially (if no two different subsets are related by inclusion).

If we need our family of subsets to be furthermore capable of having more than one copy of the same set, we can look instead at a family of indexed sets, which is a family on $I \times X$ for some indexing set I .

Definition 2.26. A *net* on X is a function $\Phi : D \rightarrow X$ from a given directed set D

Eventually in A , frequently in A

Convergence of a net

Cluster point

Directed set

In the context of filters, we typically consider the empty set to be an undesirable element for a family of sets, so that it is common to call a family of set proper if it does not contain it :

Definition 2.27. A family of subsets $\mathcal{B}$ is called *proper* if $\emptyset \notin \mathcal{B}$, and improper otherwise.

Upward closure :

Definition 2.28. The *upward closure* $\mathcal{B}^\uparrow$ of a family of sets $\mathcal{B}$

Definition 2.29. A family of subsets is *upward closed* if $\mathcal{B} = \mathcal{B}^\uparrow$.

In other words, a family of subset is upward closed if given any subset $B \in \mathcal{B}$, all the "bigger" subsets are also in $\mathcal{B}$

Directed sets

2.8 Relevance of set theory

One thing that should be noted about set theory is that while it is a popular basis for mathematical objects and almost any mathematical object can be expressed by means of sets using enough sophisticated operations on those sets, it is not necessarily a "good" representation of a given object. Set theory's main use is to talk about sets, ie collections of individual objects. While this is a powerful method that can be used for quite a lot more than what would be immediately obvious, such as modelling continuous space as a set of points, it is to be kept in mind that it is not the only possible logical system that can be employed to describe something. We will see a variety of alternative logical systems that do not rely on set theory for various fields [...]

In further chapters, we will see a variety of possible alternative definition of spaces that do not rely on the foundational notion of point, and in some cases, cannot have them, or at least not in any "direct" way

Chapter 3

Structures

One notion that will come back regularly is that of a structure. A structure is roughly the notion of functions defined on a set obeying some specific properties.

In the logical sense, a structure is a set A along with

Signature

The notion of a structure can give us that of an *isomorphism*. The point of an isomorphism, in terms of logic, is that if we have some predicate $\phi[x]$ on an object x in some structure and some isomorphism $f : S_1 \rightarrow S_2$ between two structures, the valuation of $\phi[x]$

Isomorphisms

Specifically algebraic structures

Definition 3.1. A binary operation $\cdot$ is some kind of function from the Cartesian product of two sets to another set :

$$\cdot : A \times B \rightarrow C \quad (3.1)$$

$$(a, b) \mapsto a \cdot b = c \quad (3.2)$$

The general case of algebras are abstract algebras, where we simply consider a product on some set

Definition 3.2. An *abstract algebra* A is a set equipped with a binary operation $\cdot : A \times A \rightarrow A$.

In other words, a binary operation is some kind of function on an ordered pair of elements from A

The notation of the binary operation commonly used varies wildly depending on the type of algebra that we consider, common example for two elements x, y of the algebra being

xy , $x + y$, $x \cdot y$, $x \times y$, $x \wedge y$, $x \vee y$, $[x, y]$, etc, which we will see in various algebraic structures used throughout this book.

Abstract algebras by themselves are pretty broad and include such diverse objects as groups, rings and lattices. They are usually constrained further by some additional properties

Definition 3.3. An algebraic operation is *associative* if

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad (3.3)$$

Associativity is a pretty common property of algebraic operations, being true for instance of additions, products, function composition, but will be false for say subtractions, set complements, etc

Definition 3.4. An algebraic operation is *commutative* if

$$x \cdot y = y \cdot x \quad (3.4)$$

3.1 Basic algebra

We have already seen two examples of how to construct the algebra of integers, first as a purely logical theory and then based on sets.

3.2 Lattices and semi-lattices

One simple example of an algebraic structure we can define with the objects we already defined is that of semi-lattices and lattices. Those objects are based on two algebraic operations, the join and the meet.

Definition 3.5. The *meet* $\wedge$ is an algebraic operation that is associative, commutative and idempotent :

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad (3.5)$$

Definition 3.6. A *semi-lattice*

Definition 3.7. A *lattice* $(L, \wedge, \vee)$ is an algebraic structure consisting of a set L and two commutative and associative operations $\wedge$ and $\vee$ (called the *meet* and *join* operation) obeying the absorption properties :

$$a \vee (a \wedge b) = a \quad (3.6)$$

$$a \wedge (a \vee b) = a \quad (3.7)$$

Proposition 3.8. The meet and join are idempotent operations :

$$a \wedge a = a \quad (3.8)$$

$$a \vee a = a \quad (3.9)$$

Proof. Using the absorption property and picking $b = a \vee a$,

$$a = a \vee (a \wedge (a \vee a)) \quad (3.10)$$

$$= a \vee a \quad (3.11)$$

and similarly,

$$a = a \wedge (a \vee (a \wedge b)) \quad (3.12)$$

$$= a \wedge a \quad (3.13)$$

□

The main application of lattices is that they correspond to operations that we can define on partial orders. If we have a partially ordered set $(S, \leq)$, it is a *meet-semilattice* if for any two $a, b \in S$, there exists a greatest lower bound to the set $\{a, b\}$, denoted by its join, $a \wedge b$. In other words, $a \wedge b \leq a$, $a \wedge b \leq b$, and there is no other element of the set obeying those properties that is greater than $a \wedge b$. As this property does not depend on the order in which we consider a and b , it is commutative,

The idempotence then simply indicates that the preorder relation $\leq$ is such that $a \leq a$, as there will not be a closer element to a than a itself in the preorder.

Conversely, the join operation $a \vee b$ corresponds to the least upper bound to the set $\{a, b\}$.

The fact that both the join and meet obey the same axioms stems from the fact that one can turn into the other by considering the dual relation

3.2.1 Subset lattice

One thing we saw in the previous chapter is that as structures, sets do not have a lot to offer. One algebraic structure it does offer however is that of a lattice, as the inclusion relation is a partial order, and there are indeed greatest lower bounds and lowest upper bounds to this relation. If we consider two sets A, B , their greatest lower bound will be the set that is included in both with no other such sets containing it. This is simply the intersection of those two sets, as it will

Theorem 3.9. The intersection of two sets is their meet.

Proof. We already know that $A \cap B \subseteq A, B$. If we assume a set $C \neq A \cap B$ such that $A \cap B \subseteq C$ and $C \subseteq A, B$, this means that C contains all the same elements as $A \cap B$ with some additional elements. □

Theorem 3.10. The union of two sets is their join

Proof.

□

Given any set X , and its power set $\mathcal{P}(X)$, the elements of that power set form a poset ordered by inclusion

Meet and join

Theorem 3.11. The meet and join of two elements of the power set correspond to their

The lattice of a set's power set is also a bounded lattice, where its greatest element is the set itself, and a least element, the empty set, corresponding to the equivalences that for any subset $S \subseteq X$

$$a \vee 0 = a \leftrightarrow S \cup X = \quad (3.14)$$

$$a \wedge 1 = a \leftrightarrow S \cap X = \quad (3.15)$$

[CHECK ORDER]

It is a complete lattice

Complete Heyting algebra

Law of infinite distributivity :

$$x \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} (x \wedge s) \quad (3.16)$$

Distributive lattice :

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (3.17)$$

Chapter 4

Topological spaces

Spacetime as a set does not have any more structure than its points belonging to it, or the entirety of its subsets if we consider those as well, but the powerset of a set does not contain particularly more information than its members alone. There are a lot of additional notion that we want that cannot be expressed through the languages of sets, such as that of cohesion : some points in some sense "belong together", or moreso than they do with other points. Some points may be "next to" a given subset, such as the boundary of a region, or two regions may "touch" despite sharing no points. As a set, a space has all its points on the same level, and the only thing we can say about regions is their membership, so that we can tell for instance that they are overlapping. We want to have some sort of structures to allow us to discuss such notions.

To illustrate those notions, we can consider some subsets of the plane. If we compare let's say some kind of continuous shape, a disk and an uncountable set of points sprinkled in an area (for instance the two-dimensional Cantor dust),

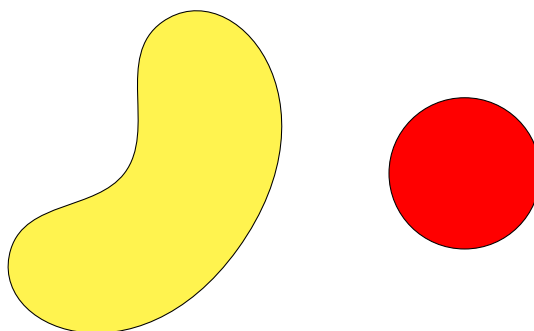


Figure 4.1: Three subsets of the plane of identical cardinality

We would expect the first two objects to have more in common than with the third, but as sets, they are isomorphic.

If we consider the set D of all points that are at a distance strictly inferior to a given value r from a central point o , we would like to say that a point at exactly a distance of r is somehow closer to D than other points, but as a set, this point is simply equivalent to every point outside of D , as $D \cap \{p\} = \emptyset$.

quasitopological spaces, approach spaces, convergence spaces, uniformity spaces, nearness spaces, filter spaces, epitopological spaces, Kelley spaces, compact Hausdorff spaces, - generated spaces

The obvious way of doing this if we start on the assumption that a space is composed of points is to look at its subsets and try to figure out what kind of structure we can impose upon them to permit us to define such properties. We should beware here however to not take too much inspiration from the topological properties of real space (or at least the topological properties we think real space has), as our definition of a topological space has to be flexible enough to allow us to define abstract spaces that may not have anything to do with it except in whatever broad notion of being a "space" that we manage to end up on.

4.1 Mereology

The source of the notion of topology comes from the more intuitive of mereology (the study of parts and wholes), and more specifically spatial mereology, where we consider the properties of parthood for space. As a logical theory, a mereology consists of a theory of a collection of regions which are ordered by a relation of inclusion.

The exact axioms of mereology as it applies to physical space are disputed, but the most common theories of mereology include the following axioms :

1. Inclusion is a partial order, so that it is reflexive, transitive and antisymmetric.
- 2.

The choice of describing spaces, and spacetime in particular, as a set of points, is not entirely uncontroversial, even going back to antiquity [1,2]. We will discuss this particular point in detail later on, but as far as mereology goes, the notion of a point is defined by an atomic region, a region which does not contain any part other than itself.

$$\text{Atom}(x) = \nexists z. z \subset x \quad (4.1)$$

If we treat our space as a set then, another axiom is the existence of atomic regions.

$$\exists x. \text{Atom}(x) \quad (4.2)$$

[Supplementation as opposed to atomism]

Density axiom

Boundaries, interior, etc

4.1.1 Mereotopology

Using the notion of points for a space will lead to quite a lot of strange behaviours, some we will see later on, and there has been some efforts to get rid of it somewhat, but on the other hand, entirely abandoning the notion leads to some very difficult to use theories that may not have the same power as the usual ones. So we will stick with the notions that we can describe points in space, and moreover, our spaces are indeed made of those points. On the other hand, we will also see mereological notions come back in the guise of frames and locales, although they will usually be implicitly regions made of points that we simply choose to ignore for the purpose.

While all the notion we'll see later on can be described somewhat without necessarily too much (or any) reference to sets, overall most of general relativity is done with this structure in mind, so that most objects in it will be described as sets with additional structures on them.

4.2 Neighbourhoods and boundaries

A basic notion we need to define for topology is that of a neighbourhood of a point $x \in X$, which is simply a subset $U \subset X$ which contains that point : $x \in U$. If we only pick that criteria for a neighbourhood, then its definition is rather simple, simply being

$$N_x = \{U \subseteq X | x \in U\} \quad (4.3)$$

For various reasons that we will see in the following sections however, this is not typically the definition we will pick for the neighbourhoods of a point, as this will encompass too many possible subsets. In particular, one outcome we need to avoid is to have the definition of a neighbourhood so large that it includes the singletons $\{x\}$ of every point of x , as this will simply give us back the powerset of X . We will instead try to define the type of properties that such neighbourhoods should obey.

As with our rules regarding mereology, it is typically assumed that the intersection of two neighbourhoods of a point x is also a neighbourhoods of x , as well as the union of two neighbourhoods (and by induction, the intersection and union of arbitrarily finitely many such neighbourhoods). It seems a

Definition 4.1. A *neighbourhood system* $\{U_i\}_{i \in I}$ at x is a filter of subsets of X which all contain x .

Adherent point, accumulation point, sequences, nets, filters, convergence, etc

Then we declare a subset U to be an *open set* if for any point $x \in U$, U is a neighbourhood of x .

One thing we will

4.3 Cohesion

Another notion we can consider for the data our subsets can carry is that of *cohesion*, telling us whether a subset we consider is of "one piece". If we consider some subset $U \subset X$, can we tell whether or not it is composed of a single piece or more, and given one point or more $p \in U$, can we determine to which of those pieces it is connected?

If we simply considered the powerset $\mathcal{P}(X)$, which as we will later find out relates to the discrete topology, there would be no difference between any subsets of X of the same cardinality. Since every individual point has its own subset in $\mathcal{P}(X)$, if we were to interpret $\mathcal{P}(X)$ as some

An atomized cloud of point will have the same status as a ball or any number of regions, which is why we say that it is totally disconnected.

If we want to talk of X as a space, we need some more specific subsets of X to be defined and to be able to express this notion of cohesion. To do this, we need to be able to talk about subsets in such a way that informs of their connectedness. Given a subset $U \subseteq X$, we define a map to some indexing set

$$\Pi : U \rightarrow \mathbb{N} \tag{4.4}$$

where here we consider $\mathbb{N}$ as the set of sets of a given cardinality, ie. we have that $2 = \{\bullet_1, \bullet_2\}$.

where each point is mapped to a given indexed element depending on the region of that space it belongs to. If we have a single blob such as a ball $B \subset \mathbb{R}^n$, this is a constant map to a single element, while if we consider some collection of different regions, these will map all points belonging to each object to different indexes of that set. This map effectively tells us given a subset whether two points are in the same region. The regions of such a subset can therefore be considered to be spanned by the inverse map $\Pi^{-1}(x_i)$ for every member of I . In the case we previously saw with the power set, this map is simply the identity map where the indexing set is the set itself, $\text{Id} : U \rightarrow U$, and therefore every point is its own region.

To define our notion of cohesion, we need to define some rules on our map Π , requiring us to figure out when two points can be considered to belong to the same element of a space. One rule we can impose is this : if we have two such connected elements $A, B \subseteq X$, their union is itself connected if the two sets are not disjoint, ie if $A \cap B \neq \emptyset$.

$$\Pi : \mathcal{P}(X) \rightarrow \mathbb{N} \quad (4.5)$$

point to piece, piece to point transformations

We will in fact define our notion of connectedness the opposite way around. This can be seen by the fact that for any subsets in our filter, we can always decompose them by considering the inverse map Pi^{-1} , which are from the condition we defined always disjoint sets. We can therefore simply say that a set is connected if there does not exist any such decomposition

The problem we had with the power set in fact stem from this fact, as if we treat every subset on an equal footing, every singleton $\{x\}$ is part of the connected components, and every two different singleton are necessarily disjoint elements, therefore any larger subset will be disconnected, completely atomizing the space. While the discrete topology is indeed one of the allowed topology we will consider, it is the smallest such topology there is and is not a very good representation of the cohesiveness of spacetime. Most topologies which we consider will not admit such small subsets to be connected.

While this definition works in the abstract for determining what a connected subset is, we need to consider what it means from the perspective of our set elements for a subset to be connected. One criteria we can look at is how we can decompose this subset. If we have a connected subset A and consider two subsets A_1, A_2 that span it, ie $A_1 \cup A_2 = A$, what can we say about the properties of these elements?

Figure 4.2: Examples of sets spanned by subsets and their connectedness

A property that we can look into is simply their overlap, whether $A_1 \cap A_2$ is empty or not, but if we consider simply the case of picking $A_2 =$

If we are considering sets as our

This is where we will the notion of convergence and boundary seen in the previous section : a connected subset is given by the property that it cannot be decomposed into two disjoint *open* subsets.

From this, we do not actually need to stick only to our connected components to define whatever collection of subsets, as we can still determine our connected components from a larger filter : if two subsets $A, B \subseteq X$ are connected components, either their union is itself a connected component, or we can determine that it is not via the existence of A, B being disjoint sets that [...]

[Is cohesion not trivial to determine in a set-based topos]

The notion of connectedness will in fact not determine much of the fundamental properties of topology as we will see, as the end result of our process to find an appropriate subset of $\mathcal{P}(X)$ to define what topology is will not entirely depend on its cohesion : if two such subsets τ_1, τ_2 have the same connected components $\tilde{\tau}_1 = \tilde{\tau}_2$, this will not guarantee the equality of those two sets unless rather stringent conditions are added.

[No non-constant map from a continuous space to a discrete spac : no motion possible, cf. Lawvere]

"By contrast, a chaotic space is so excessive in its cohesion that any point can be moved to any other point without any 'effort', that is, with no attention paid to the nature of the space-'time' which might be used to parameterize the motion."

4.4 Convergence

Another element in the notion of topology is that of convergence. We want to be able to talk about some sequences of points or regions approaching a certain final object, even if that object does not belong to the sequence. One example of this, going all the way back to Zeno of Elea [3] is the convergence of a sequence obtained by the repeated halving of a line to zero :



Figure 4.3: Convergence of a sequence of halfway points on a segment

The sequence of points $(1, 1/2, 1/4, 1/16, \dots)$ never contains the point 0

Another similar example we can look at is the convergence of a sequence of circles by considering circles of halving radiuses,

Convergence to a point

Convergence of a sequence of regions to a point

We have already seen some notions of convergence when relating to nets and filters

Convergence space

Definition 4.2. A *preconvergence space* (X, ξ) on a non-empty set X is a relation $\xi \subset X \times \text{Filters}(X)$ such that ξ is isotone

$$\mathcal{F}, \mathcal{G} \in \text{Filters}(X), F \leq G \rightarrow \lim_{\xi} \mathcal{F} \subset \lim_{\xi} \mathcal{G} \quad (4.6)$$

Any limit point of F is a limit point of a finer filter G

Definition 4.3. A preconvergence space is a *convergence space* if the relationship is centered :

$$\forall x \in X, x \in \lim_{\xi}(\{x\}^{\uparrow X}) \quad (4.7)$$

For any point in the space, the principal ultrafilter at x converges to x

4.5 Topology

From all the properties and models of a space we defined so far, we can get the usual definition of a topological space. It is important to keep in mind that there is nothing fundamentally important about the notion of a topological space. There are more general and more specific concepts that can be used in place of topological spaces, but topological spaces are a fairly broad notion that can be used to replicate quite a lot of those concepts within their own framework and are a fairly well studied topic.

Our topology should be able to give us some notion of cohesion, neighbourhood and convergence. There are a few different ways to define this, but the most common one is via the collection of *open sets*, which is how we will first define them :

Definition 4.4. A topology τ on a set A is a set of subsets of A that obeys the following properties :

- Both the empty set $\emptyset$ and the set X itself are in τ .
- For any collection X_α of members of τ , $\bigcup_\alpha X_\alpha \in \tau$.
- For any two members $X, Y \in \tau$, $X \cap Y \in \tau$.

Given a topology τ on a set X , the pair (X, τ) is called a *topological space*.

The notion of a topological space is much more general than the intuitions we tried to generate for topology from using physical space as an example (assuming that physical space is composed of points), but those intuitions can be at least replicated within that framework, as we will see. This generality will also be important as the notion of topological space is better behaved under some operations (such as quotients), and other more abstract kinds of space will fail to be similar to physical space.

Theorem 4.5. A topological space (X, τ) is a cohesive space

Theorem 4.6. A topological space is a convergence space

There are many possible topologies. The two simplest are to simply consider $\emptyset$ and X as being the only open sets, called the *trivial topology*, and at the other extreme, just

consider that every possible subset of X is an open set, called the *discrete topology*, so-called because every point $p \in X$ is an open set.

Dually to the notion of open set is that of *closed set*.

Definition 4.7. A closed set $\bar{U}$ is a subset of X such that its complement $X \setminus \bar{U}$ is an open set.

The two notions are not in opposition, as a set can be open and closed, or neither open nor closed. A set that is both open and closed is called a *clopen set*, and the simplest examples of such sets are the empty set $\emptyset$ and the entire space X , which are both open and complements of each other.

The definition of a topological space can be made using the notion of a closed set instead of an open set, simply by using De Morgan's law

Topological basis

Definition 4.8. The *closure* $\bar{S}$ of a subset $S \subseteq X$ is the intersection of all closed sets containing S , and its *interior* $\text{int}(S)$ is the intersection of all open sets containing S

$$S \subseteq S \subseteq \bar{S} \quad (4.8)$$

Definition 4.9. For a subset $S \subseteq X$, $x \in X$ is a *limit point* of S if every neighbourhood of x contains at least one other point $y \in X$ such that $x \neq y$.

Example : closure of the open disk

Closure, interior, limit points

Definition 4.10. Given a subset $S \subseteq X$ of a topological space X , its *boundary* ∂S

A common example of a topology that we will use is that of the Euclidian space topology. The standard topology for $\mathbb{R}^n$ is the topology generated by the basis of open balls on it,

Theorem 4.11. The standard topology is a topology.

Proof. The basis $\mathcal{B}_{x,r}$ covers $\mathbb{R}^n$ as we can define an open ball centered at any point, and given two overlapping balls $B_{x,r}, B_{y,r'}$, their intersection $\square$

4.5.1 Topology comparison

The same set X can be given a wide variety of topologies

4.5.2 Product topology

Definition 4.12. Given a family of topological spaces $(X_i)_{i \in I}$, the *product topology* on the set $\prod_i X_i$ is the initial topology of the projections $(\text{pr}_i)_{i \in I}$.

In other words, given the topologies τ_i , the family of sets

$$\{\text{pr}_i^{-1}(U_i) | i \in I, U_i \in \tau_i\} \quad (4.9)$$

Theorem 4.13. Given two topological spaces X, Y , the set $X \times Y$ can be given a topology called the *product topology*

Proof. □

Prove the projection is continuous

4.6 Continuous functions

Continuous functions are a class of functions between topological spaces which allows the preservation of some topological properties

Definition 4.14. A function f between two topological spaces $f : X \rightarrow Y$ is *continuous* if for any open set $U \in \tau_Y$, the pre-image of U $f^{-1}(U)$ is an open set in X : $f^{-1}(U) \in \tau_X$

Motivated from the property of preimage wrt unions and intersections, opposite category sort of thing?

A function $f : X \rightarrow Y$ is said to be continuous at a point $x \in X$ if for every neighbourhood $U_{f(x)}$ of $f(x)$, the preimage $f^{-1}(U_{f(x)})$ contains an open set of X :

$$\forall U_{f(x)} \in \mathcal{O}(Y), \exists U' \subseteq f^{-1}(U_{f(x)}), U' \in \mathcal{O}(X) \quad (4.10)$$

Ex : θ is not continuous at 0 : for every $U_{f(0)} = (1/2 - \varepsilon, 1/2 + \varepsilon)$, $f^{-1}((1/2 - \varepsilon, 1/2 + \varepsilon)) = \{0\}$ if $\varepsilon < 1/2$

Furthermore, a function is a *homeomorphism* if it is both continuous and possess an inverse that is also continuous, so if $f \in \text{Homeo}(X, Y)$,

$$f \circ f^{-1} = \text{Id}_Y \quad (4.11)$$

$$f^{-1} \circ f = \text{Id}_X \quad (4.12)$$

Example of x^3

Example 4.15. A disk is homeomorphic to the plane.

Proof.

□

Example of homeomorphism

Homeomorphisms are the isomorphisms for topological spaces, in the sense that if two topological spaces are homeomorphic, we can think of them as being identical (at least as topological spaces). This can be seen by considering the conservation of topological properties via homeomorphisms

Homeomorphisms form the set of isomorphisms in the **Top** category.

Continuous deformation

Definition 4.16. A *retraction* $r : X \rightarrow S$ of a topological space X onto a subspace $S \subseteq X$, $\iota : S \hookrightarrow X$, is a continuous map such that the map is the identity on the subset $r|_{\iota(S)} = \text{Id}_{\iota(S)}$, or equivalently, $r\iota = \text{Id}_{\iota(S)}$.

4.7 Topological properties

The topological properties of a topological space are all the properties that concern exclusively the properties we can derive from the topology.

Model theory of topology

Operations on vector spaces that count : products, coproducts, quotients, etc?

Invariant properties under homeomorphisms

Theorem 4.17. If two topological spaces X, Y are homeomorphic, any statement about X entirely constructed from the set of opens of X will also be true for Y .

4.7.1 Connected spaces

As we've seen in the section on cohesion, one of the important property that a topology can give us is the notion of connectedness of a region of a space. In terms of a topological space, we define it thusly :

Definition 4.18. A topological space (X, τ) is *disconnected* if there exists two open sets $U_1, U_2 \in \tau$ that are disjoint $U_1 \cap U_2 = \emptyset$ which cover the entire space : $U_1 \cup U_2 = X$. It is *connected* otherwise.

This gives us the notion that in some sense, a connected space requires some

Another notion of connectedness is that of path-connectedness, which is the idea that a space is path-connected if any two points can be joined by a curve, which is a fairly intuitive notion : a space is connected if one could theoretically go from one point to the other without leaving that space.

Definition 4.19. A topological space (X, τ) is *path connected* if for any two points $x, y \in X$, there exists a continuous curve $c : [0, 1] \rightarrow X$ such that $c(0) = x$, $c(1) = y$.

The two notions are unfortunately not identical, although connectedness does imply path-connectedness.

4.7.2 Separation properties

Separation properties in a topological space give us a way to differentiate points and open sets using the topology. There is some hierarchy to those properties, roughly corresponding to the objects we are separating and how we separate them.

Definition 4.20. A space is a T_0 space or a *Kolmogorov space* if for any two points $p, q \in X$ such that $p \neq q$, there exists

Definition 4.21. A space is a T_2 space (or a *Hausdorff space*) if

4.8 Compactness

As still very barebone structures, it can be hard to define some notion of a space being "finite" in some sense, since topological spaces do not have any notion of distance.

Definition 4.22. A topological space (X, τ) is *compact* if given any open cover $\{U_i\}_{i \in I}$,

$$\bigcup_{i \in I} U_i = X \quad (4.13)$$

there exists a finite subcover $\{U_i\}_{i \in I' \subset \mathbb{N}}$.

Example : $(0, 1)$ is not compact, $[0, 1]$ is compact, circle is compact, etc

Paracompact

Preservation of topological properties under continuous functions and homeomorphisms

4.9 Subspace topologies

Given a subset $S \subseteq X$ of a topological space X , we can define a topology on S via the *subset topology*

Definition 4.23. The *subset topology* τ_ι given an inclusion map $\iota : S \hookrightarrow X$ in a topological space (X, τ) is given by the set of open sets

$$\tau_\iota = \{O \cap \iota(S) \mid \forall O \in \tau\} \quad (4.14)$$

In other words, a subset of S is open if its image is the intersection of an open set in X with the image of S . Equivalently, it is the coarsest topology for which ι is continuous : open sets are $\iota^{-1}(O)$.

Theorem 4.24. For a topological subspace $Y \subseteq X$ and a subset $A \subseteq Y$, the closure of A in Y being $\text{cl}_Y(A)$, then the closure of A in X is $\text{cl}_X(A) = \text{cl}_Y(A)$.

4.10 Nerve complex

Definition 4.25. For a family of sets $B = \{U_i\}_{i \in I}$ over an indexing set I , the *nerve* of B is a set of finite subsets of I for which the intersection of the appropriate subsets in B is non-empty. That is, if $i, j, k, \dots \in I$ and

$$U_i \cap U_j \cap U_k \cap \dots \neq \emptyset \quad (4.15)$$

then $\{i, j, k, \dots\}$ is in the nerve of B . In general :

$$N(B) = \left\{ J \subseteq I \mid \bigcap_{j \in J} U_j \neq \emptyset, \text{Card}(J) \in \mathbb{N} \right\} \quad (4.16)$$

Cech nerve

Nerve complex

4.11 Quotients

Given an equivalence relation $\sim$ on a topological space (X, τ) , the *quotient space* $X/\sim$ is the set of equivalence classes of X by that relation,

$$X/\sim = \{[x] \mid x \in X, \} \quad (4.17)$$

Canonical surjection :

$$q : X \rightarrow X / \sim \quad (4.18)$$

$$x \mapsto q(x) = [x] \quad (4.19)$$

The *quotient topology* on a quotient space is the topology defined on $X / \sim$ such that a subset $U \subset X / \sim$ is open if $q^{-1}(U)$ is open in X .

Adjunction space

Example : circle, X space, projection

Theorem 4.26. The circle is the quotient of the line $\mathbb{R}$ under the equivalence relation

$$x \sim x + a \quad (4.20)$$

for some constant $a \in \mathbb{R}$.

4.12 Metric topology

Many important topological spaces are also metric spaces, on which is defined some notion of distance via a metric function :

Definition 4.27. A *metric* on a space with set X is a function

$$d : X \times X \rightarrow \mathbb{R}_{\geq 0} \quad (4.21)$$

that is symmetric ($d(x, y) = d(y, x)$), positive (for $x \neq y$, $d(x, y) > 0$), with a zero distance for a point to itself ($d(x, x) = 0$), and obeying the triangle inequality :

$$d(x, z) \leq d(y, z) + d(x, y) \quad (4.22)$$

A metric space (X, d) is then simply a set X with a metric on that space.

The standard example of a metric is the *Euclidian distance* on $\mathbb{R}^n$, which is simply given by the Pythagorean theorem :

$$d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y})^2} \quad (4.23)$$

[Illustration of the Euclidian distance with Pythagoras]

Definition 4.28. An *open ball* $B_{x,r}$ of radius r centered on a point $x \in X$ is the set of all points with a distance to x strictly inferior to r :

$$B_{x,r} = \{y \in X | d(x, y) < r\} \quad (4.24)$$

Theorem 4.29. The set of all open balls define a topological base on X

Proof. □

In addition to consider the set of continuous functions between two topological spaces, we can also consider the set of isometries between two metric spaces.

Definition 4.30. An *isometry* between two metric spaces (X, d) and (X', d') is a map $f : X \rightarrow Y$ preserving distances :

$$d(x, y) = d'(f(x), f(y)) \quad (4.25)$$

Relations between isometries and continuous functions

4.13 Topological sheaf

Definition 4.31. A *presheaf* on a topological space X is defined by a function $F : O(X) \rightarrow$ with the following properties : For every open set U , there is a set $F(U)$ (sections over U)

4.14 Lattices, locales and frames

Another way we can look at topological spaces is by considering open sets as primitive objects rather than as collections of points. As a collection of subsets, the natural relation to consider is the inclusion $O_1 \subseteq O_2$, and a natural algebra is therefore the algebra of joins and meets.

The

Lattice For a partial ordered set $(X, \leq)$, it is a lattice if each two elements has a join and a meet

join least upper bound $a \vee b$

meet greatest lower bound $a \wedge b$

prop $a_1 \leq a_2$ and $b_1 \leq b_2$ implies $a_1 \vee b_1 \leq a_2 \vee b_2$, $a_1 \wedge b_1 \leq a_2 \wedge b_2$

$a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

$$a \vee a = a \quad a \wedge a = a$$

Locale of a topology take (X, τ) , with $\Omega(X)$ the lattice of open subsets of X .

Lattice of $(\tau, \subseteq)$

For every open set A , $A \subseteq X$ and $\emptyset \subseteq A$

Least upper bound of two open sets A, B $A \cup B$?

Greatest lower bound : $A \cap B$

complete lattice

Continuous map preserves the lattice

[...]

The concept of a frame isn't entirely equivalent to that of a topological space, and this applies in both directions.

Example 4.32. If we consider the space of surjective functions $\mathbb{N} \rightarrow \mathbb{R}$

Frames and locales are op categories

Properties of frames and locales for properly separated topologies

Parallels of frames and mereology

Stone duality

Theorem 4.33.

4.15 Germs

The notion of the germ of a function on a topological space is meant to convey what it means for some properties of a function to be local. If we have various functions on the same topological space, those functions could very well have very different behaviours globally, but be similar at a given point or in a given region so that they could not be differentiated there.

To define germs, we need then to define what it means for two functions to be of similar behaviour locally. For two maps f_1, f_2 between two topological spaces, we say that they are equivalent $f_1 \sim f_2$ if there exists an open set $U_p \ni p$ such that $f_1|_{U_p} = f_2|_{U_p}$

For a locally ringed space $(X, \mathcal{O}_X)$, and a point $x \in X$ in that space, consider the net of all neighbourhoods N_x .

Chapter 5

Groups

Groups can be roughly understood as transformations on some object (the name is actually a shortened form for the older term of "group of transformations"). As a mathematical object, they are the abstraction of that transformation, in the sense that the group itself does not have anything to do with the object it is supposed to transform.

Definition 5.1. A *group* $(G, \cdot, {}^{-1})$ is a structure composed of a set G , a binary operation $\cdot : G \times G \rightarrow G$ and an inverse operation ${}^{-1} : G \rightarrow G$, such that

- There exists an element $e \in G$ called the *identity element*, which obeys the rules $g \cdot e = e \cdot g = e$
- The inverse of an element is such that $g^{-1} \cdot g = g \cdot g^{-1} = e$
- The group operation is associative : $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

If the set G is of finite cardinality $|G| \in \mathbb{N}$, we say that G is a *finite group*.

Definition 5.2. A *cyclic group* C_n of cardinality n is a group generated by a

A few common example of groups that we will see throughout this book

The cyclic group of order 2 $\mathbb{Z}_2$ is the simplest non-trivial group, simply composed of the identity and an element which is its own inverse, those two elements being generally denoted by $\{I, -I\}$. These can correspond to such actions as reversing the direction of space, sending numbers to their negatives or inverses,

Group properties

Theorem 5.3. Groups have the latin square property, where between any two elements of a group, there exists a unique third element such that the right (or left, for a possibly different third element) product of the first element by it will give the second :

$$\forall a, b \in G, \exists g. a \circ g = b \tag{5.1}$$

Proof. If we define some group element $g = a^{-1} \cdot b$, then we have

$$a \cdot g = a \cdot (a^{-1} \cdot b) \quad (5.2)$$

$$= (a \cdot a^{-1}) \cdot b \quad (5.3)$$

$$= e \cdot b \quad (5.4)$$

$$= b \quad (5.5)$$

For its uniqueness, assume that there exists another element g' such that $a \cdot g = a \cdot g' = b$.

$$g = e \cdot g \quad (5.6)$$

$$= (a^{-1} \cdot a) \cdot g \quad (5.7)$$

$$= a^{-1} \cdot (a \cdot g) \quad (5.8)$$

$$= a^{-1} \cdot b \quad (5.9)$$

$$= a^{-1} \cdot (a \cdot g') \quad (5.10)$$

$$= (a^{-1} \cdot a) \cdot g' \quad (5.11)$$

$$= e \cdot g' \quad (5.12)$$

$$= g' \quad (5.13)$$

□

Similar proof for $g \cdot a = b$

5.1 Subgroups

A *subgroup* of a group G is a subset of G closed under group operations. If we have a subgroup H of G , denoted by $H \leq G$, then the following holds :

$$\forall h \in H, \quad h^{-1} \in H \quad (5.14)$$

$$\forall h_1, h_2 \in H, \quad h_1 \circ h_2 \in H \quad (5.15)$$

$$e_G \in H \quad (5.16)$$

If the subset of G is a strict subset, we have a strict subgroup denoted by $H < G$.

Example $\mathbb{Z}_2 < \mathcal{O}(n) \subset \mathcal{O}(n+1)$, etc

Definition 5.4. Given an element $g \in G$ of a group, and a subgroup $H < G$, the left (resp. right) *coset* of H by g , denoted gH (resp. Hg), is the set of all elements of the form $gH = \{gh|h \in H\}$ (resp. $Hg = \{gh|h \in H\}$)

The set of all left (resp. right) cosets of a group is denoted by G/H (resp. H/G),

$$G/H = \{gH | g \in G\} \quad (5.17)$$

$$H \quad (5.18)$$

$$G = \{Hg | g \in G\} \quad (5.19)$$

By themselves, those cosets are simply sets, it need not have any group structure at all. If we consider the subgroup

5.1.1 Normal subgroup

Definition 5.5. A *normal subgroup* N of a group G , denoted $N \triangleleft G$, is a subgroup of G that is invariant under group conjugation, ie

$$\forall g \in G, n \in N, gng^{-1} \in N \quad (5.20)$$

Normal subgroup

Definition 5.6. The *center* $Z(G)$ of a group G is the largest subgroup composed entirely of elements of G that commute with every other elements :

$$\forall z \in Z(G), \forall g \in G, z \circ g = g \circ z \quad (5.21)$$

5.1.2 Semidirect product

Theorem 5.7. A group G is isomorphic to the semidirect product $G_1 \rtimes G_2$ if and only if $A \triangleleft G$, $B < G$, such that $A \cong G_1$, $B \cong G_2$, $A \cap B = \{e\}$ and $AG = G$.

5.2 Group properties

Definition 5.8. A group G is *abelian* if for any $g_1, g_2 \in G$,

$$g_1 \cdot g_2 = g_2 \cdot g_1 \quad (5.22)$$

The abelian character of a group can be measured by considering its *commutator* :

$$[g, h] = g^{-1}h^{-1}gh \quad (5.23)$$

where the group is abelian iff $[g, h] = e$ for any pair $g, h \in G$. The set of all commutators

$$\{g^{-1}h^{-1}gh | g, h \in G\} \quad (5.24)$$

can be used to generate a group $[G, G]$ called the *commutator subgroup*,

Commutator subgroup is a subgroup of G , it is Abelian

Any group G can be turned into an abelian group via its *abelianization* : $G/[G, G]$, $[G, G]$ the commutator subgroup

Another way to express the commutative properties of a group is via its conjugacy classes.

Definition 5.9. A group's *conjugacy classes* $\text{Cl}(G)$

Conjugacy class as it relates to non-commutativity

5.3 Group homomorphisms

Group homomorphisms are a special kind of maps between groups that preserve the group properties.

Definition 5.10. A *group homomorphism* $f : G \rightarrow H$ is a map between two groups preserving the group properties :

$$f(e_G) = e_H \quad (5.25)$$

$$f(g_1 \circ_G g_2) = f(g_1) \circ_H f(g_2) \quad (5.26)$$

$$f(g^{-1}) = [f(g)]^{-1} \quad (5.27)$$

A basic example of this would be for instance the constant map from any group to the trivial group,

$$f : G \rightarrow \{e\} \quad (5.28)$$

$$g \mapsto e \quad (5.29)$$

which fulfills all those requirements trivially. Some non-trivial examples

The *kernel* of a group homomorphism $f : G \rightarrow H$ is the preimage of its identity

$$\ker(f) = \{g \in G | f(g) = e_H\} \quad (5.30)$$

The dual of the kernel is the cokernel

Definition 5.11. The *cokernel* of a group homomorphism $f : G \rightarrow H$ is the quotient space $H/\text{Im}(f)$

Fundamental theorem on homomorphisms

Theorem 5.12. Given two groups G, H with a group homomorphism $f : G \rightarrow H$ and N a normal subgroup of G , ϕ the natural surjective homomorphism $G \rightarrow G/N$. If N is a subset of $\ker(f)$, then there exists a unique homomorphism $h : G/N \rightarrow H$ such that $f = h \circ \phi$.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow \phi & \nearrow h & \\ G/N & & \end{array}$$

Isomorphism theorems :

First isomorphism theorem

Theorem 5.13. For two groups G, H and a homomorphism $f : G \rightarrow H$, then

- $\ker(f)$ is a normal subgroup of G
- $\text{im}(f)$ is a subgroup of H
- $\text{im}(f)$ is isomorphic to $G/\ker(f)$

Second isomorphism theorem

Theorem 5.14. For a group G , a subgroup $S < G$ and a normal subgroup N , then

- $SN < G$
- N is a normal subgroup of SN
- $S \cap N$ is a normal subgroup of S
- The quotient groups $(SN)/N$ and $S/(S \cap N)$ are isomorphic

Third isomorphism theorem

Theorem 5.15. For a group G and a normal subgroup N ,

- If K is a subgroup of G , $N < K < G$, then G/N has a subgroup isomorphic to K/N .

- Every subgroup of G/N is of the form K/N for some subgroup $K < G$, $N < K < G$
- If K is a normal subgroup of G , $N < K < G$, then G/N has a normal subgroup isomorphic to K/N
- Every normal subgroup of G/N is of the form K/N for some normal subgroup $K < G$ such that $N < K < G$
- If K is a normal subgroup of G , $N < K < G$, then $(G/N)/(K/N)$ is isomorphic to G/K

”Let G be a group, and N a normal subgroup of G . The canonical projection homomorphism $G \rightarrow G/N$ defines a bijective correspondence between the set of subgroups of G containing N and the set of (all) subgroups of G/N . Under this correspondence normal subgroups correspond to normal subgroups.

This theorem is sometimes called the correspondence theorem, the lattice theorem, and the fourth isomorphism theorem. ”

5.4 Group extensions

Definition 5.16. For two groups G, N , we say that E is an *extension* of G by N if there exists a short exact sequence

$$1 \rightarrow N \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1 \quad (5.31)$$

E is a group, $\iota(N)$ is a normal subgroup of E , and the quotient $E/\iota(N)$ is isomorphic to G

Example 5.17. The *trivial extension* of two groups is the extension isomorphic to the sequence

$$1 \rightarrow N \xrightarrow{\iota} N \times G \xrightarrow{\pi} G \rightarrow 1 \quad (5.32)$$

5.4.1 Central extension

A common type of extension in group theory is the central extension :

Definition 5.18. A *central extension* of a group G by a group A is an extension E for which A is in the center of E : $A < Z(E)$

Example 5.19. The trivial extension of G by an Abelian group A is the central extension $A \times G$ given by the inclusion and projection map of group products :

$$1 \rightarrow \dots \quad (5.33)$$

Group action by central extensions

Two common types of central extensions we will commonly see will be the central extension by $\mathbb{Z}_2$, by $U(1)$. These will typically be used for extending a group action from a given space to its double cover

[Bargmann theorem?]

Example from sets?

5.5 Group action

[4]

Groups were originally conceived as transformations being done to a given mathematical objects (they were originally usually called "groups of transformations" [5]). It is therefore natural to ask how such a thing can be done from the fairly abstract theory of groups we have seen so far which have only acted on themselves.

Definition 5.20. Given a group G and a set X , a *group action* ρ is a map

$$\rho : G \times X \rightarrow X \quad (5.34)$$

$$(g, x) \mapsto \rho_g(x) \quad (5.35)$$

that preserves the properties of the group, in the sense that the identity of the group is the identity function on X and the composition of group actions is the group action of their group product :

$$\rho_e(x) = x \quad (5.36)$$

$$\rho_{g_1}(\rho_{g_2}(x)) = \rho_{g_1 \cdot g_2}(x) \quad (5.37)$$

Corrolary 5.1. The action of the inverse g^{-1} is the inverse function of the action of g

Proof.

$$\rho_g \circ \rho_{g^{-1}} = \rho_{g \cdot g^{-1}} = \rho_e = \text{Id} \quad (5.38)$$

and same reasoning for $\rho_{g^{-1}} \circ \rho_g$ □

It is always possible to find a group action of any group on any set, as can be seen by considering the trivial group action for instance :

$$\forall g \in G, \rho_g = \text{Id}_X \quad (5.39)$$

which obviously preserves the action of the identity, and $\rho_{g_1} \circ \rho_{g_2} = \text{Id}_X \circ \text{Id}_X = \text{Id}_X = \rho_{g_1 \cdot g_2}$.

Example 5.21. The action of $\mathbb{Z}_2$ on a ring like $\mathbb{Z}$ by inversion :

$$\rho_1(x) = x \quad (5.40)$$

$$\rho_{-1}(x) = -x \quad (5.41)$$

Theorem 5.22. The set of all automorphisms $\text{Aut}(X)$ on a set X forms a group by composition.

Proof. Every set possess the identity function Id_X as an automorphism, and being automorphisms, we have that for any $f \in \text{Aut}(X)$, there exists an inverse function $f^{-1} \in \text{Aut}(X)$ such that

$$f \circ f^{-1} = f^{-1} \circ f = \text{Id}_X \quad (5.42)$$

and function compositions are of course associative. $\square$

Such groups of automorphisms $\text{Aut}(X)$ have a natural group action on X simply by application, and in fact, every group action $G \times X \rightarrow X$ can be considered as a group homomorphism $G \rightarrow \text{Aut}(X)$.

Definition 5.23. For any finite set X of cardinality N , the set of all the bijections $X \rightarrow X$ forms a group called the *symmetric group* S_N

5.5.1 Types of group action

Definition 5.24. A group action ρ is *faithful* if for all $x \in X$, $g \cdot x = x$ implies $g = e_G$.

Example 5.25. The trivial group action $\rho_g(x) = x$ for all $g \in G$ is a group action but is not faithful unless G is the trivial group.

Example 5.26. The group action of the n -dimensional translation group $\mathbb{R}^n$ acting on $\mathbb{R}^n$ but preserving only one such translation :

$$\rho_{(a,b,c,\dots)}(x, y, z, \dots) = (x + a, y, z, \dots) \quad (5.43)$$

is a group action but is not faithful for any element $(0, b, c, \dots)$.

Definition 5.27. A group action is *transitive* if for any pair $x, y \in X$, there exists a transformation $g \in G$ taking x to y : $\rho_g(x) = y$.

stabilizer

Definition 5.28. The *orbit* of a group action is the set of all images of a given point under that group :

$$\text{Orb}_\rho(x) = \{x' | \exists g \in G. g \cdot x = x'\} \quad (5.44)$$

Example 5.29. If we take the action of the two-dimensional rotation group $O(2)$ on the vector space $\mathbb{R}^2$, the orbit of any point $v \neq 0$ will be a circle. [Diagram]

Example 5.30. The group acting on itself by left or right product is a group action

$$l : \quad (5.45)$$

Group action of the group on itself :

proof Left group action :

$$\rho_g(g') = g \circ g' \quad (5.46)$$

Right group action :

$$\rho_g(g') = g' \circ g \quad (5.47)$$

Adjoint action :

$$\rho_g(g') = g \circ g' \circ g^{-1} \quad (5.48)$$

Theorem 5.31. The orbit of the group action on an element of the group itself is the whole group :

$$\text{Orb}(g, \rho_G) = G \quad (5.49)$$

Proof.

□

$$a \quad (5.50)$$

Theorem 5.32. The right action, left action and adjoint action on a group are all faithful and transitive.

5.5.2 Equivariance and invariance

If we have two sets X, Y , each equipped with an action from the same group G , we say that a map between those sets $f : X \rightarrow Y$ is *equivariant* if the mapping preserves the group action. If both group actions are left actions, this means

$$f(\rho_{X,g} \cdot x) = \rho_{Y,g} \cdot f(x) \quad (5.51)$$

This is referred to as left-left equivariance. The same definition applies for any combination of type of group action, with the right-right, right-left and left-right equivariance obeying respectively

$$f(x \cdot \rho_{X,g}) = f(x) \cdot \rho_{Y,g} \quad (5.52)$$

$$f(x \cdot \rho_{X,g}) = \rho_{Y,g} \cdot f(x) \quad (5.53)$$

$$f(\rho_{X,g} \cdot x) = f(x) \cdot \rho_{Y,g} \quad (5.54)$$

homomorphisms in the category of G -sets.

Invariant map

5.5.3 Action of group extensions

5.6 Finite groups

Definition 5.33. The *cyclic group* of order n $\mathbb{Z}_n$ is the quotient group of the additive group $\mathbb{Z}/n\mathbb{Z}$, with $n\mathbb{Z}$ the ideal given by multiplication : $n\mathbb{Z} = \{nk | \forall k \in \mathbb{Z}\}$, with the equivalence relation

Rotation groups

5.7 The symmetric group

A good example of a group that will not require any structure to be defined later in this group is the symmetric group, corresponding to the group of permutations of objects in a sequence for instance.

Definition 5.34. The *symmetric group* S_X over a set X is the group of all automorphisms on X , ie. the bijective functions from X to X . The set is defined as $S_X = \text{Aut}(X)$, with the group inverse being the function inverse and the group product being the function composition, with the identity being the identity functions.

Theorem 5.35. For sets X, X' of identical cardinality, the symmetric groups are all isomorphic.

Proof.

□

We can therefore talk about *the* symmetric group for sets of cardinality n , simply denoted by S_n .

The symmetric group S_n has a group action on sequences of n elements generated

5.8 Topological groups

A group can also be equipped with a topology, associating the group structure $(G, \cdot)$ with the topological structure (G, τ) over the set G . Then a group is topological if the two structures are compatible :

Definition 5.36. A *topological group* $(G, \cdot, \tau)$ is a group for which the action of the inverse $^{-1}$ is continuous with respect to the topology, and $\cdot$ is continuous with respect to the product topology of (G, τ) .

Some examples of topological groups are groups with the discrete topology, called *discrete groups*

Another common class of topological groups is that of Lie groups, also called continuous groups, which will be seen in more details in X

1-parameter subgroup

Definition 5.37. A group H is a *1-parameter subgroup* of a topological group G if there exists a continuous group homomorphism from the additive group $(\mathbb{R}, +)$ to G :

$$\phi : \mathbb{R} \rightarrow G \quad (5.55)$$

The fact that this mapping is a group homomorphism implies that we have

$$\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) \quad (5.56)$$

If we interpret this geometrically, ϕ corresponds to a curve in the topological space. The point $\phi(0)$ corresponds to the identity of G , and for any point along this line, the

5.9 Torsors

Definition 5.38. Given a group G , a *G-torsor* is a set T with a group action $\rho : G \times T \rightarrow T$ such that the action ρ is free and transitive.

A G -torsor is in some sense the group with its identity point e_G removed. We simply have a "group operation" that can be applied to two points of our torsor

Shear map, show isomorphism as sets, etc

5.10 Groupoids

For our purpose, we will need in addition to groups a generalization called a groupoid, which is

Definition 5.39. A *groupoid* $(G, \cdot, {}^{-1})$ is a set G equipped with a unary function ${}^{-1}$ and a partial binary function $\cdot$.

Corrolary 5.2. A group is a groupoid for which the groupoid product is defined for any two objects of G .

Chapter 6

Fields and rings

Define some notion of quantity or magnitude idk

Definition 6.1. A *ring* $(R, +, \cdot)$ is a set R equipped with a binary operation $+$ with which it forms an Abelian group (the inverse being denoted by $x \rightarrow -x$), and $\cdot$, with which it is a monoid (associative and unital), along with the following distribution law :

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad (6.1)$$

$$(b + c) \cdot a = b \cdot a + c \cdot a \quad (6.2)$$

The standard example of a ring is the set of integers $\mathbb{Z}$ with addition and multiplication, hence the typical notation that we use.

Example 6.2. The set of integers modulo q , $\mathbb{Z}_q$, forms a ring by quotienting $\mathbb{Z}$ by $q\mathbb{Z}$

Proof.

□

Example 6.3. The algebra of polynomials $R[X]$ over a ring R is a ring.

Proof. Two polynomials have a sum simply by considering the sum of their coefficients of same degree :

$$\left(\sum_{i=0}^n a_i X^i\right) + \left(\sum_{i=0}^n b_i X^i\right) = \left(\sum_{i=0}^n (a_i + b_i) X^i\right) \quad (6.3)$$

□

Definition 6.4. A *field* $(F, +, \cdot)$ is a set F equipped with two closed binary operations, $+$ and $\cdot$, two neutral elements for those operations generally denoted by 0 and 1,

$$a + 0 = 0 + a = a \quad (6.4)$$

$$a \cdot 1 = 1 \cdot a = a \quad (6.5)$$

as well as two inverses for those operations, where the inverse of the product $\cdot$ is defined for every element of the field except the additive neutral element 0 :

$$a + (-a) = 0 \quad (6.6)$$

$$a \cdot a^{-1} = 0 \quad (6.7)$$

Example 6.5. The rational numbers $\mathbb{Q}$ with addition and multiplication forms a field.

The most important field for our purpose is the *real numbers* $\mathbb{R}$ equipped with addition and multiplication, which is the completion of $\mathbb{Q}$ as a field with the topology defined by the distance function $d(x, y) = |x - y|$.

Characteristic of a field

Definition 6.6. The *characteristic* $\text{Char}(R)$ of a ring is the smallest positive integer k for which the k -fold sum of 1 is equal to 0:

$$1 + \dots + 1 = 0 \quad (6.8)$$

or 0 if no such integer exists.

It is fairly uncommon in physics to have fields of characteristic higher than zero, and in particular equal to two, which is a very common condition for a lot of theorems, but as we will see for instance in the case of finite fields, such things may occur.

6.1 Ring homomorphism

Definition 6.7. A *ring homomorphism* is a map between two rings $f : R_1 \rightarrow R_2$ that preserves the ring structure :

$$f(x + y) = f(x) + f(y) \quad (6.9)$$

$$f(x \cdot y) = f(x) \cdot f(y) \quad (6.10)$$

Just as for groups, as we have some preferred elements in rings and fields, we can look at the kernel of such a homomorphism

Example 6.8.

Definition 6.9. The *kernel* of a ring homomorphism

6.2 Ideals

Just as with group and their notion of normal subgroups, fields and rings also have a notion of subsets of their algebraic structure that absorbs other elements by the use of their product.

Definition 6.10. A subring I is a left (resp. right) *ideal* of a ring R if $(I, +)$ is a subgroup of $(R, +)$, and for every $i \in I$, it absorbs multiplication from the left (resp. right):

$$i \cdot r \in I, \text{ resp. } r \cdot i \in I \quad (6.11)$$

6.3 Finite fields

Despite their most famous examples all being infinite sets, fields do not have to be infinite.

Definition 6.11. A *finite field* $\mathbb{F}_q$ of order q is a field with finite cardinality, $|\mathbb{F}_q| = q$.

In other words, we have a set F with the usual field operations $(+, \cdot, ^{-1}, -)$. They are also called Galois fields due to their origins in Galois theory, denoted by $\text{GF}(q)$.

Unlike the fields we have used so far, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, finite fields differ in that we have to consider the characteristic of the field.

Theorem 6.12. The characteristic of a finite field is a prime number.

Proof. For a finite field $(\mathbb{F}_q, +, \cdot)$, the additive group $(\mathbb{F}_q, +)$ is a finite group, therefore of finite order, $x \in \mathbb{F}_q$, $x + x + \dots + x = 0$ for some finite value, so that our field has a non-zero characteristic. If the characteristic is not prime, we have two integers $\text{Char}(\mathbb{F}_q) = ab$. We have

$$\text{Char}(\mathbb{F}_q) \cdot 1_{\mathbb{F}_q} = 0 \quad (6.12)$$

However,

$$\text{Char}(\mathbb{F}_q) \cdot 1_{\mathbb{F}_q} = (a \cdot b) \cdot 1_{\mathbb{F}_q} \quad (6.13)$$

$$= (a \cdot b) \cdot (1_{\mathbb{F}_q} \cdot 1_{\mathbb{F}_q}) \quad (6.14)$$

$$= (a \cdot 1_{\mathbb{F}_q}) \cdot (b \cdot 1_{\mathbb{F}_q}) \quad (6.15)$$

Since a field has no zero divisors, either $(a \cdot 1_{\mathbb{F}_q}) = 0$ or $(b \cdot 1_{\mathbb{F}_q}) = 0$. Hence rs cannot be the smallest possible order of the characteristic of the field. $\square$

Theorem 6.13. A finite field $\mathbb{F}_q$ must have a cardinality equal to the power of a prime number.

Proof. For any field F of prime characteristic $\text{Char}(F) = p$, there exist a unique subfield isomorphic to $\mathbb{Z}_p \dots$ $\square$

Sums and products of $\mathbb{F}_p$ are done mod p

Multiplication group $\mathbb{F}_p$ is cyclic $\mathbb{Z}_k$ for some k (the root of p)

Interpretation of $\mathbb{F}_p$ as a/b for $b \neq 0 \pmod p$ and $a/b = c/d$ if $ad - bc = 0 \pmod p$

Chapter 7

The real numbers

While simply a field as seen in the previous chapter, the real numbers $\mathbb{R}$, as the basis for most of physics and an object with a rich structure on top of its field structure, deserve its own chapter for their introduction.

We will not spend too much time defining the construction of the real line $\mathbb{R}$, as this can be found in many analysis books [6], but I will give a quick rundown of its properties.

As we've seen, the real line can be defined as a field. In addition to this, real numbers also have a total order defined on them, $\leq$, with the usual properties of a total order : it is reflexive ($a \leq a$), transitive ($a \leq b$ and $b \leq c$ imply $a \leq c$), antisymmetric ($a \leq b$ and $b \leq a$ imply $a = b$), and total ($a \leq b$ or $b \leq a$, in other words any two elements of $\mathbb{R}$ belong to this relation). In addition, it interacts with the operations via

- If $a \leq b$, then $a + c \leq b + c$
- If $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$

For convenience, we also define the other order relations on $\mathbb{R}$:

$$a < b \iff a \leq b \wedge a \neq b \tag{7.1}$$

$$a \geq b \iff ab \leq a \tag{7.2}$$

$$a > b \iff a \geq b \wedge a \neq b \tag{7.3}$$

So far, all these axioms have applied as much to the field of rational numbers $\mathbb{Q}$ as the real line $\mathbb{R}$. The fundamental difference between the two is that $\mathbb{R}$ is *complete*, as expressed in the least-upper bound axiom :

Axiom 7.1. For any non-empty subset A of $\mathbb{R}$ with an upper bound, A has a least-upper bound.

$$(A \subseteq \mathbb{R} \wedge A \neq \emptyset \wedge \exists x \in \mathbb{R}, \forall y \in A, y < x) \quad (7.4)$$

$$\rightarrow \exists x \in \mathbb{R} (\forall x \in \mathbb{R}, \forall y \in A, x \geq y \wedge \forall y \in \mathbb{R} (y < x \rightarrow \exists z \in A, y < z)) \quad (7.5)$$

An upper bound being some value x in $\mathbb{R}$ such that every element of A is strictly inferior to x , while a least-upper bound meaning that out of every possible upper bound of that set, this is the smallest possible one.

An important object in $\mathbb{R}$ is the *interval*, which is the set of every real number between two given points. There are different types of intervals depending on whether or not we include one of these values

$$[a, b] = \{x \in \mathbb{R} | x \geq a \wedge x \leq b\} \quad (7.6)$$

$$(a, b) = \{x \in \mathbb{R} | x > a \wedge x < b\} \quad (7.7)$$

$$[a, b) = \{x \in \mathbb{R} | x \geq a \wedge x < b\} \quad (7.8)$$

$$(a, b] = \{x \in \mathbb{R} | x > a \wedge x \leq b\} \quad (7.9)$$

If an interval contains those points, it is *closed* at those points (denoted by a square bracket), while if it does not contain them, it is *open* at those points, denoted by a parenthesis. Note that if $b < a$ in those definitions, we will end up with either empty intervals or just two points, but by convention we do not consider the case where $b < a$.

The real line also has a canonical metric on it, defined by

$$d(x, y) = |x - y| \quad (7.10)$$

$\mathbb{R}$ is equipped with a canonical topology, with the topological basis defined by either the set of every open intervals, or equivalently as the metric space topology of $(\mathbb{R}, d)$, since every open interval is an open ball.

For a variety of purpose, we will also consider the *extended real line*, which is the real line augmented by two additional points,

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad (7.11)$$

The ordering properties of $\pm\infty$ is that they are the lower and upper bound of $\overline{\mathbb{R}}$:

$$\forall x \in \overline{\mathbb{R}}, -\infty \leq x \leq +\infty \quad (7.12)$$

It can be equipped with the order topology with open sets (a, b) , making it a compact space (proof?)

operations :

Chapter 8

Analysis

8.1 Limits

Definition 8.1. Given a sequence $\{x_i\}_{i \in I}$, its *limit* in $\mathbb{R}$

$$\lim_{i \rightarrow \infty} x_i = x \quad (8.1)$$

8.2 Derivatives

Definition 8.2. The *derivative* of a real function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by the limit (if it exists)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (8.2)$$

Example 8.3. Any constant $f(x) = c$ has a derivative $f' = 0$

Example 8.4. The power function $f(x) = x^a$, $a \in \mathbb{Z}$,

Derivatives from infinitesimally thickened points/dual numbers/Weil algebras

8.3 Integrals

8.4 Differential equations

Peano theorem

The simplest case of first order ordinary differential equation is fairly well-known via the *Picard–Lindelöf theorem* :

Theorem 8.5. For a rectangle $D \subset \mathbb{R} \times \mathbb{R}^n$ in the space of coordinates (t, x) , with some point $(t_0, x_0) \in \text{Int}(D)$, and $f : D \rightarrow \mathbb{R}^n$ some function continuous in t and Lipschitz continuous in x , then there exists some $\varepsilon \in \mathbb{R}$ such that the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \quad (8.3)$$

has a unique solution on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$

Proof.

□

By considering the Legendre transform of a system, we can extend this theorem to the case of order k ordinary differential equations

Theorem 8.6. For a system of ordinary differential equations of order k of the form

$$y^{(n)}(t) = \quad (8.4)$$

$$y^{(n-1)}(t) = \quad (8.5)$$

Non-uniqueness example

Part II

Geometry and spacetime

The splitting of physics into various theories can be somewhat arbitrary, as almost every part of it can interact with the rest. In the case of general relativity, two aspects that we could consider separately although they are still deeply related is the geometry (and kinematics) of the theory, where we describe what spacetime is, and the dynamics, where we describe how spacetime evolves depending on its matter content (and how the matter itself will evolve). This split will not be entirely unambiguous, as some of those topics could be assigned to either, but in this first part, we will only concern ourselves with the geometric structure of spacetime.

While this section is mostly referred to as geometry, an important nuance to bring up is the difference between geometry and kinematics. A kinematic roughly speaking relates to spacetime in the same way that a geometry relates to space, although the general term for the two will still be "geometry" (we can specify "geometry of space" if we really need to distinguish the two).

Generalities on geometry

Quote by Hilbert :

"One must be able to say at all times—instead of points, lines, and planes—tables, chairs, and beer mugs."

What is geometry?

In much the same way that we consider geometry as the relations of points, lines and planes, kinematics will be the relationship between points in spacetime, the motion of free points, etc

Chapter 9

Euclidian space

An important feature of spacetime is that, while it has been shown to be non-Euclidian, it still remains approximately Euclidian in some respects (many of which we will see in details later). This has to be the case to some degree as all experiments regarding space have resulted in it obeying a Euclidian model to very good accuracy until the 20th century.

There are many properties that Euclidian space possess, and our spacetimes will either still have those properties, or at least obey them to some degree in specific circumstances.

There are quite a few ways to describe Euclidian space that highlight its different features, as well as differing levels of structures on it. The two main ways of describing Euclidian space that we will see are *synthetic geometry*, where we describe it by its fundamental objects and their properties and relations, and *analytic geometry*, where we describe it as isomorphic to products of the real number line $\mathbb{R}$ with additional structures.

9.1 Synthetic geometry

Synthetic descriptions of Euclidian geometry are the oldest ones, going all the way back to Euclid himself [7]. While Euclid's definition of Euclidian space is not particularly fitting by modern standards, being fairly incomplete as an axiomatic system, I will include its description, as they are still commonly used even today.

Euclid has quite a number of definitions, a total of 131 in all his books total, but the ones that will concern us are those of the point, the line, the straight line, the circle, the center of the circle, angles, right angles and the edges of a segment.

- A *point* is that which has no parts.
- A *line* is breadthless length.

- The ends of a line are points.
- A *straight line* is a line which lies evenly with the points on itself.
- When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.
- A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
- And the point is called the center of the circle.

Some of those definitions may actually be secretly postulates [8], so that it is important to keep them in mind. For a start, the definition of a point as a partless object takes us back to our discussion of mereology : the definition of a point (assuming any point exists in Euclidian space) tells us both that there is some mereology defined on our space and that it has an atomic mereology.

We should also note that in Euclid's definitions, what he calls lines are actually what we would today call segments.

The postulates are

- To draw a straight line from any point to any point.
- To produce a finite straight line continuously in a straight line.
- To describe a circle with any center and radius.
- That all right angles equal one another.
- That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

In more modern terms, there is a segment connecting any two points, any segment can be prolonged into a line, for any point, and any segment starting from that point, there is a circle equidistant to that point and of distance equal to that of the segment, and [parallel postulate].

The more modern common axiomatization of Euclidian geometry is the one due to Hilbert [9], and is a good example of a synthetic theory not based on set theory, as none of the objects involved will be sets (Continuity?)

For 3-dimensional Euclidian geometry, Hilbert has the three primitive objects of the points, the line and the plane. We'll denote points by letters $p, q, r, \dots$, lines by $L, M, N, \dots$, and planes by $S, T, U, \dots$. We can write that an object is a point, line, plane by the predicates $\text{Point}(p)$, $\text{Line}(L)$ and $\text{Plane}(S)$. To cut down on notation, the variables used for

the various types of objects are assumed to implicitly declare their types : If a proposition includes the variable p , this implicitly assumes that the proposition is in conjunction with $\text{Point}(p)$.

Furthermore, there are three types of relations. First are relations of *incidence* (whether one object belongs to another). If a point p lies on a line L , we will write it $I(p, L)$, and similarly if a line lies in a plane, $I(L, S)$. Second is a relation of *betweenness*, a ternary relation describing if a point lies between two other points (the line that those two points define, as we will see). if q lies between the points p and r , we will denote it by $\text{Btw}(p, q, r)$. Lastly is a relation of *congruence*, where two line segments are congruent if they are of the same size (if they can be moved rigidly to coincide).

The axioms can then be split into 5 categories : incidence, order, congruence, parallels and continuity.

The incidence axioms are :

1. For any two points p, q , $p \neq q$, there exists a line L which contains them both : $I(p, L)$ and $I(q, L)$.
2. There exists exactly one such line between two points : if $I(p, L)$, $I(q, L)$ and $I(p, L')$ and $I(q, L')$, then $L = L'$.

Order axioms

1. If a point q lies between p and r , it also lies between r and p : $\text{Btw}(p, q, r) \leftrightarrow \text{Btw}(r, q, p)$
2. If a point q lies between p and r , there is a line containing all three points : $\text{Btw}(p, q, r) \leftrightarrow \exists L. \text{Line}(L) \wedge I(p, L) \wedge I(q, L) \wedge I(r, L)$
- 3.

If A and C are two points, then there exists at least one point B on the line AC such that C lies between A and B.[7]

Of any three points situated on a line, there is no more than one which lies between the other two.[8]

Pasch's Axiom: Let A, B, C be three points not lying in the same line and let a be a line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

Congruence axioms

1. For p, q two points on a line L , and a third point p' on a line L' (possibly equal to L), then there exists a point q' on L' such that $\overline{pq}$ and $\overline{p'q'}$ are congruent.
2. A segment is congruent to itself.
3. If $\overline{pq}$ and $\overline{p'q'}$ are congruent, and $\overline{p'q'}$ is congruent with $\overline{p''q''}$, then $\overline{pq}$ is congruent with $\overline{p''q''}$.
4. For three points p, q, r on a line L , such that the two segments $\overline{pq}$ and $\overline{pr}$ only share the single point p ,

The first axiom implies the Euclidian axiom of the existence of circles,

Parallel axioms

1. For any line L and a point p not incident to L , there exists exactly one line L' contained in the plane defined by L and p such that p lies on L' and L and L' do not intersect :

$$pL \rightarrow \quad (9.1)$$

Continuity axioms

- 1.

axiom of two circles

Those axioms generalize to any number of dimensions, simply by adding more types of objects, more incidence relations, and changing some existence axioms. In particular, the fundamental notion of dimensions in a synthetic setting is that in dimension n , there exists $n + 1$ points such that no three of them are colinear.

One very important notion in Euclidian geometry that we will use and change over the course of this book is that of *parallel transport*, so that we will define it here for the first time in the context of synthetic geometry :

Definition 9.1. If we have a line L and another line L_1 secant to that one, crossing at a point $p = L \cap L_1$, the parallel transport of L_1 along L at another point q on L is another line L_2 crossing L at q such that $L_1 \parallel L_2$.

By the parallel postulate, such a line always exists and is unique, so that this parallel transport is simply a function of the two lines L, L_1 and the point $q \in L$. To harmonize with later notation, let's denote it by the transport operator,

$$\text{tra}_{p,q}(L)L_1 = L_2 \quad (9.2)$$

Euclidian geometry gives us the following theorem :

Theorem 9.2. If we start with two lines L_1, L_2 secant at p , and parallel transport L_2 along L_1 to $q \in L_1$, giving us a line L'_2 , and then parallel transport L_1 along L'_2 to a point p' , giving us a line L'_1 , the line L'_1 is parallel to the line we would get had we parallel transported the line L_1 along L_2 .

Proof. □

If we arrange our construction so that the two lines coincide, this forms a parallelogram

Construction with circles : Make two secant lines, L_1, L_2

circles

This list of axioms gives a good sense of what it means for a space to be Euclidian, and it will also in many cases give us some notion of being locally Euclidian. Spacetimes will in general maintain a lot of these properties either fully, fully within some region or approximately in some region (except for the parts concerning the extra rules relating to how time will be treated, which we will look into synthetically at 14.8).

9.2 Analytic geometry

The more common way of describing Euclidian space in modern times is analytic geometry, where we employ some correspondence between Euclidian space and the real numbers. The Euclidian space $\mathbb{R}^n$ is defined as the n -fold Cartesian product of the real numbers equipped with some additional structure. First it is equipped with the *Euclidian metric* $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by

$$d(x, y) = \sum_{i=1}^n (\text{pr}_i(x) - \text{pr}_i(y))^2 \quad (9.3)$$

Theorem 9.3. The Euclidian metric is a metric

Proof. Trivially we have that $d(x, x) = 0$ and $d(x, y) = d(y, x)$ by the parity of the square,

so that only the triangle inequality has to be proven. If we have three points x, y, z ,

$$d(x, y) = \sum_{i=1}^n (x_i - y_i)^2 \quad (9.4)$$

$$= \sum_{i=1}^n ((x_i - z_i) - (y_i - z_i))^2 \quad (9.5)$$

$$= \sum_{i=1}^n (x_i - z_i)^2 + \sum_{i=1}^n (y_i - z_i)^2 - 2 \sum_{i=1}^n (x_i - z_i)(y_i - z_i) \quad (9.6)$$

$$(9.7)$$

□

proof that it is a metric

It is also a topological space equipped with the metric topology of the Euclidian metric, called the Euclidian topology.

Theorem 9.4. The Euclidian topology is equivalent to the product topology of $\mathbb{R}$.

Theorem 9.5. There are no homeomorphisms between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$

Theorem 9.6. There is no continuous injection $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for $n > m$.

9.3 Euclidian space in general

Now that we have seen two different ways to define Euclidian space, we need to show that these two concepts are equivalent.

Proof that the analytic and synthetic axioms are identical [9]

The points of synthetic geometry are the objects p such that $\text{Point}(p)$, those of analytic geometry are objects x such that $x \in \mathbb{R}^n$. Synthetic lines are similarly L such that $\text{Line}(L)$, while analytically they are the set of points given by

$$L = \{x \in \mathbb{R}^n \mid\} \quad (9.8)$$

Incidence is defined analytically simply by inclusion : $I(p, L) \leftrightarrow p \in L$.

The betweenness $\text{Btw}(p, q, r)$ is defined analytically by

9.4 Alternatives

Defining spacetimes as locally Euclidian spaces is not entirely uncontroversial, as we can question if even locally,

Examples : Kronecker, ancient debates in incommensurability [Sextus Empiricus, Proclus]

One of the most common alternative is to give spacetimes a discrete structure instead of continuous. An example of this is simplicial manifolds, which are constructed from simplices rather than patches. We will see this in more details on the section on Regge calculus, which occurs on such spacetimes.

9.4.1 Rational spacetime

It is possible to consider other fields than $\mathbb{R}$ as the basis for spacetime. One obvious candidate being $\mathbb{Q}$.

Wildberger's rational geometry?

This is not a particularly popular alternative to locally Euclidian spaces in physics, but there have been a few axiomatizations of special relativity in first order, giving it the structure of a rational field.

9.4.2 Finite fields

Spacetimes can also be defined using *finite fields* (or *Galois fields*) $\mathbb{F}_q$, which are finite sets obeying the field properties.

Example of a Galois field

These have the additional benefit of coinciding with the incidence geometry we previously saw, as any incidence geometry of a finite number of points will coincide with a Galois field.

Rather than getting a smaller field as a basis for spacetime, it is also possible to try to go for larger ones, such as non-Archimedean fields, such as the hyperreal numbers or asymptotic numbers

Synthetic geometry

p-adic spacetime : I. V. Volovich (1987). p-adic space-time and string theory. Theoretical and Mathematical Physics, 71(3), 574–576.

Chapter 10

Vector and affine spaces

A lot of physics rests on vector and affine spaces, as most spaces are modelled on them and physical quantities are typically assumed to be vectors, or at the very least are objects that can be represented by vectors. Vector spaces tend to be quite attractive objects for actual calculations in physics, as the operations defined on them are usually quite tractable, so that it is very common to define objects on vector spaces even if it is to map them to some other kind of object afterward. This notion should be kept in mind as it will show up quite often later, where we will define some types of objects in terms of vectors even if vector spaces are much too large and redundant but easier to handle, as this will happen for gauge fields or projective objects such as quantum states.

For our definitions, let's be very formal and differentiate the objects of the field F and V , although this will typically be dropped :

Definition 10.1. A *vector space* over a field F is a set V equipped with two operation, $+_V$ and $\cdot_V$, as well as a preferred member of that set $0_V \in V$, obeying the following properties :

1. For any two members $v, w \in V$, their sum $v +_V w$ also belongs to V .
2. For $v \in V$ and some value $k \in F$, the scalar product $k \cdot_V v$ belongs to V .
3. 0_V is an identity element for $+_V$: for any $v \in V$, $v +_V 0_V = v$
4. $+_V$ is commutative : $v +_V w = w +_V v$
5. $+_V$ is associative : $(u + v) + w = u + (v + w)$
6. $\cdot_V$ is compatible with $\cdot_F$: $(a \cdot_F b)v = a \cdot_V (b \cdot_V v)$
7. $1 \in F$ is an identity element of $\cdot_V$: $1 \cdot_V v = v$
- 8.

Distributivity of scalar multiplication with respect to vector addition
 Distributivity of scalar multiplication with respect to field addition

There are generally no risks of confusion between the sum and product for the field and the vector space (especially if different variable symbols are typically used), so that they are all simply written using the same symbols : $+$ and $\cdot$. We will also not generally distinguish the zero of the field F and of the vector space V , simply denoting it 0 . Furthermore, it is common to drop the operator symbol for the scalar product $\cdot$ if there is no confusion possible and to simply denote it by concatenation : $k \cdot v = kv$.

We will overall keep the field of our vector spaces somewhat free, but overall we will mostly consider the cases of $\mathbb{R}$ and $\mathbb{C}$. In particular, for the rest of this book, the field F will be assumed to have characteristic 0, ie there is no sum of the ring's multiplicative identity (1) which will sum to zero, as this is one of the main issue in a lot of theorems if we keep our fields too general.

The prototypical example of a vector space is our Euclidian space E .

Theorem 10.2. The synthetic Euclidian space can be given a vector space structure.

Proof. Consider the synthetic Euclidian space with a set of points E . If we consider a point $o \in E$, called the origin, we can consider the line between any point $p \in E$ and the origin, $\overline{op}$. If we consider every segment $\overline{op}$ as a vector v and the trivial segment $\overline{oo}$ as the zero vector, we can define the sum of two vectors $v_1 = \overline{op_1}$ and $v_2 = \overline{op_2}$ by the respective parallel transport of v_1 by v_2 and v_2 by v_1 , those two lines intersecting in a third point p_3 such that $v_3 = \overline{op_3}$ is the sum $v_1 + v_2 = v_3$ (this is the *parallelogram law of vector addition*), and the product of a vector by a scalar is taken by considering a line a such that a and $\overline{op}$ are of ratio $k \in \mathbb{R}_{\geq 0}$, and then constructing a circle C_k of center o . The intersections of the circle with the extension of $\overline{op}$ to a line is the scalar product by k if p is in between o and p' , and $-k$ if o is in between p and p' .

proof that the vector $0 = \overline{op}$ is the identity of the sum

Proof that the parallelogram law is commutative is commutative

associative

If we prolong our segment $\overline{op}$ to a line, and construct a circle of center o and radius $\overline{op}$, the intersection between that circle and the line that is not p will be the antipodal point p' . As $\overline{op}$ and $\overline{op'}$ are colinear and of same length,

Proof that there is

Proof that the rescaling is distributive wrt both additions

proof that rescaling is associative

Proof that rescaling by a 0 length vector is the zero vector

□

This construction is slightly incomplete since we cannot actually construct segments of arbitrary length ratio with respect to $\overline{op}$, but only proportional to constructible numbers, which is a subset of the real numbers, although we can approach them arbitrarily with an unbounded number of steps.

vector will correspond to a point of that space $p \in E$, the preferred point of our space corresponds to a choice of a point $o \in E$, the sum of two vectors v, w will correspond to the parallel transport of the line $\overline{ov}$ along the line $\overline{ow}$, while the scalar product $x \cdot v$ will correspond to the intersection of the line $\overline{ov}$ with a circle of center o and radius x (in either direction depending on the sign).

Show that they obey vector properties

In analytic geometry, the operations are much simpler and correspond to the sum and scalar product as applied to the tuples of $\mathbb{R}^n$. The vector space structure is simply built by considering the sum $+: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and scalar product $\cdot: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$v + w = (v_1 + w_1, v_2 + w_2, \dots) \quad (10.1)$$

$$k \cdot v = (k \cdot v_1, k \cdot v_2, \dots) \quad (10.2)$$

proof that they form a vector space

Definition 10.3. For a vector space V ,

10.1 Basis

The basis of a vector space is the equivalent of a minimal generating set for it. In the case of a vector space, the generation is done by the concept of a span :

Definition 10.4. The *span* $\text{Span}(v_1, \dots, v_k)$ of a collection of vectors $(v_i)_{1 \leq i \leq k}$ is the set of all vectors that can be spanned using those vectors and the vector space rules, in other words they are all the vectors of the form

$$v = \sum_{i=1}^k a_i v_i \quad (10.3)$$

for $a_i \in F$

The span of some set of vectors is likely to be too large in general, as the basis is supposed to be a minimal generating set. To get the appropriate amount of generators, we need some notion to show that the vectors in the basis are the strict minimum, in the sense that some number of vectors in the basis could not generate another one.

Definition 10.5. A set of k vectors $(v_i)_{1 \leq i \leq k}$ is *linearly independent* if no vector v_i can be expressed as a linear combination of any other of the vectors. In other words, there does not exist any set of values of our field $(a_i)_{1 \leq i \leq k-1}$ such that

$$v_i = a_1 v_1 + a_2 v_2 + \dots + a_{i-1} v_{i-1} + a_{i+1} v_{i+1} + \dots + a_k v_k \quad (10.4)$$

Theorem 10.6. If a set of linearly independent vectors spans the same vector space as some set of vectors, $\text{Card}() \leq \text{Card}()$

Proof. □

Showing that they are indeed minimal. With this in mind, we can define our basis as the minimal set of generators of a vector space as follow :

Definition 10.7. A *basis* is a set of linearly independent vectors that spans the entire vector space,

$$\text{Span}((v_i)) = V \quad (10.5)$$

If the basis of a vector space V has cardinality n (finite or not), we say that V has *dimension* n , denoted by $\text{Dim}(V) = n$.

Steinitz exchange lemma

Dimension theorem for vector spaces

Theorem 10.8. For any two basis b_1, b_2 of a vector space V , they have the same dimension : $\dim(b_1) = \dim(b_2)$.

Proof. □

In other words, a vector space can only have one dimension.

Unique decomposition of vectors in a basis

A basis can also be seen as a linear map from the vector space V to $\mathbb{R}^n$:

$$\theta : V \rightarrow \mathbb{R}^n \quad (10.6)$$

$$v \mapsto \theta(v) = (v^1, v^2, \dots, v^n) \quad (10.7)$$

such that each element is a projection on the i -th basis vector

Theorem 10.9. For a vector space V , the set of all linear maps from V to F is itself a vector space.

Dual vector space

[10–14]

change of basis forms a group $GL(n) = \text{inv}L(V, V)$

10.2 Subspaces

Subspaces for vector spaces are the subobjects, being vector spaces included in some sense in other vector spaces.

Definition 10.10. For a vector space V , a linear subspace W is a subset of V under which the vector space operations are closed : for any $w_1, w_2 \in W$, $w_1 + w_2 \in W$ and $k \cdot w_1 \in W$

Every vector space contains at least two subspaces, which are the zero subspace (the set $\{0\}$), which is indeed closed since $0 + 0 = 0$ and $k \cdot 0 = 0$, and the entire space itself, which is trivially its own vector space.

If the vector space is finite dimensional of dimension n , then its subspaces will themselves at most of dimension n

Definition 10.11. The set of all subspaces of a given dimension k for a vector space V is the *Grassmannian* $\text{Gr}(k, V)$.

Example : set of all lines going through the origin on the plane.

10.3 Linear maps

The isomorphism associated with vector spaces is the linear map :

Definition 10.12. A *linear map* is an isomorphism between two vector spaces $f : V \rightarrow W$ preserving its linear structure, so that

$$f(v + w) = f(v) + f(w) \quad (10.8)$$

$$f(xv) = x \cdot f(v) \quad (10.9)$$

The set of all such linear maps is denoted by $L(V, W)$. Vector spaces and linear maps define the *category of vector spaces*, **Vec**. We can check that the identity map Id_V is indeed a linear map

$$\text{Id}_V(v + w) = v + w = \text{Id}_V(v) + \text{Id}_V(w) \quad (10.10)$$

$$\text{Id}_V(xv) = xv = x\text{Id}_V(v) \quad (10.11)$$

and that the composition of two linear maps is still a linear map

$$g(f(v + w)) = g(f(v) + f(w)) = g(f(v)) + g(f(w)) \quad (10.12)$$

$$g(f(xv)) = g(x \cdot f(v)) = x(g(f(v))) \quad (10.13)$$

The set of all linear maps between two vector spaces V, W is therefore the hom-set of those two spaces $L(V, W) = \text{Hom}(V, W)$.

Internal hom?

If we equip our vector space with a basis (e_i) , by linearity, the linear map will act on a vector $V = V^i e_i$ as

$$f(V) = f(V^i e_i) = V^i f(e_i) \quad (10.14)$$

As $f(e_i)$ is itself a vector, we can decompose it as $f^j(e_i) e_j$

For a given i, j , $f^j(e_i)$ is a scalar quantity in F , so that in a given basis, a linear map f can be represented by a collection of $\text{Dim}(V) \times \text{Dim}(W)$ elements, the matrix representation of the linear operator.

Definition 10.13. For a linear map $f : V \rightarrow W$, its *kernel* is the set of all vectors in V mapping to the zero vector in W :

$$\text{Ker}(f) = \{v \in V | f(v) = 0\} \quad (10.15)$$

Theorem 10.14. The kernel of a linear map is a subspace.

Proof. If v_1 and v_2 both belong to the kernel of f , then by linearity,

$$f(av_1 + bv_2) = af(v_1) + bf(v_2) = 0 \quad (10.16)$$

□

Example : Kernel of the identity map

Kernel of the projection on a plane or line or point

Image is a subset

$$\text{Im}(f) = V/\text{Ker}(f) \quad (10.17)$$

$$\text{rank}(f) = \text{Dim}(\text{Im}(f)) \quad (10.18)$$

Dim of kernel and image

rank and nullity of the map

Theorem 10.15.

$$\text{rank}(f) + \text{nullity}(f) = \dim(V) \quad (10.19)$$

Determinant

Coordinate independent definition :

Definition 10.16. The determinant is the unique multiplicative map $\det : L(V) \rightarrow F$ such that $\det(g) = \det(g_1, \dots, g_n)$

$$\text{GL}(n) \rightarrow \mathbb{R}^\times \quad (10.20)$$

Groups from the linear maps

Example : orthogonal group, special linear group

Stabilizer subgroup for vectors and subspaces

For vectors (in a specific basis) :

$$\text{Stab}_{\text{GL}(n),v} = \{g \in \text{GL}(n) | g(v) = v\} \quad (10.21)$$

Complementary subspaces

Traces, determinant

Trace only works in finite-dimensional vector spaces

10.3.1 Eigenvectors

Eigenvalues, eigenvectors, signature

Definition 10.17. We say that a vector v is an *eigenvector* of a linear map f if there exists some $x \in k$ such that

$$f(v) = x \cdot v \quad (10.22)$$

The value $x \in k$ is called the *eigenvalue* of this eigenvector.

Since we can always construct a basis in which any given vector is a basis element (by considering for instance some basis transformation $g \in \text{GL}(V)$ such that for any vector of the original basis, $ge_1 = v$), we can look at the form of our linear operator in such a basis. If our vector in this basis is of the form

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad (10.23)$$

Then in terms of components, the matrix operation corresponds to

$$A_j^i v^j = A_1^i = \lambda v^i = \lambda \delta_0^i \quad (10.24)$$

The column and row for the entry of that basis element are such that

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A^{n-1} & \\ 0 & & & \end{pmatrix} \quad (10.25)$$

This works for any basis vector we may choose so that this form keeps for any entry A_{ii}

Diagonalized matrix

Gauss pivot

Determinant

Definition 10.18. The *signature* (p, q, r) of a linear operator is the numbers of positive, negative, or zero eigenvalues, such that $p + q + r = n$.

Sylvester's theorem of inertia

Theorem 10.19. For any real symmetric linear operator,

10.3.2 Dual vector spaces

Definition 10.20. For an F -vector space V , its dual vector space V^* is the space of all linear maps

$$V^* = \text{Hom}(V, F) \quad (10.26)$$

where F is the field given its natural vector space structure.

Definition 10.21. The *transpose* f^T of a linear map $f : V \rightarrow W$ is a map between the duals of those spaces $f^T : W^* \rightarrow V^*$ obeying the property that for any map $g : V \rightarrow W$

10.4 Endomorphisms and square matrices

Another special kind of linear map on a vector space is the endomorphism :

Definition 10.22. A linear map from a vector space V to itself $f \in L(V, V)$ is an *endomorphism*.

Theorem 10.23. In a given basis, an endomorphism f forms a square matrix

$$f(e_i, e_j) = M_{ij} \in M(n, n) \quad (10.27)$$

Having the same source and target, endomorphisms can be chained together.

10.4.1 Kernels and images

Generate subspaces from linear operators

rank and dimension

10.5 General linear group

A special class of linear operators on a vector space are the invertible ones, the isomorphisms.

Definition 10.24. A linear map f is said to be invertible if there exists another linear map f^{-1} such that

$$f \circ f^{-1} = f^{-1} \circ f = \text{Id}_V \quad (10.28)$$

Theorem 10.25. A linear map is invertible if and only if its determinant is non-zero.

Proof. As the determinant of a composition of linear maps is the product of their individual determinants, we have

$$\det(f \circ f^{-1}) = \det(f)\det(f^{-1}) = \det(\text{Id}_V) = 1 \quad (10.29)$$

Therefore, we have that $\det(f) = \det^{-1}(f^{-1})$. This relation can only hold if $\det(f) \neq 0$. Conversely, if the determinant of a linear map is not zero, $\square$

The space of all invertible linear operators is the set of isomorphisms on a vector space, and therefore forms a group by composition, with the inverse as a group inverse, and the identity map Id_V as a group identity. This group is called the *general linear group* of the vector space V , noted by $\text{GL}(V)$.

10.5.1 Stabilizers of the general linear group

An important class of subgroups of the general linear group for a vector space V are the stabilizers of vectors and vector subspaces.

Definition 10.26. The *stabilizer subgroup* of a subspace $W \subseteq V$ of a vector space

Any two subspaces of the same dimension map to each other

10.6 Projection operators

Definition 10.27. A *projection operator* is a linear operator $P : V \rightarrow V$ that is idempotent : $P^2 = P$.

Definition 10.28. The *complementary projection* to a projection P is the operator $\text{Id}_V - P$

Complementary projection is also a projection : $(\text{Id}_V - P)(\text{Id}_V(v) - P(v)) = \text{Id}_V(v - P(v)) - P(v - P(v)) = v - P(v) - P(v) + P^2(v) = 2(\text{Id}_V(v) - P(v))$

Properties of kernel and image of idempotent maps

Decomposition of the vector space

Proposition 10.29. P is an identity on $\text{Im}(P)$: $\forall v \in \text{Im}(P), Pv = v$

Theorem 10.30. Any projection operator splits the vector space as

$$V = \text{Ker}(P) \oplus \text{Im}(P) \quad (10.30)$$

Proof. $\square$

Decomposition of vectors $v = Pv + (I - P)v$

Inversely, we have that for any splitting of the vector space into two complementary subspaces $V = W \oplus U$, there

10.7 Additive groups

Vector spaces also have the property of containing the *additive group*

10.8 Affine spaces

Vector spaces are not a good model for space, even flat space, since they have a preferred point in them in the form of the zero vector. To have a type of space without that extra assumption, we define the notion of affine space where we remove that notion :

Definition 10.31. An *affine space* A is a set associated with a vector space V such that the translation group of V has a transitive free action on A :

$$+ : V \times A \rightarrow A \quad (10.31)$$

$$(v, a) \mapsto v + a \quad (10.32)$$

The group action implies that

$$0 + a = a \quad (10.33)$$

$$v_2 + (v_1 + a) = (v_2 + v_1) + a \quad (10.34)$$

And the free transitive group action implies that for any a , the map $v \rightarrow v + a$ is bijective.

For any v , there is a bijective map $a \rightarrow v + a$ (prove)

From this we can define an inverse operation $-$ such that $a - b = v$ implies $a = v + b$

Also denoted by $b - a = \vec{ab}$, the vector connecting a to b

$$a + (b - a) = b \quad (10.35)$$

Weyl's axioms :

For any $a \in A$, $v \in V$, there exists a unique $b \in A$ such that $b - a = v$

$(c - b) + (b - a) = c - a$, this is also the parallelogram rule? AKA Chasles' formula

Commutativity of the additive group implies $b - a = b' - a'$ iff $a - a' = b - b'$

Affine space is a V -torsor

Vectorialization : choice of a point $p \in A$ gives it a vector space structure, (A, p) such that every vector is just $p - a$.

10.8.1 Affine subspaces

As with vector spaces,

Affine subspaces

10.8.2 Affine maps

Definition 10.32. An *affine map* is a map between two affine spaces $f : (A, V) \rightarrow (A', V')$ preserving the affine character, such that in particular, it induces the existence of a linear map $f : V \rightarrow V'$ (denoted here by the same symbol) such that

$$f(b - a) = f(b) - f(a) \quad (10.36)$$

$$f(a + v) = f(a) + f(v) \quad (10.37)$$

Endomorphisms : The affine space $A(A)$

Affine frame is made up of a point a_0 and a basis (e_i) of the vector space of A . The coordinates of a point $p \in A$ are the vector coordinates of $p - a_0$ in that basis.

Equivalent to $n + 1$ points : $a_0, a_1 = e_1 + a_0, \dots, a_n = e_n + a_0$

General affine group :

$$1 \rightarrow V \rightarrow \mathrm{GL}(V) \rightarrow \mathrm{GA}(V) \rightarrow 1 \quad (10.38)$$

[is it actually (from Kobayashi)]

$$1 \rightarrow V \xrightarrow{\alpha} \mathrm{GA}(V) \xrightarrow{\beta} \mathrm{GL}(V) \rightarrow 1 \quad (10.39)$$

α is a group isomorphism

$$\alpha : V \rightarrow \text{GA}(V) \quad (10.40)$$

$$v \mapsto \begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix} \quad (10.41)$$

β is the group isomorphism

$$\beta : \text{GA}(V) \rightarrow \text{GL}(V) \quad (10.42)$$

$$a = \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} \mapsto a \quad (10.43)$$

Just like for the case of a vector space, the analytic Euclidian space $\mathbb{R}^n$ is fairly easy to be shown to be an affine space.

Theorem 10.33. The synthetic Euclidian space is an affine space.

Proof. As we have already shown in 10.2 that the synthetic Euclidian space can be given the structure of a vector space, all we need to do is give it the remaining structure to make it an affine space.

Prove the translation of everything □

10.8.3 Affine combination

Definition 10.34. An *affine combination* of k vectors $(v_1, \dots, v_k)$ in an F -affine space A is a combination of those vectors up to a scalar quantity $(a_1, \dots, a_k)$:

$$\sum_{i=1}^k a_i v^i = a_1 v^1 + \dots + a_k v^k \quad (10.44)$$

such that

$$\sum_{i=1}^k a_i = 1 \quad (10.45)$$

Convex combination

Chapter 11

Constructions from vector and affine spaces

11.1 Direct sums

A basic construction on vector spaces is that of the direct sum of two vector spaces, which correspond to the intuitive notion of simply considering a pair of vector spaces.

Definition 11.1. Given two vector spaces V_1, V_2 , their direct product $V_1 \oplus V_2$ is given by the set of their Cartesian product $V_1 \times V_2$ with the vector operations

$$+ : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \rightarrow V_1 \oplus V_2 \quad (11.1)$$

$$((v_1, v_2), (w_1, w_2)) \mapsto (v_1 + w_1, v_2 + w_2) \quad (11.2)$$

As a Cartesian product, we have two projection functions

$$\text{pr}_i : V_1 \times V_2 \rightarrow V_i \quad (11.3)$$

$$(v_1, v_2) \mapsto (v_i) \quad (11.4)$$

This notion easily extends to affine spaces as well

11.2 Tensors

While a direct sum of vector spaces corresponds to a pair of those vector spaces, a tensor product of two vector space will

Tensor product properties

Given two vector spaces V, W , their *tensor product* $V \otimes W$ is another vector space

$$V \otimes W \quad (11.5)$$

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \quad (11.6)$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \quad (11.7)$$

$$r(v \otimes w) = (rv) \otimes w = v \otimes (rw) \quad (11.8)$$

Tensors are multilinear maps

Decomposable tensor

Definition 11.2. A tensor T is *decomposable* if it can be written as a tensor product of vectors and dual vectors. For a tensor $T \in T_s^r$,

$$T = \left(\bigotimes_{i=1}^r v_i \right) \otimes \left(\bigotimes_{j=1}^s \omega_j \right) \quad (11.9)$$

$$T : V \times \dots \times V \times V^* \times \dots \times V^* \rightarrow \mathbb{R} \quad (11.10)$$

While a tensor is usually defined as a multilinear map to $\mathbb{R}$, there are in fact multiple ways to consider tensors, via the *tensor-hom adjunction*. If we consider for instance a rank $(0, 2)$ tensor, $V \times V \rightarrow \mathbb{R}$,

Basis

In a given basis, tensors of rank (r, s) can be expressed like linear maps by $s \times \text{Dim}()$

Symmetric and antisymmetric tensors, symmetrization and antisymmetrization

$$T_{(\mu_1 \mu_2 \dots)} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(\mu_1, \mu_2, \dots)} \quad (11.11)$$

$$T_{[\mu_1 \mu_2 \dots]} = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} T_{\sigma(\mu_1, \mu_2, \dots)} \quad (11.12)$$

The factor is so that if T is already symmetric, its symmetrization will leave it invariant for any two indices,

$$T_{\mu\nu\sigma\dots} = T_{(\mu\nu)\sigma\dots} + T_{[\mu\nu]\sigma\dots} \quad (11.13)$$

Does not work for more than 2

Levi-Civita tensor

$$\epsilon_{\mu\nu\dots} = \begin{cases} +1 & \text{for an even permutation of } 0, 1, 2, \dots \\ -1 & \text{for an odd permutation of } 0, 1, 2, \dots \\ 0 & \text{otheriw} \end{cases} \quad (11.14)$$

11.2.1 The tensor-hom adjunction

A given tensor can be considered several different ways. For instance, given a rank $(0, 2)$ tensor, its basic interpretation is as a function taking two vectors and mapping them to the field F :

$$T : V \times V \rightarrow F \quad (11.15)$$

$$(v, w) \mapsto T(v, w) \quad (11.16)$$

However, by partial application, we can also interpret it as a function mapping a single vector to its dual space

$$T : V \rightarrow V^* \quad (11.17)$$

$$v \mapsto T(v, -) \quad (11.18)$$

We can show that this is indeed a dual vector as it is a linear map from V to F :

$$T(v, -) : V \rightarrow F \quad (11.19)$$

$$w \mapsto T(v, w) \quad (11.20)$$

To complete this, we can also trivially consider it as a map of no arguments to itself, which simply maps the empty product of V (a single element) uniquely to the original tensor itself.

$$T(-, -) : \{\bullet\} \rightarrow V^* \times V^* \quad (11.21)$$

$$\bullet \mapsto T \quad (11.22)$$

Definition 11.3. The *tensor-hom adjunction* is

11.2.2 Polynomial maps

Tensors allow us to define polynomial maps on vector spaces. Consider some tensor of rank $(0, 2)$ tensor T

$$T(X, Y) \quad (11.23)$$

If acting on the same vector, we find that

$$\begin{aligned} T((aX + bY), (aX + bY)) &= T(aX, aX) + T(aX, bY) + T(bY, aX) + T(bY, bY) \\ &= a^2T(X, X) + ab(T(X, Y) + T(Y, X)) + b^2T(Y, Y) \end{aligned} \quad (11.24)$$

This is the form of a quadratic function (if non-commutative). If the tensor is furthermore symmetric, we have

$$T((aX + bY), (aX + bY)) = a^2T(X, X) + 2abT(X, Y) + b^2T(Y, Y) \quad (11.25)$$

Any tensor of rank $(0, s)$ defines thus a monomial term of power s , however if the tensor is not symmetric, it will be equivalent to its symmetrization :

$$T(X, X) = T_{\text{sym}}(X, X) + 0 \quad (11.26)$$

We can define from this the set of monomial maps of order k between two vector spaces as

$$\text{Poly}^k = L_{\text{sym}}(V, W) \quad (11.27)$$

The set of all polynomials between two vector spaces will then be the

$$\text{Poly}(V, W) = \bigoplus_{k=0}^{\infty} \text{Poly}^k \quad (11.28)$$

with its evaluation map

$$\text{ev} : \text{Poly}(V, W) \times V \rightarrow W \quad (11.30)$$

$$(p, v) \mapsto p(v) = a_0 + a_1(v) + a_2(v, v) + \dots \quad (11.31)$$

for $a_i \in L_{\text{sym}}(V, W)$ the monomial terms making up p .

If the target vector space is a field F in particular, this is isomorphic to the polynomial ring $F[x_1, \dots, x_n]$ for n the dimension of V . This can be shown by considering a basis of V ...

11.2.3 Contraction

As we can apply vectors to dual vectors to obtain some scalar quantity, this allow us to define some natural map between the vector space $V \times V^*$ and the field it is defined on, called the *contraction* map :

$$C : V \times V^* \rightarrow F \quad (11.32)$$

$$(X, \omega) \mapsto C(X \otimes \omega) = \theta(X) \quad (11.33)$$

By linearity, this map can be extended to any element of the space of rank $(1, 1)$ tensors

We can consider the contraction map to be some $(1, 1)$ tensor. In terms of coordinates, this is given by

$$X^\mu \omega_\nu C(e_\mu \otimes \theta^\nu) = X^\mu \omega_\mu \quad (11.34)$$

This corresponds to contraction being the identity $(1, 1)$ tensor δ_μ^ν , given by

For any given vector of rank (r, s) , $r, s > 0$,

$$C_\beta^\alpha(T)(\theta^1, \dots, \theta^r, X^1, \dots, X^s) = \quad (11.35)$$

Transvection [15]

11.2.4 Abstract index notation

It is extremely common in physics to work with tensor components rather than tensors directly, as this is a very powerful notation for all kinds of tensor operations. By contrast, the usual index-free notation can be quite cumbersome and hard to parse.

To find some middle ground between the two, there is the *abstract index notation*, which is essentially just the mapping of the index-free notation directly to one resembling the component notation. So vectors $v \in V$ are written as v^i , dual vectors $\alpha \in V^*$ as α_i , and tensors can be operated on in notation directly by what would affect their indices in terms of components. For instance, the duality between vectors and dual vectors is simply expressed by raising and lowering the index (assuming a canonical bilinear form for that duality)

$$v \mapsto B(v, -) \leftrightarrow v^i \mapsto v_i \quad (11.36)$$

contractions

11.2.5 Birdtrack notation

There exists a less commonly used graphical notation tensor algebras which can sometimes be found for use in general relativity called *birdtrack notation* or *Penrose notation*. Each tensor is represented by a shape with one line coming out of it for each argument, oriented depending on the type of argument (vector or dual vector), usually up and down for vectors and dual vectors (mirroring the upper and lower indices in other notations).

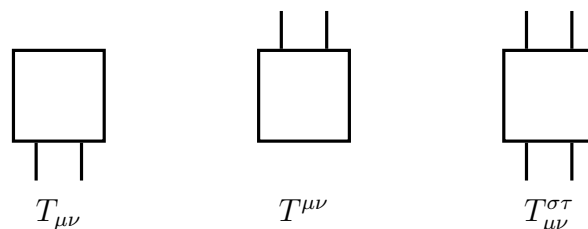


Figure 11.1

The contraction of two tensor quantities is done by connecting an upper and lower lines together

[16]

Vectors and dual vectors as directed curves

Tensors as boxes

symmetrization and antisymmetrization as bars

11.3 Complex vector spaces

As we saw in the definition, a vector space can be defined over any field F , and in particular we will often look at $\mathbb{C}$.

As vector spaces, there are no major differences between real and complex vector spaces, but it is possible to add additional structure on a complex vector space in the interpretation of $\mathbb{C}$ as a product of the real line.

First is the real structure, which shows the correspondence between complex vector spaces of dimension n with real vector spaces of dimension $2n$:

Definition 11.4. A *real structure* on a complex vector space V is an antilinear involution σ

$$\sigma(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1^* z_1 + \lambda_2^* z_2 \quad (11.37)$$

The real structure is the equivalent of the complex conjugate for the vector space, and this map allows us to select a "real" part of the vector space simply by picking the set of vectors unaffected by it, ie the set

$$V_{\mathbb{R}} = \{v \in V | \sigma(v) = v\} \quad (11.38)$$

This is in fact an eigenspace of σ as an antilinear operator

, making $V_{\mathbb{R}}$ a vector subspace of V

Similarly, the imaginary subspace

Imaginary subspace

Reality structure

Its converse is the complex structure, which can be used to map real vector spaces of dimension $2n$ to complex ones of dimension n .

Definition 11.5. A *linear complex structure* J on a vector space V is a linear map such that $J^2 = -\text{Id}_V$.

Conjugate

Complexification of a vector space

$$V_{\mathbb{C}} = V \times \mathbb{C} \quad (11.39)$$

11.4 Bilinear forms

A bilinear form is a bilinear map from an F -vector space to its field F , ie :

Definition 11.6. A *bilinear form* is a rank $(0, 2)$ tensor

$$B : V \times V \rightarrow F \quad (11.40)$$

From the tensor-hom adjunction, we also have that they define maps between vectors and dual vectors,

$$B : V \rightarrow V^* \quad (11.41)$$

$$v \mapsto B(v, -) \quad (11.42)$$

We say that the bilinear form is *degenerate* if there exists any non-zero vector X for which there exists non-zero vectors Y such that $B(X, Y) = 0$. It is nondegenerate otherwise. Degeneracy is a condition on the kernel of the map, $\text{Ker}(B) \subset V \times V$ is not the zero vector space

A bilinear form allows us to define a notion of orthogonality on a vector space :

Definition 11.7. Two vectors $X, Y \in V$ are said to be *orthogonal* with respect to a bilinear form B if $B(X, Y) = 0$.

A given vector X is called *isotropic* with respect to B if $B(X, X) = 0$. The set of all such vectors is called the *isotropic cone*.

If B is symmetric, $B(X, Y) = B(Y, X)$ Unique if B is symmetric

Theorem 11.8. Every alternating bilinear form is skew-symmetric

Proof. For any two vectors $v, w \in V$, we can consider the product $B(v + w, v + w)$, which is zero by its alternating character. Therefore, by bilinearity,

$$B(v + w, v + w) = B(v, v) + B(v, w) + B(w, v) + B(w, w) \quad (11.43)$$

$$= 0 + B(v, w) + B(w, v) + 0 \quad (11.44)$$

$$= 0 \quad (11.45)$$

Therefore, $B(v, w) = -B(w, v)$. □

11.4.1 Subspaces

Definition 11.9. Given a subspace $W \subset V$, its *orthogonal complement* $W^\perp$ is the set of all vector for which their product by the bilinear form with any vector $w \in W$ is zero :

$$W^\perp = \{v \in V | \forall w \in W, B(v, w) = 0\} \quad (11.46)$$

11.4.2 Orthonormal basis

Definition 11.10. A basis $\{e_i\}$ is called *orthonormal* if for any two elements of the basis, they obey the relation

$$B(e_i, e_j) = \delta_{ij} \quad (11.47)$$

Orthogonal basis

11.5 Metric tensor

A non-degenerate symmetric bilinear form g of signature (p, q) defines a *metric tensor* on the vector space V . The vector space equipped with this tensor, (V, g) , is a

By the Gram-Schmidt process, we know that all such bilinear forms can be expressed in some basis $\{e_i\}$ by the canonical metric

$$g = \sum_{i=1}^p \theta_i \otimes \theta_i - \sum_{j=p+1}^q \theta_j \otimes \theta_j \quad (11.48)$$

The name of a metric tensor stems from the case where $(n, 0)$, in which case the metric tensor defines a norm, which is the usual Euclidian norm :

$$\|X\| = \sqrt{g(X, X)} \quad (11.49)$$

which, for the canonical form of the metric, reduces to

$$\|X\| = \sqrt{\sum_{i=1}^n (X^i)^2} \quad (11.50)$$

Theorem 11.11. $\|X\|$ defines a norm on V .

Proof. 1.

2. Triangle inequality : If we have $\|X + Y\|$,

$$\|X + Y\| - \|X\| - \|Y\| = \sqrt{g(X + Y, X + Y)} - \sqrt{g(X, X)} - \sqrt{g(Y, Y)} \quad (11.51)$$

□

Therefore, a metric tensor of signature $(n, 0)$ defines a metric space, hence the name.

Two vectors X, Y such that $B(X, Y) = 0$ are said to be *orthogonal*.

Theorem 11.12. Orthogonality is preserved by linearity.

Proof. If we have two vectors Y_1, Y_2 orthogonal to X , any linear combination of them is orthogonal :

$$B(X, \alpha Y_1 + \beta Y_2) = \alpha B(X, Y_1) + \beta B(X, Y_2) \quad (11.52)$$

$$= 0 \quad (11.53)$$

□

This means that orthogonal vectors form subspaces, called *orthogonal subspaces*.

Dimension of the orthogonal subspace in a given signature

The metric tensor g allows the definition of a *projection operator* on any non-isotropic vector

$$\text{proj} : \ker(\| - \|) \times V \rightarrow V \quad (11.54)$$

$$(u, v) \mapsto \text{proj}_u(v) = \frac{g(u, v)}{g(u, u)} u \quad (11.55)$$

A metric tensor can be used to define an *orthonormal basis*. A basis (e_i) is orthonormal if for any two vectors e_i, e_j in the basis, they obey the property

$$g(e_i, e_j) = \pm \delta_{ij} \quad (11.56)$$

The sign of the inner product $g(e_i, e_i)$ being such that they obey the metric's signature : there are p vectors of positive norm and q vectors of negative norms.

We can always construct such an orthonormal basis via the *Gram-Schmidt process* :

Theorem 11.13. Given n non-isotropic vectors (v_i) forming a basis on V , there exists an orthonormal basis $\{e_i\}$ constructed from those vectors.

Proof. The basis is constructed by induction by first taking the basis vector

$$e_1 = \frac{v_1}{\|v_1\|} \quad (11.57)$$

which is normalized so that $g(e_1, e_1) = 1$. now given a set of k basis vectors $(e_1, \dots, e_k)$, the $(k + 1)$ -th basis element is constructed by

$$e_{k+1} = \quad (11.58)$$

□

Cauchy-Schwarz inequality :

Lemma 11.1.

$$|\langle v, w \rangle|^2 \leq \langle v, v \rangle \langle w, w \rangle \quad (11.59)$$

Euclidian l_2 norm, canonical or induced norm

11.6 Decomposition

The decomposition (or factorization) of a matrix is the splitting of a given matrix into more matrices of a specific type, such that their product give us the original matrix. If we have a matrix A , its decomposition will therefore be something of the form

$$A = BC \dots \quad (11.60)$$

They tend to be useful tools for the study of the matrix groups for the transformations of vector spaces.

There are many such decompositions possible. One of the most famous one being the polar decomposition :

Theorem 11.14. Any real or complex matrix A can be decomposed into a unitary matrix U and a positive semi-definite Hermitian matrix P :

$$A = UP \quad (11.61)$$

Proof.

□

Theorem 11.15. If A is invertible, this decomposition is unique and P is positive definite.

Generalized polar decomposition

In the real case, we also have the *QR decomposition*:

Theorem 11.16. Any matrix A can be decomposed as a product $A = QR$, of Q an orthonormal matrix and R an upper triangular matrix

"If A is invertible, then the factorization is unique if we require the diagonal elements of R to be positive."

For any $g \in GL(n)$, there exists a unique upper triangular positive matrix R and orthogonal matrix Q

11.7 Graded vector spaces

A G -graded vector space is a vector space that can be decomposed into a direct sum of vector spaces according to some set G , ie it is of the form

$$V = \bigoplus_{i \in G} V_i \quad (11.62)$$

and we have a *degree map*, which associates vectors purely in a given subspace V_i to the element of G

$$\deg : V \rightarrow G \quad (11.63)$$

$$(0, 0, \dots, 0, v_g, 0, \dots, 0) \mapsto g \quad (11.64)$$

which associates

Grade projection operator :

$$\langle - \rangle_{(-)} : V \times G \rightarrow V \quad (11.65)$$

$$(v, g) \mapsto \langle v \rangle_g \quad (11.66)$$

Properties :

$$v = \sum_{g \in G} \langle v \rangle_g \quad (11.67)$$

Linear :

$$\langle av + bw \rangle_g = a \langle v \rangle_g + b \langle w \rangle_g \quad (11.68)$$

Projection is

$$\langle \langle v \rangle_g \rangle_g = \langle v \rangle_g \quad (11.69)$$

for $g, h \in G$, $g \neq h$, $\langle \langle v \rangle_g \rangle_h = 0$

Image : $\langle V \rangle_g = \text{im}(\langle - \rangle_g)$

Tensor product law : if G is a monoid, Vect^G is a monoidal category

$$(V \otimes W)_g = \bigoplus_{h \cdot k = g} V_h \otimes W_k \quad (11.70)$$

The most common type of graded vector space is the $\mathbb{Z}$ -graded vector space

$$V = \bigoplus_{k \in \mathbb{Z}} V_k \quad (11.71)$$

Elements V_k are homogeneous of degree k , $\deg(v) = k$ for $v \in V_p$.

Finite type if $\dim(V_k) < \infty$ for all k

linear map is of degree p if $f(V_q) \subseteq W_{p+q}$ for all q .

Morphism is a linear map of degree zero

$$\text{Hom}(V, W) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_k(V, W) \quad (11.72)$$

r -suspension $(s^r V)_n = V_{n-r}$. s is the 1-suspension, s^{-1} the -1 -suspension, $(s^{-1} V)_n = V_{n+1}$

Dual : $\#V$ such that

$$\langle V \rangle_k = \text{Hom}_k(V, F) \quad (11.73)$$

Transpose of a map f of degree $|f|$: ${}^t f : \#W \rightarrow \#V$

Tensor product :

$$(V \otimes W)_n = \bigotimes_{p+q=n} V_p \otimes W_q \quad (11.74)$$

$$() \quad (11.75)$$

11.8 Exterior algebra

There are a few different ways to construct the exterior algebra of a vector space. One of the most common one is to consider the exterior product on this vector space :

Definition 11.17. The *exterior product* of two vectors is a binary product on V to a vector space $\bigwedge^2 V$ such that

$$v \wedge v = 0 \quad (11.76)$$

Using this, we can generalize the wedge product of two vectors so that the wedge product becomes a closed operation in a larger vector space $\bigwedge^\bullet V$, the exterior algebra.

Definition 11.18. The *exterior algebra* $\bigwedge V$ of a vector space V is an algebra generated by V (there's an inclusion map $V \hookrightarrow \bigwedge V$) and the exterior product $\wedge$ such that, for any three elements of the algebra $\omega_1, \omega_2, \omega_3 \in \bigwedge V$,

- The algebra is associative : $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$
- The exterior product is bilinear : $(a\omega_1 + b\omega_2) \wedge \omega_3 = a\omega_1 \wedge \omega_3 + b\omega_2 \wedge \omega_3$ and $\omega_1 \wedge (a\omega_2 + b\omega_3) = a\omega_1 \wedge \omega_2 + b\omega_1 \wedge \omega_3$
- For any element v of V , $\wedge$ is alternating : $v \wedge v = 0$

The exterior algebra $\bigwedge V$ can also be defined as the quotient algebra of the tensor algebra $T(V)$ by the ideal generated by elements of the form $v \otimes v$, so that the exterior algebra is the equivalence class of tensors up to symmetric elements of that tensor (in particular they are all equivalent to the representative of the symmetric tensor 0). If we have a rank 2 tensor,

Theorem 11.19. The exterior algebra is the quotient $\otimes V / I(v \otimes V)$

Proof. The tensor algebra is already associative and bilinear by construction, so that we only need to show that it is alternating for vectors. Almost by definition, we have

$$v \otimes v = 0 + v \otimes v \quad (11.77)$$

Therefore, $v \wedge v$ is in the equivalence class of 0. □

The exterior algebra is furthermore a graded algebra. We define the k -th order exterior algebras $\bigwedge^k V$ as the algebra generated by elements of the form

$$\bigwedge_{i=1}^k v_i \quad (11.78)$$

the 0-th order product simply being the unit element of the field $1 \in F$, so that $\bigwedge^0 V = F$.

$$\bigwedge V = \bigoplus_{k=0}^{\infty} \bigwedge^k V \quad (11.79)$$

Theorem 11.20. $\bigwedge^k$

The exterior algebra of a vector space has a natural graded vector space structure

In a given basis of V , $\{e_i\}_{1 \leq i \leq n}$, the

For elements of the exterior algebra that are simply exterior products of vectors, there is a natural interpretation of those elements as k -dimensional oriented parallelotopes (the k -dimensional generalization of parallelograms).

Theorem 11.21. If a set of k vectors has a zero wedge product,

$$\bigwedge_{i=1}^k v_i = 0 \quad (11.80)$$

then all those vectors fail to be linearly independent.

Proof.

□

The space of oriented parallelograms can be embedded in the space $\bigwedge^k V$

Zero wedge product is equivalent to the vectors being linearly dependent

Theorem 11.22. If a set of k vectors has a zero wedge product,

$$\bigwedge_{i=1}^k v_i = 0 \quad (11.81)$$

then all those vectors fail to be linearly independent.

Proof.

□

wedge $v_1 \wedge \dots \wedge v_k \neq 0$ means the vectors form a signed basis of the span

Exterior algebra of a dual vector space $\bigwedge V^*$

11.8.1 Duality pairing

Duality pairing?

$$\langle \cdot, \cdot \rangle : \bigwedge^k V \times \bigwedge^k V^* \rightarrow \bigwedge^0 V \quad (11.82)$$

11.8.2 Hodge duality

The Hodge star is the exterior algebra version of complementary subspaces. In a vector space V equipped with a nondegenerate symmetric bilinear form B , the Hodge dual of a k -form which defines a subspace W will roughly correspond to a form corresponding to its orthogonal complement $W^\perp$.

If we have some k -vectors $\alpha \in \bigwedge^k V$ spanning a linear subspace $W \subseteq V$, the Hodge dual $\star\alpha$ will be a k -vector spanning a complementary subspace $W^\perp$, orthogonal to W with respect to the bilinear form.

$$\star : \bigwedge^k \rightarrow \bigwedge^{(n-k)} \quad (11.83)$$

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega \quad (11.84)$$

11.8.3 The interior product

interior product $\lrcorner$, adjoint to the exterior product

$$\iota : V \times \bigwedge^k V^* \rightarrow \bigwedge^{k-1} V^* \quad (11.85)$$

$$\iota_X : \Lambda^k V \rightarrow \Lambda^{k-1} V \quad (11.86)$$

$$\omega \mapsto \omega(X, -) \quad (11.87)$$

$$(\iota_X \omega)(X_1, X_2, \dots, X_{k-1}) = \omega(X, X_1, X_2, \dots, X_{k-1}) \quad (11.88)$$

For a 1-form,

$$\iota_X \alpha = \alpha(X) \quad (11.89)$$

for β a p -form and γ a q -form :

$$\iota_X(\beta \wedge \gamma) = (\iota_X \beta) \wedge \gamma + (-1)^p \beta \wedge (\iota_X \gamma) \quad (11.90)$$

Graded Leibniz rule

$$\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega \quad (11.91)$$

Hodge star

Show that the dual of some synthetic object is a form

11.9 Symmetric algebra

On the flipside of the exterior algebra, which is composed from antisymmetric tensors, the *symmetric algebra* SV is made of symmetric tensors, using the *symmetric tensor product*, defined as

$$v \circ w = v \otimes w + w \otimes v \quad (11.92)$$

Ideal generated by $v \otimes w - w \otimes v$

Basis

11.10 Polyvectors

With our exterior algebra, we can form the space of *multivectors*, also called *polyvectors*, which is the vector space composed of every member of the exterior algebra

$$\Lambda V = \bigoplus_{k=0}^{k=n} \Lambda^k V \quad (11.93)$$

Blades

A special kind of polyvector is the case where every element $\alpha_i \in \Lambda^i V$ is such that any element α_i is contained within the successor α_{i+1} .

11.11 Flags

[17–20]

Flags as sequences of subspaces

Definition 11.23. A *flag* in a vector space V is a sequence of subspaces $(F_i)_{i \in I}$, I some subset of the range from 0 to n

$$F_{i_1} \subset F_{i_2} \subset \dots \subset F_{i_k} \quad (11.94)$$

The signature of the flag is the tuple of each dimension of the subspaces, $(\text{Dim}(F_0), \text{Dim}(F_1), \dots, \text{Dim}(F_k))$. For instance, the name "flag" derives from the case of flags $(0, 1, 2, n)$, composed of a point, a line going through that point, and a plane that contains that line, in the analogy the flagpole being the line and the flag itself being the plane. All flags of course include the subspace that is the entire space V , as any subspace is always trivially included in V , so that it can be ignored from the list of subspaces. The same is true of the zero dimensional subspace, as there is only one such subspace so that its inclusion does not influence such results.

Example 11.24. Every k -dimensional subspace is isomorphic to a $(0, k, n)$ flag

If a flag contains a subspace of every dimension in V , it is referred to as a *full flag*.

Every flag has an adapted basis for which given some element F_k of the flag are such that the first k basis vectors also form a basis of that subspace.

The stabilizer subgroups of $\text{GL}(n, V)$ leaving flags invariant are called the *parabolic subgroups*. We will denote them by the notation of a flag of signature σ having the parabolic subgroup $P(\sigma)$. In particular, as any linear transformation leaves the zero subspace and the full space invariant, the parabolic subgroup $P(0, k, n)$ is the stabilizer of k -dimensional subspaces. For brevity in such cases, we will simply denote it by $P(k)$.

Borel subgroup

Theorem 11.25. The stabilizer subgroup of a full flag is the Borel subgroup $B \subset \text{GL}(V)$.

Theorem 11.26. The stabilizer subgroup of a flag of signature $(d_0, d_1, \dots)$

maximal torus

Schubert cells?

11.12 Invariant theory

Definition 11.27. For a set X and a group G with action ρ , an element $x \in X$ is G -invariant if $\rho_g(x) = x$.

Definition 11.28. For a set X and a group G with action ρ , a subset $S \subseteq X$ is G -stable if $\rho_G(S) \subseteq S$.

Case of some linear group

If we have some function $f : X \rightarrow Y$, both under some group action ρ^X, ρ^Y of the same group G ,

[15, 21–23]

Rational and polynomial concomitants

Independent of the basis

Theorem 11.29. If two tensors T_1, T_2 are invariant, then so are

1. Their sum $T_1 + T_2$
2. Their tensor product $T_1 \otimes T_2$
3. Any of their contractions $C(T_1 \otimes T_2)$

Corrolary 11.1. This implies that symmetrization, antisymmetrization and transvection are also invariant.

If a tensor A is concomitant with B and B is concomitant with A , A and B are said to be equivalent

Theorem 11.30. Any invariant polynomial map under $O(p, q)$ is composed from inner products of the vectors.

11.13 Topological vector space

Definition 11.31. A *topological vector space* is a vector space $(V, +, \cdot)$ endowed with a topology τ such that $+$ and $\cdot$ are continuous functions with respect to the product topology on $V \times V$ and $F \times V$.

A common topology on real finite dimensional vector space will be the standard topology on $\mathbb{R}^n$ via the isomorphism to any such vector space. Given some basis $\{e_i\}$, we can

define that topology on a finite dimensional vector space via the pullback topology along $e_i : V \rightarrow \mathbb{R}^n$.

For a real vector space V , a subset $S \subset V$ is a *convex set* if for any two vectors $v, w \in C$ and any real number $t \in [0, 1]$,

$$(1 - t)v + tw \in S \quad (11.95)$$

Definition 11.32. A *locally convex topological vector space* (LCTVS)

11.14 Function spaces

In addition to the usual vector spaces that are simply isomorphic to K^n for whatever field they are modelled on, which is the case for every finite dimensional vector space, quite a lot of important function spaces are themselves vector spaces, simply due to the definition of addition and multiplication upon them. If we have a set of functions $f : S \rightarrow X$ such that X is a vector space, we can define the sum of two functions $f, g \in [S, X]$ by

$$f + g \rightarrow \forall s \in S, (f + g)(s) = f(s) + g(s) \quad (11.96)$$

Example 11.33. A typical example of a function vector space is the space of continuous real functions $C(\mathbb{R}, \mathbb{R})$, as the sum of two continuous functions or the scalar multiplication of a continuous function is itself a continuous function.

Function spaces are typically much less well-behaved than the examples we've seen so far as they are typically infinite dimensional, and do not necessarily admit a reasonable basis.

In fact even finite dimensional vector spaces can be considered as function spaces. Any n -dimensional F -vector space is isomorphic to the set of functions from sets of cardinality n to F :

$$v : S_n \rightarrow F \quad (11.97)$$

$$(1, 2, \dots, n) \mapsto \quad (11.98)$$

11.15 Modules

A more general notion to that of vector spaces is *modules*, where we do not require F to be a field, but can be any ring R .

Example 11.34. The space of vector fields is a $C^\infty(M)$ -modules.

Chapter 12

Algebras

As opposed to the more general algebraic structures

12.1 Algebras over a field

The word "algebra" will quite often refer to what are more selectively called algebras over a field, which are vector spaces given an algebraic structure.

Definition 12.1. An *algebra* A is an F -vector space V or an R -module M equipped with a product $\cdot : V \rightarrow V \rightarrow V$ that is left and right distributive, and compatible with the

$$(ax + by) \cdot z = a(x \cdot z) + b(y \cdot z) \quad (12.1)$$

$$x \cdot (ay + bz) = a(x \cdot y) + b(x \cdot z) \quad (12.2)$$

Just as with other algebraic structures, we can give various properties to our algebras.

Definition 12.2. An algebra is *abelian* if the product is commutative :

$$x \cdot y = y \cdot x \quad (12.3)$$

Example 12.3. Any field F (such as $\mathbb{R}$ or $\mathbb{C}$) considered as a vector space with its multiplication forms an algebra.

Beyond this basic example, we have already seen examples of algebras previously, such as the algebra defined by the exterior algebra. defined purely in algebraic term without a geometric interpretation, it is also referred to as the Grassmann algebra :

Definition 12.4. The n -dimensional *Grassmann algebra* over F Λ_n is the algebra generated by a set of n anti-commuting generators ζ_a ,

$$\zeta_a \zeta_b = -\zeta_b \zeta_a \quad (12.4)$$

Definition 12.5. The algebra of polynomials $F[X_1, X_2, \dots]$ is the algebra over the field F with generators $X_1, X_2, \dots$ [Finite number]

Example : $\mathbb{R}[X]$

$$p = a_0 + a_1X + a_2X^2 + \dots \quad (12.5)$$

Evaluation map

Formal series $F[[X]]$

12.2 Graded algebra

If an algebra is based on a graded vector space, it is a *graded algebra* if the product is such that

$$V_g \cdot V_{g'} \subseteq V_{g+g'} \quad (12.6)$$

That is, the product of two homogeneous elements of degree g and g' is a homogeneous element of degree $g + g'$. In terms of a $\mathbb{Z}$ -graded algebra, this means

$$V_k \cdot V_{k'} \subseteq V_{k+k'} \quad (12.7)$$

12.3 Differential algebras

As we have seen, vector spaces can be function spaces, such as spaces of functions of various types from $\mathbb{R}$ to itself, and those functions themselves form algebras of their own, such as the simple algebra of real functions.

One type operator those algebras have is a derivation, which we can abstract into an operator on algebras by the use of the Leibniz rule :

Definition 12.6. A *derivation* on an algebra A is a function $D : A \rightarrow A$ that is linear

$$D(av_1 + bv_2) = aD(v_1) + bD(v_2) \quad (12.8)$$

and obey the *Leibniz rule* :

$$D(vw) = vD(w) + D(v)w \quad (12.9)$$

A differential algebra (A, D) is then simply the pairing of algebra A and a derivation on this algebra D .

Example 12.7. The algebra of smooth functions on $\mathbb{R}$ with the usual derivative form a differential algebra.

Differential graded algebra

12.4 Subalgebras and ideals

Definition 12.8. A *subalgebra* is a subset of an algebra A , $S \subseteq A$, which is closed under the algebra : it is a vector subspace[?] and for $x, y \in S$, $x \cdot y \in S$.

Definition 12.9. An *ideal* I of an algebra A is a subalgebra

12.5 Category of algebras

Category **Alg** of algebras and algebra homomorphisms

Empty algebra, algebra of one object

Chapter 13

Projective spaces

Given a real vector space V , the associated *projective space* $P(V)$ is the quotient space of that vector space (minus the zero vector) by the rescaling of each vector. In other words, for any vector $v \in V$, $v \neq 0$, the projectivization of that vector is the equivalence class generated by its rescaling up to some non-zero factor,

$$[v] = \left\{ \frac{v}{k} \mid k \in \mathbb{R}_{\neq 0} \right\} \quad (13.1)$$

A good way to visualize this is to consider the representatives of this equivalence class which are the vectors themselves divided by their norm, $v/\|v\|$, as any other vector colinear to it will simply be that vector up to factor, although every projective vector actually contains two such unit vectors, some given vector $v/\|v\|$ and its inverse, $-v/\|v\|$.

We therefore have some equivalence relation $v \sim v'$ for $[v] = [v']$, and the quotient space of V by this relation is the projectivized vector space $P(V)$:

$$P(V) = V / \sim \quad (13.2)$$

If we are given a map from our vector space (minus the zero vector) to its projectivization,

$$P : V \setminus \{0\} \rightarrow P(V) \quad (13.3)$$

$$v \mapsto [v] \quad (13.4)$$

Then the preimage of each projective vector is in fact the straight line kv going through 0 with the zero vector removed. If we consider the space of all such lines (the Grassmannian $\text{Gr}(V, 1)$), we can define some isomorphism between PV and $\text{Gr}(V, 1)$ by completing the line kv to include zero (which we can do uniquely), or in the reverse direction, mapping any of the non-zero point of the line kv to $[v]$.

$$PV \cong \text{Gr}(V, 1) \quad (13.5)$$

As every projective point $P(V)$ corresponds to a line of V going through the origin, $P(V)$ is therefore isomorphic to the space of 1-dimensional subspaces, the Grassmannian $\text{Gr}(1, V)$.

Projective spaces can also be considered as the compactification of vector spaces via the addition of hyperplanes at infinity. If we have an n -dimensional vector space V^n , there is a subset of its projective space $P(V)$ which is isomorphic to an $(n - 1)$ dimensional vector space $V^{(n-1)}$ via the stereographic map

Angular interpretation of projective space

CW complex $P^n(V) \cup P^{n-1}(V) \cup \dots$

Plane in V^{n+1} corresponds to a curve in P^n , or alternatively, a curve in P^n corresponds to a curve in V^n in the compactification interpretation

What is the interpretation of the motion group for these two

Projective subvarieties

Intersection of projective subvarieties is a projective subvariety

[24]

Pappus theorem

Desargues theorem

Affine projective space

Definition 13.1. For an affine space A over F based on a vector space V , the *projective closure* $\hat{A}$ of A is

$$\hat{A} = P(V \times F) \quad (13.6)$$

The affine space A embeds in its projective closure by picking a given point a via the map j_a :

$$j_a : A \rightarrow \hat{A} \quad (13.7)$$

$$b \mapsto [(b - a, 1)] \quad (13.8)$$

The remaining part of $\hat{A}$ is given by the hyperplane at infinity, $P(V \times 0)$.

13.1 Homogeneous coordinates

Homogeneous coordinates

$$\mathbb{R}^{n+1} \rightarrow P(V) \quad (13.9)$$

Projective basis

13.2 Crossratio

Projective spaces do not admit a notion like that of a metric, as all vectors of differing norm are identified to a single point, and different points in the projective space will roughly correspond to lines separated by a given angle

[Diagram comparing points on the projective line and triangles on the plane]

Definition 13.2. The *cross-ratio* of four points $p, q, r, s \in A$ in an affine metric space is the ratio of the ratios $pr : qr$ and $ps : qs$:

$$(p, q; r, s) = \frac{pr : qr}{ps : qs} \quad (13.10)$$

$$= \quad (13.11)$$

13.3 Projective groups

The *projective linear group* $\mathrm{PGL}(n)$ is an extension of the general linear group $\mathrm{GL}(n)$

Central extension

$$Z = D = \mathbb{R}^\times \hookrightarrow \mathrm{GL}(V) \xrightarrow{\det} \mathrm{PGL}(V) \quad (13.12)$$

For any linear transformation $g \in \mathrm{GL}(V)$, any action on a vector $v \in V$, $v' = g \cdot v$, has a corresponding transformation on the projective space via $[v'] = [g \cdot v]$. These transformations may however

There are also a larger projective group that can be of importance, called the *projective semilinear group*, that can be of importance,

Semilinear group : linear transformation up to automorphism of the field K :

$$\Gamma L(V) = GL(V) \rtimes \text{Aut}(K) \quad (13.13)$$

$$\text{P}\Gamma L(V) = \text{PGL}(V) \rtimes \text{Gal}(K/k) \quad (13.14)$$

$$\begin{array}{ccc} GL(V) & \xrightarrow{i} & \Gamma L(V) \\ \downarrow & & \downarrow \\ \text{PGL}(V) & \xrightarrow{i} & \text{P}\Gamma L(V) \end{array}$$

However, as the set of field automorphisms of $\mathbb{R}^n$ is the trivial automorphism, this is simply the projective linear group itself. This will however come into play later when we consider finite geometry with finite fields.

An important property of the projective group is that if we consider the pullback of its action on the original vector space,

$$\rho_g : G \times P(V) \rightarrow PV \rightarrow \quad (13.15)$$

We can also consider the corresponding group of affine transformations, in which

$$\mathbb{R}^\times \rightarrow \text{PGA}(V) \rightarrow \text{GA}(V) \quad (13.16)$$

The projective transformations preserve the crossratio

13.4 Operations on projective spaces

Segre embedding

13.5 Synthetic projective geometry

Like Euclidian space as a vector and an affine space, it is possible to define projective space purely in synthetic terms.

The most obvious semantic one can get from those axioms of projective space is at the complementation of Euclidian space with a point at infinity, although there is of course a similar interpretation with projective space as the space of all lines through a point.

1. Every line contains at least 3 points
2. Every two distinct points p and q lie on a unique line pq
3. If pq and rs intersect, then so do pr and qs (if all distinct points)

Hyperplane at infinity

Homogeneous coordinates

$$h : V \rightarrow P(V) \tag{13.17}$$

projective line

Projective spaces lack the Euclidian notion of congruence

$$(A, B; C, D) = \frac{AC : BC}{AD : BD} \tag{13.18}$$

Plücker embedding

13.6 Category of projective spaces

The category of projective spaces **Proj**

Chapter 14

Minkowski space

The modelization of spacetime as an affine (or vector) space is done via the concept of *Minkowski space*, which is an affine space equipped with a specific bilinear form, its metric tensor. [25]

Mathematically there is not much more to add to the definition of Minkowski space (at least as far as its description as an affine space goes), so that we can simply define it as

$$\text{Mink}(n) = (A^{1,(n-1)}, \eta) \quad (14.1)$$

Minkowski space is typically considered for the case $n \geq 2$, as the lower dimensional cases do not have much in common with the usual case (the one dimensional case is isomorphic to basic geometry on the line).

From this we will look into the specific properties of this space that we can define.

14.1 Lorentz inner product

Definition 14.1. For an n -dimensional real vector space V , an *Lorentzian inner product* is a nondegenerate symmetric bilinear map $g : V \times V \rightarrow \mathbb{R}$ of signature $(1, (n - 1))$.

Definition 14.2. A *Lorentzian vector space* (V, g) is a real vector space equipped with a Lorentzian inner product g .

By the Gram-Schmidt process, we know that there exists an orthonormal basis of V in which the basis elements obey the properties

$$g(e_0, e_0) = -1 \quad (14.2)$$

$$g(e_i, e_j) = \delta_{ij} \quad (14.3)$$

$$g(e_0, e_i) = 0 \quad (14.4)$$

This means that, given the dual basis $\{\theta^\mu\}$, the metric tensor can be expressed in an orthonormal basis as

$$g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^n \theta^i \otimes \theta^i \quad (14.5)$$

Since g is nondegenerate, it furnishes an isomorphism $V \rightarrow V^*$ by $x \mapsto \eta(x, \cdot)$. We will commonly denote η by $\langle \cdot, \cdot \rangle$. We define the norm as $|x| = |\langle x, x \rangle|^{1/2}$.

The components of this inner product in the basis $\{e_\mu\}$ are expressed by $\eta_{\mu\nu} = \eta(e_\mu, e_\nu)$, given by the relation

$$\eta(v^\mu e_\mu, w^\nu e_\nu) = v^\mu w^\nu \eta(e_\mu, e_\nu) = v^\mu w^\nu \eta_{\mu\nu} \quad (14.6)$$

14.2 Symmetries of Lorentz vector spaces and the Poincaré group

We will define the group of all symmetries that keep the norm of the metric tensor invariant by the Lorentz group $O(1, n)$. To construct it, consider first the transformation $\Lambda \in GL(n, \mathbb{R})$. We want the subgroup obeying

$$\mathbf{Y}^T \eta \mathbf{X} = (\mathbf{Y})^T ({}^T \eta) (\mathbf{X}) \quad (14.7)$$

If we require that

$${}^T \eta = \eta \quad (14.8)$$

It is easy to see that the group

This is equivalent to the invariance of the norm $g(x, y)$

$$(\Lambda x)^T \Lambda^T g \Lambda (\Lambda y) = x^T g y \quad (14.9)$$

$$\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \quad (14.10)$$

$$\eta_{\mu\nu} v^\mu v^\nu \rightarrow \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\Lambda^\alpha{}_\rho v^\rho) (\Lambda^\beta{}_\sigma v^\sigma) \quad (14.11)$$

The members of the Lorentz group will be defined as the linear transformations that leave the metric invariant.

$$O(1, n) = \{ \in \text{Mat}_{n \times n} \mid \eta = \eta \} \quad (14.12)$$

in which case we have

$$\begin{aligned} \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\Lambda^\alpha{}_\rho v^\rho) (\Lambda^\beta{}_\sigma v^\sigma) &= \eta_{\alpha\beta} (\Lambda^\alpha{}_\rho v^\rho) (\Lambda^\beta{}_\sigma v^\sigma) \\ &= \eta_{\rho\sigma} v^\rho v^\sigma \end{aligned} \quad (14.13)$$

which shows that it indeed preserves the Lorentzian inner product.

Proposition 14.3. The Lorentz group is a Lie group.

Proof. As a matrix group, the Lorentz group is a subset of $\mathbb{R}^{n^2}$, and as the matrices are symmetric, a subset of $\mathbb{R}^{n(n+1)/2}$. Something something preimage theorem. $\square$

Lie group with Lie algebra of the Lorentz algebra

4 connected components to the Lorentz group : Connected to the identity, time-reversed, space-reversed and time and space reversed

$$\begin{array}{ccc} O(n, 1) & \longrightarrow & SO(n, 1) \\ \downarrow & & \downarrow \\ O^\uparrow(n, 1) & \longrightarrow & SO^\uparrow(n, 1) \end{array}$$

Figure 14.1

14.2.1 Lie algebra of the Lorentz group

14.2.2 Representation of the Lorentz group

Connected to the identity :

$$\Lambda^\mu{}_\nu = e^{i\omega J} \quad (14.14)$$

Complexified basis of the Lie algebra :

$$\begin{aligned} A_i &= \frac{J_i + iK_i}{2} \\ B_i &= \frac{J_i - iK_i}{2} \end{aligned}$$

Proposition 14.4. The algebra basis $\{A_i\}$ and $\{B_i\}$ are each basis of the $\mathfrak{su}(2)$, and commute with each other.

Proof.

$$\begin{aligned} [A_i, A_j] &= \frac{1}{4} [J_i + iK_i, J_j + iK_j] \\ &= \frac{1}{4} ([J_i, J_j] + i[K_i, J_j] + i[J_i, K_j] - [K_i, K_j]) \end{aligned} \quad (14.15)$$

□

14.2.3 Subgroups of the Lorentz group

Identity, $\{\pm I\}$, discrete rotations and boosts, $O(k)$, $O(k, 1)$

14.3 Classification of vectors

Vectors in a Lorentzian vector space can be classified by the sign of their norm.

- timelike if $g(v, v) > 0$
- null if $g(v, v) = 0$
- spacelike if $g(v, v) < 0$

We will also say that a vector is causal if it is either null or timelike, that is, $g(v, v) \geq 0$.

Proposition 14.5. We have the following relations among timelike and null vectors:

1. Two timelike vector are never orthogonal

2. A timelike vector is never orthogonal to a null vector
3. Two null vectors are orthogonal if and only if they are proportional.

Proof. 1. Clear from 1.4 (a).

2. Clear from 1.4 (b).

3. Let l_1 and l_2 be null and orthogonal. Fix a unit timelike vector u ; we have

$$l_1 = \lambda_1 u + y_1 \quad \text{and} \quad l_2 = \lambda_2 u + y_2,$$

where $\langle u, y_i \rangle = 0$. Orthogonality of l_1 and l_2 gives $-\lambda_1 \lambda_2 + \langle y_1, y_2 \rangle = 0$. Since $\lambda_1^2 = g\langle y_1, y_1 \rangle$ and $\lambda_2^2 = \langle y_2, y_2 \rangle$, we obtain

$$\langle y_1, y_1 \rangle \langle y_2, y_2 \rangle = \langle y_1, y_2 \rangle^2.$$

Since g is positive definite on $\mathbb{R}y_1 \oplus \mathbb{R}y_2$, we have $y_2 = \lambda y_1$. This implies $\lambda_2 = \lambda \lambda_1$, and we are done.

□

14.3.1 Subspaces

As an n -dimensional vector space, a Lorentz vector space admits a number of subspaces. They can be classified by the class of vectors it contain.

Definition 14.6. Let $V \subset W$ be a subspace. We have three possibilities:

1. W is *spacelike* if $g \upharpoonright W$ is positive definite.
2. W is *null* or *lightlike* if $g \upharpoonright W$ is strictly positive semi-definite.
3. W is *timelike* if neither (i) or (ii) hold.

If $W \subset V$ is a subspace, we define $W^\perp$, the orthogonal subspace, to be $\{v \in V : \langle w, v \rangle = 0 \ \forall w \in W\}$.

Proposition 14.7. Let $W \subset V$ be a subspace.

1. W is timelike if and only if $W^\perp$ is spacelike.
2. W is spacelike if and only if $W^\perp$ is timelike.
3. W is null if and only if $W \cap W^\perp \neq \{0\}$, or equivalently, if and only if $W^\perp$ is null.

Proof. 1. W contains a unit timelike vector w and V splits as $S \oplus \mathbb{R}w$. Since g can be negative definite on a subspace of at most dimension 1, we conclude that g is positive definite on S . The forward direction follows. For the converse, suppose $W^\perp$ is spacelike. We have a splitting $V = W \oplus W^\perp$. So if $v \in V$ is timelike then $v = w + w'$ where $w \in W$ and $w' \in W^\perp$. Then $\langle w, w \rangle = \langle v, v \rangle - \langle w', w' \rangle < 0$, so w is timelike and hence W is timelike.

2. Follows from (a) and $W^{\perp\perp} = W$.

3. W lightlike implies W contains a lightlike vector w_0 , but no timelike vector. Then for all $a \in \mathbb{R}$ and $w \in W$, $\langle w + aw_0, w + aw_0 \rangle = \langle w, w \rangle + 2a\langle w, w_0 \rangle \geq 0$. Choosing a negative enough, we see that $\langle w, w_0 \rangle = 0$ for all $w \in W$, so $w_0 \in W \cap W^\perp$. On the other hand, if $0 \neq w_0 \in W \cap W^\perp$, then w_0 is null. Since W cannot contain a timelike vector by (a), W is null. The other part follows from $W^{\perp\perp} = W$.

□

14.3.2 Light cones

The timelike vectors of a Lorentz vector space can be split in two categories

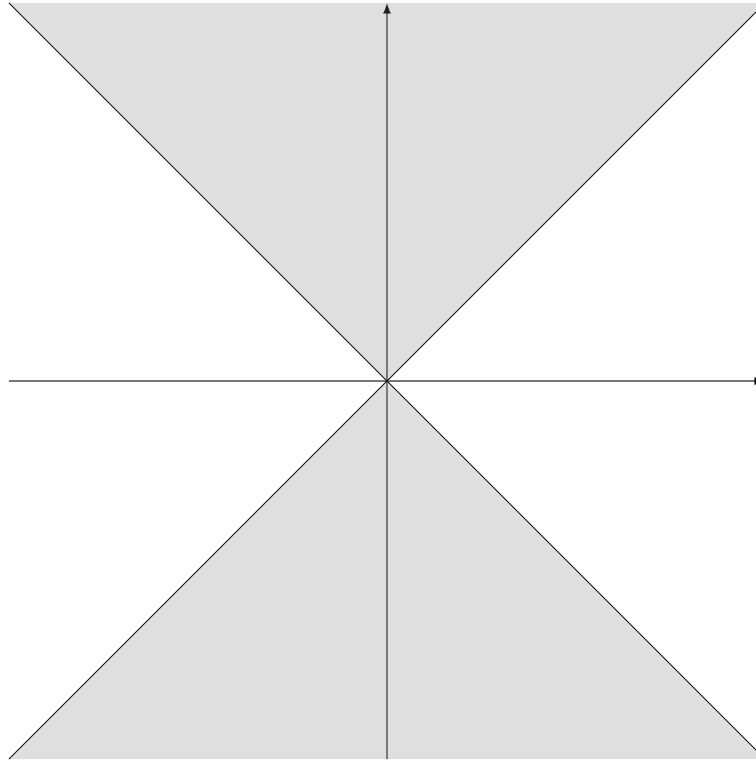


Figure 14.2: The light cone in $(\mathbb{R}^2, \eta)$

Definition 14.8. for a vector v , $v^\perp$ is the set of all orthogonal vectors to v defined by $v^\perp = \{u \in T_p\mathcal{M} | g(u, v) = 0\}$

Theorem 14.9. if u is timelike, the subspace $u^\perp$ is spacelike.

Proof. If we have a timelike vector u and a vector v such that $g(u, v) = 0$, and we switch the basis of $T_p\mathcal{M}$ to (u, v, x, y) , the new metric will be $(g(u, u), g(v, v), g(x, x), g(y, y))$. The signature will be correct only if v is spacelike, as Sylvester's law dictates that a change of basis cannot change the signature. $\square$

Theorem 14.10. There are two disjoint sets of timelike vectors at p .

Proof. If we pick any timelike vector u in $T_p\mathcal{M}$, let's define the future lightcone :

$$\mathfrak{T}^+(u) = \{v \in T | g(u, v) < 0\} \quad (14.16)$$

And the past lightcone

$$\mathfrak{T}^-(u) = \{v \in T | g(u, v) > 0\} \quad (14.17)$$

Obviously $\mathfrak{T}^+(u) \cap \mathfrak{T}^-(u) = \emptyset$, and $g(u, v) = 0$ is impossible since only spacelike vectors are orthogonal to timelike vectors. $\square$

We will call timelike (resp. null, causal) vectors in C^+ future-directed timelike (resp. null, causal) vectors, and past-directed if they are in C^- .

14.3.3 Properties

Proposition 14.11. The sum of two timelike vectors ξ_1, ξ_2 belonging to the same light cone is in the same lightcone.

Proof. By bilinearity, we have

$$g(\xi_1 + \xi_2, \xi_i) = g(\xi_1, \xi_i) + g(\xi_2, \xi_i) \quad (14.18)$$

One of those terms will be simply the norm of a timelike vector (negative) while the other will be the product of our original two vectors which will be in the same lightcone so that $g(\xi_1, \xi_2) < 0$, therefore a sum of two negative terms which will be negative. The three vectors ξ_1, ξ_2 and $\xi_1 + \xi_2$ are in the same lightcone. $\square$

Definition 14.12. A tensor T is said to be future (resp. past) if $T(u_1, \dots, u_n) \geq 0$ (resp. ≤ 0) for all future-directed vectors u_i . A causal tensor is a tensor which is either future or past.

Generalizes to T_s^r tensors and tensor fields.

14.4 Geometry in Minkowski space

As quite a lot of the geometry of Euclidian space rests on its metric properties, quite a few theorems of geometry have to be modified in Minkowski space, in particular the Pythagorean theorem :

Theorem 14.13. For any three points $A, B, C \in \text{Mink}$ such that AB and BC are orthogonal ($\eta(AB, BC) = 0$), their interval is given by the formula

$$\|AB\|^2 + \|BC\|^2 = \|AC\|^2 \quad (14.19)$$

The change isn't immediately noticeable but the norm on those intervals are not positive definite and therefore can go in the opposite direction that one could imagine. It can be difficult in particular to interpret the lengths on a spacetime diagram due to this as this feature cannot be replicated in a diagram.

A classic example of this is to consider a timelike interval AB and a spacelike interval BC orthogonal to one another, for instance by trying to picture some kind of diagram representing an observer going along the trajectory AB , at a given distance BC of some other point emitting a signal in its direction along the path AC .

[diagram]

14.5 Basis of the tangent space

As the product of two vectors remain constant under Lorentz transformation, so will the basis remain orthonormal under it. This will allow us to define new more appropriate basis.

We may then define timelike (resp. null, spacelike) vectors in this new orthonormal basis as $v = -|v|\partial_0$ with ∂_0 a timelike vector, $v = |v|\partial_1$ with ∂_1 a spacelike vector, or $v = \partial_u$, with ∂_u a null vector.

$$v = v^0\partial_0 + \vec{v} \cdot \vec{e} \quad (14.20)$$

For a spacelike vector : Lorentz boost to the $\vec{v}$ direction

$$v \rightarrow \quad (14.21)$$

Canonical form of a timelike, null and spacelike vector : $\xi = \pm\partial_t$, $k = \pm(\partial_t + \partial_{x_1})$, $s = \partial_{x_1}$

every vector can be expressed in that form up to rotation, dilation and reflection of the basis

14.6 Analysis on vectors

In addition to its affine space structure, Minkowski space is also traditionally given the canonical topology of $\mathbb{R}^n$. This topology is unrelated to anything like a metric topology given by the metric tensor η , as it does not actually define a metric.

Topology on Minkowski space

Theorem 14.14. Any null vector k can be expressed as the convergence of a sequence of timelike vectors.

Proof. If we write k in its canonical form $k = \partial_t + \partial_x$, we can use the sequence

$$\xi_n = \partial_t + (1 - n^{-1})\partial_x \quad (14.22)$$

which is always of negative norm $|\xi_n| = -n^{-1}$. It can be easily seen that this sequence of vectors converges to the desired null vector.

$$|\xi_n - k| = |-n^{-1}\partial_x| = n^{-1} \quad (14.23)$$

□

14.7 Compactification and projectivization

Just like any vector space, we can look at the projective version of Minkowski space and its compactification.

Definition 14.15. The projective Minkowski space PMink is the quotient of Minkowski space by the rescaling of its vectors,

$$\text{PMink} = \text{Mink} / \sim \quad (14.24)$$

As with the usual case of projective space, this also comes in an oriented version OMink where we only consider rescaling by positive factors.

As Minkowski space is simply an affine space

[Affine projective space or vector projective space??]

The projective Minkowski space as a vector space has the interpretation of a (half-)sphere with o

Theorem 14.16. The set of timelike vectors on PMink is a disk

Proof.

□

This projective space is also called the *celestial sphere*

The projectivization of Minkowski space as an affine space should furnish us with a compactification of Minkowski space as it does in the Euclidian case, but while this is technically true, giving us a compact space,

[What happens to the metric if we try?]

Compactification of Minkowski space [26]

14.8 Synthetic Minkowski space

[27, 28]

Just like in the case of Cartesian space $(\mathbb{R}^n, \delta)$ being isomorphic to Euclidian space E , there exists a synthetic version of Minkowski space. While the synthetic Minkowski space is rarely brought up explicitly, it is very common in relativistic arguments to rely on very geometric arguments with the use of diagrams. For this reason it can be useful to define explicitly what the synthetic case is and how it relates to the analytic case.

Synthetic Minkowski space Mink is a set composed of spacetime *events* as points, $p \in \text{Mink}$, as well as the other usual structures of Euclidian space such as lines, planes, volumes, etc. If we follow the scheme of Euclidian geometry, we will consider the $(3+1)$ -dimensional case, with lines, planes and volumes. Just as in the Euclidian case, this naturally generalizes to higher dimension by changing a few specific axioms.

The mirror of the signature of the analytic Minkowski space in the synthetic case is that we have different types of objects. There are no specific kinds of lines beyond that which we wish to define in the Euclidian case, but Minkowski space will distinguish three kinds of lines, timelike lines, spacelike lines and null lines.

On a topological level, there is no fundamental differences between Minkowski space and Euclidian space, so that all the axioms relating to incidence (at least those involving non-specific types of line, planes, etc) will remain the same as before. This can be shown simply from the fact that both remain real affine spaces. The axioms relating to the dimensionality of the space such that the minimal number of non-collinear points are also unaffected by the metric.

As the congruence axioms relate to the distance function, those will change and really define the difference between the various types of objects.

For any two lines of the same type L_1, L_2 , we have a congruence relation $\text{Cong}_{\text{Type}}(L_1, L_2)$.

Lightlike segments $\overline{pq}$ are congruent to the segment $\overline{pp}$

Between, order, congruence, angles,

Chapter 15

Locally Euclidian spaces

15.1 Plots, charts and locally Euclidian spaces

The sense in which our spaces are locally Euclidian is that there exists some functions mapping subsets of our spaces to subsets of the Euclidian spaces $\mathbb{R}^k$

Definition 15.1. A *parametrization* of a set X is a map $P : U \subset \mathbb{R}^n \rightarrow X$ for some $n \in \mathbb{N}$.

If we consider some particular value of n , we talk of an n -parametrization. The set of all parametrizations on X is denoted by $\text{Param}(X)$

$$\text{Param}(X) = \bigcap_{n \in \mathbb{N}} \quad (15.1)$$

For $p \in X$, denote the unique zero-parametrization $\mathbf{p} : \{0\} \rightarrow X$, $\mathbf{p}(0) = p$.

There exists parametrizations for any space as we did not specify much constraints on those functions, so that we are going to have some more conditions on this. The usual basic condition that we impose is that a locally Euclidian spaces locally has similar topological properties to $\mathbb{R}^k$ for a given k . To maintain such topological properties, we therefore need that for any point of our space $p \in X$, there exist an open neighbourhood U_p which is homeomorphic to some open set $\mathcal{O} \subset \mathbb{R}^k$, as homeomorphisms preserve local topological properties. This is done via the notion of a chart [29] :

Definition 15.2. A *chart* on a topological space S is a homeomorphism $\phi : U \rightarrow \mathcal{O}$, from an open set U of S to an open set $\mathcal{O}$ of $\mathbb{R}^n$. The chart is C^k if the function ϕ is C^k .

The condition of the existence of a chart will already impose some constraints on our space. If there exist a chart for any open set, the space obeys all the local topological properties of $\mathbb{R}^n$: it is locally connected, locally Hausdorff and locally metrizable.

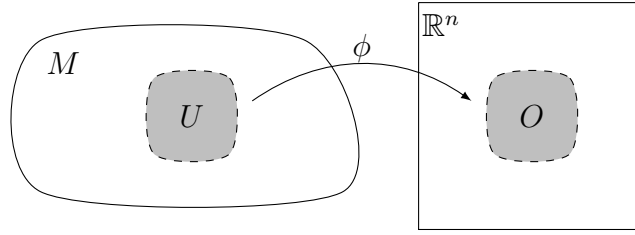


Figure 15.1: Chart on a manifold

Theorem 15.3. A space covered by charts is locally connected.

Proof.

□

15.2 Smooth spaces

Beyond having our space be locally Euclidian, we would like those charts

The largest possible category we could make for locally Euclidian spaces, as a topos, is that of a sheaf on the site of Cartesian spaces,

$$\mathbf{Smooth} = \mathbf{Sh}(\mathbf{CartSp}) \quad (15.2)$$

Presheaf on Cartesian spaces

15.3 Diffeological spaces

A very broad category of locally Euclidian space is that of diffeological spaces, which is a topological space (X, τ) equipped with a *diffeology*

While we will not typically consider diffeological spaces throughout this book, it has quite a lot of benefits over the category of manifolds that we will later see, one of them being that it is Cartesian closed : any set of maps between two diffeological space is itself a diffeological space. It is also closed under quotients,

While not as well-behaved as manifolds, diffeological spaces will allow us to still use a lot of the machinery of manifolds to a variety of other spaces that we will have to use later on, such as function spaces $C^\infty(M)$, sections of fiber bundles $\Gamma(E)$ and configuration spaces of physical theories $\mathbf{Conf}$, allowing us to cast a lot of the theory in the same terms as manifolds by curves, vectors, linear maps and so on.

Axioms of diffeology :

1. *Covering* : $\mathcal{D}$ contains the constant parametrizations for any $p \in X$

$$\mathbf{p} : \mathbb{R}^n \rightarrow X \quad (15.3)$$

$$x \mapsto \mathbf{p}(x) = p \quad (15.4)$$

for every $n \in \mathbb{N}$.

2. *Locality* (sheaf condition?) : For a parametrization $P : O \rightarrow X$, if for every point $x \in O$, there exists an open neighbourhood $U \ni x$ such that $P|_U$ belongs to D , then P belongs to D .
3. *Smooth compatibility* : For every element $P : O \rightarrow X$ of D , for every real domain $O' \subset \mathbb{R}^n$, for every $f \in C^\infty(O', O)$, $P \circ f \in D$.

A *diffeological space* (X, D) is a set X equipped with a diffeology D .

Wire diffeology :

Orbifolds

Theorem 15.4. For two diffeological spaces X, X' , and their smooth maps $C^\infty(X, X')$ with the evaluation map

$$\text{ev} : C^\infty(X, X') \times X \rightarrow X' \quad (15.5)$$

$$(f, p) \mapsto f(p) \quad (15.6)$$

a *functional diffeology* is any diffeology on $C^\infty(X, X')$ for which ev is smooth.

v. standard functional diffeology

Example 15.5. The space of smooth function $C^\infty(\mathbb{R})$ is a diffeological space.

Many generalizations of manifolds are diffeological spaces, such as manifolds with boundaries, Frechet manifolds, Banach manifolds or Hilbert manifolds. We will use these if necessary, but much of the basic machinery of differential geometry will be able to be defined with simply diffeological spaces.

15.4 Category

Category of sheaves $X \in \text{Sh}(S)$ on a site S (local model space), diffeological spaces are concrete sheaves, a plot- is an element $X(U) \in \text{Set}$ for $U \in S$, therefore by Yoneda a morphism $U \rightarrow X$

Chapter 16

Manifolds

A *manifold* is a certain type of locally Euclidian space. It can be defined by a compatibility condition between different charts

1. For every point $p \in M$, there exists an open neighbourhood $U_p \ni p$ such that there exists a chart $\phi : U_p \rightarrow \mathcal{O} \subset \mathbb{R}^n$
2. Every chart in the atlas must be *compatible* with all other charts.

The compatibility of two charts (U_α, ϕ_α) and (U_β, ϕ_β) is given by a transition function $\psi_{\alpha\beta}$ defined on the neighbourhood $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by

$$\psi_{\alpha\beta} = \phi_\beta|_{U_{\alpha\beta}} \circ \phi_\alpha^{-1}|_{\phi_\alpha^{-1}(U_{\alpha\beta})} \quad (16.1)$$

such that $\psi_{\alpha\beta}$ is a homeomorphism, ie a C^0 function on those subsets of $\mathbb{R}^n$.

Compatibility is generally defined for $U_{\alpha\beta} \neq \emptyset$, although if the intersection is empty, we simply obtain that the transition chart is the empty function, which is vacuously always a homeomorphism, thereby automatically fulfilling that condition.

$$\psi_{\alpha\beta} : \mathcal{O}_\alpha \subset \mathbb{R}^n \rightarrow \mathcal{O}_\beta \subset \mathbb{R}^n \quad (16.2)$$

If we consider the charts as describing coordinates of a neighbourhood, the transition function corresponds to a change of coordinate on those same neighbourhoods.

Cech nerves

The transition charts of a manifold obey the *cocycle condition*

This definition of manifold does allow one pathological example : the empty set $\emptyset$ is a manifold vacuously, with its chart simply being the map from itself (the only member of its topology) to the empty subset of $\mathbb{R}^n$, for *any* dimension n . If we are to consider the empty set as a manifold, it will be a manifold of every dimension.

Overall we will not generally consider the empty set as a manifold in most circumstances, but it does exist and has some specific uses. The empty set manifold is the initial object of the category of manifolds. If we consider manifolds with boundaries, the inclusion of manifolds in manifolds with boundaries simply has their boundary (which is a manifold) be the empty set. It is overall useful for circumstances where we would expect either a result to be a manifold or nothing, in which case we can simply put them all together as the same type.

We will however not generally be including it for ie, classifications of manifolds. The empty set will not be in the classifications of n -dimensional manifolds, it will not be an example of a homogeneous manifold, or a

Manifolds are of constant dimensions and unique

A manifold is said to be of *dimension* n if all the charts of its atlas are to Cartesian spaces of the same dimension.

Theorem 16.1. All manifolds are of a specific dimension n , except for the empty set.

Proof. Inverse thm □

One thing we can do if we wish for a better behaved atlas is to select different types of neighbourhoods and transition functions.

Theorem 16.2. For any coordinate chart (ϕ, U) in an atlas, we can replace it by a coordinate chart $(\phi|_{U'}, U')$ for a subset U' as long as the resulting atlas still covers the entire manifold.

Proof. If we replace the atlas $\{\dots, (\phi, U), \dots\}$ by $\{\dots, (\phi|_{U'}, U'), \dots\}$, assuming that the atlas still forms a cover of the manifold M , we only have to check the gluing condition □

Restriction of the map, decomposition of the region into other regions, further homeomorphism to another region

With this, we can replace our manifold's atlas with any number of different conditions on the

Theorem 16.3. A manifold can be entirely covered by an atlas of

1. Connected components
2. Simply connected components

3. Open sets homeomorphic to the open ball $B_{0,r}$
4. Open sets homeomorphic to $\mathbb{R}^n$

Proof. 1. For any open set with k connected components, we have k open sets covering the same regions

2. Any non-simply connected component of $\mathbb{R}^n$ can be decomposed into simply connected elements

3. Any region of $\mathbb{R}^n$ can be covered by countably many open balls

4. Open balls can be mapped to $\mathbb{R}^n$

□

Coordinate line : map $\gamma : \mathbb{R} \rightarrow M$ obeying that $\partial \text{pr}_\mu \circ \phi \circ \gamma = 0$ if $\mu \neq$ the coordinate or 1 otherwise

Theorem 16.4. Manifolds have an empty boundary.

Proof. As there exists an open set around every point $p \in M$

□

16.1 Spherical coordinates

While this is a pretty rigorous definition of coordinates on a manifold, it is very common that coordinates will be used a bit more loosely as well, usually to include some closed coordinate lines, such as the case of polar coordinates on the plane :

$$\phi : \mathbb{R}^2 \rightarrow S^1 \times \mathbb{R}_{\geq 0} \quad (16.3)$$

$$(x, y) \mapsto (\text{atan2}(x, y), x^2 + y^2) \quad (16.4)$$

where atan2 is the usual function to obtain the polar angle of two points on the plane :

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases} \quad (16.5)$$

This chart can somewhat be made into an ordinary manifold chart, by mapping S^1 into some charts on S^1 such as the stereographic charts, but *it is not a proper chart as it fails to be a homeomorphism*. This can easily be seen by just considering the subset $(0, 2\pi) \times \{0\}$, which all map to $(0, 0)$, this map therefore fails to be injective at the origin.

It will work if the origin is removed and we switch to charts of S^1 , but it still remains very common to see it used even outside of this case. Just be aware that due to this, it is possible that some theorem relying on the chart being a homeomorphism may fail when using such charts, such as for instance the determinant of the metric tensor being zero at the origin of the plane :

$$\delta = dr^2 + r^2 d\theta \rightarrow \det(\delta) = r^2 \quad (16.6)$$

In general in n dimensions, we can see *hyperspherical coordinates*, which are given by the coordinates $(r, \varphi_1, \varphi_2, \dots, \varphi_{n-1})$

$$x_1 = r \cos(\varphi_1) \quad (16.7)$$

$$x_2 = r \sin(\varphi_1) \cos(\varphi_2) \quad (16.8)$$

$$x_3 = r \sin(\varphi_1) \sin(\varphi_2) \cos(\varphi_3) \quad (16.9)$$

$$\dots \quad (16.10)$$

$$x_{n-1} = r \sin(\varphi_1) \sin(\varphi_2) \sin(\varphi_3) \dots \sin(\varphi_{n-2}) \cos(\varphi_{n-1}) \quad (16.11)$$

$$x_n = r \sin(\varphi_1) \sin(\varphi_1) \sin(\varphi_2) \dots \sin(\varphi_{n-2}) \sin(\varphi_{n-1}) \quad (16.12)$$

Metrics that have some parts of them expressed in hyperspherical coordinates will use the hyperspherical volume element,

$$d\Omega^{n-1} = \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \dots \sin(\varphi_{n-2}) d\varphi_1 d\varphi_2 \dots d\varphi_{n-1} \quad (16.13)$$

One important property of hyperspherical coordinates that we will often use is that those coordinates obey the hypersphere's equation, $\sum x_i^2 = r^2$:

$$\sum_{i=1}^n x_i^2 = r^2 \left[\sum_{i=1}^{n-1} \cos^2(\varphi_i) \prod_{j=1}^{i-1} \sin^2(\varphi_j) \right] + r^2 \prod_{j=1}^n \sin^2(\varphi_j) \quad (16.14)$$

This can be shown by induction to be true, by first considering the 2-dimensional case where it is simply

$$x_1^2 + x_2^2 = r^2 (\cos^2(\varphi_1) + \sin^2(\varphi_1)) = r^2 \quad (16.15)$$

and then by assuming it to be true for n dimensions and checking it for $n + 1$:

$$\sum_{i=1}^{n+1} x_i^2 = r^2 \left[\sum_{i=1}^n \cos(\varphi_i) \prod_{j=1}^{i-1} \sin(\varphi_j) \right] + r^2 \prod_{j=1}^{n+1} \sin(\varphi_j) \quad (16.16)$$

$$= r^2 \left[\sum_{i=1}^{n-1} \cos(\varphi_i) \prod_{j=1}^{i-1} \sin(\varphi_j) \right] \quad (16.17)$$

$$+ r^2 \left[\cos(\varphi_n) \prod_{j=1}^{n-1} \sin(\varphi_j) \right] + r^2 \sin(\varphi_n) \prod_{j=1}^n \sin(\varphi_j) \quad (16.18)$$

$$(16.19)$$

[CHECK]

16.2 Smooth structures

In addition to their atlases, the manifolds we will be interested will have a more specific atlas of charts with smooth transition maps :

Definition 16.5. A *smooth manifold* is a manifold with a smooth atlas, ie a subset $\mathcal{A}_S$ of the atlas $\mathcal{A}$ that itself forms an atlas such that every transition map within that atlas is smooth.

The smooth atlas $\mathcal{A}_S$ is called the *smooth structure* of the manifold.

This choice of a subset is not unique, as it is possible to find two such subsets such that each subset is maximal in the sense that there are no more compatible charts that could be added : if we tried to add another chart from one atlas to the second, some of their transition maps with the second atlas would not be smooth.

In general there exists a smooth structure that seems more "natural" than others for a given manifold, and this is referred to as the canonical smooth structure. If a manifold has multiple smooth structures, the smooth structures that differ from the canonical smooth structure are referred to as *exotic smooth structures*.

While exotic smooth structures do exist, their construction tends to be extremely difficult, so that we will not give examples here, but as they still have some consequences in general relativity, a discussion on them has been deferred to chapter X.

We will not consider any non-smoothable C^0 manifold throughout this book. As the physics involved with spacetime relies quite heavily on the use of derivatives, which are defined with the use of the derivability of the charts, C^0 manifold will not allow us to

model such things. They are furthermore fairly difficult to even build explicitly and rare to encounter in general.

A few example of manifolds

The canonical example of a manifold is that of Euclidian space $\mathbb{R}^n$ itself, which can be given a manifold structure by the single chart $(\mathbb{R}^n, \text{Id}_{\mathbb{R}^n})$, with the map

$$\text{Id}_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \text{Id}_{\mathbb{R}^n} \quad (16.20)$$

$$x \mapsto x \quad (16.21)$$

The identity map is indeed bijective, continuous and smooth, so that this defines an appropriate C^∞ -manifold.

Sphere is a manifold

16.3 Manifolds as diffeological spaces

Theorem 16.6. Manifolds are diffeological spaces.

Due to their very similar definitions, we have the convenient property that many properties of manifolds will also hold for diffeological spaces. A property of a manifold will in general hold for a diffeological space if it does not depend on the cocycle condition or the fixed dimension of its charts.

16.4 Equivalence of manifolds

The atlas poses the problem that the same manifold could be covered by two different atlases, even $\mathbb{R}^n$. To remedy this, we will need some way of comparing manifolds with different atlases. There are two ways to do this. The first one is to define an equivalence relation

Definition 16.7. Two C^k atlases on the same set are C^k -equivalent if their union is also a C^k -atlas.

For instance, if we consider the two atlases $(\mathbb{R}^n, \phi_1)$ and $(\mathbb{R}, \phi^2)$, those are equivalent if $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ are both C^k on $\mathbb{R}^n$. We can pick such an example as simply $\phi_1 = \text{Id}$ and $\phi_2 = \text{Id} + a$ which translates every point, in which case the transition functions are $\text{Id} + a$ and $\text{Id} - a$, which are both smooth functions.

The union of two atlases may be of different regularity C^{k_1} , C^{k_2} , in which case the resulting atlas will be at best of regularity $C^{\min(k_1, k_2)}$

Example : identity chart versus sgn chart?

The other method is to equip every manifold with a complete atlas.

Definition 16.8. A C^k atlas is complete if every possible C^k chart that intersects it with a C^k transition function is included in it.

The equivalence between the two is simply that if we consider the equivalence class of every C^k -equivalent atlas on a manifold, the complete atlas is the union of all those atlases. A non-complete C^k atlas will then just be a representative of a complete atlas. We will usually just refer to a representative of the atlas for simplicity, as these are never finite for any manifold of dimension > 0 .

Definition 16.9. A C^k manifold is a topological space associated with a C^k complete atlas.

A C^0 manifold is also called a topological manifold (so called because it only has a topological structure and no differential structure). A C^∞ manifold is also called a smooth manifold.

Generally speaking, we will not look at any manifolds other than C^0 (although very rarely) and C^∞ . This is due to the smoothability of C^k -manifolds [30] :

Theorem 16.10. For a C^k -manifold M with a C^k atlas $\mathcal{A}$, $k \geq 1$, there exists for every $k < k' \leq \infty$ a compatible $C^{k'}$ atlas $\mathcal{A}'$, $\mathcal{A}' \subset \mathcal{A}$, unique up to $C^{k'}$ diffeomorphism.

16.5 Orientation

16.6 Topological properties

Being locally Euclidian will generally restrict the topological properties of manifolds quite drastically, at least for the local properties, as many of those properties will be preserved under homeomorphisms. A property will be called local if it is defined using either a neighbourhood or neighbourhood basis around every point. A list of common properties of subsets of $\mathbb{R}^n$ which transfer over as local properties of manifolds is that all manifolds are locally connected (as we can always define an atlas of connected open sets), locally metrizable (from the canonical metric of $\mathbb{R}^n$) and locally normal.

All manifolds are *first-countable*. First countability is defined as follow

Definition 16.11. A space X is first countable if every $p \in X$ has a countable number of neighbourhoods $\{N_i\}$ of p such that for any neighbourhood $N \ni p$, there exists an $i \in \mathbb{N}$ such that $N_i \subset N$

s this property is inherited directly from $\mathbb{R}^n$.

In addition, we say that a manifold is *second-countable* if it admits a countable basis, that is

Definition 16.12. A manifold is second-countable if there exists a countable set of open sets $\{\mathcal{B}_i\}$ (the basis) such that every open set U is the union of a subset of the basis.

Manifolds are not guaranteed to be second countable, second countability being a global property, and there are indeed examples of non-second countable manifolds such as the long line $\mathbb{L}$ or the Prüfer manifold. We will use the example of the long line here, which is the simplest such manifold that can be constructed, and is fairly representative of what happens when a manifold is not second countable.

Before we construct the long line, let's first consider this construction of the real line instead : take the half interval $[0, 1)$, and consider the disjoint union of this half-interval for every integer $n \in \mathbb{N}$. Every member of this disjoint union can therefore be written as the tuple of an integer and an element $x \in [0, 1)$, (n, x) . There is an easy enough bijection between this and the positive real numbers as

$$(n, x) \mapsto y = n + x \quad (16.22)$$

$$y \mapsto ([y], y - [y]) \quad (16.23)$$

If we furthermore give it the lexicographical order

Definition 16.13. The *long line* $\mathbb{L}$ is a manifold constructed by considering a set of copies of the half-open interval $[0, 1)$

The analytical properties of the long line are hard to prove without spending quite a bit of time going into various theorems concerning [cardinals?], but we can give a few theorems about it :

- Every continuous function on $\mathbb{L}$ is [...]

All of these theorems make it fairly unfit to perform anything resembling physics on it. The long line is fairly representative of the behaviour of non-second countable manifolds in general [31]. They cannot be given a metric, [etc]. Try as one might, there are no use one can find for non-second countable manifolds in the context of general relativity, so that we can simply forget about them for the rest of this book.

As far as separation properties go, every manifold is T_1

proof

Manifolds are not guaranteed to be Hausdorff on the other hand, and it is in fact possible to define a wide variety of non-Hausdorff manifolds. We will generally consider manifolds to be Hausdorff unless otherwise stated, although we will also look in 76 at what non-Hausdorff spacetimes look like.

Paracompact

Manifolds are also not guaranteed to be metrizable, but by Smirnov's metrization theorem, any topological space that is Hausdorff, locally metrizable and paracompact is metrizable. Therefore if our manifold is both paracompact and Hausdorff, as most of the manifolds we will consider are, it is also metrizable.

16.7 Dimensions

With this link between manifolds and $\mathbb{R}^n$, we can define the dimensionality of a manifold by referring to the dimensionality of the domain of its charts.

Definition 16.14. A manifold is said to have *dimension* n , noted $\dim M = n$, if any of its chart is to a subset of $\mathbb{R}^n$.

Since homeomorphisms between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ can only occur if $n = m$ (outside of the empty set), this means that a connected manifold of dimension n has all of its charts to subsets of $\mathbb{R}^n$. This can be seen by considering any two charts on a connected manifold

[Take the nerve complex?]

In other words, any manifold that isn't the empty set will only have a single dimension n . The empty set, which has the complete atlas $\{\emptyset, \emptyset\}$ will be a manifold of every dimension $n \in \mathbb{N}$.

In particular, this implies

Proposition 16.15. A non-empty manifold M cannot be homeomorphic to a manifold N unless $\dim M = \dim N$.

16.7.1 Manifolds of dimension 0

The set $\mathbb{R}^0$ is simply the singleton $\{0\}$. There is then obviously only one connected zero-dimensional manifold (outside of the empty set), up to homeomorphism, defined by the chart

$$(U, \phi) = (\{*\}, \langle *, 0 \rangle) \quad (16.24)$$

The unique chart is simply mapping the unique point $*$ to 0 itself : $\phi(*) = 0$. Its diffeomorphism group is simply the identity $\text{Id}(0) = 0$, as the only possible automorphism on a singleton. Any other 0-manifold with point $*$ ' (if we decide for some reason that there can be more than one different "type" of point) is then simply homeomorphic to this via the explicit homeomorphism $f(*) = *'$, $f^{-1}(*) = *$.

If we drop the requirement that the manifold be connected, any disconnected 0-manifold will then just be a set of such charts, classified only by their cardinality.

16.7.2 Manifolds of dimension 1

Two simple examples of manifolds of dimension 1 are the line, which has the canonical chart $(\mathbb{R}, \text{Id}_{\mathbb{R}})$, and the circle, which has for instance the following two charts covering the manifold : $((-1, 1), \phi_1)$ and $((-1, 1), \phi_2)$, where the intervals $(-1, 0)$ for each neighbourhood overlap and so do the $(0, 1)$ interval, with transition charts

$$\tau_{12}(x) = \begin{cases} -1 + x & \text{for } x \in (-1, 0) \\ 1 - x & \text{for } x \in (0, 1) \end{cases} \quad (16.25)$$

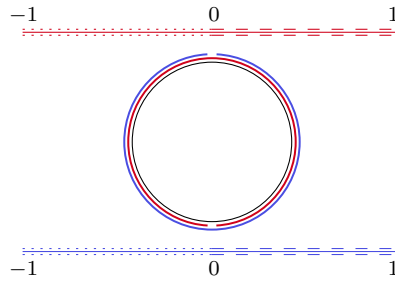


Figure 16.1: Charts on a circle

A common way of defining the circle as a manifold is to consider it as an embedded manifold in $\mathbb{R}^2$, via

$$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} \quad (16.26)$$

with two horizontal lines intersecting it as the north and south pole,

$$L_{\pm} = \{(x, y) \in \mathbb{R}^2 | x = \pm 1\} \quad (16.27)$$

in which case they will correspond to the two charts we saw, via the *stereographic projection*

$$(\pm 1, y) \mapsto (,) \quad (16.28)$$

Proposition 16.16. There are only two Hausdorff, second-countable, connected manifolds of dimension one.

Proof. □

16.7.3 Manifolds of higher dimensions

Dimension 0 and 1 will be the only trivial cases for the classification of manifolds (although if the manifold isn't assumed paracompact, there are 4 1-dimensional manifolds, and if it is not assumed Hausdorff, there is an uncountable number of them). More details on the classification of manifolds of higher dimensions will be present later in the book.

Beyond dimension 1, manifolds become much more difficult to classify. It is still possible to classify manifolds in dimension 2 [32]. In particular, for compact connected manifolds, we have that every 2-manifold is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes, being classified by their orientability $o \in \mathbb{Z}_2$ and genus $g \in \mathbb{Z}$.

The classification of manifolds of dimension 3 is more complex but remain possible. As every 3-manifold is triangulizable [33], they can be classified by their combinatorial type

Impossibility of classification for $n > 4$

16.8 More general manifolds

Definition 16.17. A *manifold with boundaries* is a topological space with an atlas $\phi_\alpha : U_\alpha \rightarrow \mathcal{O}_\alpha \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^{(n-1)}$, the upper half space.

Unlike the case of manifolds, a manifold with boundaries can contain boundary points, as the name implies.

Theorem 16.18. Any point $p \in M$ mapping to the boundary of the upper half space, $\phi(p) \in \{0\} \times \mathbb{R}^{(n-1)}$, is a boundary point of the manifold, $p \in \partial M$.

Since the interior of the half space is itself diffeomorphic to $\mathbb{R}^n$ simply by mapping $\mathbb{R}_{>0}$ to $\mathbb{R}$ via some map such as $\tan$ [something], every manifold can be given the structure of a manifold with boundaries via the composition

Theorem 16.19. The boundary ∂M of a manifold with boundary M is a manifold of dimension $\dim(M) - 1$.

Definition 16.20. A *manifold with corners* is a topological space with an atlas $\phi_\alpha : U_\alpha \rightarrow \mathcal{O}_\alpha \subset (\mathbb{R}_{\geq 0})^n$, the positive orthant.

Definition 16.21. An *orbifold*

conifolds

16.9 Manifold construction

We will need to construct quite a few manifolds for our purposes throughout this book, so that some techniques for constructing them can be useful.

It should first be noted that there is not in general any way to generate all possible manifolds, even theoretically. The classification problem of manifolds of dimension ≥ 4 can be shown to be impossible to solve from computability arguments.

16.9.1 Product manifolds

Given two manifolds M, N , with their own topologies, atlas and differential structure, there exists a product manifold $M \times N$, which is set-wise the Cartesian product of the two

Theorem 16.22. If M and N are manifolds of dimension m and n , there is a manifold $M \times N$ of dimension $n + m$.

Proof. The charts

$$\phi_M \times \phi_N : U_M \times U_N \rightarrow O_M \times O_N \subseteq \mathbb{R}^{n+m} \quad (16.29)$$

$$(p, q) \mapsto (\phi_M(p), \phi_N(q)) \quad (16.30)$$

maps points of the resulting manifold to open sets (with the product topology) of $\mathbb{R}^{n+m}$. □

A manifold $M = N^k$ will correspond to the k -fold product manifold $N \times N \times \dots \times N$, of dimension $k \dim N$.

16.9.2 Submanifold

Inclusion map $\iota : S \hookrightarrow M$

The easiest case of a submanifold to consider is to simply consider an open set $U \subseteq M$ and give it the manifold structure constructed by either the same coordinate charts if $U_\alpha \cap U = \emptyset$, or the coordinate charts $(U_\alpha \cap U, \phi|_U)$ otherwise.

16.9.3 Quotient manifolds

If we have some equivalence relation $\sim$ on a manifold, the quotient space $M/\sim$, as a topological space, does exist, but it is not guaranteed to be a manifold.

A basic counterexample to this is to consider the equivalence relation given by the retraction of the plane $\mathbb{R}^2$ on the cross $(x=0) \cup (y=0)$, for instance by mapping every point for which $y \neq 0$ to the y axis : $\forall y \neq 0, (x, y) \mapsto (x, 0)$, and the identity map otherwise. The quotient topology define is that any open set on $\mathbb{R}^2$ not overlapping the center $(0, 0)$ corresponds to an open set on the two axis made of unions of intervals on them, while the center has open sets over it which are never homeomorphic to any subset of $\mathbb{R}$

Theorem 16.23. The cross is never homeomorphic to a Euclidian space.

Proof. If an open set U at the center of the cross is homeomorphic to a subset $\mathcal{O} \subseteq \mathbb{R}^2$, there is a homeomorphism $f(U) = \mathcal{O}$. As we've seen, we can always find a restriction such that $\mathcal{O}$ is an open ball. If we remove the central point, the restriction $f(U \setminus (0, 0))$ will also be a homeomorphism. however, $U \setminus (0, 0)$ has 4 connected components, but the map to $\mathcal{O}$ minus a point will give either two connected component (in 1 dimension) or only one. As homeomorphisms preserve the number of connected components, there is no such diffeomorphism/. $\square$

We therefore need to add some specific conditions if we wish for our quotient manifold to be a manifold.

Theorem 16.24. The quotient space of a manifold is a manifold if the graph of the equivalence relation is a submanifold of $M \times M$ and the projection $\text{pr}_1 : E \rightarrow M$ is a submersion

quotient by a Lie group action

examples : toruses, contractions, quotients by involutions

Example 16.25. The torus T^n is homeomorphic to the quotient of the Euclidian space $\mathbb{R}^n$ by the group action of the translation

$$(x, y, z, \dots) \rightarrow (x + a, y + b, z + c, \dots) \quad (16.31)$$

for some $(a, b, c, \dots) \in \mathbb{R}^n$

Example 16.26. The projective space $P\mathbb{R}^n$ is the quotient of the n -sphere S^2 by the antipodal group action

Proof.

$\square$

16.9.4 Covers

Definition 16.27. A *covering map* $\phi : \tilde{M} \rightarrow M$ is a smooth map such that for every point $p \in M$, there exists a connected neighbourhood U_p that is evenly covered by ϕ : all connected components of $\phi^{-1}(U_p)$ are mapped onto U_p via a diffeomorphism.

The identity is always a covering map

Theorem 16.28. For a connected manifold M , the cardinality of $\phi^{-1}(p)$ (which is countable) is the same for all points.

Proof. O'Neill □

Multiplicity of the covering

Covering of curves

Theorem 16.29. Every connected manifold M has a covering map from a simply connected manifold.

Definition 16.30. A *cover* of a manifold M

16.9.5 Manifold surgery

Surgery is the process of building new manifolds by stitching together existing manifolds. This covers a few such operations, most of them related to the concept of gluing.

[34]

Definition 16.31. Given a manifold (not necessarily connected) with boundaries M , with the boundary containing two homeomorphic elements, $\partial S_1, \partial S_2$, with a homeomorphism $h : \partial S_1 \rightarrow \partial S_2$, the *gluing* of those boundaries is the space obtained by taking the equivalence relation $\sim$ defined by :

- If p does not belong to either boundary, then $p \sim p$ and no other point.
- If $p \in \partial S_1$, $p \sim h(p)$
- If $p \in \partial S_2$, $p \sim h^{-1}(p)$

and then taking the quotient space $M / \sim$.

Theorem 16.32. The gluing of such a manifold with boundary is a manifold with boundary, or if those two elements were the only elements of its boundary, a manifold, of regularity as high as the smallest regularity of M and h .

Proof. The proof of the existence of tubular neighbourhood is delayed til chap. X. $\square$

One type of manifold surgery is the *connected sum*, where we get consider some original manifolds (possibly with boundaries, corners, etc) M_1 and M_2 , and two maps from the open n -disk to each of those manifolds

$$\iota_i : D \rightarrow M_i \quad (16.32)$$

Given two manifolds M_1, M_2 , remove an open disk D_1, D_2 from each manifold, giving them two boundaries $\partial D_1, \partial D_2$ with the $S^{(n-1)}$ topology.

Palais theorem

Definition 16.33. A *prime manifold* is a manifold that cannot be constructed as a non-trivial connected sum of two other manifolds.

A more general case is the cut and paste procedure, where we do not have a unique subset to remove, a unique homeomorphism or the requirement of having two different manifolds.

Definition 16.34. A *cut and paste manifold* is a manifold with boundaries such that, for two connected components of this boundary, ∂S_1 and ∂S_2 , there exists a homeomorphism $h : \partial S_1 \rightarrow \partial S_2$

Theorem 16.35. The cut and paste of a manifold is a manifold

Proof. $\square$

16.10 Complex manifolds

We will use the idea of a complex manifold much less commonly in this book, but as it will come up, we have to define it

Definition 16.36. A *complex manifold* is a topological space (M, τ) such that for any point $p \in M$, there exists a neighbourhood U that is homeomorphic to $\mathbb{C}^n$, and for any two open sets in the atlas U_α, U_β with some overlap $U_{\alpha\beta}$, the transition function

$$\tau_{\alpha\beta} = \phi_\beta|_{U_{\alpha\beta}} \circ \phi_\alpha^{-1}|_{\phi_\alpha^{-1}(U_{\alpha\beta})} \quad (16.33)$$

is holomorphic.

Example 16.37. The complex space $\mathbb{C}^n$ is a complex manifold

Proof. Much like the case of $\mathbb{R}^n$, this is trivially true by considering the atlas of one patch $(\mathbb{C}^n, \text{Id}_{\mathbb{C}^n})$, whose only transition function with itself is the identity function, which is holomorphic. $\square$

Example 16.38. The Riemann sphere $\hat{\mathbb{C}}$ composed of the compactification of the complex plane by the addition of a point at infinity, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, is a complex manifold.

Proof. As the Riemann sphere is topologically simply a sphere, the same method can be used to show it to be a complex manifold, where we consider the two patches of $(\mathbb{C}^n, \text{Id}_{\mathbb{C}^n})$, and the patch of the "other pole" of the sphere, consisting of every point of the Riemann sphere except for 0, $(\hat{\mathbb{C}}, \phi)$, with the map

$$\phi(z) = z^{-1} \quad (16.34)$$

which maps ∞ to 0 and every other point to their inverse. The transition functions of those two charts is then simply defined on $\mathbb{C} \setminus \{0\}$, via

$$\tau_1 = \text{Id}_{\mathbb{C} \setminus \{0\}} \circ z|_{\mathbb{C} \setminus \{0\}} \quad (16.35)$$

$$\tau_2 = z^{-1}|_{\mathbb{C} \setminus \{0\}} \circ \text{Id}_{\mathbb{C} \setminus \{0\}} \quad (16.36)$$

As both z and z^{-1} are holomorphic on $\mathbb{C} \setminus \{0\}$, it is a complex manifold. $\square$

Example 16.39. The complex projective space $\mathbb{C}P^n$ is a complex manifold.

Chapter 17

Maps between manifolds

The morphisms in our categories of C^k -manifolds are the C^k -maps between manifolds

$$f : M \rightarrow N \tag{17.1}$$

The set of C^k functions between two manifolds will be written as $C^k(M, N)$, or $C^\infty(M, N)$ if smooth. If the target manifold is the real line $\mathbb{R}$, we will simply write it as $C^k(M, \mathbb{R}) = C^k(M)$, the space of real functions on M , also called the space of scalar functions on M .

The function space $C^\infty(M)$ has the structure of a vector space, given by the sum of its values :

$$(f + g)(p) = f(p) + g(p) \tag{17.2}$$

17.1 Pullback

Given any two morphisms $f : M \rightarrow N$ and $g : N \rightarrow P$, we have the composition of those two maps given by $g \circ f : M \rightarrow P$. This is also called the *pullback* of a map.

In the case of manifolds,

17.2 Topology

[35]

There are two main topologies that are commonly used for the space of maps between

two manifolds, the Whitney topology, and the compact-open topology. As the Whitney topology relies on the definition of jets, we will defer its definition until X.

Definition 17.1. The *compact-open topology* of the function space $C(X, Y)$, for two topological spaces X, Y

For a compact set $K \subset X$ and an open set $U \subset Y$, $V(K, U)$ is the set of functions $f \in C(X, Y)$ such that $f(K) \subset U$.

$V(K, U)$ forms a subbase of the compact-open topology.

Metric spaces

Calculus on function spaces

Gâteaux derivative

Definition 17.2. A *Gâteaux derivative* on a Banach space is the derivative defined by

$$\frac{\delta F}{\delta_h \phi} = \lim_{\varepsilon \rightarrow 0} \frac{F(\phi + \varepsilon h) - F(\phi)}{\varepsilon} \quad (17.3)$$

This v. Frechet derivatives

17.3 Smooth functions

Continuous functions between manifolds are defined the same way as for any topological space. A function $f : M \rightarrow X$ to any topological space is continuous if $f^{-1}(U)$ is an open set.

C^k functions, C^∞ functions

Space of smooth functions $C^\infty(M)$

The space of smooth functions $C^\infty(M)$ forms a vector space in the usual way for a function space to a vector space, via the identification

$$(af + bg)(p) = af(p) + bg(p) \quad (17.4)$$

Frechet derivatives

Frechet spaces

Banach space

17.4 Manifolds as locally ringed spaces

Another description of manifolds is that of *locally ringed spaces*, which can be useful in some context where we want to consider the behaviour of functions on our manifolds.

Definition 17.3. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ on X , its *structure sheaf*.

In our case, the topological space will be the manifold itself, while the sheaf of rings will be formed by C^k functions on open neighbourhoods of the manifold.

[They do form a sheaf]

Theorem 17.4. There is an isomorphism between manifolds as topological spaces with a Euclidian atlas and manifolds as a locally ringed space.

Proof.

□

17.5 Isomorphisms of manifolds

The isomorphisms between two manifolds are composed of morphisms which admit an inverse morphism, so that for some isomorphism $f : M \rightarrow N$, there exists another isomorphism $f^{-1} : N \rightarrow M$ such that their composition is the identity morphism.

$$f \circ f^{-1} = \text{Id}_N \quad (17.5)$$

$$f^{-1} \circ f = \text{Id}_M \quad (17.6)$$

In the category of topological manifolds, such isomorphisms are *homeomorphisms*, noted by $\text{Homeo}(M, N)$

If the isomorphism is furthermore an endomorphism, that is a morphism from an object to itself $f : M \rightarrow M$,

some specific subsets of such maps are the *homeomorphism group* $\text{Homeo}(M)$ and the *diffeomorphism group*, $\text{Diff}(M)$, both being the automorphism groups of respectively topological manifolds and smooth manifolds. We can also consider the group of C^k homeomorphisms in general, but due to the smoothability argument we saw, there is less focus on this.

The diffeomorphism group is composed of every endomorphisms of **SmoothMan** $f : M \rightarrow M$

[36]

As the name implies, the diffeomorphism group is a group under function composition, as the composition of two diffeomorphism is itself a diffeomorphism, and diffeomorphism are by definition invertible. If we take the identity map Id_M as the identity, then we have all the structure of a group.

The diffeomorphism group is typically infinite dimensional for any manifold of dimension > 0 , except for very pathological cases [31]. A simple way to see this is that if manifolds are Hausdorff, they are *homogeneous*, which is to say that for any two points $p, q \in M$, there exists a diffeomorphism carrying p to q . For a given p this means that there exists at least one different diffeomorphism for every point of the manifold, which will be of cardinality $2^{\aleph_0}$. [show that there is at least one diffeomorphism per polynomial or something]

Due to this, it cannot be a manifold in the typical sense of the word, although it can be a generalization of a manifold such as a Frechet manifold or ILH manifold. It is also a diffeological space.

As an infinite dimensional manifold, we can apply plots for it of arbitrarily high dimensions. Most commonly used in there are the 1-parameter subgroups, which are roughly curves in the diffeomorphism group :

$$f_t : \mathbb{R} \rightarrow \text{Diff}(M) \quad (17.7)$$

Support of diffeomorphisms

Topology (compact open, Whitney?)

The diffeomorphism group is in general not a connected space

Compact diffeomorphism

Show that $\text{Diff}_c^k(M)$ is a normal subgroup [36]

Diffeomorphism leaving a point invariant

Mapping class group

One-parameter families of diffeomorphism lead to the possibility of a derivative

$$\partial_t \phi_t \quad (17.8)$$

Isotopic diffeomorphisms

Diffeomorphism from vector fields or not

oriented or not

One-parameter families of diffeomorphisms define curves

17.6 Pseudogroups

Just like we can consider the atlas $\mathcal{A}$ on M with the isomorphism group $\text{Homeo}(\mathbb{R}^n)$, and the atlas $\mathcal{A}_S$ with the isomorphism group $\text{Diff}(\mathbb{R}^n)$, we can generalize this notion to more restricted groups of transformation. This is done using the notion of a pseudogroup :

Definition 17.5. A *pseudogroup* Γ on a topological space X is a set of homeomorphisms, Γ such that

- The domains of those homeomorphisms cover $X : \bigcup_{f \in \Gamma} \text{Dom}(f) = X$
- The restriction of $f : U \rightarrow V$ to an open set $U' \subseteq U$ is also in Γ
- If two functions $f, g \in \Gamma$ are such that $\text{Im}(f) = \text{Dom}(g)$, then the composition $g \circ f \in \Gamma$.
-

From this definition, trivially the diffeomorphisms and homeomorphisms of the (smooth) atlas of a manifold form a pseudogroup

17.7 Bump functions and partition of unity

As smooth functions on a manifold have the structure of a sheaf,

A bump function is a function on a manifold that vanishes outside of a compact region, which is useful to generalize local theorems to global ones. Definitions of bump functions may vary a bit, but we will use the most common one for working with manifolds :

Definition 17.6. A *bump function* at p is a smooth function of compact support, that is, for a given compact set $\text{supp}(f)$, all points $q \notin \text{supp}(f)$ are such that $f(q) = 0$, such that there exists a neighbourhood U of p where $f(U) = 1$.

We can construct a bump function on $\mathbb{R}^n$ [37], starting with the function on $\mathbb{R}$:

$$f(x) = \begin{cases} \exp(-\frac{1}{x}) & x < 1 \\ 0 & x \geq 1 \end{cases} \quad (17.9)$$

This function is non-negative, smooth and positive for $x > 0$. Then we define

$$g(x) = \frac{f(x)}{f(x) + f(1-x)} \quad (17.10)$$

which is non-negative, smooth and has value 1 for $x \geq 1$, and 0 for $t \leq 0$. We then obtain a bump function by mirroring it

$$h(x) = g(2+x)g(2-x) \quad (17.11)$$

The bump function $\Pi(x^\mu)$ on $\mathbb{R}^n$ will then simply be a product of those functions.

$$\Pi(x^\mu) = h(x_1)h(x_2) \dots h(x_n) \quad (17.12)$$

More generally, we can define a bump function at any point y^μ with compact support on the ball $B_{y,R}$ by translating and rescaling this function.

$$\Pi_{y,R}(x^\mu) = \Pi\left(\frac{2\|x-y\|}{R}\right) \quad (17.13)$$

This can be generalized to arbitrary (Hausdorff) manifolds by the following theorem

Theorem 17.7. For a Hausdorff differential manifold, for every point p and every open set $U \ni p$, there's a bump function f with support $\text{supp}(f) \subset U$ that is equal to 1 in an open neighbourhood V of p , $V \subset U$.

Proof. For an atlas $\{(U_\alpha, \phi_\alpha)\}$ on the manifold, with an index $\beta \in I$ such that $q \in U_\beta$, consider the neighbourhood $O' = \phi_\alpha(U \cap U_\beta)$, with the coordinates $y = \phi_\beta(p)$. For $R, \epsilon > 0$, consider the open ball $B_{y,R+\epsilon}$ of center y and radius $R+\epsilon$ in O' . This is always possible for a small enough radius since O' is open. We will also need the open balls $B_{y,R/2}$, $B_{y,R}$ and the closed ball $\bar{B}_{y,R}$. By Heine-Borel, $\bar{B}_{y,R}$ is compact.

We then have in M the sets

$$\begin{aligned} V &= \phi_\beta^{-1}(B_{y,R/2}) \\ W &= \phi_\beta^{-1}(B_{y,R}) \\ K &= \phi_\beta^{-1}(\bar{B}_{y,R}) = \bar{W} \end{aligned}$$

The bump function will then be defined by

$$f(p) = \begin{cases} \Pi_{y,R}(\phi_\beta(p)) & p \in U_\beta \\ 0 & p \notin U_\beta \end{cases} \quad (17.14)$$

Then we have that $f = 1$ in V , which is an open set of M . By the Hausdorff condition, if K is closed in U_β , it is also closed in M , so that $\text{supp } f = K \subset U_\beta$. W is open in U_β , hence $W \cap U_\alpha$ is open for any α .

[blablabla]

Hence $f(p)$ is smooth on M .

□

We can also use this theorem in a different way, by considering some smooth function in a neighbourhood V

In addition to having for any neighbourhood U a bump function of compact support $\text{supp}(f) \subseteq U$, we can find a bump function for which the support of the value 1 is larger than U . [Lee?]

One useful thing that can be done using a partition of unity is to

One important tool for the study of manifolds, to help make local structures into global ones, is the partition of unity.

Definition 17.8. A partition of unity on a manifold $\mathcal{M}$ is a set of smooth functions $\{f_\alpha | \alpha \in A\}$, $f_\alpha : M \rightarrow [0, 1]$ such that

- for every point p , there is a neighbourhood of p that intersects only a finite number of $\text{supp}(f_\alpha)$
- $\sum_\alpha f_\alpha(p) = 1$

A particular example of a partition of unity is if we have a cover $\{U_\alpha\}$. If the partition of unity $\{f_\alpha\}$ is such that $\text{supp } f_\alpha \subset U_\alpha$, we say that the partition of unity is subordinate to this cover.

Theorem 17.9. For a Hausdorff, second-countable manifold, every open cover has a countable partition of unity subordinate to it.

Proof.

□

17.8 Submanifolds

Given an n -dimensional manifold and some manifold of dimension $k < n$, we wish to be able to talk about

Chapter 18

Categories

A common tool in modern mathematics is that of category theory, which is used to describe mathematical objects themselves. It is a useful tool to consider for the various layers of structure that one can apply to a theory, as well as comparing different types of mathematical objects that still share some common property.

The foundations of mathematical theory are always a bit difficult to work out since there isn't any way to define any theory as somewhat fundamental apart from all others. If we look at pillars of mathematics such as set theory, category theory, logic, model theory, etc, none of these theories really stand on their own. The axioms and rules of all such theories depend on model theory, their semantics often depend on logic, they typically involve objects which belong to "collections", which are typically sets or classes, and so on. This is a typical problem of metamathematics and there is not much one can do about it. When we talk about those metatheories, we need to use informal notions that are only later defined rigorously, but those notions will be defined on those foundations themselves, making them somewhat circular in the end.

I won't get too deeply into this issue as this is something one can easily get lost into, but despite category theory being our way of defining sets and functions here, category theory also has similar concepts, which are collections and functions defined on those collections. A collection is roughly the same as a class in the set theory sense, and functions on it are simply functions between classes. The exact type of classes we allow will change what type of category theory we are dealing with. To start with, we will be dealing only with categories where all the collections are classes, in the ZFC [Grothendieck universe?] sense of the term. [38]

With this out of the way, let's define what a category is :

Definition 18.1. A *category* $\mathbf{C}$ is a collection of *objects*, denoted by $\text{Obj}(\mathbf{C})$, and a collection of *morphisms*, denoted by $\text{Mor}(\mathbf{C})$ on which collections we have three functions, the *source* map s and target map t , both functions from morphisms to objects :

$$s, t : \text{Mor}(\mathbf{C}) \rightarrow \text{Obj}(\mathbf{C}) \quad (18.1)$$

and the composition map $\circ$ mapping two morphisms to another morphism

$$\circ : \text{Mor}(\mathbf{C}) \times \text{Mor}(\mathbf{C}) \rightarrow \text{Mor}(\mathbf{C}) \quad (18.2)$$

$$(f, g) \mapsto g \circ f \quad (18.3)$$

Categories are often represented graphically as graphs, where the vertices represent the objects and

The objects of a category are often what we consider to be mathematical objects

and *morphisms*, which are often (although not always) maps between those objects.

Each morphism $f \in \text{Mor}(\mathbf{C})$ has two attributes called its *source* s and its *target* t , which are both functions from $\text{Mor}(\mathbf{C})$ to $\text{Obj}(\mathbf{C})$. A morphism $f : A \rightarrow B$ has the source A and target $B : s(f) = A, t(f) = B$. If the morphism f is a function in the traditional sense of the word, the source and target will be its domain and codomain. The morphisms of a category obey the following properties :

1. For any two morphisms $f, g \in \text{Mor}(\mathbf{C})$, with $t(f) = s(g)$, there also exists a morphism $g \circ f : s(f) \rightarrow t(g)$, the *composition* of those morphisms.
2. Composition is associative :
3. For any object $A \in \text{Obj}(\mathbf{C})$, there exists an *identity morphism* Id_A , pointing to itself : $s(\text{Id}_A) = A, t(\text{Id}_A) = A$, that is a left and right unit : for any morphism $f : A \rightarrow B$,

$$\text{Id}_B \circ f = f = f \circ \text{Id}_A \quad (18.4)$$

Some example of categories that we will use throughout this chapter are **Set**, the category of sets, with sets as objects and functions as morphisms, **Vec**, the category of vector spaces, with vector spaces as objects and linear maps as morphisms, **Top**, the category of topological spaces, with topological spaces as objects and continuous functions as morphisms, or **Man**, the category of manifolds, with manifolds as objects and homeomorphisms as morphisms. The category of sets is in particular a good model to keep in mind, as quite a lot of concepts in category theory can be easily modeled on it, although it is to be remembered that not all categories can be understood in the sense of objects being some sort of sets and morphisms some sort of functions on those sets. Sets and their category (in particular the proofs between the usual set theory and their categorical description) are detailed in chapter 2.

Beyond categories that are simply sets and functions with extra structures, another common type of category is that of objects with some kind of preorder relation $\leq$. A preorder is defined by the properties that for any objects A, B, C , we have that $A \leq A$, and if $A \leq B$ and $B \leq C$, then $A \leq C$, which are exactly the properties that morphisms must obey. Therefore any set with a preorder, such as a partially ordered set, totally ordered set or directed set, defines its own category where the objects need not be sets and the

morphisms are not functions. Some basic example of these would be the total order of integers $\mathbb{Z}$ or real numbers $\mathbb{R}$, or the preorder of a collection of sets by inclusion.

[diagram of subset inclusion]

As our categories are secretly built from classes and functions, one thing we can look at are the subsets involved with our functions. In particular, if we look at the preimages of the source and target map by a single object X , $s^{-1}(\{X\})$ and $t^{-1}(\{X\})$, these are the set of all morphisms of source X

Definition 18.2. The *hom-set* $\text{Hom}_{\mathbf{C}}(X, Y)$ for two objects X, Y in a category $\mathbf{C}$ is the collection of all morphisms with source X and target Y .

Example 18.3. The hom-set between two sets A, B in **Set** is the set of all functions from A to B , the hom-set between two topological spaces X, Y in **Top** is the set of all continuous functions $\text{Hom}(X, Y) = C(X, Y)$, the hom-set between two vector spaces V, W in **Vec** is the set of all linear maps $L(V, W)$.

Example 18.4. The hom-set of a partial order is either empty (the two objects cannot be compared) or has one element (the source is smaller than the target).

Internal Hom

The *opposite category* $\mathbf{C}^{\text{op}}$ of a category $\mathbf{C}$ is a category sharing the same objects $\text{Obj}(\mathbf{C}) = \text{Obj}(\mathbf{C}^{\text{op}})$ but with the morphisms pointing in the reverse direction :

$$\forall f \in \text{Mor}(\mathbf{C}), \exists f^{\text{op}} \in \text{Mor}(\mathbf{C}^{\text{op}}), t(f) = s(f^{\text{op}}) \wedge s(f) = t(f^{\text{op}}) \quad (18.5)$$

Without getting into the more complex categories we have foreshadowed, we can give a few simple example of categories which can be entirely described in the basic formalism of category theory..

The *empty category* $\mathbf{E}$ is simply the category with no objects in it, and therefore no morphisms.

The *trivial category* $\mathbf{1}$ is the category of a single object, $\text{Obj}(\mathbf{1}) = \{\bullet\}$, and a single morphism $\text{Id}_{\bullet}$. Similarly, we can define the *discrete categories*, which are a collection of objects for which the only morphisms are the identity morphisms

$$\forall X \in \text{Obj}(\mathbf{C}), \text{Hom}(X, X) = \{\text{Id}_X\} \quad (18.6)$$

$$\forall X, Y \in \text{Obj}(\mathbf{C}), X \neq Y \rightarrow \text{Hom}(X, Y) = \emptyset \quad (18.7)$$

$$(18.8)$$

If the discrete category has a finite number of objects, it is written as $\mathbf{n}$, with n the cardinality of its objects. For instance the category $\mathbf{2}$ is defined by the diagram

[diagram]

Another common example of categories where we cannot interpret its objects as sets and its morphisms as functions is a poset category, which is interpreted as a partial order relation on objects.

Example 18.5. A *poset* P , composed belonging to the category of **Poset**, is a collection of objects with morphisms denoted by $\leq$ (a morphism between an object A and B is denoted by $A \leq B$), such that

Example 18.6. The usual sets of numbers such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ all form a total order category.

$\mathbb{Z}$ as a total order is simply a category composed of a countable infinite number of objects for which there are

Subcategories

18.1 Epimorphisms and monomorphisms

Among the various morphisms a category can possess, two important types of morphisms are the epimorphisms and the monomorphisms. They correspond to a generalization of the notion of injections and surjections for functions on sets.

Definition 18.7. An *epimorphism* $f \in \text{Mor}(\mathbf{C})$ is a morphism $f : X \rightarrow Y$ such that for any two morphisms $g_1, g_2 : Y \rightarrow Z$ that can be composed to the left of it, we have that

$$g_1 \circ f = g_2 \circ f \rightarrow g_1 = g_2 \quad (18.9)$$

Definition 18.8. A *monomorphism* $f \in \text{Mor}(\mathbf{C})$ is a morphism $X \rightarrow Y$ such that for any two morphisms $g_1, g_2 : Y \rightarrow Z$ that can be composed to the right of it, we have that

$$f \circ g_1 = f \circ g_2 \rightarrow g_1 = g_2 \quad (18.10)$$

Definition 18.9. An *isomorphism* is a morphism f for which there exist another morphism f^{-1} with the property that f^{-1} is a left and right inverse : $f \circ f^{-1} = \text{Id}$ and $f^{-1} \circ f = \text{Id}$.

A basic example of an isomorphism is the identity of an object, which is trivially its own inverse.

The point of monomorphisms and epimorphisms is that, while for sets, which are defined by their elements, injections and surjections are defined by the elements they map, for objects in a category, they are meant to be defined by how they interact with every other object. A surjective function has that its image is the entire codomain, while an epimorphism is such that

Definition using retracts (monomorphism, split monomorphism and regular monomorphism)

Unlike in the case of sets, it can be possible for a morphism to be an epimorphism and a monomorphism without being an isomorphism, although the converse is true :

Theorem 18.10. An isomorphism f is an epimorphism and a monomorphism.

Proof.

□

As we will prove properly in the category of Set, epimorphisms and monomorphisms indeed correspond to injections and surjections, and furthermore, this will also be true for any category that is "built from sets".

Theorem 18.11. If f is an epimorphism in $\mathbf{C}$, it is a monomorphism in $\mathbf{C}^{\text{op}}$ and vice-versa.

18.1.1 Subobjects

Just as injections can be seen as sending an object inside of another, we can also consider monomorphisms to be defined similarly. A monomorphism of sets is an injection, a monomorphism of vector spaces is a vector subspace, a monomorphism of topological spaces is a topological subspace, all kinds of objects we consider to be somewhat "inside" the other.

[Illustration]

When we consider such objects nestled inside other objects, there are two manners in which we can consider them : we can view it as the monomorphism $\iota : X \rightarrow Y$, or if we are working with sets, we can try to consider the image $\iota(X)$, which is actually inside the set Y itself. When we are working with categories however, we may not have such concepts available directly (there's no universal notion of elements inside the objects of a category), but we can find similar ones. The difference between the injection and its image in the case of sets is that different injections may have the same image. If we have two injections ι, ι' , we could have that those two will map different points of X to different points of Y , but the total image of those functions are still identical. The same idea may be true if we have two injections from different objects X_1, X_2 into Y .

Definition 18.12. A *subobject* S of a category $\mathbf{C}$ is an equivalence class of monomorphisms up to isomorphisms.

18.2 Higher categories

As category theory is the theory of mathematical structures, and as a mathematical structure itself, we can define a category for categories. This is the category $\mathbf{Cat}$, whose

objects are called 0-categories (which are what we called a category so far), with the morphisms between them being functor between categories.

[...]

We can unravel such structures so as to keep the various levels of analysis of our categories. By recursion, an n -category will therefore be a category whose morphisms are $(n - 1)$ -morphisms, morphisms between two $(n - 2)$ -morphisms. At its base, a 0-morphism is simply an object, a 1-morphism is a map between two objects, a 2-morphism is a map between two 1-morphisms (also called a *homotopy*), etc.

homotopies

18.3 Functors and natural transformations

A *functor* is a map between two categories. For two categories $\mathbf{C}$ and $\mathbf{D}$, given a functor $F : \mathbf{C} \rightarrow \mathbf{D}$, every object $X \in \mathbf{C}$ has an associated object in $\mathbf{D}$, $F(X) \in \text{Obj}(\mathbf{D})$, and every morphism $f : X \rightarrow Y$ in $\text{Mor}(\mathbf{C})$ has an associated morphism $F(f) : F(X) \rightarrow F(Y)$ in $\text{Mor}(\mathbf{D})$, in such a way that the structures of the categories are preserved, that is,

$$\forall X \in \text{Obj}(\mathbf{C}), F(\text{Id}_X) = \text{Id}_{F(X)} \quad (18.11)$$

$$\forall f, g \in \text{Mor}(\mathbf{C}), F(f \circ g) = F(f) \circ F(g) \quad (18.12)$$

For instance, if we pick a simple category $\mathbf{C}$ made of two objects,

$$\text{Id}_A \hookrightarrow A \xrightarrow{f} B \rightrightarrows \text{Id}_B$$

An example of a functor $F : \mathbf{C} \rightarrow \mathbf{C}'$ would be, acting on the morphism $f : A \rightarrow B$,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow F & & \downarrow F \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

just like any function, we are not restricted to mapping our objects and morphisms one to one, so that another functor we could conceive of would be something like

Example 18.13. The identity functor $\text{Id}_{\mathbf{C}}$ that maps every object and morphism of a category to themselves. Trivially every categorical structure is preserved.

Example 18.14. The *constant functor* $\Delta_Y : \mathbf{C}_1 \rightarrow \mathbf{C}_2$, for some object $Y \in \text{Obj}(\mathbf{C}_2)$, is the functor mapping every object $X \in \text{Obj}(\mathbf{C}_1)$ to Y , and every morphism $f \in \text{Mor}(\mathbf{C}_1)$ to the identity morphism of Y , Id_Y .

$$\Delta_Y : \mathbf{C}_1 \rightarrow \mathbf{C}_2 \quad (18.13)$$

$$X \mapsto Y \quad (18.14)$$

A generalization of functors is that given by the distinction between *covariant functors* and *contravariant functors*.

Given the various linear orders of numbers we defined, we can define a few functors representing maps between them if they either preserve order or reverse it.

Theorem 18.15. For any two posets P_1, P_2 , a map $f : P_1 \rightarrow P_2$ is a covariant functor if the map preserves order, $A_1, B_1 \in P_1, A_1 \leq B_1 \rightarrow f(A_1) \leq f(B_1)$, and a contravariant functor if it reverses them : $A_1, B_1 \in P_1, A_1 \leq B_1 \rightarrow f(B_1) \leq f(A_1)$

Proof. □

For instance, the inclusion between integers and real numbers

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{R} \quad (18.15)$$

This is a functor as

Translation on a linear order

As functors are fundamentally functions on the underlying set structure of the category, we can define the typical function properties for functor. The equivalent of an injection and a surjection are that of a faithful functor and a full functor. If we consider a functor $F : \mathbf{C} \rightarrow \mathbf{D}$, its action on the morphisms of $\mathbf{C}$ induces a mapping between the hom sets of any two objects in $\mathbf{C}$:

$$F_{X,Y} : \text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{D}}(F(X), F(Y)) \quad (18.16)$$

Definition 18.16. A *faithful functor* is a functor for which the maps $F_{X,Y}$ for every pair of object is injective on the hom sets.

In other words, if a functor is faithful,

[faithful if $t(f) = t(g)$, $s(f) = s(g)$ and $F(f) = F(g)$ implies $f = g$]

It remains possible that a faithful functor may not map objects injectively. For instance, while the forgetful functor from **Top** to **Set** that sends topological spaces to their underlying sets is faithful, two different sets may have a different topology defined on them.

Similarly, two morphisms in C may map to the same morphism in D . If two different objects in C are mapped to one object in D , their identity map will also do so for instance.

[Conservative faithful functor if it reflects isomorphisms]

Definition 18.17. A *full functor* is a functor for which the maps $F_{X,Y}$ for every pair of object is surjective on the hom sets.

If a functor is both full and faithful, it is called a *fully faithful functor*

Theorem 18.18. If $F : \mathbf{C} \rightarrow \mathbf{D}$ is a fully faithful functor, $\mathbf{C}$ is a full subcategory of $\mathbf{D}$.

Definition 18.19. A functor $f : \mathbf{C} \rightarrow \mathbf{C}$ from a category to itself is an *endofunctor*.

Example 18.20. The opposite functor maps a given category to its opposite category ${}^{\text{op}} : C \rightarrow C^{\text{op}}$, sending every objects to themselves and every morphism to themselves, but reversing the source and target of each morphism. As the identity morphisms shares their source and target, they remain unaffected by the functor, so that we only have to check the behaviour of arrows up to composition. If we have $f : A \rightarrow B$ and $g : B \rightarrow C$, we have a composed morphism $g \circ f : A \rightarrow C$. In the opposite category, we therefore have $f^{\text{op}} : B^{\text{op}} \rightarrow A^{\text{op}}$ and $g^{\text{op}} : C^{\text{op}} \rightarrow B^{\text{op}}$, with composition $f^{\text{op}} \circ g^{\text{op}} : C^{\text{op}} \rightarrow A^{\text{op}}$. The action of the functor will then be

$$(g \circ f)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}} \quad (18.17)$$

A very common type of functor, although generally not defined formally, is the *forgetful functor*, which maps categories to other categories by "forgetting" some of their properties. For instance if we have a group that can be represented by the set G with the group product $\cdot$, the forgetful from the category of groups to the category of sets will map the group $(G, \cdot)$ to the set G :

$$F : \mathbf{Grp} \rightarrow \mathbf{Set} \quad (18.18)$$

$$(G, \cdot) \mapsto G \quad (18.19)$$

And the group homomorphisms will be mapped to the appropriate set homomorphisms.

Definition 18.21. The Hom *functors* are two classes functors defined by partial application of the Hom function. For two objects $A, B \in \mathbf{C}$, we have the covariant Hom functor

$$\text{Hom}_C(A, -) : \mathbf{C} \rightarrow \mathbf{Set} \quad (18.20)$$

which sends objects of C to the set of functions from A to those objects and morphisms from $f : X \rightarrow Y$ to its composition $f \circ g$ by the morphisms $g \in \text{Hom}(A, X)$, $\text{Hom}_C(A, f) : \text{Hom}_C(A, X) \rightarrow \text{Hom}_C(A, Y)$, and the contravariant Hom functor

$$\text{Hom}_C(-, B) : \mathbf{C} \rightarrow \mathbf{Set} \quad (18.21)$$

Example 18.22. In a category which includes $\mathbb{R}$ as an object, the $\text{Hom}(-, \mathbb{R})$ functor will be the set of real-valued functions.

An intuitive way to consider the structures in our categories that can be found in most mathematics is to consider the objects in our categories as tuples. It is common for instance to consider the manifold in general relativity as a tuple of the form $(M, \tau, \mathcal{A}, g)$, with M the manifold as a set, τ its topology (as in the set of its open sets), $\mathcal{A}$ its smooth atlas and g its metric tensor. In this case, the forgetful functor will simply drop various terms of this tuple, and the morphisms between the objects will have to fulfill various conditions :

Natural transformation

A *natural transformation* η is a mapping between two functors $\eta : F \Rightarrow G$

$$\begin{array}{ccc} A & & F(A) \xrightarrow{\eta_A} G(A) \\ \downarrow f & \Longrightarrow & \downarrow F(f) \quad \downarrow G(f) \\ B & & F(B) \xrightarrow{\eta_B} G(B) \end{array}$$

There are two ways of composing natural transformations together,

Whiskering

Definition 18.23. Given a natural transformation $\eta : F \rightarrow G$ between two functors $F, G : C \rightarrow D$, and a third functor to another category $H : D \rightarrow E$, the whiskering of η by H is the natural transformation $H\eta : H \circ F \rightarrow H \circ G$ such that for every object $X \in C$, we have the relation

$$(H\eta)_X = H(\eta_X) \quad (18.22)$$

Cone and co-cone

Definition 18.24. For two functors $F, G : C \rightarrow D$, if F is a constant functor $F = \Delta_X$, $X \in C$, then a natural transformation $\eta : F \rightarrow G$ is a *cone* over G , while if G is a constant functor $G = \Delta_Y$, $Y \in C$, then $\eta : F \rightarrow G$ is a *cone* under G .

Show the diagrams

A very nice property of some categories for which the set of objects and objects correspond to a set and its elements is that functors will then simply be functions on that set. [Be more specific bc that's true of everything]

Theorem 18.25. For a category $\mathbf{C}$ for which

18.4 Comma categories

Categories have a distinction between the objects and the morphisms of a category, but it can be useful to look at a category where the objects themselves are the morphisms of another category.

A simple example of this is the arrow category

Definition 18.26. The *arrow category* $\text{Arr}(\mathbf{C})$ of a category $\mathbf{C}$ is a category for which every object corresponds to a morphism $f : A \rightarrow B$ in $\mathbf{C}$, and morphisms correspond to pairs of morphisms in $\mathbf{C}$ obeying

Slice category $\mathbf{C}/c$

Definition 18.27. For a category $\mathbf{C}$ and an object within that category $x \in \mathbf{C}$, the *slice category* $\mathbf{C}/x$ of $\mathbf{C}$ over x

18.5 Universal properties

One of the motivation of category theory is the use of universal properties, where we try to define mathematical objects by their properties instead of their composition.

Part of this was a reaction to the use of set theory to define mathematical objects. For instance, the real numbers are defined by recursion by the following rules. For von Neumann ordinals, numbers are defined first by equating zero with the empty set,

$$0 = \emptyset \tag{18.23}$$

And any number is then defined by the set which contains its predecessors :

$$n = \{0, 1, 2, \dots, n-1\} \tag{18.24}$$

Or, by a recursive rule,

$$0 = \emptyset \tag{18.25}$$

$$n+1 = n \cup \{n\} \tag{18.26}$$

While this construction does allow to derive the rules of arithmetic properly, it is however not the unique way of doing so. For instance Russell defined them as the equivalence

class of all sets of the same cardinality, which can be performed without referring to the numbers themselves by schemes such as

$$0 = \{A \mid \nexists x. x \in A\} \quad (18.27)$$

$$1 = \{A \mid \quad (18.28)$$

$$\exists x. \forall y. x \neq y \wedge x \in A\} \quad (18.29)$$

Zero is the set of all sets containing no elements, 1 is the set of all sets containing an element x such that all other elements y differing from x are not in the set, etc etc.

Zermelo system

The idea of universal properties is that mathematical objects such as numbers should not depend on any specific construction, but only

Yoneda's lemma

18.6 Limits

In category theory, a limit or a colimit are roughly speaking a construction on a category. For some given set of objects $A, B, \dots$ in our category C , and some morphisms between them, a limit or colimit of those objects will be some construction performed using those. Those constructions can be quite different, but overall, a limit will often be like a "subset", while a colimit is more of an "assemblage" of those.

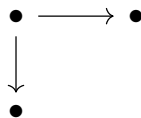
A (co)limit is done using an indexing category, which is roughly the "shape" that our construction will take. An indexing category is a small category, ie it has a number of objects and morphisms small enough that you could fit them into sets. Typically they are fairly simple ones. As we are only interested in their shape, it's common to denote the objects by simple dots. Examples include the discrete categories of n elements $\mathbf{n}$,



The span category :



and the cospan :



and the parallel pair :



To do our constructions, we need to send this diagram's shape into our category C , which we do by using some functor $F : I \rightarrow C$, producing a diagram :

Definition 18.28. A *diagram* of shape I into C is a functor $F : I \rightarrow C$.

The image of this functor in C will then be some subset of C that "looks like" I , although we are not guaranteed that the objects and morphisms of I will be mapped injectively into C (this simply corresponds to cases where our construction will use the same object or morphisms several times).

Let's now consider the constant functor $\Delta_X : I \rightarrow C$, which for $X \in C$ sends every object of I to X . If we can find a natural transformation between Δ_X and our diagram $F : I \rightarrow C$, we will have either a cone over F $\eta : \Delta_X \Rightarrow F$ or a cone under F $\eta : F \Rightarrow \Delta_X$.

Definition 18.29. The *limit* of a diagram $F : I \rightarrow M$ is an object $\lim F \in \text{Obj}(C)$ and a natural transformation $\eta : \Delta_{\lim F} \rightarrow F$, such that for any $X \in \text{Obj}(C)$ and any natural transformation $\alpha : \Delta_X \rightarrow F$, there is a unique morphism $f = \Delta_X \rightarrow \lim F$ such that $\alpha = \eta \circ F$. The cone of $\Delta_{\lim F}$ over F is the *universal cone* over F .

In other words, if we pick any object X in our category C and define some collection of morphisms from X to other objects

Let's consider for instance the case of the trivial category $\mathbf{1}$. Any functor $F : \mathbf{1} \rightarrow C$ is simply a choice of an object in C , mapping $\bullet$ to $F(\bullet) = A$, ie it is just the constant functor Δ_A for some A . A natural transformation $\eta : \Delta_X \rightarrow F$ is then simply $\eta : \Delta_X \rightarrow \Delta_A$, and conversely, $\eta : F \rightarrow \Delta_X$ is $\eta : \Delta_A \rightarrow \Delta_X$. The components of this natural transformations are simply a morphism from X to A (and a morphism from A to X).

Limit and colimit

Types of indexing category I and their limit and colimit :

Definition 18.30. Given the empty category $\mathbf{0}$, the limit $\text{Lim} F$ of a diagram $F : \mathbf{0} \rightarrow C$ is its *terminal object*, and the colimit is its *initial object*.

As there exists only one functor from the empty category to any other category (the empty functor), the initial and terminal objects do not depend on specific objects and are simply special objects of the category.

Terminal object t there exists only one morphism from $c \in C$ to t

Initial and terminal objects occur in quite a lot of important categories, and tend to be somewhat similar objects. In **Set**, the

In **set** : singleton, the unique function is the constant function

In **top** : One point space with the indiscrete topology. Only one continuous function, the constant function

In **Group** : Group with one element, the constant homomorphism

Ring : zero ring

Vect : zero vector space

poset : greatest element

N : zero

Definition 18.31. For a discrete category $\mathbf{n}$, the limit of any functor $F : \mathbf{n} \rightarrow C$ is the *product* of the objects, while its colimit is the *coproduct* of those objects.

Any functor of a discrete category of n elements to C simply picks out n objects in C . If we look at any cone over those objects and the universal cone it has to cycle through,

As the name implies, the product in a category tends to be some notion of a product, ie the association of two objects together.

In **Set**, the limit is the Cartesian product of the sets picked, while the cone is the projection maps of that product. Similarly, in **Top**, the limit is the topological product of those two spaces.

group : direct product of groups

Poset : the greatest lower bound, or meet [cf locale]

N : the GCD of the numbers

Example of product, element, empty set, etc

$$\begin{array}{ccc}
 A & \xrightarrow{\phi} & B \\
 \downarrow \textcolor{red}{\eta} & & \downarrow \textcolor{red}{\psi} \\
 C & \xrightarrow{\textcolor{blue}{\eta}} & D
 \end{array}$$

Definition 18.32. For the parallel pair category $\bullet \rightrightarrows \bullet$, the limit of a diagram $F : I \rightarrow C$ is the *equalizer* of the two morphisms between the two objects, and the colimit is the *coequalizer*.

Example : If the category C has a zero object 0 , with a unique morphism from and to any other object,

18.6.1 Distributive and linear categories

Definition 18.33. A category with finite products and coproducts is a *distributive category* if the product $\times$ and coproduct $+$ have the canonical distributivity isomorphism

$$(X \times Y) + (X \times Z) \rightarrow X \times (Y + Z) \quad (18.30)$$

Equivalent : the functor $X \times -$ preserves coproducts

Example : **Set** is distributive

$$(A \times B) \sqcup (A \times C) \cong A \times (B \sqcup C) \quad (18.31)$$

Distributive categories are "like a space"?

linear categories are "like numbers"

18.7 Equivalences and adjoints

Like many objects in mathematics, it is possible to try to define some kind of equivalence between two categories. Like most such things, we try to consider two mappings between our categories. Let's consider two categories C, D , and two functors F, G between them,

$$C \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} D \quad (18.32)$$

The usual process of finding equivalent objects in such cases is to have those two maps be inverses of each other, ie $FG = \text{Id}_D$ and $GF = \text{Id}_C$. The composition functor FG

maps every object and morphism of D to itself and likewise for GF on C , so that in some sense, the objects X and $F(X)$ are the same objects, and likewise, $f : X \rightarrow Y$ and $F(f) : F(X) \rightarrow F(Y)$ represent the same morphism.

If two categories admit such a pair of functors, we say that they are *isomorphic*. While this is the most obvious definition of equivalence, it is in practice not commonly used, as very few categories of interest are actually isomorphic, and this is generally considered too strict a definition in the philosophy of category theory. A good example of this overly strict definition is that the product of two objects done in a different order will not be isomorphic.

If we weaken the notion of equivalence, we can look at the case where our two functors are merely isomorphic to the identity, $FG \cong \text{Id}_D$ and $GF \cong \text{Id}_C$, where there exists a natural transformation η taking FG to Id_D .

$$C \qquad D$$

Triangle identities

For every objects $X \in C, Y \in D$, the components of the relevant natural transformations are

$$\text{Id}_{F(Y)} = \epsilon_{F(Y)} \circ F(\eta_Y) \quad (18.33)$$

$$\text{Id}_{G(X)} = G(\epsilon_X) \circ \eta_{G(X)} \quad (18.34)$$

$$F \dashv G$$

Adjunction between two categories

Example 18.34. A basic non-trivial example of adjoint functors is the even and odd functors. If we consider $\mathbb{Z}$ as a linear order category, with $\leq$ as its morphisms, functors are its order-preserving functions. Two specific functors that we have are the even and odd functors, give by

$$\forall k \in \mathbb{Z}, \text{Even}(k) = 2k, \text{Odd}(k) = 2k + 1 \quad (18.35)$$

These do indeed preserve the order so that for the unique morphism $k_1 \leq k_2$, it is mapped to the unique morphism $2k_1 \leq 2k_2$ and similarly for the odd functor. The corresponding "inverse" functor is the functor mapping any integer to the floor of its division by 2 :

$$\lfloor -/2 \rfloor : \mathbb{Z} \rightarrow \mathbb{Z} \quad (18.36)$$

Even is then the left adjoint and Odd the right adjoint of $\lfloor -/2 \rfloor$. The unit and counit of Even are :

$$\epsilon_l = f \circ \lfloor -/2 \rfloor \quad (18.37)$$

[...]

This is however not an equivalence, as the floor function is not strictly an inverse of the even and odd functor (and not being a faithful functor to begin with), as $\lfloor (2n)/2 \rfloor = \lfloor (2n+1)/2 \rfloor$, and we have

Example 18.35. Take the two linear order categories of $\mathbb{Z}$ and $\mathbb{R}$, with their elements being the objects and their order relations are the morphisms. The inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$, mapping $n \in \mathbb{Z}$ to its equivalent real number, is a functor

We can try to define left and right adjoints for it, by finding two functions $f, g : \mathbb{R} \rightarrow \mathbb{Z}$ for which there exists

- A left and right counit $\epsilon_l : f\iota \rightarrow \text{Id}_{\mathbb{Z}}$ and $\epsilon_r : \iota g \rightarrow \text{Id}_{\mathbb{R}}$
- A left and right unit $\eta_l : \text{Id}_{\mathbb{R}} \rightarrow \iota f$ and $\eta_r : \text{Id}_{\mathbb{Z}} \rightarrow g\iota$

And all these must obey the triangle identities

If we take for instance the left adjoint, we need that our function f be such that there exists a natural transformation between the identity on $\mathbb{R}$ ($\text{Id}_{\mathbb{R}}(x) = x$) and our function f reinjected into $\mathbb{R} : \iota(f(x))$. For every $x, y \in \mathbb{R}$, there's a morphism

$$\begin{array}{ccc} x & & x \xrightarrow{\eta_{l,x}} \iota(f(x)) \\ \downarrow \leq & \xRightarrow{\eta} & \downarrow \leq \\ y & & y \xrightarrow{\eta_{l,y}} \iota(f(y)) \end{array}$$

In the context of our linear order, this means that for any two numbers x, y such that $x \leq y$, we have $x \leq \iota(f(x))$, $y \leq \iota(f(y))$ and $\iota(f(x)) \leq \iota(f(y))$

$$x \leq y \leq \iota(f(y)) \tag{18.38}$$

and for the counit, we need a natural transformation between the mapping of an integer into $\mathbb{R}$ and then back into $\mathbb{Z}$ via f with $f(\iota(n))$, and the identity on $\mathbb{Z}$, $\text{Id}_{\mathbb{Z}}(n) = n$. For every n, m , $n \leq m$,

$$\begin{array}{ccc} n & & f(\iota(n)) \xrightarrow{\epsilon_{r,n}} n \\ \downarrow \leq & \xRightarrow{\epsilon} & \downarrow \leq \\ m & & f(\iota(m)) \xrightarrow{\epsilon_{r,m}} m \end{array}$$

We have the condition that if we inject n into $\mathbb{R}$, its left adjoint will be such that $f(\iota(n)) \leq n$, $f(\iota(m)) \leq m$ and $f(\iota(n)) \leq f(\iota(m))$. If we take the case $m = n + 1$ and ignoring the injection ι for now, this means that $f(n) \leq f(n + 1) \leq n + 1$ and $f(n) \leq n$. As

$f(n) \leq n + 1$ cannot be equal to $n + 1$, $f(n)$ can only be equal to n or smaller. If we pick the case $n - 1 \leq n$ instead, the natural transformation implies $f(n - 1) \leq f(n) \leq n$ and $f(n - 1) \leq n - 1 \leq n$, so that $f(n - 1) < n$ and

Triangle identities : for any $n \in \mathbb{Z}$,

$$\text{Id}_{\iota(n)} = \iota(\epsilon_{l,n}) \circ \eta_{l,\iota(n)} \quad (18.39)$$

The components of this natural transformations give us that, if we transform our integer n to a real and back,

$$\begin{array}{ccccc} x & & x & \xrightarrow{\eta_{l,x}} & \iota(f(x)) \\ \downarrow \leq & \xRightarrow{\eta} & \downarrow \leq & & \downarrow \iota(f(\leq)) \\ y & & y & \xrightarrow{\eta_{l,y}} & \iota(f(y)) \end{array} \quad \xRightarrow{\epsilon}$$

18.8 Elements

One fundamental feature of set theory that does not feature fundamentally in category theory is the notion of element, where some object can be said to belong to a set, denoted by $x \in A$.

18.9 Monoids and products

For quite a lot of different types of mathematical objects, we wish to define some kind of operation between two such objects. We may want to have tensor products of vector spaces, Cartesian products of sets, direct sums of groups, etc etc.

Just as we would in set theory, to define some operation on categories, we first define their products :

Definition 18.36. A *product category* $C \times D$ is a category constructed from two categories C and D such that $\text{Obj}(C \times D) = \text{Obj}(C) \times \text{Obj}(D)$

A binary operation on a category is then defined as some map from a product category to the category itself. The classic example of such an operation gives rise to the concept of a monoidal category :

Definition 18.37. A *monoidal category* $(M, \otimes, I, \alpha, \lambda, \rho)$ is a category M equipped with a bifunctor

$$\otimes : C \times C \rightarrow C \quad (18.40)$$

called the *tensor product*, an object I called the *unit object*, a natural isomorphism α

$$\alpha \quad (18.41)$$

18.10 Frames and locales

Localic frames

18.10.1 Points

Locales are a rather broad generalization of spaces, so that we need to find out what points mean within that context, as it will be quite often the type of space we consider.

As Euclid mentions, the fundamental property of a point is that it has no further constituent part. In terms of a locale, we can consider the frame constituted of a single point as an example : $\{0 \leq 1\}$, also called the frame of boolean values as it is isomorphic to the algebra generated by boolean values, ie we have

$$0 \wedge a = 0 \tag{18.42}$$

$$a \wedge 0 = 0 \tag{18.43}$$

$$1 \wedge 1 = 1 \tag{18.44}$$

$$0 \vee 0 = 0 \tag{18.45}$$

$$a \vee 1 = 1 \tag{18.46}$$

$$1 \vee a = 1 \tag{18.47}$$

$$\tag{18.48}$$

0 is empty in this context : as a frame there are no elements inside it (it is the greatest lower bound of inclusion with any element). 1 has only one element ()

To look at what points are for any locale, we need to consider similar properties, but of course typically points do not belong to the locale of most topological spaces. We do know however that topological spaces have a notion of convergence of the filter of neighbourhoods of a point converging to that point

18.11 Spaces

To define the notion of space with categories, we will have to first define some kind of notion parallel to that of a topological space. First we need to find a way to decompose a space into subobjects via the mean of a cover :

Definition 18.38. A *cover* on an object X of a category C is a collection of morphisms $\{f_i : U_i \rightarrow X\}_{i \in I}$, for U_i other objects in C .

Regular epimorphism, coverage, site, sieves

Definition 18.39. A *sieve* S

A sieve's basic notion is that it selects a

Definition 18.40. A *site* (C, J)

Nets? Čech nerves?

18.12 Sheaves and toposes

Presheaf, sheaf, topos,

Definition 18.41. An S -valued *presheaf* on a category C is a functor from the opposite category C^{op} to S :

$$F : C^{\text{op}} \rightarrow S \quad (18.49)$$

Without application, this definition is a little too abstract to be of much use. The main application for us in which it is to be understood is in the context of C being the category of some spaces, often the category of topological spaces, **Top**, with continuous maps as morphisms. A class of those morphisms is the inclusion map $\iota : U \hookrightarrow V$, where U is a subset of V (or more specifically, $\iota(U)$ is). Applying the opposite functor to this in the category **Top**^{op}, we obtain the opposite morphism

$$\iota^{\text{op}} : V^{\text{op}} \rightarrow U^{\text{op}} \quad (18.50)$$

Once we apply our presheaf functor on this, we obtain

$$F(\iota^{\text{op}}) : F(V^{\text{op}}) \rightarrow F(U^{\text{op}}) \quad (18.51)$$

Our presheaves will usually map those regions of the topological space to some structures of interest, like function spaces defined on those regions. The sheaf's functor gives us the property of a restriction map from the inclusion map:

Definition 18.42. A *restriction map* ρ_{UV}

Definition 18.43. A *representable presheaf* is a presheaf that is naturally isomorphic to the hom functor $\text{hom}_C(-, X)$ for some object $X \in C$.

This gives us the notion of a presheaf as a set of functions on our objects. For instance, a useful presheaf to consider is the presheaf of real functions. If we consider the category of Cartesian spaces

$$\mathbf{PSh}(\mathbf{CartSp}_{\mathbf{Diff}}) \quad (18.52)$$

Category of presheaves on a category :

$$\mathbf{PSh}(C) \quad (18.53)$$

Presheaf on the category of open sets

Important cases where the category of a presheaf differs from **Set** are presheaves of rings $C \rightarrow \mathbf{Ring}$, such as the smooth functions we've seen, as well as more specific cases of rings such as

Sheaf on a group : stacks

A sheaf is a presheaf along with some property

Definition 18.44. A *sheaf*

Sheaves will have many applications throughout this book, as they will be used for a

Sheaf over a site, topos

Topos has finite limits and colimits, finitely complete category

It admits initial and terminal objects, products and coproducts, equalizers and coequalizers

It has an exponential, for all $X, Y \in C$, $X^Y \in C$

Subobject classifier Topos are like sets

Gros topos v. petit topos

Stalks

18.12.1 Intensive and extensive quantities

18.13 Enriched categories

As we've seen, categories are fundamentally based on sets, in particular in the case of the hom-set between two objects,

$$\text{hom}(X, Y) \quad (18.54)$$

This description may however not be the most structure we can apply to a given hom-set, however. As the hom-set can often be understood as a set of functions between the objects, we can already find many examples of larger structures that may apply there : many function spaces are topological spaces, where we can furnish the set of all functions with a topology. They may have the structure of a metric space. Linear maps between vector spaces are themselves linear maps.

These are examples of what are called *enriched categories*

Definition 18.45. An *enriched category* E over a category C

By that definition, every category is already an enriched category over the category of sets.

18.14 Duality

One important notion that will come up a lot in physics is the notion of *duality*, where two objects of different categories are in some sense equivalent. This comes up in particular in the notion of equivalence between geometries and algebras, such as

Yoneda embedding

Map from any category C to the category $[C^{\text{op}}, \mathbf{Set}]$, the category of functors from the opposite category to sets, the hom map : $X \mapsto \text{hom}(-, X)$. Yoneda embedding :

$$: C \rightarrow [C^{\text{op}}, \mathbf{Set}] \quad (18.55)$$

$$X \mapsto \text{hom}(-, X) \quad (18.56)$$

Dual : co-Yoneda embedding

$$^* : C \rightarrow [C, \mathbf{Set}]^{\text{op}} \quad (18.57)$$

$$X \mapsto \text{hom}(X, -) \quad (18.58)$$

Isbell conjugate : any functor $F : C^{\text{op}} \rightarrow \mathbf{Set}$ has an Isbell conjugate $F^* : C \rightarrow \mathbf{Set}$

$$F^*(X) = \text{hom}(F, (X)) \quad (18.59)$$

and conversely, any functor $G : C \rightarrow \mathbf{Set}$

$$G^*(X) = \text{hom}(* (X), G) \quad (18.60)$$

Isbell duality : the two duality functors $*$ from $[C, \mathbf{Set}]^{\text{op}}$ to $[C^{\text{op}}, \mathbf{Set}]$ and vice-versa are adjoints. $\text{hom}(F^*, G)$ and $\text{hom}(F, G^*)$ are isomorphic.

Special case : if $F^{**} = F$

Presheaves and copresheaves

18.15 The categories so far

The previous chapters we have seen give us quite a lot of examples of various objects which can be made into categories.

18.15.1 The category of sets

The category of sets, denoted $\mathbf{Set}$, is our first proper example of a category. As a category, its objects are the sets. We know that for various reasons [Russell etc], the collection of all sets does not form a set in most axiom systems, so we will only say here that the objects of $\mathbf{Set}$ is the class of all sets :

$$\text{Obj}(\mathbf{Set}) = V \quad (18.61)$$

And the morphisms of $\mathbf{Set}$ is the class of all functions on those sets :

$$\text{Mor}(\mathbf{Set}) = \{\forall A, B \in \text{Obj}(\mathbf{Set}), f : A \rightarrow B\} \quad (18.62)$$

As functions, those easily verify the laws of composition and the existence of an identity. The source and target map are simply the domain and image of the function, which are indeed objects of the category.

Theorem 18.46. Injective functions are monomorphisms and surjective functions are epimorphisms.

Proof.

□

From this, we have that the isomorphisms of **Set** are the bijections between sets, and therefore every pair of sets in **Set** is isomorphic if they have the same cardinality.

We have a few special properties for sets that we need to express using category theory, the first of which being that of the belonging relationship, which is related to the subobject classifier of the category **Set**. The subobject classifier corresponding to the usual notion of belonging is the set $\Omega = \{0, 1\}$ (if we write it in pure set, $\Omega = \{\emptyset, \{\emptyset\}\}$, although any two element set will be isomorphic of course).

Boolean category

Since topoi are objects modelled to look like sets, the category of sets **Set** is obviously a topos itself.

First, **Set** has finite limits and colimits. Some of the important

Theorem 18.47. The empty set is an initial object.

Proof.

□

Theorem 18.48. Sets with one elements are terminal objects

Proof.

□

Well-pointed topos

As a category, the important properties of are its initial object, the *empty set* $\emptyset$, and its terminal objects, the singleton sets.

complete

The objects belonging to a spacetime as a set (and in geometrical spaces in general) are called *points*.

Subsets as subobjects

18.15.1.1 Concrete categories

Most of modern mathematics was constructed in such a way that many objects in it were defined to be sets with extra properties (this was due to the popularity of set theory as a basis for mathematics following [...]). This idea can be found in category theory under the definition of a concrete category

Definition 18.49. A *concrete category* is a category for which there exists a faithful functor to the category of sets.

In other words, every object in a concrete category can be made to correspond to a set in such a way that for any two objects X, Y , the hom set $\text{Hom}(X, Y)$

. The functor itself may not be unique, as there could be different ways to model a given category with different sets, but given one such functor $U : C \rightarrow \text{Set}$, every object of C has a canonical set associated to it. The functor U is an example of a *forgetful functor*, where all the structures of the category C are forgotten to leave only the set of points of whatever category we're dealing with.

Quite a lot of categories we will consider throughout this book are concrete categories, and usually have an explicit construction in terms of sets, such as Vec , Grp , etc. There are examples of categories which cannot be made into concrete categories [homotopy spaces]

18.15.2 Category of topologies

The category of topological spaces, denoted by **Top**, has topological spaces as objects and continuous functions as morphisms. This is easy enough to show to be a category since quite a lot of the proof can be done by modelling it on the proof for sets, as the composition of continuous functions is also a continuous function, and the identity is always a continuous function. Beware however that while we are modelling our category of topological spaces on some sets with structure, the category **Top** could be defined independently of such notions and it is only by convention that we use its representation as sets, as it makes a lot of the proofs easier.

Since we are still using functions, we can translate a lot of the properties for sets here. As the empty set is itself a topological space, and continuous functions will always be a subset of all functions on sets, there is also a unique continuous function from the empty topological space to any other topological space, the empty map, making it an initial object of the category. And conversely, the topological space of one element admits only one continuous map from any topological space to itself (the map projecting all elements to that one, which is continuous since the only non-empty open set is the whole space $\{\bullet\}$ and its preimage is the whole space X which is also open), the terminal object.

No zero object

The categorical product of two topological space is simply their

Theorem 18.50. The n -fold product topology is the

Product is the product topology

Adjunction

If we consider the forgetful functor sending a topological space to their set of points, $\Gamma : \mathbf{Top} \rightarrow \mathbf{Set}$, we can consider its left and right adjoint

Left adjoint : We have our forgetful functor as a right adjoint and another undetermined functor f ,

$$\Gamma : \mathbf{Top} \rightarrow \mathbf{Set} \quad (18.63)$$

$$f : \mathbf{Set} \rightarrow \mathbf{Top} \quad (18.64)$$

Together, they must define a unit and counit, unit $\eta : \text{Id}_{\mathbf{Top}} \rightarrow \Gamma f$ counit $\epsilon : f\Gamma \rightarrow \text{Id}_{\mathbf{Set}}$ natural isomorphism $\Phi : \text{hom}_{\mathbf{Top}}(f-, -) \rightarrow \text{hom}_{\mathbf{Set}}(-, \Gamma-)$

Unit : If we have two topological spaces with a continuous function $f : X \rightarrow Y$, the components of the unit are such that

Counit

Right adjoint

Disc $\dashv$ Γ $\dashv$ CoDisc

The discrete functor $\text{Disc} : \mathbf{Set} \rightarrow \mathbf{Top}$ has itself a left adjoint,

topological concrete category : Given a functor $U : C \rightarrow D$, C is a topological category over D if for any algebra X and any family of spaces $S_i \in C$ and homomorphisms $f_i : X \rightarrow U(S_i)$, there's a space T such that $U(T) = X$ and maps $m_i : T \rightarrow S_i$ such that $U(m_i) = f_i$, and for any space T' , homomorphism $g' : U(T') \rightarrow X$, and maps $m'_i : T' \rightarrow S_i$, and if every composite $g' \circ f_i = U(m'_i)$, then there is a unique map $n : T' \rightarrow T$ such that $U(n) = g'$ and $n \circ m_i = m'_i$

example : $F : \mathbf{Top} \rightarrow \mathbf{Set}$

18.15.2.1 Grothendieck topology and sites

18.15.3 The category of groups and groupoids

Every group can by itself be expressed by a category in itself, with a single object (the "group" itself), and a morphism for every element of the group. The composition law of the group stems simply from the group operation, with its associativity representing the associativity of the product, the identity element being the identity morphism, and the inverse imposing that any morphism has an inverse morphism.

From this definition, it seems that quite a lot of groups could simply stem from subcategories of a given categories

Theorem 18.51. Every set of endomorphisms defines a group

Proof. Given some object X in a category C , the endomorphisms $\text{End}(X)$ contain the identity morphisms, inverses for any of them, and composition by category theory, thereby forming a group. $\square$

The category of all groups, denoted by **Grp**, can therefore be viewed as a 2-category, where each object is the category defined by a specific group, and every morphism is a functor between those categories. If we ignore the 2-category structure, **Grp** is simply a category whose objects are groups and whose morphisms are group homomorphisms.

Since the groups as we defined them are built from sets, it is trivially a concretizable category, with the forgetful functor

$$U : \mathbf{Grp} \rightarrow \mathbf{Set} \quad (18.65)$$

$$(g, \circ) \mapsto g \quad (18.66)$$

Trivial group $\{e\}$ is the zero object, trivial homomorphism is the zero morphism

[40]

Free functor adjoint to free groups

Theorem 18.52. The forgetful functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ has a free functor left adjoint F mapping sets to their free groups.

Proof. $\square$

Kernel of a group as the equalizer

18.15.4 The category of rings

18.15.5 The category of vector spaces

The category of vector spaces **Vect** is composed of vector spaces as objects and their linear maps as morphisms. If we only consider isomorphic objects in that category, and we consider the subcategory of vector spaces over a given field, this allows us to represent the first few objects in an orderly fashion by dimension :

$$\begin{array}{c} V^0 \xrightarrow{L(V^0, V^1)} V^1 \xrightarrow{L(V^1, V^2)} V^2 \longrightarrow \dots \\ \quad \quad \quad \curvearrowright \\ \quad \quad \quad L(V^0, V^2) \end{array}$$

where the arrows are all the possible linear maps between each vector space of a given dimension.

The zero vector space is both an initial object and a terminal object, making it a zero object.

The notion of an adjoint functor comes from the category of vector spaces, as the adjoint operator that we saw are an example of the action of such functors. If we consider the

Dual vector space as Isbell duality

18.16 The category of Cartesian spaces

Euclidian space in categorical terms is given by the categories of Cartesian spaces, with objects being $\mathbb{R}^n$ for every $n \in \mathbb{N}$. The Cartesian space $\mathbb{R}^n$ has a lot of structure on it, and due to this, there are many categories it can be in, depending on which of those we consider. This translates to a variety of different categories and different morphisms on $\mathbb{R}^n$.

There is the category of topological Cartesian spaces $\mathbf{CartSp}_{\text{Top}}$, where we consider $\mathbb{R}^n$ with its canonical topology and its morphisms are the continuous real maps.

$\mathbf{CartSp}_{\text{Lin}}$ for real vector spaces

$\mathbf{CartSp}_{\text{Aff}}$ for real affine spaces

$\mathbf{CartSp}_{\text{Smooth}}$ for the smooth category of cartesian spaces

18.16.1 The category of manifolds

The category of C^k -manifolds is given by manifolds as its set of objects, and the C^k -homeomorphisms between them. In particular, the category of smooth manifolds, denoted **SmoothMan**, has smooth manifolds as its objects and the diffeomorphisms between them as morphisms.

Initial object : empty manifold, Terminal object : 0-manifold

Product and coproduct

As we've seen, manifold products are well defined, and those are indeed products in the categorical sense. If we have two manifolds M_1, M_2

equalizer and coequalizer in trouble,

Quotients are not closed on manifold, as can be seen by considering the quotient sending $\mathbb{R}^2$

Therefore not all colimits exist in **Man**, fails to be a topos

18.17 Spacetime categories

There are many structures and categories that are involved in physics, but as this is the main focus of the book, let's focus on the example of spacetime as an object of study.

In a categorical sense, we can construct a spacetime geometry on a hierarchy of various categories, where the spacetime gains more and more structure as we build it. One usual scheme for this is to consider spacetime first as a set, as a topological space, as various locally Euclidian spaces, ending on smooth manifolds, and then adding various additional structures on this, such as metrics, connections, etc.

Every morphism on the spacetime object in a different category defines a structure on it : points, open sets, plots, charts

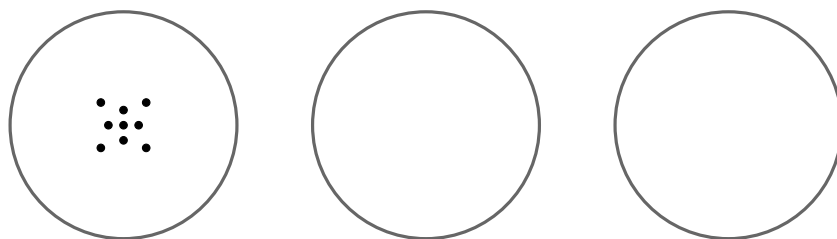


Figure 18.1: A given spacetime in various categories

Chapter 19

Curves

Curves tend to be one of the fundamental object of study in general relativity, as many objects will be modelled by curves, mathematical structures will be used to describe curves or predict their motion.

We can define curves in a few different ways, first simply as map from the real numbers to the manifold.

Definition 19.1. A *curve* γ is a continuous map from a connected subset of the real line $I \subset \mathbb{R}$ (omitting the trivial cases of the empty set $I = \emptyset$ and the singleton $I = \{x\}$) to the manifold, $\gamma : I \rightarrow M$.

This definition gives as the space of all curves the entire set of continuous functions $C(I, M)$ for all possible intervals of $\mathbb{R}$:

$$\mathcal{C}(M) = \bigcup_{I \subset \mathbb{R}} C(I, M) \quad (19.1)$$

For a variety of examples, we will consider curves in some coordinate patch. Given a curve $\gamma : I \rightarrow M$, its coordinate representation in the coordinate patch (U, ϕ) with coordinates $x^\mu = \text{pr}_\mu \circ \phi$ will be $x^\mu \circ \gamma$. For simplicity, this will usually be written by a set of n functions denoted by x^μ :

$$x^\mu : \mathbb{R} \rightarrow \mathcal{O} \subset \mathbb{R}^n \quad (19.2)$$

$$\lambda \mapsto x^\mu(\lambda) = (\text{pr}_\mu \circ \phi \circ \gamma)(\lambda) \quad (19.3)$$

This can present the issue of giving us a disconnected curve. Consider the case of a curve doing a full turn and a neighbourhood that does not contain this turn

The preimage of $\text{Im}(\gamma) \cap U$ will be two disconnected components of $\mathbb{R}$. We will in general only consider the coordinate representation of a curve to be a map from a connected

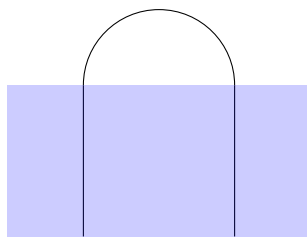


Figure 19.1: Curve with disconnected components in a neighbourhood

component of that preimage to the open set. If we have two neighbourhood in this case, one containing the turn and the other not, the decomposition of the curve into coordinate representations will be given by three such curve segments.

A possible problem of the definition of a curve simply as a continuous function is that it will include a variety of pathological curves. If we take the simple example of the manifold $\mathbb{R}^2$ with its identity chart $\text{Id}_{\mathbb{R}^2}$ in Cartesian coordinates, we can get such curves as the trivial curve

$$x^\mu(\lambda) = 0 \quad (19.4)$$

Or curves that will reduce to this curve in some parts

$$x^\mu(\lambda) = \begin{cases} (\lambda, 0), & \lambda < 0 \\ (0, 0), & \lambda \in [0, \varepsilon] \\ (\lambda - \varepsilon, 0) & \lambda > \varepsilon \end{cases} \quad (19.5)$$

These curves are not fundamentally "wrong", but there are circumstances where we will wish to have more specific notions of a curve that conform more with our natural intuition for them. The class of curves obeying this is regular curves :

Definition 19.2. A curve is said to be *regular* if for any coordinate representation,

$$\frac{d}{d\lambda}\gamma(\lambda) \neq 0 \quad (19.6)$$

This guarantees that the curve will not map some subset of I to identical points [proof?]

(This notion will later relate to the more general notion of embedded submanifolds, for which it is nothing but the one-dimensional case)

Curves in general can also

Hilbert space filling curve? torus space filling curve?

Some other problematic curves are for instance the topologist's sine curve,

$$x^\mu(\lambda) = (\lambda, \sin(\lambda^{-1})) \quad (19.7)$$

Does the map have to be bijective etc

For more regular curves, we can consider the space of C^k curves, which are curves such that their coordinate representation $x^\mu : I \subset \mathbb{R} \rightarrow \mathcal{O} \subset \mathbb{R}^n$ is a C^k function with the usual derivative for real functions in every coordinate chart.

Theorem 19.3. If a curve is represented by a C^k function in a chart (U_α, ϕ_α) , it is a C^k function in every overlapping chart.

Proof. If our curve has coordinate representation $\phi_\alpha \circ \gamma$ in U_α , and we consider another chart (U_β, ϕ_β) , its coordinate representation in that chart will be

$$\phi_\beta \circ \gamma = \phi_\beta \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \gamma \quad (19.8)$$

$$= \psi_{\beta\alpha} \circ (\phi_\alpha \circ \gamma) \quad (19.9)$$

As the transition functions of a smooth manifold are smooth, the composition of those two functions will be a C^k function. $\square$

Note that this relies on the transition maps being smooth, so that this will only apply to charts within the smooth structure. If we used the case of C^r charts and C^k curves, the resulting function would change to $C^{\min(k,r)}$.

Another way to define curves in a manifold is to consider them as 1-submanifolds, in which case they are defined by the space of maps from 1-manifolds to M . To give the appropriate extent of the possible types of curves, we will consider manifolds with boundaries, which in the 1-dimensional case give us four possible manifolds : the line L , the circle S^1 , the ray L^+ and the segment I .

The definition as maps of intervals on $\mathbb{R}$ coincides with this if we consider coordinate charts on either L , L^+ or I , as they are mapped respectively from (a, b) , $[a, b)$ or $[a, b]$, with the open ends of the interval being possibly at infinity. A given coordinate chart on a 1-manifold is referred to as a parametrization of the curve.

Definition 19.4. For a curve $\gamma : L \rightarrow M$, a parametrization h is a coordinate chart

$$h : L \rightarrow I \subseteq \mathbb{R} \quad (19.10)$$

$$(19.11)$$

such that the parametrized curve takes values in $\mathbb{R}$ to points in M via

$$\gamma \circ h(\lambda) = \gamma(a) \in M \quad (19.12)$$

A transition map of the 1-manifold for a curve is a *reparametrization*

Example

Just as diffeomorphism on a manifold, reparametrizations form a group which is simply the diffeomorphism group of the 1-manifold in question, such as the diffeomorphism group of the line for instance, with its usual split

$$\text{Diff}(L) = \text{Diff}^+(L) \cup \text{Diff}^-(L) \quad (19.13)$$

The group of orientation-preserving diffeomorphisms will in particular be important as contrarily to most manifolds, the line (as well as half-line and segment) can be given an *order* relation that stems from the order on $\mathbb{R}$: we can assign a choice of order between any two points by convention and this choice will define an order for every point of the curve.

Take two points $a, b \in L$ and define $a < b$. We can find some parametrization of the curve where this is reflected by the order of $\mathbb{R}$, as the map $\mathbb{R} \rightarrow L$ is bijective and [total order, only two possible orders]

This ordering is preserved by any reparametrization in $\text{Diff}^+(L)$

[proof]

Any reparametrization in $\text{Diff}^+(L)$ is therefore called *orientation preserving*, while any reparametrization in $\text{Diff}^-(L)$ is called *orientation reversing*.

This ordering defines a natural orientation on a curve [...]

Real line $\mathbb{R}$ v. affine line A v. projective line $P\mathbb{R}$

Difference between L and $\mathbb{R}$, category of **SmoothMan** and **CartSp**

The indiscriminate use of $\mathbb{R}$ as the model space for curve can sometimes be misleading, as $\mathbb{R}$ can represent a space with quite a variety of structures on it : $\mathbb{R}$ is a topological space, it is a totally ordered set, a metric space, has a smooth structure, is a measure space, generally has preferred points (such as 0), comes with its own scale $[0, 1]$, has natural algebraic structures, and so forth. While these can be useful in a variety of context, these tend to be a bit too much structure for the basic definition of a curve. We will in general try to use the 1-manifold L (and other related spaces such as S and I) instead as the fundamental model of a curve.

Curves themselves can have a little too much structure, as two different curves can have the same image if they are related by a reparametrization.

$$\text{Im}(\gamma) = \text{Im}(\gamma \circ f) \quad (19.14)$$

Proof that paths are equivalent to images of curves

Theorem 19.5. The equivalence class of curves up to reparametrization is isomorphic

to the image of that curve.

Proof.

□

The notion to talk about those equivalence classes of curves is called a path (1-dimensional subobjects)

Definition 19.6. A path $[\gamma]$ is an equivalence class of curves related by the reparametrization group

From the atlas of a manifold $\mathcal{A}$, we can define curves as maps $\mathbb{R} \rightarrow \mathbb{R}^n$ obeying the cocycle property.

Definition 19.7. A set of maps over coordinate neighbourhoods (c_i) , $c_i : \mathbb{R} \rightarrow U_i$, with coordinate representations $x_i = \phi_i \circ c_i : \mathbb{R} \rightarrow \mathbb{R}^n$, obeying the cocycle property that for any two such maps,

$$c_i \circ c_j^{-1} \quad (19.15)$$

defines a curve on the manifold M .

Curves can also be viewed as a topological subspace of the product manifold $\mathbb{R} \times M$, where we consider the subset given by $\{(\lambda, p) | \gamma(\lambda) = p\}$

Affine line

Curve orientation?

Definition 19.8. A curve is said to be *orientable* if it admits a coordinate representation for which every transition map is orientation-preserving.

Example 19.9. Any curve on $\mathbb{R}^n$ is orientable

Example 19.10. A curve wrapping around the Möbius strip is not orientable.

Defines some order on the curve?

Examples of non-continuous curves, non- C^1 curves, non- C^{0-} curve

19.1 Orientation

19.2 Paths

Definition 19.11. A *path* is an equivalence class of curves by the equivalence class of the diffeomorphism group. For some curve $\gamma : I \rightarrow M$, the path with γ as a representative is

$$[\gamma] = \{\gamma \circ f | f \in \text{Diff}(I)\} \quad (19.16)$$

19.3 Endpoints

The endpoints of a curve are defined rather simply if our curve is defined on a closed interval $[a, b]$, simply as $\gamma(a)$ and $\gamma(b)$ (the image of the boundary $\gamma(\partial[a, b])$), but in the open case, things may fail to be so easy to deal with. If the domain of definition of the curve admits a closure that can be continuously extend the curve, this would be easy enough, ie if we had some open domain (a, b) , we could look at the compactification $[a, b]$, but such extensions may not be possible.

For instance if we consider some curve in Euclidian space $\gamma : [0, \infty) \rightarrow M$ with $\gamma(0) = 0$, and we then consider the same manifold with 0 deleted, with the inclusion map $\iota : M \setminus \{0\} \rightarrow M$, there is a corresponding curve $\gamma' : (0, \infty) \rightarrow M \setminus \{0\}$ such that $\iota \circ \gamma' = \gamma|_{\text{Im}(\iota)}$.

Example from Penrose's book [41]

To define what the endpoint of a curve properly is, we need to use the convergence property of sequences to a boundary through a continuous map : as the boundary b of (a, b) is an accumulation point of some sequence $(x_n)_{\mathbb{N}}$

Definition 19.12. A point $p \in M$ is a *terminal accumulation point* of λ if for every open neighbourhood U of p , and for every $\lambda_0 \in S$, there exists a $\lambda \in S$, $\lambda > 0$ such that

Definition 19.13. p is an *endpoint* of a curve γ if for every neighbourhood U of p there exists a $\lambda_0 \in S$ such that for all $\lambda > \lambda_0$, $\gamma(\lambda) \in U$.

While every endpoint is an accumulation point, the converse isn't true. For instance, take in $\mathbb{R}^2$ the curve

$$\begin{aligned} y(\lambda) &= \sin(\lambda^{-1}) \\ x(\lambda) &= \lambda \end{aligned}$$

0 is an accumulation point since for the open set B_r , for $\lambda_0 < r$ and $\lambda_0 = [2n\pi]^{-1}$, $\gamma(\lambda_0) \in B_r$, but if $r < 1$, this is not true for $\lambda_0 < r$ and $\lambda_0 = [(2n+1)\pi]^{-1}$. The curve leaves and re-enters any neighbourhoods of 0 no matter how close.

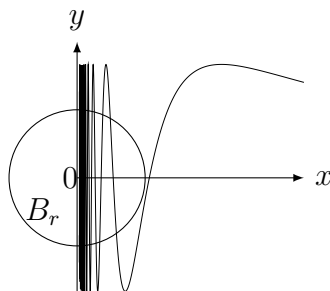


Figure 19.2: $\sin(x^{-1})$ with no endpoint

In general, we will only consider curves for which the terminal accumulation points are endpoints.

A curve γ is called endless if it has no endpoints.

A curve γ is *inextendible* if there's no curve that contains it as a proper subset.

Proposition 19.14. Inextendible curves have no endpoints.

Proof. If a curve has an endpoint $p \in M$, we can find a neighbourhood U of p . In a small enough neighbourhood, we can extend the □

From the examples we've seen already, the curve in Euclidian space minus a point does not have a terminal accumulation point at $\lambda = 0$, and is not extendible. [something about the boundary idk]

4.24 means that since $\text{cl}(\gamma(I)) = \gamma(\text{cl}(I))$ and we deleted the endpoint, then the boundary in the subspace topology is such that the boundary is empty, therefore no endpoint [check the nuances between boundaries and limit points here]

[proof]

extendible curve

A simple way to create inextendible curves on a manifold is to consider a new manifold after removing a closed set $\bar{S}$ from it, such that the original curve γ intersected with that region.

To prove : if we have a curve which is defined at $p \in M$ ($\exists \lambda_0 \in I, \gamma(\lambda_0) = p$), and we remove the point p from the manifold, the connected curves in the subspace topology of that new manifold have no endpoints at p

[Punctured neighbourhood, cf. here]

If we have some curve $\gamma : L \rightarrow M$, and we then consider some inclusion map of $M \setminus S$ with S a closed set of M , $\iota : M \setminus S \rightarrow M$, the curve

find the inverse $\gamma(M \setminus S)$, map it to $M \setminus S$

Look at the boundaries of $\gamma(M \setminus S)$, show they are not part of $M \setminus S$

Sequences idk

19.4 Loops

A special type of curves that is commonly considered in differential geometry is the loop :

Definition 19.15. A *loop* is a continuous map from the circle S^1 to the manifold,

$$\gamma : S^1 \rightarrow M \quad (19.17)$$

or alternatively, a continuous map from the interval $[a, b]$ such that the two endpoints of the curve are identical : $\gamma(a) = \gamma(b)$.

As we originally defined our curve to be any map from a connected one dimensional manifold (possibly with boundaries) to M , it can be a good idea to split this categorization between curves on one hand $L \rightarrow M$ and loops $S^1 \rightarrow M$.

Definition 19.16. The *loop space* ΩM of a manifold M is the space of all loops of a manifold M .

We also have to consider pointed loops :

Definition 19.17. A *pointed loop* (p, γ) is a loop paired with a point $p \in M$ such that $p \in \text{Im}(\gamma)$.

While a pointed loop can have its point corresponding to any point of the loop, generally speaking, we consider p to be the image of the origin of our parametrization. If our loop is the image of $[0, 1]$ with boundaries identified, the point p will be $\gamma(0) = \gamma(1)$. The space of pointed loops at p , $\Omega_p(M)$, is then the space of all such loops.

19.5 Curves and diffeomorphisms

A 1-parameter family of diffeomorphisms f_t connected to the identity will generate a curve via the evaluation map

$$I \times \text{Diff}(M) \times M \rightarrow M \quad (19.18)$$

$$(t, f, p) \mapsto \text{ev}(f_t, p) = \gamma(t) \quad (19.19)$$

For a given diffeomorphism and point, this will indeed give us a curve $f_t(p) : I \rightarrow M$, with $f_0(p) = p$. If f_t is continuous in t in the [Whitney, compact open?] topology,

What class of diffeomorphism curves lead to what type of curve

Family of curves generated by the full diffeomorphism action ϕ_t on a set U

19.6 Frölicher spaces

Definition 19.18. A *Frölicher space* is a triple made of a manifold M , the space of its smooth functions $C^\infty(M)$,

[42]

The space of curves and the space of scalar functions are in some sense dual to each other

Chapter 20

Jets, vectors and vector fields

One important part of spacetime is our ability to do calculus on it, much like we did in the affine space of $\mathbb{R}^n$, as this underpins a lot of the actual calculations we perform in physics. But our manifold is not a vector space, has no natural notion of vectors or transporting vectors around, and is not a metric space. If we consider for instance a derivative on $\mathbb{R}^n \rightarrow \mathbb{R}$, in the direction of v

$$\nabla_v f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (20.1)$$

we need some way to perform a subtraction between f at x and at $x+h$, since the value of the limit will actually be evaluating a converging sequence of such values.

norm for vector calculus version

While all those notions do exist in the context of $\mathbb{R}^n$, to which it is locally diffeomorphic, it remains to be seen how well they will translate to the context of a manifold, as we would like everything to remain independent of the specific chart we are using to do our computations.

Naive derivative

20.1 Jets

Jets [43] serve to help define a notion of derivatives in a purely geometric way, without reference to a specific coordinate system. And just as the various notions of derivatives are built on the notion of derivatives with respect to a single variable, we must first look at the one-dimensional case here, which is the jets of curves.

In a geometric sense, a jet starts with an equivalence relation between two curves. As the

ordinary derivative of a function can be interpreted by the tangent line to that function's graph, we need something equivalent.

Two curves γ_1, γ_2 are said to be in k -th order contact at two given parameters λ_1, λ_2 if, for every smooth function $\phi \in C^\infty(M)$, the derivatives of $\phi \circ \gamma_1, \phi \circ \gamma_2 : \mathbb{R} \rightarrow \mathbb{R}$ are equal up to order k at the values λ_1, λ_2 :

$$\forall 0 < i < k, \frac{d^i}{d\lambda^i} [\phi \circ \gamma_1(\lambda)] \Big|_{\lambda=\lambda_1} - \frac{d^i}{d\lambda^i} [\phi \circ \gamma_2(\lambda)] \Big|_{\lambda=\lambda_2} = 0 \quad (20.2)$$

While we do technically need to keep two different values of the parameters (after all, the ranges of those two curves are not guarantee to even overlap), in practical terms we can always reparametrize our curves to at least shift them so that both parameters are equal to zero without loss of generality, as very few properties depend on the parametrization of a curve up to affine transformation. It will at the very least not influence their contact :

Theorem 20.1. If two curves γ_1, γ_2 are in contact at λ_1, λ_2 , their reparametrizations $\gamma_1 \circ h_1, \gamma_2 \circ h_2$ are in contact at $h_1^{-1}(\lambda_1), h_2^{-1}(\lambda_2)$

Proof. This is obviously true at order 0, so that we need to prove it inductively for other orders. If this is true for contact of order k , let's consider the $k+1$ - order contact. By the chain rule, we have that

$$(\phi \circ \gamma \circ h)' = (\phi \circ \gamma)' \circ h \circ h' \quad (20.3)$$

□

We will therefore simply define contact on our new reparametrized curves,

$$\frac{d^i}{d\lambda^i} [\phi \circ \gamma_1(\lambda) - \phi \circ \gamma_2(\lambda)] \Big|_{\lambda=0} \quad (20.4)$$

Due to this, the notion of jets will overall not be terribly influenced by our choice of the origins of our two curves, so that we can freely translate them to both coincide at zero. We will usually assume without loss of generality that the curves are appropriately reparametrized so that their point of contact is 0 for both

Theorem 20.2. The contact of two curves is only determined by their local behaviour, ie if two curves γ_1, γ_2 are in contact at p , two other curves γ'_1, γ'_2 with the same germ at p will also be in contact.

Proof.

□

A particular example of the behaviour of curves in contact is the smooth function defined within a coordinate neighbourhood by $x^\mu = \text{pr}_\mu \circ \phi : M \rightarrow \mathbb{R}$, corresponding to the μ -th component of the coordinate map. This function $x^\mu \in C^\infty(U)$ can be expanded to a smooth function on M as we've seen in [X]. This means that if two curves are in k -th order contact, their coordinate representations will also share the same derivatives. It will in fact be enough to prove the other way around :

Theorem 20.3. If two curves have the same coordinate derivatives in a given chart ϕ up to order k , then they are in k -th order contact.

Proof. If we have our coordinate derivative equal,

$$\left. \frac{d^i}{d\lambda^i} [x^\mu \circ \gamma_1(\lambda) - x^\mu \circ \gamma_2(\lambda)] \right|_{\lambda=0} = 0 \quad (20.5)$$

Let's consider a function $f : N \rightarrow \mathbb{R}$, with coordinates $f \circ x^\mu$.

$$\frac{d^k}{d\lambda^k} (f \circ \gamma) = (f \circ x^{-1} \circ x \circ \gamma)' \quad (20.6)$$

$$= \quad (20.7)$$

□

The two notions are therefore equivalent. In particular, the derivative at order 0 being the function itself, if two functions are at 0-th order contact at λ , they touch each other at that parameter : $\gamma_1(\lambda) = \gamma_2(\lambda)$, as we have $x^\mu \circ \gamma_1(\lambda) = x^\mu \circ \gamma_2(\lambda)$ for every dimension.

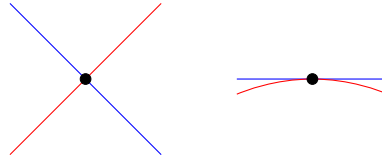


Figure 20.1: On the left : two curves in 0-th order contact at p , on the right : two curves in first order contact at p

Theorem 20.4. $\sim_k$ is an equivalence relation.

Proof. Trivially $\gamma \sim_k \gamma$ for any k since

$$\phi \circ \gamma(\lambda) - \phi \circ \gamma(\lambda) = 0 \quad (20.8)$$

$\gamma_1 \sim_k \gamma_2$ is equivalent to $\gamma_2 \sim_k \gamma_1$ just from the anticommutativity of subtraction which will not affect the zero result, and the transitive property stems from □

The contact between two curves is a property preserved by C^k maps :

Theorem 20.5. If $\gamma_1 \sim_k \gamma_2$, then for every C^k map $f : M \rightarrow N$, $f \circ \gamma_1 \sim_k f \circ \gamma_2$.

Proof. ϕ being a scalar function on N implies $\phi \circ f$ is a scalar function on M , and as a scalar function, this implies that

$$\frac{d^i}{d\lambda^i} [\phi \circ f \circ \gamma_2 - \phi \circ f \circ \gamma_1] = 0 \quad (20.9)$$

□

[44, 45]

Given this, we say that two maps between the same manifolds $f, g : M \rightarrow N$ have the same k -jet at $p \in M$ if for every curve $\gamma : L \rightarrow M$, $\gamma(0) = p$, we have $f \circ \gamma = g \circ \gamma$, written as $j_p^k(f) = j_p^k(g)$.

A way to see the more usual meaning of derivatives from this is to consider some coordinate chart ϕ on M , and to pick coordinate lines defined by that chart. The n coordinate lines crossing p will make each jet correspond to the n derivatives with respect to each coordinates

$$[\phi \circ f \circ \gamma(\lambda) - \phi \circ g \circ \gamma(\lambda)] \Big|_{\lambda=0} = 0 \quad (20.10)$$

A jet is then the equivalence class of all the possible functions sharing that relation at the point p . The set of all possible k -jets between two manifolds M, N is denoted by $J^k(M, N)$.

For any jet $X = j_p^k f \in J^k(M, N)$, we define two maps α, β , with α the *source* map

$$\alpha : J^k(M, N) \rightarrow M \quad (20.11)$$

$$X \mapsto \alpha(X) = p \quad (20.12)$$

mapping the jet to the point where it is considered at, and β the *target* map

$$\beta : J^k(M, N) \rightarrow N \quad (20.13)$$

$$X \mapsto \beta(X) = f(p) \quad (20.14)$$

mapping it to the value of the function at a point.

Two important classes of subsets of the jet space $J^k(M, N)$ are the subsets of jets with a given source point $p \in M$, denoted by $J_p^k(M, N)$, and jets with a given target point

$q \in N$, denoted by $J^k(M, N)_q$. Any jet of the form $j_p^k f$ will therefore only belong to $J_p^k(M, N)$ if $p = p'$, and the same goes for $J^k(M, N)_q$ with the condition that $f(p) = q$. In particular a function f can only have its jet in $J^k(M, N)_q$ if q is in the image of f .

The intersection of those two subsets is the subset of jets with a given source and target :

$$J_p^k(M, N)_q = J_p^k(M, N) \cap J^k(M, N)_q \quad (20.15)$$

Define the case $k = 0$ Define how it works with C^k functions

Jets of fixed source do not depend on the topology of M , so for any subset it is equivalent. In particular, $J_p^k(M, N) = J_p^k(\mathbb{R}^n, N)$ [Check if it's true for any other cases]

As defining

Theorem 20.6. For two functions $f, f' : M \rightarrow N$ and $g, g' : N \rightarrow P$ such that $j_p^k f = j_p^k f'$ and $j_q^k g = j_q^k g'$ with $f(p) = f'(p) = q$, then $j_p^k(g \circ f) = j_p^k(g' \circ f')$.

Proof.

□

Composition of jets

Invertible jets

Theorem 20.7. A jet is invertible if and only if its 1-jet $\pi_1^k X$ is invertible.

Proof. Implicit function theorem

□

Theorem 20.8. For a diffeomorphism $f : M \rightarrow M'$ and a map $g : N \rightarrow N'$

$$J^k(f, g) : J^k(M, N) \rightarrow J^k(M', N') \quad (20.16)$$

$$X \mapsto (j_q^k g) \circ X \circ (j_q^k g)^{-1} \quad (20.17)$$

$$p = \alpha(X), q = \beta(X)$$

J^k is a function on the product category of manifolds

Jets and polynomials

Composition law of jets [35]

20.1.1 Euclidian jets

As every manifold is locally Euclidian, it is useful to look at the case of Euclidian jets, the space $J^k(\mathbb{R}^n, \mathbb{R}^m)$.

In the Euclidian case, we can set a global Cartesian coordinate chart by the identity map, in which case a rather simple form of coordinate expression for jets is simply

$$j^k f = \quad (20.18)$$

we can directly use the curves given to us by the Cartesian coordinates [show that it's enough]

$$j_x^k f = (x, f(x), \frac{\partial f(x)}{\partial x^\mu}) \quad (20.19)$$

Given any C^k function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Definition as the quotient of the ideal of germs with 0 derivatives of order $k + 1$

Theorem 20.9. The jets of fixed source and target $J_0^k(\mathbb{R}^n, \mathbb{R}^m)$ form the set of [...]

20.1.2 Jet coordinates

Coordinate charts are themselves maps between manifolds, $\phi : M \rightarrow \mathbb{R}^n$, so that jets will also apply to them. The coordinate form of a given function

$$\mathbb{R}^n \xrightarrow{\phi^{-1}} M \xrightarrow{f} N \xrightarrow{\psi} \mathbb{R}^n \quad (20.20)$$

We can consider the Euclidian jet of the function $\psi \circ f \circ \phi^{-1}$,

$$j_x^k(\psi \circ f \circ \phi^{-1}) = \quad (20.21)$$

Diffeomorphisms on the coordinate jet

In particular, if we call the intersection point $\gamma_1(0) = \gamma_2(0) = p$, if we pick an open set $U \ni p$ with a coordinate map $\phi_U : U \rightarrow \mathcal{O} \subset \mathbb{R}^n$, taking the components of this coordinate map $x^\mu = \text{pr}_\mu \circ \phi_U : U \rightarrow \mathbb{R}$, this implies that for any local coordinate system, all the components of their derivatives up to order k will be equivalent.

The equivalence class of curves at k -th contact at the same point p , if, given a representative curve γ , will be the set of equivalences $[\gamma, p]$, forms the tangent bundle of order k at p , denoted $T_p^k M$, with the tangent bundle of order k being their disjoint union

$$T^k M = \sqcup_{p \in M} T_p^k M \quad (20.22)$$

the bundle being $\pi^k : T^k M \rightarrow M$ such that $\pi^k([\gamma, p]) = p$.

For an atlas $\{U_\alpha, \phi_\alpha\}$

This will in particular be true for diffeomorphisms, and therefore the contact relationship is a diffeomorphism invariant.

Under a curve reparametrization $\gamma' = \gamma \circ h$, we have that

$$\left. \frac{d^k}{d\lambda^k} [\gamma'(\lambda)] \right|_{\lambda=0} = \left. \frac{d^{k-1}}{d\lambda^{k-1}} [(\dot{\gamma} \circ h(\lambda)) \dot{h}(\lambda)] \right|_{\lambda=0} \quad (20.23)$$

The k -th order contact can also be applied to more general maps between manifolds. For two maps $f, g : M \rightarrow N$, those functions are in k -th order contact such that for every curve $\gamma : I \subset \mathbb{R} \rightarrow M$, with $\gamma(0) = p$,

In the case of the spaces $\mathbb{R}^p$, we will consider the jet spaces denoted by

$$A_{p,q}^k = J^k(\mathbb{R}^p, \mathbb{R}^q) \quad (20.24)$$

Roughly corresponding to the space of polynomials from

An important class of jet spaces on a manifold M is the bundle of affine m^k -velocities, denoted $AT_m^k M$, such that

$$AT_p^k M = J^k(\mathbb{R}^m, M) \quad (20.25)$$

An element of $AT_m^k M$ is denoted by the jet $j^k X$ of some map $X : \mathbb{R}^m \rightarrow M$

For $k = 1$, this corresponds to

$$\pi_0^1 : J^1(\mathbb{R}, M) \rightarrow M \quad (20.26)$$

Affine bundle atlas : ATM has bundle

The jet space between two manifolds $J^k(M_1, M_2)$ of dimensions n_1, n_2 has a natural bundle structure defined by

$$(J^k(M_1, M_2), (\alpha, \beta), M_1 \times M_2, A_{n_1, n_2}^k) \quad (20.27)$$

and, trivially, this can also define a bundle structure over M_1 or M_2 by restricting the projection to either α or β . If we pick the manifold over M_2 by using β as our projection function, the jets with target $p \in M_2$ form the fiber at p , $F_p = J(M_1, M_2)_p$

[...]

Lift of an object : for any p -brane, there exists a lift of it to the bundle of $(p+1)^k$ -velocities by taking the map $X : \Xi \rightarrow M$ and defining the lift

$$X^{\uparrow k} = j^k(X) \quad (20.28)$$

such that $X^{\uparrow k} : \Xi \rightarrow T_{p+1}^k M$. In particular, for a curve, $\gamma^{\uparrow k}$ maps to the k -th order tangent bundle $T^k M$.

Bundle of velocities, as affine bundles and vector bundles

Given our tangent bundle $TM = J^1(\mathbb{R}, M)$, we will consider three prolongations of it. First is the non-holonomic prolongation, the double tangent bundle

$$T^2 M = T(TM) = J^1(\mathbb{R}, J^1(\mathbb{R}, M)) \quad (20.29)$$

The double tangent bundle has the obvious properties one could expect from a tangent bundle,

with local coordinates $(x, v, \dots)$

We can consider two subbundles of $T^2 M$, the first being the semi-holonomic prolongation of the tangent bundle.

The last prolongation will be the holonomic one, also called the second order tangent bundle

Spray : vector field $H \in \Gamma(TTM)$ such that $(\pi_{TM})_*$

Semisprays, sprays, connections

jet group

A jet $X \in J^k(M, N)$ is said to be invertible if there exists another jet $X^{-1} \in J^k(N, M)$

such that $X \circ X^{-1} = j^k()$

...

We can now consider the space of functions from $\mathbb{R}^p$ to our manifold, $C^\infty(\mathbb{R}^p, M)$. Most important among these, we will look at the case $p = 1$ (the space of curves), and $p = n$ (which can correspond to coordinate patches). The case $p = 0$ will simply be the spacetime points and any $1 < p < n$ case will have some use for submanifolds.

Velocity bundle as an affine space

While we could consider jets of arbitrary orders, second order will be usually the highest one we will need to consider, as it will relate to connections.

$$\bar{T}^2 M = J^2(\mathbb{R}, M) \quad (20.30)$$

20.1.3 Jets from ringed spaces

As manifolds can also be considered not as topological spaces with an atlas but as locally ringed spaces, we can also consider our jets from this perspective.

A manifold M , as a locally ringed space $(M, \mathcal{O}_M)$ with the sheaf of ring $\mathcal{O}_M$ from coordinate neighbourhoods U to the rings $C^k(U)$

Diagonal map $\Delta_n : M \rightarrow \prod_{i=1}^{n+1} M$. $\mathcal{I}$ is the sheaf of smooth germs which vanish on Δ_n for $0 < n \leq k$. The pullback $\Delta_n^*(\mathcal{I}/\mathcal{I}^n)$ is the sheaf of k -jets. [46]

20.1.4 Jets from Weil algebras

Another way of considering jets on a manifold is via the perspective of Weil algebras, which are in some sense the formalization of infinitesimal elements.

The basic example of a Weil algebra is that of the *dual numbers* $[\varepsilon]$, the extension of the ring $\mathbb{R}$ by the element ε obeying $\varepsilon^2 = 0$

$$a + \varepsilon b \quad (20.31)$$

Quotient $\mathbb{R}[X]/\langle X^2 \rangle$

$$(a + \varepsilon b)(c + \varepsilon d) = ac + (ad + bc)\varepsilon \quad (20.32)$$

This allows us to define the differentiation

$$f'(x) = f() \quad (20.33)$$

For any analytic function,

$$f(x + \varepsilon y) = f(x) + yf'(x)\varepsilon \quad (20.34)$$

This can be generalized to a wide variety of derivatives via the notion of a Weil algebra [47]

Definition 20.10. A *Weil algebra* A is a commutative unital $\mathbb{R}$ -algebra

Infinitesimally thickened points

Prolongation of smooth functions

20.2 Vectors and dual vectors

Vectors correspond to the first jet of a 1-submanifold, or a curve in other words, with fixed source point. As the global topology of the space does not matter to the jet space of fixed source, we can simply consider it as the jet of the real line $J_0^1(L, M)$, as the vectors of any other type of curve will be equivalent to some jet from a curve $\gamma : L \rightarrow M$.

Their jet space $J_0^1(L, M)$ is referred to as the *tangent bundle* (we will see later 27 on what a bundle is exactly), denoted by TM . If we restrict the space of those jets to jets of constant target, $J_0^1(L, M)_p$, corresponding to jets of curves mapping $\lambda = 0$ to $\gamma(0) = p$, this is the *tangent space* at p , $J_0^1(L, M)_p = T_pM$. The tangent space at p will correspond to all the possible directions and magnitude of a curve at that point,

As a set, the tangent bundle is therefore the disjoint union of all the tangent spaces at every point of M :

$$TM = \bigsqcup_{p \in M} T_pM \quad (20.35)$$

Theorem 20.11. The tangent space T_pM has the structure of a real n -dimensional vector space for every p .

Proof. If we ignore the equivalence relation for now, the elements of T_pM can be considered as the set of derivatives of every possible curve going through p at $\lambda = 0$ in some coordinate chart (U, ϕ_U) . For a given curve and chart, a vector is

$$v_{\gamma, U, p} = \left(\frac{d}{d\lambda} \phi_U \circ \gamma(\lambda) \right) |_{\lambda=0} \quad (20.36)$$

□

Old stuff :

We can then define a tangent vector in the usual way for curves in $\mathbb{R}^n$. The tangent vector to the curve γ in the chart U at the point $p = \gamma(u)$ is defined by

$$v_{\gamma,U,p} = \left(\frac{d}{d\lambda} \phi_U \circ \gamma(\lambda) \right) \Big|_{\lambda=u} \quad (20.37)$$

with $v \in \mathbb{R}^n$. Because of this, any manifold is required to be at least C^1 to admit vectors at every point.

For the expression in another chart V such that $V \cap U \neq \emptyset$,

$$v_{\gamma,V,p} = \left(\frac{d}{d\lambda} \phi_V \circ \gamma(\lambda) \right) \Big|_{\lambda=u} = \left(\frac{d}{d\lambda} \tau_{UV} \circ \phi_U \circ \gamma(\lambda) \right) \Big|_{\lambda=u} \quad (20.38)$$

So that, by the chain rule, we have

$$v_{\gamma,V,p} = \frac{d\tau_{UV}}{d\phi_U(p)} v_{\gamma,U,p} \quad (20.39)$$

If the chart U has the coordinate system $x^\mu(p) = \phi_U^\mu(p)$ and the chart V has the coordinate system $y^\mu(p) = \phi_V^\mu(p)$, the transition map will be the expression of y^μ in terms of x^μ , $\tau_{UV}(p) = y^\mu(x^\nu(p))$. This corresponds to

$$\frac{d\tau_{UV}}{d\phi_U(p)} = \frac{\partial y^\mu(x)}{\partial x^\nu} \quad (20.40)$$

This is the Jacobian matrix of the transformation of coordinates, as one could expect from a change of coordinates.

If we take a slightly more general transformation, with the use of a diffeomorphism $f \in \text{Diff}(M)$, the coordinate expression of the curve in $f(M)$ will be $f \circ \phi_U \circ \gamma(\lambda)$

A tangent vector at $p \in M$ is an equivalent class of vectors v where two curves at p give rise to the same vector in the same chart if $v_{\gamma,V,p} = v_{\gamma',V,p}$, and vectors in two different charts are equivalent if they obey the transformation law.

The set of all such tangent vectors at p is the tangent space, denoted $T_p M$.

Proposition 20.12. The tangent space has the structure of a vector space.

Proof. Check that tangent vectors obey tangent space rules :

- The product of a tangent vector X by a scalar $a \in \mathbb{R}$ is a tangent vector : Consider the reparametrization of a curve with tangent vector X by $\lambda \rightarrow a\lambda$. By the chain rule, aX is the tangent vector of this curve.
- The sum of two tangent vectors is a tangent vector : Consider the curve in a small enough neighbourhood of p formed by the sum of the components of $\gamma_1(\lambda)$ with tangent X_1 and $\gamma_2(\lambda)$ with tangent X_2 . By the linearity of the derivative the resulting curve has tangent vector $X_1 + X_2$.

□

Proposition 20.13. The vector space structure is preserved by diffeomorphism.

Proof. bla

- Multiplication by a scalar : If we have
- Sum of two vectors :

□

20.2.1 Basis

Since the tangent space at p is a (finite-dimensional) vector space, it can be equipped with a basis. Arbitrary basis are noted by a set of n tangent vectors (e_μ) . We can show that given coordinates in a patch around p , there exists a basis formed from them, called the coordinate basis, or holonomic basis.

For a point p in a chart (U, ϕ) , consider the family of n curves γ_ν defined by the coordinates $\phi^\mu \circ \gamma_\nu(\lambda) = (\phi^0(p), \phi^1(p), \dots, \phi^\nu(p) + \lambda, \dots, \phi^{n-1}(p))$, where only the ν -th coordinate varies with respect to λ (this correspond to the axis of that coordinate). This is called a coordinate curve. With this we can define a set of n tangent vectors :

Definition 20.14. A coordinate (or holonomic) basis vector ∂_ν at p for the coordinate chart (U, ϕ) is the tangent vector defined by the tangent of the ν -th coordinate curve going through p .

Theorem 20.15. The set of all coordinate basis vectors at p forms a basis for the tangent at p .

Proof. For any curve γ for which a derivative exists at p , since the curve is C^1 at p , by Taylor's theorem, we can expand around λ_p , the value for which $\gamma(\lambda_p) = p$:

$$\phi \circ \gamma(\lambda) = \phi(p) + (\lambda - \lambda_p)X(\lambda_p) + (\lambda - \lambda_p)^2 h(\lambda_p) \quad (20.41)$$

Since $X(\lambda_p)$ can be decomposed in some basis of $\mathbb{R}^n$, pick the canonical basis $\{e_\mu\}$ and then

$$\phi(p) + \lambda X(\lambda_p) = \phi(p) + \sum_{\mu=0}^n \lambda X^\mu e_\mu = \sum_{\mu=0}^n X^\mu \phi \circ \gamma_\mu(\lambda) \quad (20.42)$$

So that the derivative becomes

$$\begin{aligned} X &= \left[\sum_{\mu=0}^n X^\mu \frac{d}{d\lambda} (\phi \circ \gamma_\mu(\lambda)) + 2(\lambda - \lambda_p) \frac{d}{d\lambda} h(\lambda_p) \right]_{\lambda=\lambda_p} \\ &= \sum_{\mu=0}^n X^\mu \partial_\mu \end{aligned}$$

□

Hence any tangent vector can be decomposed, in Einstein notation, as $X = X^\mu \partial_\mu$, with X^μ called the (holonomic) components of the tangent vector. By some abuse of notation, X^μ will often be referred to as the vector itself.

The coordinate basis vectors transform in the same way as any vector

Transformation of components :

$$X^\mu \partial_\mu \rightarrow X^\mu J^{\mu'}_{\mu} \partial_{\mu'} \quad (20.43)$$

One thing to keep in mind is that while the basis defined by a coordinate system (U_α, ϕ_α) exists independently of any coordinates (as those are simply vectors which are a coordinate-independent notion), the notion of these being a *holonomic* basis is. If we have two basis $(e_\mu), (e'_\mu)$ associated with ϕ, ϕ' , those two will be a basis of the tangent bundle no matter the coordinates that we pick (or even without any coordinates at all), but those will only form a holonomic basis respectively relative to the coordinates ϕ or ϕ' .

holonomic coordinates, holonomic frames and coframes

We can then define vector fields as the associations of a vector to every point of the manifold. While it is possible to define C^k vector fields with this definition of vectors (for instance by defining them as the vectors associated to a C^k foliation of the manifold by

curves), it will be much simpler to do so with the definition of vectors as derivatives in the next section.

The space of all vector fields is denoted by $\mathfrak{X}(M)$

continuous, C^k vector fields

Definition 20.16. The *support* $\text{Supp}(X)$ of a vector field X is the close of the set on which its complement vanish :

$$\text{Supp}(X) = \{p \in M | X_p \neq 0\} \quad (20.44)$$

If the support of a vector field is compact, it is said to be a *compactly supported vector field*.

Integral curves

Flow of vector fields

Flow box theorem

Theorem 20.17. For a nowhere vanishing vector field X , there exists a coordinate chart (U, ϕ) such that X is the holonomic basis of the first coordinate of the coordinate map ϕ .

Proof.

□

kinematic v. geometric v. analytic vector fields

Vectors as derivations

Cattaneo for equivalence

Vectors as derivatives of diffeomorphisms

Exponential map

Vector fields form an $\mathbb{R}$ module

Affine tangent bundle ATM

Regularity : Define C^k vector fields and dual vector fields

20.2.2 Dual vectors

Just as vectors in a vector space have a corresponding notion of dual vectors as linear maps, a similar notion applies to vectors and vector fields on a manifold. Trivially we can define

As geometric objects, vectors and dual vectors on a manifold are connected via a duality similar to those on a vector space. While vectors are the jets of maps taking the form of curves $L \rightarrow M$, dual vectors are the jets of maps taking the form of scalar functions $M \rightarrow L$.

Definition 20.18. A *dual vector* is a first jet of a scalar function $f \in C^1(M)$

20.2.3 Left and right derivatives

As we will also consider curves that are only piecewise C^1 , it will be useful to also require the usual notion of left and right derivative.

The left tangent vector $X^- = \dot{\gamma}^-$ is the vector defined by, in the coordinate patch U ,

$$X^-(\lambda) = \lim_{h \rightarrow 0^-} \frac{1}{h} [\phi_U \circ \gamma(\lambda + h) - \phi_U \circ \gamma(\lambda)] \quad (20.45)$$

while the right tangent vector $X^+ = \dot{\gamma}^+$ is the vector defined by, in the coordinate patch U ,

$$X^+(\lambda) = \lim_{h \rightarrow 0^+} \frac{1}{h} [\phi_U \circ \gamma(\lambda + h) - \phi_U \circ \gamma(\lambda)] \quad (20.46)$$

In the case of a piecewise C^1 curve, we may have some point $p = \gamma(\lambda)$ that $X^+(\lambda) \neq X^-(\lambda)$. Such a point is called a corner of the curve. Further properties on corners will be defined once the metric tensor is defined.

20.3 Higher dimensional jets

The typical jets of interest are either the 1-jets, corresponding to curves and vectors, or the n -jets, corresponding to neighbourhoods and frames. However jets can be of any dimension

The space of jets $J^k(S, M)$ for an r -dimensional manifold

20.4 Contact elements

One issue of jets is that they involve the parametrization of the submanifold considered, something which can be problematic in some circumstances. We would not necessarily expect two curves with the same image in the manifold to be different, depending on the

context, but jets vary up to diffeomorphism on the submanifold. For $f \in \text{Diff}(S)$ and $g : S \rightarrow M$, by the chain rule :

$$j^k(f \circ g) = j^k(f) \circ j^k(g) \quad (20.47)$$

For instance in the case of curves that we saw, the reparametrization by some diffeomorphism $h \in \text{Diff}(L)$ gives us

$$j^k(\gamma \circ h) = j^k(\gamma) \circ j^k(h) \quad (20.48)$$

A *contact element* $[X]$ is an equivalence class of jets $X \in J^k(M, N)$ up to the group of diffeomorphisms of the source manifold, $\text{Diff}(M)$.

Contact element, manifold of jet immersions, manifold of jets of submanifolds, grassmannian manifold

Equivalence classes of immersed k -dimensional submanifolds m -dimensional subspaces of tangent spaces
Equivalence classes of non-zero decomposable tangent k -vectors

Just as there are four main possible types of curves we can consider with the isomorphisms of L , there are similarly four types of contact elements we can consider. The usual jet $j^k f$, the contact element $[j^k f]$, the oriented contact element $[j^k f]^\pm$, and the [direction jet?]

In the case of vectors, a contact element will correspond to the direction of a vector without considering its norm

There are in fact four kinds of jets we can consider, depending on the equivalence class we define with respect to the group under consideration : either the trivial group $\{e\}$, the group of orientation-changing transformations $\mathbb{Z}_2$, the group of orientation-preserving diffeomorphisms $\text{Diff}^+(M)$, or both as the full diffeomorphism group $\text{Diff}(M)$. These will correspond, in the 1-dimensional case, to vectors, unoriented vectors (also called line elements), oriented 1-distributions and 1-distributions.

Theorem 20.19. The space of r -contact elements of first order of fixed source and target is isomorphic to the space of Grassmannians $\text{Gr}(r, \mathbb{R}^n)$.

Proof. □

Corrolary 20.1. The space of 1-contact elements of first order, the 1-distributions, is isomorphic to $\text{Gr}(1, \mathbb{R}^n) \cong \mathbb{R}P^n$.

Due to this, the full set of contact elements with fixed source is also called the *Grassmann bundle*

$$G_d M = [J_0^1(\mathbb{R}^d, M)] \quad (20.49)$$

Definition 20.20. The *blow-up* of a vector

Blow up of a vector into a norm and 1-distribution.

20.5 Prolongations

Given some k -th order jet of a map, we can always extend it by considering the jet of this jet in several ways. The most general prolongation we can do is that of the *non-holonomic jet*, where we simply map a jet $X \in J^k(M, N)$ to another $X' \in J^r()$

As the jet of a function contains the function itself, we can

$$j^r j^k \quad (20.50)$$

The more obvious case is that of *holonomic jets*, where the k -jet is simply mapped to an r -jet

The two prolongations are related in the sense that every

Holon. v. semi-holon. v. non-holonomic

Weil algebra?

20.6 Vectors as derivations

[48] A seemingly unrelated notion is that of derivations.

Definition 20.21. A *derivation* on an algebra A of is a linear map $D : A \rightarrow A$ that satisfies the Leibniz rule :

$$D(fg) = D(f)g + fD(g) \quad (20.51)$$

The derivation we are concerned with here is on the algebra of scalar functions $C^\infty(M)$.

Set of all derivations : $\text{Der}(C^\infty(M))$, the $C^\infty(M)$ module of derivations.

Derivations as $J^1(M, \mathbb{R})$?

Derivations on a function vanishing outside a region is zero outside that region [cattaneo, wald, etc]

Theorem 20.22. There exists a $C^\infty(M)$ -linear isomorphism between the space of derivations $\text{Der}(C^k(M))$ on a manifold and the space of C^k vector fields $\mathfrak{X}(M)$:

$$\tau : \mathfrak{X}(M) \rightarrow \text{Der}(C^k(M)) \quad (20.52)$$

Proof. □

Definition 20.23. A vector field X is said to be a C^k vector field for $k \in \mathbb{N} \cup \{\infty\}$ if for any smooth scalar function $f \in C^\infty(M)$, $X[f] \in C^k(M)$.

20.7 The pushforward

If we have a map ϕ between two manifolds M and N , we can map scalar fields to N via the pullback. In addition, it is possible to uniquely map vector fields via the pushforward, or differential map.

Definition 20.24. For a map $\phi : M \rightarrow N$ a C^k map between two C^k manifolds, and a C^{k-1} vector field X on M , the pushforward of X by ϕ is done by the *differential map* $d\phi$, defined by

$$\begin{aligned} d\phi : TM &\rightarrow TN \\ [d\phi(X)](f) &\mapsto X(\phi^*f) \end{aligned}$$

Described by the commuting diagram

$$\begin{array}{ccc} TM & \xrightarrow{d\phi} & TN \\ \downarrow \pi_{TM} & & \downarrow \pi_{TN} \\ M & \xrightarrow{f} & N \end{array}$$

Figure 20.2: Diagram of a differential map

The differential map is also sometimes noted by ϕ_* or Tf , but we will stick with the $d\phi$ notation here.

If we get a basis ∂_μ in a coordinate patch of TM and a basis ∂_a in a coordinate patch of TN , the components of the differential maps are

$$[df(X^\mu \partial_\mu)] = [df(\partial_\mu)]^a X^\mu \partial_a \quad (20.53)$$

Written as

$$[df(\partial_\mu)]^a = df_\mu^a \quad (20.54)$$

$\dim(M) \times \dim(N)$ dimensional matrix

Definition 20.25. The rank of a map f at p is the dimension of its image

$$\text{rank}_p(f) = \dim(\text{Im}(T_p f)) \quad (20.55)$$

if f is of maximal rank, $T_p f$ is injective if $\dim(M) \leq \dim(N)$, surjective if $\dim(N) \leq \dim(M)$, f is resp. an immersion, submersion

Rank is lower semicontinuous, $T_p f$ is of maximal rank on an open neighbourhood. If f is an immersion/submersion at p , it is locally injective/surjective around p

If both, local diffeomorphism.

Immersion/submersion if immersion/submersion at every $p \in M$.

Properties of the differential :

$$d : C^\infty \rightarrow \mathfrak{X}^*(M) \quad (20.56)$$

d is linear, obey the Leibniz rule, chain rule

Definition 20.26. The *pushforward* df of a map between two manifolds $f : M \rightarrow N$

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \pi_{TM} \downarrow & & \downarrow \pi_{TN} \\ M & \xrightarrow{f} & N \end{array}$$

Rank

Isomorphisms and Jacobians

Example of curve

Example of diffeomorphism

The pushforward allows us to define the *tangent functor*, from the category of manifolds to the category of tangent manifolds,

$$T : \mathbf{Man} \rightarrow \quad (20.57)$$

proof that it is a functor

20.8 The Lie bracket

As vectors fields can be defined to be derivations on the space of smooth functions, it is easy to attempt to define some kind of algebra on them. We can look at the object given by some product XY defined by

$$(XY)[f] = X(Y(f)) \quad (20.58)$$

This is not however a proper derivation, as we can see by looking at its application to products :

$$X(Y(fg)) = X(Y(f)g + fY(g)) \quad (20.59)$$

$$= X(Y(f)g) + X(fY(g)) \quad (20.60)$$

$$= X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y(g)) \quad (20.61)$$

$$= XY(f)g + fXY(g) + Y(f)X(g) + X(f)Y(g) \quad (20.62)$$

The composition of two such derivations does not obey the Leibniz rule, and therefore is not itself a derivation (or at least one of first order).

If we look at it in terms of coordinates,

[...]

XY is a second jet?

Most general term of second order that will be a vector

$$\begin{aligned} \alpha XY(fg) + \beta YX(fg) + \gamma X(fg) + \delta Y(fg) &= \alpha XY(f)g + \alpha fXY(g) \\ &\quad + (\alpha + \beta)Y(f)X(g) \\ &\quad + (\alpha + \beta)X(f)Y(g) \\ &\quad + \beta YX(f)g + \beta fYX(g) \\ &\quad + \gamma X(f)g + \gamma fX(g) + \quad (20.63) \end{aligned}$$

$$= \quad (20.64)$$

To obtain the Leibniz identity, we'd need to get rid of those $(\alpha + \beta)$ terms, so that we have $\alpha + \beta = 0$. Without loss of generality we can pick $\alpha = 1$, $\beta = -1$. Other values of α and β can be part of the algebra by considering a scalar multiplication of $[X, Y]$ which is also part of the algebra.

The tangent space at a given point $T_p M$ therefore has an algebraic structure given by the *Lie bracket* :

Definition 20.27. The Lie bracket $[X_p, Y_p]$ is an algebraic operation defined between two vectors $X_p, Y_p \in T_p M$ given by the identity

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad (20.65)$$

In terms of jets of a curve, [Interpretation as flow]

The Lie bracket obeys the *Jacobian identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \quad (20.66)$$

For the holonomic basis ∂_μ given by the coordinates x^μ , the value of the Lie bracket

$$[\partial_\mu, \partial_\nu] \quad (20.67)$$

Many of the brackets we will see throughout this book will be either Lie brackets or some generalization thereof.

$$[f_* X, f_* Y] = f_* [X, Y] \quad (20.68)$$

"If X, Y two vector fields then $[X, Y]$ is the vector field corresponding to the derivation $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$, $\mathcal{L}_{[X, Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$

20.9 Vector fields and diffeomorphisms

As we've seen, every 1-parameter family of diffeomorphism connected to the identity defines a curve when applied to a single point, and a family of curves (possibly trivial) in general. If applied to a neighbourhood, it will generate a family of curve going through

Each of these curves define a vector, so that we can define a vector for every such family of diffeomorphism by

$$v_{\phi_{t'} p} = \frac{d}{dt} \phi_t(p) \Big|_{t=t'} \quad (20.69)$$

The inverse map of this process is the exponential map :

Definition 20.28. The *exponential map* is a map from vector fields $\mathfrak{X}(M)$ to the diffeomorphism group $\text{Diff}(M)$

20.10 Lie derivatives and vector flow

While vectors were constructed from the jets of curves, we have not proven yet the converse that a vector field will correspond in some sense to curves possessing such jets. This is what the concept of flow and integral lines will give us.

For any vector field X and a chosen point $p \in M$, we can find in some neighbourhood U of p a curve γ such that $\dot{\gamma}$ is everywhere equal to the value of X . This is the curve obeying the differential equation

$$X(\gamma(\lambda)) = \dot{\gamma}(\lambda) \quad (20.70)$$

With initial conditions $\gamma(0) = p$, $\dot{\gamma}(0) = X(p)$. In terms of a given coordinate patch around p , this is the differential equation

$$X^\mu(x^\alpha(\lambda)) = \frac{d}{d\lambda}x^\mu(\lambda) \quad (20.71)$$

By Peano's theorem, if X^μ is a continuous function of $x^\mu(\lambda)$, there exists a solution in some neighbourhood $\lambda \in (-\varepsilon, \varepsilon)$. By the Picard–Lindelöf theorem, if X^μ is uniformly Lipschitz continuous, that curve is locally unique given some initial point p such that $\gamma(0) = p$, $\gamma'(0) = X(p)$.

This curve is invariant under coordinate change, as changing the coordinates will simply change the vector field up to the Jacobian

[...]

This uniqueness allows us to define the flow of a vector field :

Definition 20.29. Given a vector field on $U \subset M$, the *flow* of a point p by the vector field X at t is the point $\gamma(t)$ of the integral curve γ of tangent X for $\gamma(0) = p$. It is noted by the function

$$\begin{aligned} \Phi : M \times \mathfrak{X}(M) \times I &\rightarrow M \\ (p, X, t) &\mapsto \text{Fl}_t^X(p) = \gamma(t) \end{aligned}$$

where I is the range of $\mathbb{R}$ around 0 for which this curve is defined.

The reason for this interval not being necessarily $\mathbb{R}$ can be seen if we look at some 1-dimensional examples. If we have some vector fields on the line $\mathfrak{X}(L)$, given in Cartesian coordinates,

$$X = f(x)\partial_x \quad (20.72)$$

Example of why the flow is only defined on an interval

Definition 20.30. A vector field is *complete* if for any $p \in M$, there is an integral curve with its domain being $\mathbb{R}$.

Theorem 20.31. Compactly-supported vector fields are always complete.

Smooth vector fields on compact manifolds are always complete

We can drop the dependence of our flow on the initial point of our manifold to instead

[49]

Properties of flow :

$$\text{Fl}_0^X = \text{Id} \quad (20.73)$$

If we have $t, s \in I$ such that $t + s \in I$, we also have

$$\text{Fl}_{t+s}^X = \text{Fl}_t^X \circ \text{Fl}_s^X \quad (20.74)$$

If $t, -t \in I$, we also have

$$\text{Fl}_{-t}^X = (\text{Fl}_t^X)^{-1} \quad (20.75)$$

since we have that $t + (-t) = 0$, therefore $\text{Fl}_t^X \circ \text{Fl}_{-t}^X = \text{Fl}_{-t}^X \circ \text{Fl}_t^X = \text{Id}_M$.

$\mathbb{R} \times M$

For an object ω

$$\mathcal{L}_X \omega = \left[\frac{d}{dt} \exp(tX)^* \omega \right]_{t=0} \quad (20.76)$$

Relate Lie derivative and integration of vector fields flows etc

Cartan's magic formula

$$\mathcal{L}_X = d\iota_X + \iota_X d \quad (20.77)$$

$$\mathcal{L}_X\omega = d\iota_X\omega + \iota_Xd\omega \quad (20.78)$$

Involutive distributions

20.11 The jet functors

Jets can be defined in categorical terms

The *jet functor*

20.12 Jets on diffeological spaces

Quite a lot of what we have defined so far has not used any properties specific to manifolds, and could therefore be used in the more general case of diffeological spaces.

Vectors on function spaces etc

Chapter 21

Tensor fields and differential forms

21.1 Dual vector fields

As with vector spaces, we have a dual for the space of vector fields $\mathfrak{X}(M)$. A dual vector at a given point p will simply be the same dual vector as for a vector space V , a vector morphism $\omega_p : T_p M \rightarrow \mathbb{R}$. A *dual vector field* similarly will be a linear map from vector fields to scalar fields,

$$\alpha : \mathfrak{X}(M) \rightarrow C^\infty(M) \quad (21.1)$$

which, at a point, will evaluate to a dual vector on $T_p M$.

The space of dual vector fields is the space of 1-forms,

$$\Omega^1 M = \text{Hom}_{C^\infty}(\mathfrak{X}(M), C^\infty(M)) \quad (21.2)$$

We saw previously that we can define the pushforward of a manifold map $f : M \rightarrow N$. In the case of a scalar function $f : M \rightarrow \mathbb{R}$, we obtain a special case of this,

$$\begin{array}{ccc} TM & \xrightarrow{df} & T\mathbb{R} \\ \pi_{TM} \downarrow & & \downarrow \pi_{T\mathbb{R}} \\ M & \xrightarrow{f} & \mathbb{R} \end{array}$$

As the tangent bundle of a vector space, $T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$. df therefore gives us

Definition 21.1. The *differential* $d : C^\infty(M) \rightarrow \Omega^1 M$ is a map from scalar functions to 1-forms, such that

$$df \quad (21.3)$$

Dual jets : elements of $J^k(M, \mathbb{R}^r)$

21.2 Tensor fields

As in the case of vector spaces, we can define tensor product of vector fields and dual vector fields to form tensor fields

Definition 21.2. The *tensor product* of

Regularity class of a tensor field

Theorem 21.3. If a tensor field T is of class C^k , it can be expressed by a set of C^k scalar functions $T^{\mu\nu\cdots}_{\alpha\beta\cdots}$.

21.3 Differential forms

Just as in the vector space case, we can define the exterior product for 1-forms :

Definition 21.4. The *exterior product* $\wedge : \Omega^1 M \times \Omega^1 M \rightarrow \Omega^2 M$ is a

The spaces defined by the exterior product form the space of forms on the manifold, $\Omega^\bullet M$

forms and the levi-civita tensor

Definition 21.5.

exterior derivative

Definition 21.6. The *exterior derivative* of order k , $d : \Omega^k M \rightarrow \Omega^{k+1} M$, is a differential operator on the space of k -forms $\Omega^k(M)$ such that

1. The exterior derivative of 0-forms (real functions) is its differential df
2. For a 0-form f , $d(df) = 0$
3. For $\alpha \in \Omega^p M$, and $\beta \in \Omega^k M$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \quad (21.4)$$

(Graded derivative)

Theorem 21.7. The expression of the

Proposition 21.8. The property $d(d\alpha) = 0$ extends for α any k -form.

Proof. As any k -form can be decomposed in some appropriate basis as

$$\alpha = f \bigwedge_{i=1}^k dx_i \quad (21.5)$$

we have that

$$d\alpha \quad (21.6)$$

□

Multivector fields, p -vector fields

Definition 21.9. The *Schouten bracket*

21.4 The Cartan form

A section of the jet bundle could technically resemble just about anything by simply considering some random components that obey the matching conditions on overlaps, as jets can be defined point-wise. However, this would not correspond to our notion that jets correspond in some sense to a map and its derivatives. If we had for instance the jet field consisting of a constant function and some non-zero derivatives, we would not consider this to be in the spirit of its meaning. This jet field $X \in J^k(M, N)$ would not correspond to the jet of any map $f : M \rightarrow N$ such that $X = j^k f$.

To deal with this, we need some consistency condition on the jet bundles. This notion is given by the Cartan distribution

Definition 21.10. For a jet space $J^k(M, N)$, the Cartan distribution $\mathcal{C}$ in the tangent bundle $TJ^k(M, N)$ is a distribution on the tangent bundle $TJ^k(M, N)$ that is spanned by the

Contact form

Theorem 21.11. In terms of local coordinates, a contact form θ on the first jet space $J^1(M, N)$ is given by

$$\theta^a = dy^a - u_\mu^a dx^\mu \quad (21.7)$$

This corresponds to the notion that for the first jet of a function, the contact form puts the first jet coordinate of the section in correspondence with the first derivative of the value of the function.

Example 21.12. For the map $f : \mathbb{R} \rightarrow \mathbb{R}$, where we put the coordinates x and y on the source and target space, and y' for the first jet, the contact form is

$$\theta = dy - y' dx \quad (21.8)$$

Chapter 22

Submanifolds I

Submanifolds are an important part of spacetime, but due to this, they will require to be talked about a few different times, as we will need a few elements that we will only see later on to talk about them fully, but we also cannot skip out talking about them much longer. In this first part, we will first discuss what type of submanifolds exist and their properties.

We briefly saw earlier what a submanifold is, at least in its broadest definition, and we will now see in more details the various kinds of submanifolds one can find as well as what happens to the manifold structures on them.

The curves and paths that we have seen, as well as their jets and contact forms, already quite a nice overview of the kind of behaviours that we should expect from submanifolds.

We can roughly define submanifolds in two ways : first as a subset $S \subset M$ which is endowed with a manifold topology, or as a mapping $f : S \rightarrow M$ (this roughly parallels the notion of a subobject versus a monomorphism in the category of manifolds). What makes S a submanifold is that its topology should in some way depend on that of M .

All submanifolds are defined in the same manner : a manifold N is a submanifold of the manifold M if there exists an inclusion map $\iota : N \hookrightarrow M$. The subset $f(N) \subset M$ is then called the submanifold. The map f is not generally required to be injective, as will be seen with some examples.

22.1 Immersions, submersions and embeddings

As we saw previously in the case of curves (which are simply the case of 1-manifolds), leaving the space of all submanifolds to $C(S, M)$ is in general much too large for most purpose. This space will comprise subspaces $\text{Im}(f)$ that are of the wrong dimension, $\dim(\text{Im}(f)) < \dim(S)$, subspaces that will self-intersect, and subspaces that will fail to have the same subspace topology as the topology of S itself.

There are no general definition of what a proper submanifold is, as the various kinds that we can obtain can have their use in various circumstances. We will therefore have to define a few

Definition 22.1. A smooth map $f : M \rightarrow N$ is an *immersion* if $df(p)$ is a bijection for all $p \in M$.

As a commutative diagram

Equivalent : Jacobian $d\phi_p$ has rank m relative to some coordinate system, a coordinate system $\{y^\mu\}$ of N there are integers such that $\{y^{i(\nu)}\}_{1 \leq \nu \leq n}$ is a coordinate system on a neighbourhood $p \in M$.

Theorem 22.2. An immersed submanifold $f : S \rightarrow M$ has a pushforward df of constant rank at any given point $p \in S$ equal to the dimension of the manifold S : $\text{Rank}(df|_p) = \text{Dim}(S)$

Proof.

□

Roughly speaking an immersion is a map for which the image has the same number of dimension at every point, and where moving in the original space S corresponds to some movement in the image $\text{Im}(f)$. The case of curves that we saw previously gives us a good example, as regular curves ($\gamma' \neq 0$) are immersions :

A degenerate curve will not be one though

[also counterexample of a curve turning around $x = \tau$ for $\tau \in (-\infty, 0)$, $x = -\tau$ for $\tau \in (0, +\infty)$.]

Immersion still allow for some possible issues like self-intersections or almost self-intersections [example of a map of $(0, 1)$ where the boundaries are the same point], such that the topology given by the map f fails to be isomorphic to that of the subspace topology. To give us a notion of

Definition 22.3. An *embedding* of a manifold S in M is an immersion $f : S \rightarrow M$ such that f is a homeomorphism.

The submanifold topology is the same as the subspace topology

Example : embedding of $(a_1, a_2, \dots, a_m) \rightarrow (a_1, a_2, \dots, a_m, 0, 0, \dots)$

Finally a notion dual to that of an immersion exists

Definition 22.4. A smooth map $f : M \rightarrow N$ is a submersion if

Injection maps $\iota : M \hookrightarrow M'$, diffeomorphic on $\iota^{-1}(M')$

Definition 22.5. A subset N is a k -dimensional submanifold of M , $k \leq n$, if there is an atlas of N $\{(U_\alpha, \phi_\alpha)\}$ such that for all α where $U_\alpha \cap N \neq \emptyset$,

Example of manifold from submanifolds : The n -sphere S^n as a submanifold of $\mathbb{R}^{n+1}$

Whitney embedding theorem :

Theorem 22.6. Any smooth, Hausdorff and second-countable manifold of dimension n can be smoothly embedded in $\mathbb{R}^{2n}$.

Given a manifold subset $S \subseteq M$, with an inclusion map $\iota : S \rightarrow M$

Manifold is a submanifold if S is a topological subspace of M and the inclusion map is smooth and $d\iota$ is bijective

Adapted coordinate system

Given a coordinate system on S given by some open set U_S and a coordinate chart $\phi_S : U_S \rightarrow \mathcal{O}_S \subseteq \mathbb{R}^k$, we can

Chapter 23

Foliations 1

A special type of structure on a manifold is a foliation, which is the notion of splitting a given manifold into different manifolds of lower dimensions. The simplest example of this is the foliation of the plane by lines, for instance

[Diagram]

[50]

Definition 23.1. A C^k foliation $\mathcal{F}$ of codimension k of a manifold M of dimension n is a partition of M into connected disjoint submanifolds $\{L_\alpha\}$ (called *leaves*) along with an atlas of *foliated charts* $\{(U_\beta, \phi_\beta)\}$ such that, for any leaf L_α , its connected components with any foliated neighbourhood $L_\alpha \cap U_\beta$ us such that $\text{pr}^\mu \circ \phi_\beta(L_\alpha \cap U_\beta)$ is constant for $\mu > (n - k)$

[Diagram]

Equivalent definition :

Definition 23.2. A foliation $\mathcal{F}$ of codimension k of a manifold of dimension n is given by an atlas $\mathcal{A}_\mathcal{F}$ such that the transition maps

$$\tau_{\alpha\beta} : \mathbb{R}^{n-k} \times \mathbb{R}^k \rightarrow \mathbb{R}^{n-k} \times \mathbb{R}^k \quad (23.1)$$

are such that the projection of the transition map on the second factor $\mathbb{R}^k$ depends only on the second

$$y' = \tau_{\alpha\beta}(y) \quad (23.2)$$

Show that the foliation does a cover of the manifold by submanifolds of dimension k

Example 23.3. For a smooth submersion $f : M \rightarrow B$ with a surjective derivative map at every point, $f^{-1}(p)$ forms a family of disjoint submanifolds for every $p \in B$.

23.1 Complementary foliations

Transversal mapping

Definition 23.4. For two foliations $\mathcal{F}_1, \mathcal{F}_2$ of codimension k_1, k_2 on the same manifold M , we say that $\mathcal{F}_1$ is *transversal* to $\mathcal{F}_2$, denoted $\mathcal{F}_1 \pitchfork \mathcal{F}_2$, if $k_1 + k_2 \leq n$ and any two leaves L_1, L_2 intersect transversally if $L_1 \cap L_2 \neq \emptyset$, the tangent spaces of their leaves span the full tangent bundle :

$$T_p L^1 \oplus T_p L^2 = TM_p \quad (23.3)$$

Where L_1, L_2 are the leaves going through p .

Example of transversal foliations of the plane by curves, example of a non-transversal foliation

23.2 Integrability

Just like for any object with a global and local version, we can wonder about the integrability of foliations. While every foliation defines a field of distributions, the reverse problem of integrability is to find a foliation from a given distribution.

Definition 23.5. A k -distribution Δ is said to be *integrable* if there exists a k -foliation of the manifold M with the distribution Δ everywhere tangent to the leaves.

It turns out that not every distribution is integrable, as can be seen by this example :

Example 23.6. In Euclidian space $\mathbb{R}^3$ with Cartesian coordinates, the distribution spanned by the two vector fields

$$X_1 = \partial_1 + x_2 \partial_3 \quad (23.4)$$

$$X_2 = \partial_2 \quad (23.5)$$

ie, $\Delta = \text{Span}(X_1, X_2)$, is not integrable.

Proof. If Δ is integrable, let's consider the leaf $\mathcal{L}_0$ containing the origin. At this point, the distribution is simply $\alpha \partial_1 + \beta \partial_2$. This leaf must contain the integral curve of X_1 going through the origin, □

[Show something something about subfoliations first]

Frobenius theorem

Integrability by k -blades instead of k -distributions

Corrolary 23.1. Every 1-distribution is integrable.

Proof. This simply follows from every vector in Δ being colinear, therefore $[X_1, X_2] = 0$ for any two vectors, which is always in the distribution. $\square$

Gives a parametrization mb

Godbillon-Vey invariants

Chapter 24

Algebraic topology

Algebraic topology is as the name indicates the application of algebra to topology, where we typically consider some functor from the category of topological spaces to some category of algebraic objects such as groups or rings.

$$F : \mathbf{Top} \rightarrow \mathbf{Alg} \quad (24.1)$$

It is a useful tool for the study of topological spaces and their properties invariant under continuous deformations (homeomorphisms).

24.1 Homotopy

Homotopy is the study of the deformation of continuous map. If we take two continuous maps $f, g : X \rightarrow Y$ for two topological space X, Y , we would like to know if in some sense those two maps can be deformed into one another. This is generally done by simply trying to construct a larger continuous map with those two as edge cases : [51]

Definition 24.1. For two continuous functions $f, g : X \rightarrow Y$, X, Y two topological spaces, a *homotopy* is a continuous map

$$H : X \times [0, 1] \rightarrow Y \quad (24.2)$$

$$(x, t) \mapsto H_t(x) \quad (24.3)$$

(with the canonical topology for $[0, 1]$ and product topology on $X \times [0, 1]$) such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

The space $X \times [0, 1]$ is also called the *cylinder* of the space X , $\text{Cyl}(X)$, named after the case of the space being S^1 in which case it is an actual cylinder.

Using partial application, we can consider a homotopy to be a continuous deformation between the two functions by partial evaluation, $H_t : C(X, Y) \rightarrow C(X, Y)$

If we consider for instance f, g to be two curves in our topological spaces, $X = [0, 1]$, and Y some bog standard space like $\mathbb{R}^n$, our two functions will represent a deformation of the curve defined by f to that of g .



Figure 24.1: Two homotopic curves

A homotopy between the two will therefore represent some kind of plane figure of the form

Deformation retraction, Mapping cylinder

If there exists a homotopy between two functions $f, g \in C(X, Y)$, we say that they are *homotopic*, denoted by $f \cong g$.

Theorem 24.2. Homotopy forms an equivalence relation.

Proof. Every function is homotopic to itself, $f \cong f$, simply by considering the constant function

$$h : X \times I \rightarrow Y \quad (24.4)$$

$$(x, t) \mapsto h_t(x) = f(x) \quad (24.5)$$

And the relationship is symmetric as well. If we have some homotopy h such that $f \cong g$, we can define the inverse homotopy

$$h_t^{-1} : X \times I \rightarrow Y \quad (24.6)$$

$$(x, t) \mapsto h_t^{-1}(x) = h_{1-t}(x) \quad (24.7)$$

which is still a continuous map and does indeed give $h^{-1}(0) = g$, $h^{-1}(1) = f$. If we have two homotopies h_t^1, h_t^2 for the functions $f_1 \cong f_2$ and $f_2 \cong f_3$, we can define a third homotopy h_t^{12} by

$$h_t^{-1} : X \times I \rightarrow Y \quad (24.8)$$

$$(x, t) \mapsto h_t^{-1}(x) = h_{1-t}(x) \quad (24.9)$$

□

A very common type of homotopy we will consider in particular are the homotopies of subspace maps such as curves and surfaces, where we consider the homotopies of maps $C(S, M)$. This will give us more intuitive example than the general case of topological spaces.

Example 24.3. Two curves γ_1, γ_2 are homotopic if their maps $\gamma_1, \gamma_2 : L \rightarrow M$ are homotopic.

endpoint-preserving homotopies

homotopies between homotopies between homotopies etc : ∞ -groupoid

A specific kind of function is the constant function $f(x) = c$, corresponding geometrically to a degenerate figure being a single point. If a function is homotopic to the constant function, we say that it is *null-homotopic*, and the

Contractions

The word homotopy as applied to a space will concern more specifically the behaviour of maps from k -spheres S^k to the topological space under homotopy. The exact object we use doesn't matter much up to homeomorphism, as we are only interested in its topological properties, so that we could use any space homeomorphic to S^k , such as the k -cube, $\partial[0, 1]^k$. We would like to put some algebraic structure on those maps, which we will build using the concatenation of those spheres :

Definition 24.4. If we consider the function space of maps from the pointed k -sphere (S^k, a) (the k -sphere with a base point $a \in S^k$) to a pointed topological space (X, x) (with a base point $x \in X$), its k -th *homotopy group* at x $\pi_k(X, x)$ is the group of

Theorem 24.5. If the space X is path-connected, the homotopy group $\pi_k(X, x)$ is independent of the point x considered :

$$\forall x, x' \in X, \pi_k(X, x) = \pi_k(X, x') \quad (24.10)$$

Proof. For any homotopy group $\pi_k(X, x)$, we can consider all the maps pointed at x and replace them by maps pointed at x' simply by concatenating a k -sphere retracted to a line connecting x and x' and its inverse connecting x' and x . $\square$

As the group is independent of the base point, we can simply write it out as $\pi_k(X)$ for a path connected topological space.

For the manifold M , $\pi_k(M)$ is the set of homotopy classes of maps $f : S^k \rightarrow M$

The 0-th homotopy group $\pi_0(X)$ is then simply the set of maps from the zero-dimensional sphere, composed of two points, to the manifold. Such a function will then map the base point of the 0-sphere to the base point of the manifold, and the other point to another point of the manifold. A homotopy of such a function will be a continuous map $h : S^0 \times I \rightarrow X$ with

The first homotopy group $\pi_1(M)$, also called the *fundamental group*, is the homotopy group of maps of loops into the space

The computation of homotopy group is not necessarily trivial to work out, but there are a few simple cases that we can investigate, first of all being Euclidian space :

Example 24.6. Euclidian space $\mathbb{R}^n$ has for all homotopy groups $\pi_k(\mathbb{R}^n) = I$

Proof. Since it is a topological vector space, the map $h_t(p) = (1 - t) \cdot p$ is continuous and maps every point of the manifold to 0 at $t = 1$, hence all spheres are homotopically equivalent to a point. $\square$

Loop space

In categorical terms, the ho

Seifert-van Kampen theorem

24.1.1 Homotopy quotient

Borel construction

24.1.2 Homotopies in categories

24.2 Sequences and chain complexes

Chain complexes are not an entirely algebraic topological tool, as they are more general than that, but as they are a very common one for it, it can be useful to introduce them here.

A *chain complex* $(A_\bullet, d_\bullet)$ is a sequence of objects (generally groups or modules) $(A_i)_{i \in \mathbb{Z}}$, along with morphisms between them, $d_i : A_i \rightarrow A_{i+1}$, obeying the property $d_i \circ d_{i+1} = 0$, the zero map sending elements of A_i to the zero element of A_{i+2} , such as the neutral element or the trivial module.

$$\dots A_{-2} \xleftarrow{d_{-1}} A_{-1} \xleftarrow{d_0} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3} A_3 \dots \quad (24.11)$$

A cochain complex is a similar structure, but with arrows reversed,

An element $a \in A_i$ for some i of a chain or cochain complex is called a *chain* or *cochain*. For a chain a such that $a \in \ker(d_i)$, a is called a *cycle*, so that a cycle a obeys the property

$$da = 0 \quad (24.12)$$

The set of all i -th cycles is denoted by $C_i = \ker(d_i)$. As it is the kernel of a group homomorphism, it is itself a group^{5.13}, and a normal subgroup of A_i (or a submodule if A_i is a module).

If a chain $a \in A_i$ is in the image of A_{i+1} , such that $a \in \text{im}(A_{i+1})$, it is called a *boundary*, obeying the property

$$\exists a' \in A_{i+1}, a = da' \quad (24.13)$$

The set of all i -th boundaries is denoted by $B_i = \text{im}(d_{i+1})$, and as the image of a group, it is itself a group.

Similarly for cochains we define *cocycles* as cochains belonging to the kernel of d_i , $c_i \in \ker(d_i) \subseteq A^i$.

A special case of a chain element is the case where it is 0. As a zero object, there exists exactly one map $A \rightarrow 0$ and $0 \rightarrow A$ for any A , so that any boundary operator or differential operator will have to be this object, in both case the constant zero map 0 mapping any element of the space to 0. As all elements are mapped to zero, we have that $0 : A \rightarrow B$ has the kernel $\ker(0) = A$, and its image is always $\text{im}(0) = \{0\}$.

Any such zero map will in fact always be an operator. Consider the sequence

$$A \xrightarrow{d} B \xrightarrow{0} 0 \quad (24.14)$$

Then we always have $d \circ 0 = 0$

From these properties, we can always form a chain and cochain just from a few objects and morphisms obeying the properties, simply by extending to $\mathbb{Z}$ with zero objects. If we have some finite sequence

$$A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_k \quad (24.15)$$

We can extend it into a chain complex by

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_k \rightarrow 0 \rightarrow 0 \rightarrow \dots \quad (24.16)$$

A chain is bounded if there exists two integers $a, b \in \mathbb{Z}$, $a < b$, such that every A_i for $i < a$ and $i > b$ is the trivial object

If there are only three non-zero objects in a sequence,

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0 \rightarrow 0 \rightarrow \dots \quad (24.17)$$

this is called a *short exact sequence*

$$\dots 0 \xleftarrow{d_{-2}} A \xleftarrow{d_{-1}} B \xleftarrow{d_0} C \xleftarrow{d_1} 0 \dots \quad (24.18)$$

$$0 \hookrightarrow B_n \hookrightarrow Z_n \hookrightarrow C_n \quad (24.19)$$

Given a chain complex $A_\bullet$ in which the objects are vector spaces, we can define an underlying differential graded vector space denoted by $A^\sharp$, such that

$$A^\sharp = \bigoplus_{i \in \mathbb{Z}} A_i \quad (24.20)$$

in which case the differentials d_i are maps of degree -1 . The same applies to cochains except in this case, the differentials are of degree $+1$.

”The category of chain complexes of vector spaces carries the tensor product of chain complexes and a braiding which makes it a symmetric monoidal category. The corresponding commutative monoids are the differential graded-commutative algebras.”

An important theorem for chains is the *splitting lemma* :

Theorem 24.7. Given a short exact sequence

$$0 \rightarrow A \xrightarrow{q} B \xrightarrow{r} C \rightarrow 0 \quad (24.21)$$

the following statements are equivalent :

1. There exists a morphism $t : B \rightarrow A$ such that $t \circ q$ is the identity on A
2. There exists a morphism
- 3.

Proof.

□

Snake lemma 

Theorem 24.8. Snake lemma

Category of chains : $A^\bullet \in \mathbf{Ch}$

product of complexes

Chain complex map : functor

24.2.1 Bicomplexes

Chain complexes can themselves be the objects of a chain complex, where every object in the chain is a different chain complex. Each object is denoted by $C_{i,j}$, the i -th object of the j -th chain complex.

[diagram]

From X , we have that every such chain can be considered as a graded module (or group?), and every chain complex map is our

Trivial chain complex :

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \dots \quad (24.22)$$

24.3 Cell complex

A cell complex is an abstraction of a polytope (the n -dimensional generalization of a polygon), objects made of various n -dimensional elements which contain lower dimensional elements in some specific ways.

Depending on the properties we wish the cell complex to obey, there are quite a large number of possible definitions of a cell complex. We can allow for instance complexes which are "free floating" on a given vertex :

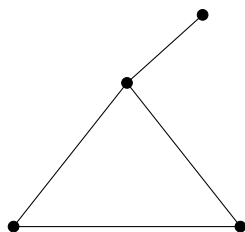


Figure 24.2: Some examples of simplexes

The complex can have disconnected elements, it can have self-intersections, be a degenerate object.

Definition 24.9. An *abstract simplicial complex* is a set X closed under subsets.

In other words, if we have $A \in X$, and $S \subset A$, then $S \in X$. Due to this, $\emptyset \in X$ for any simplicial complex.

Each element of X is called a simplex. If $A \in X$ and $\text{Card}(A) = 1$, then A is called a *vertex*, corresponding to a point of the polytope. The dimension of a simplex is defined as

$$\dim(A) = \text{Card}(A) - 1 \quad (24.23)$$

The simplex of various dimensions are named by their polytope equivalents :

- -1 -simplex : Empty set
- 0 -simplex : Point (vertex)
- 1 -simplex : Line segment (side)
- 2 -simplex : Triangle
- 3 -simplex : Tetrahedron
- 4 -simplex : 5-cell

Beyond which there are no common specific name for simplices, usually simply called n -cells.

The closure under subsets mean that for any k -simplex in the complex, all of its lower dimensional k' -simplexes are connected in some sense. For a tetrahedron, not only do all the vertices belong to the complex, but also every edges connecting any two of those vertices, every face connecting any three of those lines, and so on.

Example 24.10. A square can be decomposed as a simplicial complex made of two triangles. If we denote the four sides by $\{a, b, c, d\}$, the simplicial complex will be

$$\begin{aligned} X = \{ & \{a, b, c\}, \{b, c, d\}, \\ & \{a, b\}, \{b, c\}, \{c, d\}, \{a, c\}, \{b, d\}, \\ & \{a\}, \{b\}, \{c\}, \{d\} \\ & \emptyset \} \end{aligned} \quad (24.24)$$

where the first line is composed of the two triangles, the second line is the three sides of each triangles (including one side in common, bc), its four vertices and the empty set. It can be checked that every subset of each element is itself an element of the complex.

Proposition 24.11. An n -simplex possesses $\binom{n+1}{k+1}$ k -simplices for $k \leq n$.

Proof. This is simply a combinatorial argument : the set of all k -simplices is simply the set of all choices of $(k + 1)$ elements out of the $(n + 1)$ elements of the simplex. $\square$

Abstract simplicial complex lack the structure necessary to say much about them.

24.3.1 Geometric simplicial complex

24.3.2 Δ -complexes

24.3.3 CW complexes

CW complexes (named for Closure-finite, weak topology) are a generalization of simplicial complexes

A CW complex X starts from a discrete collection of points, $X^0 \subset X$.

Definition 24.12. A *singular n -simplex* is a continuous mapping σ_n from an n -simplex Δ^n to a topological space X ,

$$\sigma_n : \Delta^n \rightarrow X$$

Triangulation of a space into CW complexes

24.4 Homology

Beyond a few trivial cases, it can be difficult to fully compute the homotopy of a space. For this purpose, it is common to use instead the abelianization of the homotopy group, called its *homology group*.

The relation between

Given a chain complex $(A^\bullet, d_\bullet)$, its associated k -th degree *homology group* is

$$H_k = \ker(d_k) / \text{im}(d_{k+1}) \quad (24.25)$$

An element of the k -th degree homology group is therefore the equivalence class of a cycle of A_k up to a boundary :

$$H_k = \{[a] \in A_k \mid da = 0 \wedge [a] \sim a \circ da'\} \quad (24.26)$$

If we are considering an additive group, this means

$$H_k = \{[a] \in A_k \mid da = 0 \wedge [a] \sim a + da'\} \quad (24.27)$$

$$0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0 \quad (24.28)$$

Theorem 24.13. If two topological spaces are homeomorphic, they share the same homology.

Proof.

□

Homology is a functor $H : \mathbf{Top} \rightarrow \mathbf{Ch}$

24.4.1 Singular homology

Homotopy equivalence of homology : n -homotopy is singular homology with different operation on loops (homology is abelian)

n -simplex : Set of $n + 1$ points in $\mathbb{R}^n$ that do not lie in a hyperplane of dimension less than n (all vectors $v_i - v_j$ are linearly independent)

Linear independence :

$$\sum_i s_i v_i = \sum_i t_i v_i \wedge \sum_i s_i = \sum_i t_i \rightarrow s_i = t_i \quad (24.29)$$

Set of all points $t_i v^i$ such that $t_i \geq 0$: convex hull of the simplex

For an *ordered simplex*, the ordering kept in check : $s = [v_0, v_1, \dots]$, determines the orientation of the edges

σ_p the p -simplex with coordinates δ_{ki} for every k : $(1, 0, 0, \dots)$, $(0, 1, 0, \dots)$, standard p -simplex

every p -simplex is given by a homeomorphism $f : \sigma_p \rightarrow s$

Singular simplex in X : continuous function $\phi : \sigma_p \rightarrow X$

The set of all n -simplices on a topological space X , $\sigma_n(X)$

The singular chain complex $C^k(X)$ of a topological space X is the free abelian group generated by the k -simplices σ^k

Boundary operator ∂

Singular h

Example 24.14. The singular homology of a single point $\{*\}$

Proof. There is only one 0-simplex, the point itself, in

□

Example 24.15. As Euclidian space $\mathbb{R}^n$ is contractible, its singular homology is that of a point :

$$H_i(\mathbb{R}^n) = H_i(\{*\}) \quad (24.30)$$

$$= \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases} \quad (24.31)$$

Example 24.16. The homology of the sphere

24.4.2 Betti numbers

The *Betti number*

$$b_k(X) = \text{rank}(H_k(X)) \quad (24.32)$$

Interpretation

24.4.3 The Euler characteristic

The Euler characteristic χ is a quantity defined for compact manifolds by the alternating sum of their Betti numbers, if they form a converging sum.

$$\chi = \sum_{i=0}^n (-1)^i b_i \quad (24.33)$$

Properties :

$$\chi(M \sqcup N) = \chi(M) + \chi(N) \quad (24.34)$$

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N) \quad (24.35)$$

$$\chi(M \# N) = \chi(M) + \chi(N) - \chi(S^n) \quad (24.36)$$

$$\chi(M \times N) = \chi(M)\chi(N) \quad (24.37)$$

For a k -sheeted covering space $\bar{M} \rightarrow M$

$$\chi(\bar{M}) = k\chi(M) \quad (24.38)$$

Fiber bundle $\pi E \rightarrow M$:

$$\chi(E) = \chi(M)\chi(F) \quad (24.39)$$

24.4.4 Homology ring

Cup product

24.4.5 Homological algebra

24.4.6 Relative homology

Short exact sequence of singular chains

$$0 \rightarrow C^\bullet(S) \rightarrow C^\bullet(M) \rightarrow C^\bullet(M)/C^\bullet(S) \rightarrow 0 \quad (24.40)$$

Excision

24.5 Cohomology

Just as a homology is a sequence of groups associated with a chain, a *cohomology* is a sequence of groups associated with a cochain $(A^\bullet, d)$, where each i -th cohomology group H^i corresponds to the quotient given by.

$$H^k = \ker(d_k)/\text{im}(d_{k+1}) \quad (24.41)$$

24.5.1 Singular cohomology

The most basic type of cohomology is the dual one to singular homology, called the *singular cohomology*.

Take the singular chain, fix an abelian group A , replace the group C_i by its dual $C_i^* = \text{Hom}(C_i, A)$ and the dual homomorphism to ∂_i :

$$d_{i-1}^* : C_{i-1}^* \rightarrow C_i^* \quad (24.42)$$

Cochain complex (C_i^*, d_i)

The *singular cohomology* made of the i -th singular cohomology groups $H^i(X, A)$

24.5.2 Čech cohomology

Definition 24.17. For a family of sets $C = (A_i)_{i \in I}$, the *nerve* of C is the set of finite subsets of I which have a non-empty intersection.

$$N(C) = \left\{ J \subseteq I \mid \bigcap_{j \in J} U_j \neq \emptyset, \text{ Card}(J) \in \mathbb{N} \right\} \quad (24.43)$$

For a topological space X with an open cover $C = (U_i)$,

Nerves of the atlas

24.5.3 de Rham cohomology

The spaces of k -form naturally forms a cochain of $C^\infty(M)$ -modules $(\Omega^\bullet M, d)$

$$\dots \rightarrow 0 \rightarrow \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1} M \xrightarrow{d} \Omega^n M \xrightarrow{d} 0 \rightarrow \dots \quad (24.44)$$

de Rham cohomology

Augmented complex :

$$\dots \rightarrow 0 \rightarrow \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1} M \xrightarrow{d} \Omega^n M \xrightarrow{d} 0 \rightarrow \dots \quad (24.45)$$

Poincaré lemma

Theorem 24.18. For a simply connected manifold M , the de Rham cohomology $H_{\text{dR}}^\bullet(M)$ vanishes for $k \geq 1$, ie :

$$\dots \rightarrow 0 \rightarrow \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1} M \xrightarrow{d} \Omega^n M \xrightarrow{d} 0 \rightarrow \dots \quad (24.46)$$

Chapter 25

Lie groups

[52]

A special kind of manifold that has a lot of uses in the context of differential geometry is that of *Lie groups*. In addition to the usual structures of a (smooth) manifold, a Lie group also has a group structure, that is, every point of the manifold corresponds to a group element, with the appropriate operations defined on them.

We can define a Lie group as either a group G , given the structure of a manifold with its group operations $\cdot$ and $^{-1}$ smooth with respect to the manifold's smooth structure, or we can define it as a manifold G with some smooth group operations.

Category **LieGrp** of Lie groups and smooth group homomorphisms

As every point leads to a group action on every other point, a Lie group defines a group action on itself. There are three such actions often defined, which are the *left action* l_g , *right action* r_g , and *adjoint action*. They are defined in much the same way that their name implies :

$$l_g g' = g \cdot g' \quad (25.1)$$

$$r_g g' = g' \cdot g \quad (25.2)$$

$$\text{Ad}_g g' = g \cdot g' \cdot g^{-1} \quad (25.3)$$

Since every element of this group action is a group element, we have that for a given g , the action is bijective : if $l_g p = gp = p'$,

As a bijective and smooth group action, every group action of the Lie group on itself is therefore a diffeomorphism, $l_g, r_g, \text{Ad}_g \in \text{Diff}(G)$.

”the action of any group on itself by left multiplication is free”

Simply transitive : transitive and free

Diffeomorphisms generated by the Lie group action

[53]

Category of Lie groups : Objects are Lie groups and morphisms are smooth homomorphisms

Forgetful functors to groups and smooth manifolds

Theorem 25.1. Any connected Lie group is diffeomorphic to the product of a maximal compact subgroup and $\mathbb{R}^n$ for some n .

Lie groups deformation retract onto their maximal compact subgroups

25.1 Matrix Lie groups

Most Lie groups of interests are matrix Lie groups, in that they can be represented by a subspace of the square matrix space $L(V, V)$. This is in particular true of the general linear group $GL(n, F)$, which is isomorphic to the space of invertible matrices, and the largest such possible Lie group for a given vector space V .

Definition 25.2. A *matrix Lie group* is a subgroup G of $GL(n, F)$ such that for any converging sequence of matrices $\{A_i\}_{i \in \mathbb{N}}$ in G , either A_i converges to a value in G or its limit is not invertible.

Example 25.3. The subgroup $GL(n, \mathbb{Q})$ of matrices with rational values is a non-closed subgroup of $GL(n, \mathbb{R})$.

Example 25.4. The subgroup of matrices of the form $\text{diag}(\exp(it), \exp(iat))$ for a irrational is a non-closed subgroup of $GL(n, \mathbb{R})$. ("irrational lines on a torus")

25.2 A few Lie groups

25.2.1 The translation group

The simplest example of a Lie group is Euclidian space $\mathbb{R}^n$, which is the Lie group of translations. To avoid some confusion in its treatment, we will call it T^n here. In Cartesian coordinates, the group action is performed by

$$\cdot : T^n \times T^n \rightarrow T^n \quad (25.4)$$

$$((x, y, z, \dots), (a, b, c, \dots)) \mapsto (x + a, y + b, z + c, \dots) \quad (25.5)$$

The inverse of a group element is done the obvious way by $(-a, -b, -c, \dots)$, with the zero coordinates as a neutral element. As we need a specific neutral element, this does mean that the Lie group isn't only the manifold $\mathbb{R}^n$, but in fact the pointed manifold $(\mathbb{R}^n, 0)$, with 0 a point chosen as the neutral element. The diffeomorphisms generated by the group action are simply, for a group element $t = (a, b, c, \dots)$,

$$f_t : T^n \rightarrow T^n \quad (25.6)$$

$$(x, y, z, \dots) \mapsto (x + a, y + b, z + c, \dots) \quad (25.7)$$

T^n is a matrix Lie group

Matrix :

$$g_{(x,y,z,\dots)} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & x \\ 0 & 1 & 0 & \dots & 0 & y \\ 0 & 0 & 1 & \dots & 0 & z \\ \vdots & \dots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad (25.8)$$

25.2.2 The general linear groups

As shown in 10.5, $\text{GL}(n, F)$ has a group structure, but we also need to show it having a manifold structure.

As it can be shown to be isomorphic to the space of invertible matrices, every element of $\text{GL}(n, F)$ can be considered as a point of F^{n^2} . The general linear group is then simply the inverse map of

$$\text{GL}(n, F) = \det^{-1}(F_{\neq 0}) \quad (25.9)$$

As the determinant is a continuous function on F^{n^2} [polynomial], and since $F_{\neq 0}$ is an open set in $\mathbb{R}$'s topology, $\text{GL}(n, F)$ is an open set in F^{n^2} , and therefore itself a manifold.

In the real case of $\text{GL}(n, \mathbb{R})$, since $\mathbb{R}_{\neq 0}$ is discontinuous with two connected components, the general linear group also has that property. The two connected components are written as $\text{GL}^+(n, k)$ and $\text{GL}^-(n, k)$, with $\text{GL}^+(n, k)$ the element connected to the identity, as the identity matrix I has determinant 1.

Many of the groups that we'll see (any groups that are groups acting linearly on a vector space) will be subgroups of the general linear group

Trivially of course, the general linear group is a matrix Lie group, via the usual identification of linear maps with matrices.

25.2.3 The dilation group

The dilation group D is the group of transformations of vectors leaving them in their own subspace. In other words, for $g \in G$ and its representation ρ_g on V ,

$$\text{Span}(v) = \text{Span}(\rho_g(v)) \quad (25.10)$$

In one dimension this span is simply preserved by scalar multiplication by a non-zero real number (multiplication by zero would just give us the zero subspace). The dilation group is therefore isomorphic to the group of multiplications in $\mathbb{R}$, $\mathbb{R}^\times$, as we can check by using the group action of $\mathbb{R}^\times$ on a vector space

$$\rho : \mathbb{R}^\times V \rightarrow V \quad (25.11)$$

$$(a, v) \mapsto av \quad (25.12)$$

which obeys the appropriate relations etc

The dilation group is in fact isomorphic to the one-dimensional general linear group, $D \cong \text{GL}(1, \mathbb{R})$, making it also disconnected with two components. The element connected to the identity is the orientation-preserving dilation group, corresponding to multiplications by positive numbers : $D^+ \cong \mathbb{R}_{>0}^\times$

The dilation group is a subgroup of $\text{GL}(n, \mathbb{R})$, as a linear operation on any real vector space V , and is the center of $\text{GL}(n, \mathbb{R})$. Due to this it is also sometimes written as Z (short for $Z(\text{GL}(n, \mathbb{R}))$)

25.2.4 The rotation groups

The rotation groups $\text{O}(p, q)$ that preserve the metric tensor of signature $\text{O}(p, q)$ are Lie groups

25.2.5 The special linear group

The *special linear group* $\text{SL}(n, k)$ is the group of linear transformations with determinant 1 :

$$\det(\rho_g) = 1 \quad (25.13)$$

Therefore, the special linear group (as a matrix group) is the preimage of 1 for the map $\det : \mathrm{GL}(n, F) \rightarrow \mathbb{R}_{\neq 0}$.

General linear group splits as $\mathrm{SL}^{\pm}(n) \times D^{\pm}$

25.3 Group properties

As a group, a Lie group can have all the usual group properties.

Theorem 25.5. A connected Abelian Lie group is isomorphic to $T^k \times S^l$.

Maximal torus

Conjugacy "For G a connected compact Lie group, then its conjugacy classes are in bijection with the quotient $T/W(G, T)$ of its maximal torus by the action of the Weyl group. "

orbits

Cosets

quotient groups

Fundamental theorem on homomorphisms

Definition 25.6. A Lie group G is *simple* if it is connected, non-Abelian, and has no nontrivial connected normal subgroups.

The definition of a simple Lie group does not quite match that of a simple group. This is due to the fact that the common definition stems from

Simple, semisimple, reductive Lie groups

Definition 25.7. A group H is a *normal subgroup* of a group G if

Normal subgroups

Group properties : simple, semi-simple

Center of a group

Definition 25.8. The *central extension* of a Lie group G by some Lie group A is a Lie group E obeying the short exact sequence

$$1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1 \quad (25.14)$$

such that A is in the center of E : $A \subset Z(E)$.

Two important cases of central extensions are the central extension by $U(1)$, $\mathbb{R}$, $\mathbb{Z}_2$, by dilation group?

Central extension

25.4 Lie algebras

As smooth manifolds, the Lie groups have a tangent bundle TG , equipped with the traditional Lie bracket. The difference with your typical manifold is that for a given Lie group, the algebra of vectors defined at each point is isomorphic.

The *Lie algebra* of a Lie group is the algebra defined by the Lie bracket at the neutral element of the group, ie the Lie algebra $\mathfrak{g}$ of a Lie group G is the algebra defined by $(T_e G, [\cdot, \cdot])$

Lie algebras can also be defined independently of Lie groups, where a Lie algebra $\mathfrak{g}$ is defined as a vector space V equipped with an alternating bilinear map satisfying the Jacobi identity, ie :

$$[aX_1 + bX_2, Y] = a[X_1, Y] + b[X_2, Y] \quad (25.15)$$

$$[X, aY_1 + bY_2] = a[X, Y_1] + b[X, Y_2] \quad (25.16)$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad (25.17)$$

which the Lie bracket does indeed obey.

The different operators on the Lie group can be lifted to the Lie algebra using the push-forward. If we take the morphism $\mu : M \times M \rightarrow M$, this will lift at the identity to

$$d\mu|_{(1,1)} : T_{(1,1)}(M \times M) \cong T_1 M \oplus T_1 M \rightarrow T_1 M \quad (25.18)$$

Therefore it is lifted to a Lie algebra morphism

$$\mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g} \quad (25.19)$$

Sum of the algebra

Likewise for the inverse,

$$d^{-1}|_1 : \mathfrak{g} \rightarrow \mathfrak{g} \quad (25.20)$$

Show that this correspond to $X \rightarrow -X$

"In order to detect a possible noncommutativity of G on the infinitesimal level at 1 (that is, in terms of finitely many derivatives at 1), we therefore have to turn to second-order derivatives at 1." [54]

Given the adjoint map $\text{Ad}_x : y \mapsto xyx^{-1}$ (conjugation), noncommutativity of G means that Ad_x is not the identity.

Adjoint mapping : $d\text{Ad}_x|_1 : \mathfrak{g} \rightarrow \mathfrak{g}$

The group action of the Lie group's own action on itself defines a specific set of vector fields

Definition 25.9. A *left-invariant vector field* is a vector field whose pullback under left action is itself evaluated at the point of that left action.

$$(l_g)_*X = X \circ l_g \quad (25.21)$$

Examples

A basic example of a left-invariant vector field is the case of the translation group, where every constant vector field $X(x, y, \dots) = X_0$ is a left-invariant vector field. In the canonical coordinates, the pushforward by the translation g is the identity map, mapping the vector components at the original point p to the same ones at l_gp :

$$(l(a, b, \dots))_*X = \quad (25.22)$$

Therefore any left-invariant vector field has to be such that their translations are invariant in that coordinate system, ie

$$d \quad (25.23)$$

Example for $\text{SO}(2)$:

Theorem 25.10. All left-invariant vector fields are smooth.

Proof. □

Theorem 25.11. The set of all left-invariant vector fields is a real Lie algebra isomorphic to the Lie algebra $\mathfrak{g}$ of G .

Proof. Proof that it is a real vector space by showing that linearity preserves the left-invariant character [cf. Frank Warner p. 85] □

Left invariant forms : $\delta l_g \omega = \omega$

Vector space of left-invariant k -forms $E_l^k(G)$

$$E_l^\bullet(G) = \bigoplus_{k=0}^{\dim(G)} E_l^k(G) \quad (25.24)$$

Left-invariant forms are smooth

Maurer-Cartan form : map $\omega : T_g G \rightarrow T_e G$. For $v \in T_g G$,

$$\omega(v) = (l_{-g})_* v \quad (25.25)$$

Theorem 25.12. The Lie algebra is isomorphic to the algebra of the Lie bracket at every point of the Lie group.

Proof. □

In a geometric manner, we can look at the 1-parameter subgroups of the Lie group. If we pick some

The association of an algebra to every Lie group is a functor called the *Lie functor* Lie , which maps the category of Lie groups to the category of finite dimensional Lie algebras, and also mapping Lie homomorphisms to Lie algebra homomorphisms.

$$\text{Lie}(G) = \mathfrak{A}, \text{ Lie}(f : G \rightarrow H) = \mathfrak{f} : \mathfrak{g} \rightarrow \mathfrak{h} \quad (25.26)$$

Prove it

Exponential map

Identity :

$$\det(e^A) = e^{\text{Tr}(A)} \quad (25.27)$$

Theorem 25.13. If a 1-parameter family of diffeomorphisms $\phi : G \rightarrow G$ stems from a 1-parameter family of points in G (ie a curve on G) via Lie group action, then the corresponding vector field [...]

Definition 25.14. The direct sum of two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ is the direct sum of their vector spaces $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the Lie bracket

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]) \quad (25.28)$$

Decomposition of a Lie algebra into direct sum of two Lie subalgebras : if $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{g}$, and for all $X_1 \in \mathfrak{g}_1, X_2 \in \mathfrak{g}_2$ and $[X_1, X_2] = 0$, then $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$

Structure constants

Definition 25.15. For a Lie algebra with a basis $\{e_i\}$, the *structure constants* of the Lie algebra are the components of the tensor defined by the Lie bracket,

$$[e_i, e_j] = C^k_{ij} e_k \quad (25.29)$$

25.4.1 Adjoint representation

As every group has the adjoint action $g^{-1}hg$, Lie groups also have such a group action on themselves. As the group product and inverse are diffeomorphisms, so is this adjoint action, and it defines an automorphism the same way that the group product does by composition. For any element $g \in G$, we define the inner automorphism

$$\Psi_g : G \rightarrow G \quad (25.30)$$

$$h \mapsto g^{-1}hg \quad (25.31)$$

and the function $\Psi_{(-)}$ itself is a map from the Lie group G to the space of its automorphisms :

$$\Psi_{(-)} : G \rightarrow \text{Aut}(G) \quad (25.32)$$

The differential map of this automorphism is therefore an automorphism of the tangent space,

$$d\Psi_g : TG \rightarrow TG \quad (25.33)$$

And from this we define the *adjoint representation* of G by considering the evaluation of this map at the neutral element e ,

$$\text{Ad}_g = (d\Psi_g)_e : T_e G \rightarrow T_e G \quad (25.34)$$

[Show that Lie group automorphisms map to Lie algebra automorphisms]

$$\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}) \quad (25.35)$$

$$g \mapsto \text{Ad}_g \quad (25.36)$$

Adjoint action : $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) = \text{End}(\mathfrak{g})$, $[-, -]$, as $\text{ad}_x(y) = [x, y]$

Jacobi identity :

$$[\text{ad}_x, \text{ad}_y](z) = \text{ad}_{[x, y]}(z) \quad (25.37)$$

Theorem 25.16. The adjoint map $\text{ad}_{(-)}$ is a derivation on $\mathfrak{g}$

Solvable Lie algebra, derived algebra, derived series

25.4.2 Coadjoint orbits

25.4.3 Killing forms

If $\mathfrak{g}$ is finite-dimensional, we can define a bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow F$ via

$$B(x, y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y) \quad (25.38)$$

This is the *Killing form*. From the cyclic property of the trace, we can see that it is a bilinear symmetric form on the vector space $\mathfrak{g}$.

We say that the Killing form is *invariant* if it is invariant under automorphisms of the form $s \in \mathfrak{g} : B(s(x), s(y)) = B(x, y)$

$$B([x, y], z) = B(x, [y, z]) \quad (25.39)$$

Theorem 25.17. For any simple Lie algebra, any invariant symmetric bilinear form is a multiple of B .

"The Cartan criterion states that a Lie algebra is semisimple if and only if the Killing form is non-degenerate."

"The Killing form of a nilpotent Lie algebra is identically zero."

"If I, J are two ideals in a Lie algebra $\mathfrak{g}$ with zero intersection, then I and J are orthogonal subspaces with respect to the Killing form."

"The orthogonal complement with respect to B of an ideal is again an ideal."

"If a given Lie algebra $\mathfrak{g}$ is a direct sum of its ideals $I_1, \dots, I_n$, then the Killing form of $\mathfrak{g}$ is the direct sum of the Killing forms of the individual summands."

In a basis $\{e_i\}$:

$$B_{ij} = B(e_i, e_j) \quad (25.40)$$

$$B = B_{ij} e_i \otimes e_j$$

25.4.4 Quadratic Lie algebras

Definition 25.18. A Lie algebra $(\mathfrak{g}, [-, -])$ equipped with a non-degenerate symmetric bilinear form $(-, -) : \mathfrak{g} \times \mathfrak{g} \rightarrow F$ invariant under adjoint action :

$$() \quad (25.41)$$

is a *quadratic Lie algebra*.

Theorem 25.19. A semisimple Lie algebra with its Killing form B forms a quadratic Lie algebra.

25.4.5 Universal enveloping algebra

When defined in terms of matrices, the Lie algebra is sometimes written as the commutator of the two matrices

$$[X, Y] = XY - YX \quad (25.42)$$

While correct in this context, it can mislead as to the exact nature of the Lie algebra, as the individual products XY and YX have no meaning within the context of the Lie algebra. However, there does exist a larger algebra that contains all the terms of polynomial forms with the Lie algebra as a subalgebra, called the universal enveloping algebra of the Lie algebra.

Definition 25.20. The *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $(\mathfrak{g}, [-, -])$ is an algebra on the vector space $\mathfrak{g}$ such that

$$[X, Y] = XY - YX \quad (25.43)$$

Universal enveloping algebra unique up to isomorphism

Isomorphism to

Definition 25.21. For a Lie algebra $\mathfrak{g}$ and B a non-degenerate bilinear form on $\mathfrak{g}$, invariant under adjoint action of $\mathfrak{g}$ on itself :

$$B(\text{ad}_X Y, Z) + B(Y, \text{ad}_X Z) = 0 \quad (25.44)$$

Given $\{X_i\}$ any basis of $\mathfrak{g}$, and its dual basis $\{X^i\}$ with respect to B ($B(X_i, -) = X^i$, the *Casimir element* Ω for B is the element of the universal enveloping algebra $U(\mathfrak{g})$ given by

$$\Omega = X_i X^i \quad (25.45)$$

This definition does not depend on the basis, as under a change of basis $A_j^i X_i$,

$$X'^i = B(A_j^i, -) \quad (25.46)$$

and

Theorem 25.22. The Casimir element commutes with every element of the universal enveloping algebra.

If B is the Killing form

For a rep,

$$\rho(\Omega) = \sum_i \rho(X_i) \rho(X^i) \quad (25.47)$$

”Since for a simple Lie algebra every invariant bilinear form is a multiple of the Killing form, the corresponding Casimir element is uniquely defined up to a constant. For a general semisimple Lie algebra, the space of invariant bilinear forms has one basis vector for each simple component, and hence the same is true for the space of corresponding Casimir operators. ”

Relation between UEA and left invariant differential operators on G , the Casimir element is the Laplacian

"By definition any member of the center of the universal enveloping algebra commutes with all other elements in the algebra. By Schur's Lemma, in any irreducible representation of the Lie algebra, any Casimir element is thus proportional to the identity. The eigenvalues of all Casimir elements can be used to classify the representations of the Lie algebra (and hence, also of its Lie group)."

Chevalley-Eilenberg algebra

Casimir invariants

25.5 Invariant differential operators

Just as in the case of manifolds, differential operators on G can be represented by vector fields $X \in \mathfrak{X}(G)$, in this case making them Lie algebra-valued fields. For some function f on G , we have the differential operator defined by

$$X(f) = X^i e_i(f) \quad (25.48)$$

in which case the basis elements will be

Definition 25.23. A left/right *invariant differential operator* D on a Lie group G is a differential operator defined by a left/right invariant vector field on G .

25.6 Quotients

Given a

25.7 ...

[...] Lie algebra is matrix valued

In the case of a matrix Lie group, the exponential map of a vector is the actual exponential :

$$e^X = \sum_{j=0}^{\infty} \frac{X^j}{j!} \quad (25.49)$$

Theorem 25.24. The series 25.49 converges for every $X \in M_n(F)$ and is a continuous function of X .

Hilbert-Schmidt norm :

$$\|X\| = (\text{Tr}(X^\dagger X))^{\frac{1}{2}} \quad (25.50)$$

Theorem 25.25. $\exp(tX)$ is a smooth curve in $M_n(\mathbb{C})$.

Lie product formula

Left-invariant, right-invariant, bi-invariant metric

25.8 Root systems and weights

Definition 25.26. Given a finite dimensional real vector space V with the Euclidian metric $(-, -)$, a root system Φ is a set of non-zero vectors $\{v_i\}$ obeying the following properties :

- The roots span the vector space : $\text{Span}(\Phi) = V$.
- If a vector α belongs to Φ , the only scalar multiples of that vector belonging to the root system are α and $-\alpha$
- Root systems are closed under reflection in that for any root α , every root β must have a corresponding root being its reflection through the orthogonal hyperplane $\text{Span}(\alpha)^\perp$:

$$\beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha \in \Phi \quad (25.51)$$

- The projection of one root onto another is an integer or half-integer :

$$\exists k \in \mathbb{Z}, 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = k \quad (25.52)$$

Given a root system Φ , the *rank* $\text{rank}(\Phi)$ is the dimensionality of the vector space generated by Φ :

$$\text{rank}(\Phi) = \dim(\text{Span}(\Phi)) \quad (25.53)$$

Example 25.27. The basic example of a root system is the rank 1 system called A_1 , defined by two vectors on the line, $(-\alpha, \alpha)$. This trivially spans the line, is closed under the reflection by the "hypersurface" $\alpha^\perp$ which is just the zero subspace, and the integrality condition is simply $2(\pm\alpha, \alpha)/(\alpha, \alpha) = \pm 2$, which is always verified.

Root system isomorphism

Our A_1 root system can be defined for a variety of vectors α , but all are isomorphic under rescaling.

The root system A_1 can be used to generate basic root systems that correspond to the mirrored basis vectors of the orthonormal basis of any real vector space, i.e. for any vector space V of dimension n with orthonormal basis $(e_1, e_2, \dots, e_n)$, there exists a corresponding root system $(\pm e_1, \pm e_2, \dots, \pm e_n)$

Example 25.28. The root system A_2 is a root

•—•

root system, dynkin etc

25.9 Representation theory

[21]

Definition 25.29. A *group representation* of G on a vector space V is a Lie group homomorphism to the general linear group of that vector space :

$$\Pi : G \rightarrow \mathrm{GL}(V) \quad (25.54)$$

Alternatively, a representation is a group action on a vector space

$$\rho : G \times V \rightarrow V \quad (25.55)$$

such that ρ_g is a linear map on V .

A lot of the Lie groups we've already seen have been built from linear maps on a vector space, so that they have a canonical representation with the identity homomorphism. For instance the canonical representation of $\mathrm{GL}(V)$ as a Lie group is simply itself as a space of linear maps.

A representation is *real* or *complex* if the associated vector space V is real or complex, and the same goes for any other field. It is finite dimensional if V is, and infinite dimensional likewise.

Similarly to the group representation, a *Lie algebra representation* is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \quad (25.56)$$

or alternatively, [...]

The same descriptors of real, complex, finite or infinite dimensional apply in the same circumstances.

If a representation is bijective, it is a *faithful representation*.

Theorem 25.30. Every Lie group admits a trivial representation on the zero dimensional vector space.

Proof. We can simply define the map to the zero-dimensional general linear group

$$\Pi : G \rightarrow \text{GL}(0) \cong \{e\} \quad (25.57)$$

$$g \rightarrow e \quad (25.58)$$

This map is obviously mapping e to e , inverses to inverses (e again), and preserve composition (since $e \circ e = e$). It is furthermore linear : $\rho_g(\alpha 0 + \beta 0) = e(\alpha 0 + \beta 0) = \alpha e(0) + \beta e(0) = \alpha 0 + \beta 0 = 0$ Same goes for Lie algebras $\square$

Just from the cardinality of the groups, we can see that the trivial representation is never faithful except for the identity Lie algebra $\{e\}$.

Rep. for matrix Lie groups Rep for tensor products : $g(v \otimes w) = gv \otimes gw$, $g(v \oplus w) = (gv \oplus gw)$

Dual rep: $\rho_G^*(g) = \rho_G(g^{-1})^\top$

Obeys $\langle \rho_G^*(g)\alpha, \rho(g)v \rangle = \langle \alpha, v\lambda, \rangle$

Rep of $\text{Hom}(V, W)$: rep as $\text{Hom}(V, W) = V^* \otimes W$

Definition 25.31. For a representation $\rho_g^k : \bigwedge^k V \rightarrow \bigwedge^k V$, the representation $\rho_g^{(n-k)*} : \bigwedge^{n-k} V^* \rightarrow \bigwedge^{n-k} V^*$ is isomorphic

Definition 25.32. For a representation Π of G on V , a subspace $W \subset V$ is called *invariant* if for $w \in W$, $\Pi(g)w$ is still in W . This invariant subspace is *nontrivial* if $W \neq \{0\}$ and $W \neq V$.

Example of a non-trivial invariant subspace : consider the representation of the general linear group $\text{GL}(V)$ acting on the vector space $V \oplus V$ via

$$\rho_g : V \oplus V \rightarrow V \oplus V \quad (25.59)$$

$$(v_1, v_2) \mapsto (\rho_{g_1} v_1, \rho_{g_2} v_2) \quad (25.60)$$

As a matrix :

$$\rho_g v = \begin{pmatrix} \rho_{g_1} & 0 \\ 0 & \rho_{g_2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (25.61)$$

Then the subspaces $\{0\} \oplus V_2$ and $V_1 \oplus \{0\}$ are nontrivial invariants :

$$\rho_g(0, v_2) = (\rho_{g_1} 0, \rho_{g_2} v_2) = (0, \rho_{g_2} v_2) \in \{0\} \oplus V_2 \quad (25.62)$$

Take the canonical representation of $\mathcal{O}(n)$ on a real vector space V , and consider the tensor space $V \otimes V$.

$$V \otimes V \cong \Lambda^2 V \oplus S^2 V \quad (25.63)$$

Definition 25.33. A representation is called *irreducible* if it acts on a vector space with no nontrivial invariant subspaces

Indecomposable v. irreducible

Intertwining operator

Schur's lemma

irreps

Central charge

Casimir

Cartan-Weyl basis, Chevalley basis

Weyl's theorem on complete reducibility

Weyl's unitarian trick

25.9.1 Young tableaux

One useful tool for the classification of some representation is Young tableaux.

Definition 25.34. A *Young diagram* is the unordered integer partition of a finite set into subsets, so that a partition is the

25.10 Representation theory of specific Lie groups

This section will just be the application of the methods of representation theory that we've seen to the various groups that we will commonly use throughout this book.

25.10.1 Representation theory of the translation group

25.10.2 Representation theory of the dilation group

25.10.3 Representations of the general linear group

Standard rep : identity map

$$\rho_{(-)} : \mathrm{GL}(V) \times V \rightarrow V \quad (25.64)$$

$$(g, v) \mapsto g(v) \quad (25.65)$$

Dual rep : ρ_g^{-T}

Determinant rep : $\Delta^P : g \mapsto \det(g)^p$

Polynomial : classification by Young tableaux

Schur-Weyl duality

Theorem 25.35. Given a vector space V on which the general linear group $\mathrm{GL}(V)$ acts via its canonical representation, the irreducible representations of the tensor space

$$V^k = \bigotimes^k V \quad (25.66)$$

which carries both the diagonal group action from $\mathrm{GL}(V)$:

$$\rho_{\mathrm{GL}(V)} : G \times V^k \rightarrow V^k \quad (25.67)$$

$$(g, (v_1, \dots, v_k)) \rightarrow (g \cdot v_1, \dots, g \cdot v_k) \quad (25.68)$$

as well as the permutation representation from the symmetric group S_k

$$\rho_{S_k} : S_k \times V^k \rightarrow V^k \quad (25.69)$$

$$(\sigma, (v_1, \dots, v_k)) \mapsto (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \quad (25.70)$$

are classified by the

Determinant reps

25.10.4 Representations of the unitary group

A lot of the further proofs for the representation theory of various important Lie groups depend on the representations of the unitary groups, so that it will be best to start with this one.

Representation of $SU(2, \mathbb{C})$

25.10.5 Representations of the rotation group

25.10.6 Representations of the Lorentz group

As the kinematic group of spacetime, the representation theory of the Lorentz group is one of the most important in general relativity.

25.10.7 Representations of the conformal group

25.11 Group contraction

A group contraction is in some sense the limit of a Lie group as one of its algebra generator tends to a certain value, usually 0 or ∞ , so that we contract that generator. The classic example of this is the case of the action of the rotation group $SO(3)$ on a sphere S^2 as one increases the radius of that sphere to infinity. If we consider the action of $SO(3)$ on the sphere embedded in $\mathbb{R}^3$,

$$\rho_g v = \exp(\theta^i L_i) v \quad (25.71)$$

where our vector v is a point on a sphere of radius R ,

Basis of $SO(3)$ as the motion group on the sphere, rescale generators by some dimensions [55]

This has the physical interpretation of the rotation group acting on the sphere in such a way that a rotation of $\theta_x, \theta_y = 1$ corresponds to a translation along the sphere by a distance of d , with θ_z will be the rotation of the observer [redo the coordinates].

[56, 57]

Inönü-Wigner contraction of Lie groups

For a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$, consider some parametrized change of basis $U_j^i(\lambda)$, such that

$$U_j^i(\lambda = 1) = \delta_j^i \quad (25.72)$$

$$\det(U_j^i(\lambda = 0)) = 0 \quad (25.73)$$

So that at $\lambda = 0$, the transformation is singular, reaching a point outside of $\text{GL}(\mathfrak{g})$.

The transformation of the structure constants is

$$[Y_i, Y_j] = c_{ij}^k Y_k \quad (25.74)$$

$$c_{ij}^k(\lambda) = U_i^r(\lambda) U_j^s(\lambda) c_{rs}^t(1) (U^{-1})_t^k(\lambda) \quad (25.75)$$

If the limit

$$\lim_{\lambda \rightarrow 0} c_{ij}^k(\lambda) \quad (25.76)$$

exists, the new structure constants characterize a Lie algebra that may not be isomorphic to the original.

Coordinate-free version : Saletan contraction [58]

$U(\lambda)$ is an injective mapping that preserves the commutation bracket of one algebra $\mathfrak{g}$ onto another $\mathfrak{g}'$

$$[a', b'] = c' \quad (25.77)$$

$$[Ua, Ub] = Uc \quad (25.78)$$

$$U^{-1}(\lambda)[U(\lambda)a, U(\lambda)b] = c \quad (25.79)$$

If the limit

$$\lim_{\lambda \rightarrow 0} U^{-1}(\lambda)[U(\lambda)a, U(\lambda)b] \quad (25.80)$$

exists, then we have a new structure.

Example : for a Lie algebra $\mathfrak{g} = V_R \oplus V_N$, consider the change of basis

$$U(\lambda) = \begin{pmatrix} I & 0 \\ 0 & \lambda I \end{pmatrix} \quad (25.81)$$

then

$$U(\lambda)X_i^R = X_i^R \quad (25.82)$$

$$U(\lambda)X_i^N = \lambda X_i^N \quad (25.83)$$

Therefore we have

$$c_{ab}^c(\lambda) = \lambda^k c_{ab}^c(1) \quad (25.84)$$

For some constant k equal to the number of covariant indices from V_N minus the number of covariant indices in V_N . If we have one index from V^N and two from V^R ,

$$[Y_a^R, Y_b^R] = c \dots Y_c \quad (25.85)$$

The structure constants will only converge if $c_{\alpha\beta}^k(1) = 0$.

Then we have the relations

$$c'_{\alpha\beta}{}^\gamma = c_{\alpha\beta}{}^\gamma \quad (25.86)$$

$$c'_{\alpha\beta}{}^k = c_{\alpha\beta}{}^k = 0 \quad (25.87)$$

$$c'_{\alpha j}{}^\gamma = \lambda c_{\alpha j}{}^\gamma \rightarrow 0 \quad (25.88)$$

$$c'_{\alpha j}{}^k = c_{\alpha j}{}^k \quad (25.89)$$

$$c'_{ij}{}^\gamma = \lambda^2 c_{ij}{}^\gamma \rightarrow 0 \quad (25.90)$$

$$c'_{ij}{}^k = \lambda^1 c_{ij}{}^k \rightarrow 0 \quad (25.91)$$

V_R is closed under commutation in $\mathfrak{g}(0)$ V_R forms a subalgebra in $\mathfrak{g}(1)$ V_N forms a subspace in $\mathfrak{g}(0)$ V_N is an abelian invariant subalgebra in $\mathfrak{g}(1)$

$\mathfrak{g}(1)$ is nonsemisimple, $\mathfrak{g}(0)$ may be simple, semisimple or nonsemisimple.

Theorem 25.36. For $\mathfrak{g}(0)$ a Lie algebra and $U(\lambda)$ a singular transformation when $\lambda \rightarrow 0$,

with

$$\mathfrak{g}(0) = V_R \oplus V_N \quad (25.92)$$

$$U(0)V_R = V_R \quad (25.93)$$

$$U(0)V_N = 0 \quad (25.94)$$

Then $\mathfrak{g}(0)$ can be contracted with respect to V_R iff V_R is closed under commutation. V_R then forms a subalgebra in $\mathfrak{g}(0)$ and $\mathfrak{g}(1)$, V_N is an abelian invariant subalgebra of $\mathfrak{g}(1)$, and $\mathfrak{g}(1)$ is nonsemisimple.

Some examples : $\mathrm{SO}(n)$ obeys

$$X_{ij} = x^i \partial_j - x^j \partial_i \quad (25.95)$$

$$[X_{ij}, X_{rs}] = X_{is} \delta_{jr} + X_{jr} \delta_{is} - X_{ir} \delta_{js} - X_{js} \delta_{ir} \quad (25.96)$$

Consider the basis change given by

$$\forall j \neq n, Y_{ij} = x^i \partial_j - x^j \partial_i \quad (25.97)$$

$$X_{in} = \lim_{R \rightarrow \infty} x^i \partial_j - x^j \partial_i \quad (25.98)$$

Example 25.37. The standard example of a group contraction is the contraction of the rotation group $\mathrm{SO}(3)$ to the Euclidian group $\mathrm{Euc}(2)$, generally considered as the limit of the action of $\mathrm{SO}(3)$ on a sphere of radius R as we consider the limit $R \rightarrow \infty$, or more accurately, in the limit where the distance defined by the angle of rotation becomes negligible when compared to the radius.

Example 25.38. A useful example of group contraction is the case of the Lorentz group.

25.12 Integration of Lie algebras

As the Lie functor Lie gives us a map from the category of Lie groups to the category of Lie algebras, we can also look into its adjoint functors, which associate Lie groups to a given Lie algebra.

It is quite easy to show that broadly speaking, there is no unique correspondence between Lie algebras and Lie groups, as the one dimensional case already shows : both the translation group T^1 and rotation group $\mathrm{SO}(2) \cong S^1$ have the same trivial Lie algebra over $\mathbb{R}$, so that we cannot hope to find a unique inverse in that sense.

However, given more specific conditions on the nature of the Lie groups, it is possible to find such an adjoint, which is given by the third Lie theorem :

Theorem 25.39. For every finite dimensional real Lie algebra $\mathfrak{g}$, there exists a Lie group G such that $\mathfrak{g} = \mathrm{Lie}(G)$.

25.13 Algebraic topology and Lie groups

Theorem 25.40. Every Lie group G has a commutative fundamental group $\pi_1(G)$.

For the exterior algebra of the dual Lie algebra, $\bigwedge^k \mathfrak{g}^*$, with the operator δ

$$B^k(\mathfrak{g}) = \text{Im}(\delta) \quad (25.99)$$

$$Z^k(\mathfrak{g}) = \text{Ker}(\delta) \quad (25.100)$$

k -th cohomology group is $H^k(\mathfrak{g}) = Z^k(\mathfrak{g})/B^k(\mathfrak{g})$ (de Rham cohomology of the Lie group)

For abelian G , $H^k(\mathfrak{g}) = \bigwedge^k \mathfrak{g}^*$

Whitehead lemma

Theorem 25.41. If G is a finite-dimensional semisimple Lie group and R is a [...], $H^2(\mathfrak{g}, R) = 0$.

25.14 Properties of our Lie groups

Now that we've seen the broad theory of Lie groups, let's give a look at various common example of Lie groups and their properties.

25.14.1 The translation group

The most common groups that we will have to consider for general relativity are the isometry groups of Minkowski space, the *Lorentz group* and the *Poincaré group*.

[59]

Given the n -dimensional space $\mathbb{R}^n$ with a symmetric bilinear form η of signature (p, q) ($p + q = n$), the *indefinite orthogonal group* $O(p, q)$ is the group of every invertible linear transformation, $O(p, q) \subset GL(n, \mathbb{R})$ preserving this form :

$$\forall \Lambda \in O(p, q), \Lambda \eta \Lambda = \eta \quad (25.101)$$

Proof of group, Lie group

Give the manifold structure

Affine isometries on the space for Poincaré group

Proof that Poincaré group is

$$\text{Poinc}(n) = \mathbb{R}^n \rtimes \mathcal{O}(n-1, 1) \quad (25.102)$$

25.15 Parabolic subgroups

Algebraic groups, Zariski topology

[60, 61]

For $\mathfrak{g}$ a real semisimple Lie algebra (for the general case see Lie Algebras and Algebraic Groups by Patrice Tavel and Rupert W. T. Yu),

Definition 25.42. The *Borel subgroup* of an algebraic group G

We will not look quite that deeply into parabolic subgroups and we will mostly consider the cases associated to the general linear group $\text{GL}(V)$ and other associated groups.

Theorem 25.43. Every stabilizer subgroup of a flag in a vector space V is a parabolic subgroup of $\text{GL}(V)$.

Proof. Given a flag F of signature $\sigma = (d_0, d_1, \dots)$, its stabilizer subgroup $P(\sigma)$ is the subgroup of $\text{GL}(V)$ leaving each subspace of the flag invariant. The action of $\text{GL}(V)$ on a flag F is given by

$$\rho_g F = (gF_1, gF_2, \dots) \quad (25.103)$$

□

Theorem 25.44. The parabolic subgroups

Chapter 26

Klein geometries

Klein groups were a concept invented by Felix Klein in the framework of his Erlangen program [62], with the goal of classifying the many new geometries that were created in the 19th century. One idea that came out of this program was that each geometry could be described by the *principal group* of its transformations, the group of all transformations leaving the properties of the geometry invariant, and the subgroup of that group leaving a point of the geometry fixed (the stabilizer of a point). As Klein only considered homogeneous geometries where the properties remained invariant at every point of the space, the stabilizer group can be uniquely defined by a single group.

Klein geometries are defined a step further by ignoring the original geometry as the primitive object and instead deriving it from those two abstract groups.

Definition 26.1. A *Klein geometry* is a pair of two Lie groups (G, H) , G the principal group and H a closed subgroup of G , such that the left coset space G/H (the space of the geometry) is connected.

We can check that as a space, the action of H on $X = G/H$ leaves it indeed invariant

G/H is a manifold of dim. $g - h$

Theorem 26.2. The action of G on G/H is transitive

What can we say about the homogeneous space? Given a point $p \in G/H$, there is a map $\pi :$

Theorem 26.3. There is a bijection [continuous? smooth?] between G/H and any manifold M with the appropriate actions [sharp]

Bundle short sequence

$$H \hookrightarrow G \rightarrow G/H \tag{26.1}$$

Two methods to work out : Show that given a space M , G acts on M in a transitive way, the stabilizer is a closed subgroup of G , or perform the computation of the quotient directly

The classic example of a Klein geometry is Euclidian space itself, with the Klein geometry $(E(n), O(n))$.

show that $E(n)/O(n) \cong \mathbb{R}^n$

Action of $E(n)$ on $\mathbb{R}^n$:

$$gv = \quad (26.2)$$

Minkowski space is itself a Klein geometry, defined by the pair of groups $(\text{Poinc}(n), \text{Lor}(n))$,

de Sitter space, anti de Sitter space, sphere

Other variants of model spaces of $\mathbb{R}^n$

Theorem 26.4. For a Lie group G and a closed Lie subgroup $H < G$, there exists a unique smooth manifold structure on the coset space G/H

26.1 Homogeneous spaces identities

Theorem 26.5. If H is a deformation retract of G , $\pi_i(G) \cong \pi_i(H)$, and by the sequence $H \rightarrow G \rightarrow G/H$, all $\pi_i(G/H)$ are trivial, making G/H contractible.

Theorem 26.6. The homogeneous space of the general linear group $GL(n)$ by the orthogonal group $O(p, q)$ is

$$GL(n)/O(p, q) \cong \quad (26.3)$$

Proof. From the QR decomposition, we have that any □

26.2 Projective geometry

The projective space $P\mathbb{R}^n$ as the Klein geometry $(PGL(n+1), H)$ for the stabilizer subgroup

$$H \quad (26.4)$$

Chapter 27

Fiber bundles

27.1 Definitions

A bundle is a way to add structures to a manifold M , by means of attaching to every point of the manifold additional entities. The most general notion of a *bundle* is to give a surjective map $\pi : E \rightarrow M$ between two sets, where the *fiber* at $p \in M$ is the preimage $\pi^{-1}(p) \subset E$.

For most of our cases of interest, we will only consider *fiber bundles*, where the conditions on each object are more restrictive. For a fiber bundle, E and M must both be manifolds, with π a continuous surjection between the two :

$$\pi : E \rightarrow M \tag{27.1}$$

The manifold E is called the *total space* (the original manifold plus the structures added to it), M is the *base manifold*, and π is the *projection function*. To be properly called a fiber bundle, the projection function must also obey *local trivialization* : for each point $p \in M$, there exists a neighbourhood $U \ni p$ with a homeomorphism

$$\psi : \pi^{-1}(U) \rightarrow U \times F \tag{27.2}$$

such that $\pi = \phi \circ \text{pr}_1$.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times F \\ \pi|_U \downarrow & & \downarrow \text{pr}_1 \\ U & \xrightarrow{\text{Id}} & U \end{array}$$

$\pi^{-1}(U)$ is homeomorphic to $U \times F$, and every local trivialization is therefore a local fiber respecting C^k diffeomorphism.

We can form a *fiber bundle atlas* from local trivializations, by a set of trivializations (U_α, ψ_α) for which $\bigcup_\alpha U_\alpha = M$. Just like a manifold atlas, we have transition functions between two local trivializations,

$$\psi_\alpha \circ \psi_\beta^{-1}(p, f) = (p, \psi_{\alpha\beta}(p, f)) \quad (27.3)$$

Cocycle condition

A bundle can thus be expressed by the triplet (E, π, M) , generally noted as $\pi : E \rightarrow M$. It can also be noted by the total space E , or, since two different bundles can have the same total space and base space, π . As there will be no confusion here, we will simply use E . If we need to specifically indicate the fiber, we will also occasionally write it as $F \hookrightarrow E \rightarrow M$.

Example 27.1. The prototypical example of a fiber bundle is the product of two manifolds, $E = B \times F$, with projection map $\pi = \text{pr}_1$. π is a surjection by its status as a projection and continuous in the product topology. Every point $p \in B$ is part of the open set B which has inverse projection $\pi^{-1}(B) = B \times F$, which is trivializable by the identity map.

[Illustration]

Example 27.2. Example of a bundle that isn't a fiber bundle

Example 27.3. Example of a bundle with a typical fiber that's not locally trivializable

As with manifolds, a fiber bundle is C^k if all of its manifolds E, B, F are C^k manifolds, and its projection π and local trivializations $\{(U_\alpha, \psi_\alpha)\}$ are all smooth functions. This is also true of a *smooth fiber bundle* if all those objects are smooth.

As a manifold, the local trivialization lets us form a specific class of coordinates on E . If we consider the maximal C^k atlas of E , and the corresponding C^k atlases on M and F ,

Smooth fiber bundle

Bundle coordinates

Bundle atlas

Definition 27.4. A *coordinate bundle* is a fiber bundle (π, E, M, F) with a trivializing atlas $\{\psi_\alpha, U_\alpha\}_{\alpha \in I}$ such that U_α covers M , for which the bundle transition functions

$$\psi_{\alpha\beta} = \psi_\beta^{-1} \psi_\alpha \quad (27.4)$$

is C^k and, evaluated at a point $p \in U_\alpha \cap U_\beta$, belongs to some transformation group G of the fiber F , the group of the bundle [Effective group action]

Those transition maps are coordinate transformations of the bundle

They obey the cocycle condition such that, for $p \in U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\psi_{\alpha\beta}\psi_{\alpha\gamma} = \quad (27.5)$$

Equivalence of coordinate bundles

Just like in the case of manifolds, we can therefore also define a fiber bundle as either an equivalence class of coordinate bundles, or alternatively, we can consider the maximal bundle atlas $\mathcal{A}_B$.

27.2 Structure groups

The bundle atlas of a given bundle will always have some transition functions valued in $\text{Homeo}^k(F)$ ($\text{Diff}(F)$ for smooth bundles), which will not in general be a very well-behaved space, but for most of the fiber bundle that we consider, we can pick a smaller bundle atlas for which there exists a much smaller group in which our transition functions can be valued. If there exists a Lie group G with an action on F $\rho : G \times F \rightarrow F$, giving us a group homomorphism $G \rightarrow \text{Diff}(F)$,

(G, λ) bundle

The simplest case of a fiber bundle is the *trivial bundle* (or *product bundle*), defined by the total space $E = M \times F$ with the projection

$$\text{pr}_1 : M \times F \rightarrow M \quad (27.6)$$

$$(p, f) \mapsto \text{pr}_1(p, f) = p \quad (27.7)$$

The local trivialization for such a bundle is then true for any open set of M by using the identity map :

$$\psi_U : \quad (27.8)$$

Bundles of contractible fibers are trivial (cf. Steenrod)

27.3 Bundle morphisms

Bundle morphisms are the functions preserving the bundle structure

Definition 27.5. A *bundle morphism* between two bundles $\pi_1 : E_1 \rightarrow M_1$ and $\pi_2 : E_2 \rightarrow M_2$ is a pair of C^k maps $f : M_1 \rightarrow M_2$, $\phi : E_1 \rightarrow E_2$ such that $\pi_2 \circ \phi = f \circ \pi_1$

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

A bundle morphism over a common base is the case where $M = M_1 = M_2$, $f = \text{Id}_M$.

In particular, a *bundle isomorphism* if there exists an inverse morphism

27.4 Bundles as locally ringed spaces

As we've seen, a fiber bundle's definition is quite close to that of a manifold, with more specific kinds of atlases and transition functions.

[Locally free sheaves?]

27.5 Sections

If we consider our bundles as a way to attach information to the manifold, a section of that bundle is a specific instance of information on the manifold, with one point in the fiber associated with every point of the manifold.

Definition 27.6. A *section* of a fiber bundle $\pi : E \rightarrow M$ is a map $\sigma : M \rightarrow E$ such that $\pi \circ \sigma = \text{Id}_M$.

If we have some local trivialization ψ , for some point $p \in M$ with a trivialization neighbourhood U_p , then in terms of coordinates, our section can be seen locally as a map from the manifold to the fiber

$$\psi \circ \sigma : M \rightarrow U \times F \tag{27.9}$$

$$p \mapsto (p, f) \tag{27.10}$$

If furthermore we define some coordinate maps ϕ_U on M and ϕ_{U_F} on some neighbourhood of F where our section is defined, we obtain the expression of our section in coordinates :

$$(\text{pr}_\mu \circ \phi_{U,p}, \text{pr}_\nu \phi_{U',f}) \circ \psi \sigma(p) = (x^\mu, y^\mu) \quad (27.11)$$

Sections do not necessarily exist, and specific types of sections may also fail to exist, but we always have the guarantee of at least a *local section*, which is a section defined on an open set of the bundle atlas.

Theorem 27.7. Every section of a trivial bundle $\pi : M \times F \rightarrow M$ is a function $f : M \rightarrow F$.

Proof. As a product of manifolds, we have that for any function $s : M \rightarrow M \times F$, there exist two functions $s_1 = \text{pr}_1 \circ s$ and $s_2 = \text{pr}_2 \circ s$. As by the definition of a section $\text{pr}_1 \circ s = \text{Id}_M$, our section is simply defined by the function

$$f : M \rightarrow F \quad (27.12)$$

$$p \mapsto f(p) = \text{pr}_2 \circ s(p) \quad (27.13)$$

and the original section can be reconstituted simply as $p \mapsto (p, f(p))$. $\square$

Local v. global sections

As fiber bundles are [sheaves], we can also define sections in terms of an assemblage of local sections, which is in essence what we often do in physics, as we typically treat sections as what they look like on a trivializable neighbourhood. That is, if we have some open set $U \subseteq M$, and an open cover $\{U_\alpha\}$ of U which are all trivializable neighbourhoods of a bundle E , we can define a section of E over U as a set of sections $\{s_\alpha\}$, $s_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha) \cong U_\alpha \times F$ such that

Theorem 27.8. A global section $s : M \rightarrow E$ is an embedding of M into E .

Proof. $\square$

In other words, if

Right inverse

Retraction

27.6 Vertical and horizontal bundles

As manifolds themselves, the total spaces of fiber bundles possess a tangent bundle (if they are of regularity C^1 or better), and generally jet spaces of all types and orders. Due to the structure of a fiber bundle, the tangent bundle decomposes into subspaces. The canonical one is the vertical space of a bundle :

Definition 27.9. The *vertical space* $V_e E$ at a point $e \in E$ of a bundle E is the subspace of the tangent space $T_e E$ defined by curves lying entirely in the fiber $\pi^{-1}(\pi(e))$.

Unlike the vertical space, the horizontal space is not uniquely defined, simply being a complementary choice of the tangent bundle. We say that the subspace W of $T_p E$ is horizontal if $W \oplus V_p E = T_p E$.

As a subspace, along with the possibility of defining it with an appropriate $\dim(F)$ -distribution on E or the corresponding number of vectors, it is possible to define it with a

Selection of horizontal space, what you need, projection operator

Defining the horizontal space with a form

In coordinate form

Sequence

$$0 \rightarrow VE \longrightarrow TE \longrightarrow \pi^* TM \rightarrow 0 \quad (27.14)$$

27.7 Subbundle

A *subbundle* E' of a bundle E is a subset of the total space E , with some injection function $\iota : E' \hookrightarrow E$, such that there also exists a bundle structure on E' over M obeying $\iota \circ \pi = \pi'$:

$$\pi(\iota(e')) = \pi'(e) \quad (27.15)$$

check

Vertical and horizontal space splitting?

27.8 Sequences of bundles

Sequences of bundles

Exact sequence of fiber bundles :

$$0 \rightarrow E' \xrightarrow{\phi} E \xrightarrow{\psi} E'' \rightarrow 0 \quad (27.16)$$

0 is the trivial vector bundle $M \times \{0\}$, exactness is $\text{Im}(\phi) = \text{Ker}(\psi)$

Exact if section $s : M \rightarrow E''$ s.t. etc

27.9 Double fibrations

It is common to consider the case of a fiber bundle with more than one base space, such as the case of the *double fibration*, where we have two such base spaces.

$$M_1 \leftarrow E \rightarrow M_2 \quad (27.17)$$

The simplest case of this is simply the trivial bundle $M_1 \times M_2$, which has the simple double fibration

$$M_1 \xleftarrow{\text{pr}_1} M_1 \times M_2 \xrightarrow{\text{pr}_2} M_2 \quad (27.18)$$

This gives rise to the notion of correspondence between

Polarities idk

see also span, horocycle, correspondence

27.10 Pullback bundles

Given a fiber bundle $\pi : E \rightarrow M$ and some manifold morphism $f : N \rightarrow M$, the *pullback bundle* of π by f , noted as $f^*\pi : f^*E \rightarrow N$,

$$f^*E = \{(p, e) \in N \times E \mid f(p) = \pi(e)\} \quad (27.19)$$

The fiber has the typical fiber of E over N , or the image of f in M

Theorem 27.10. The pullback bundle f^*E has the typical fiber F over N .

Proof. The fiber of f^*E is given by $f^*\pi^{-1}(p)$ □

A common example of a pullback bundle is the pullback bundle over a curve, for the map $\gamma : L \rightarrow M$.

Pullback bundle : $\pi : E \rightarrow M, \gamma^*\pi : E \rightarrow L$

For any bundle section $\sigma \in \Gamma(E)$, we can define a *pullback section*

27.11 Category of fibered manifolds

Fiber bundles together with the bundle morphisms between them form the *category of fibered manifolds*, **FibMan**.

Chapter 28

Vector and affine bundles

Some of the most common types of fiber bundles are fiber bundles meant to attach either vector or affine spaces over the manifold, called vector and affine bundles.

28.1 Vector bundle

The basic idea of a vector bundle is that its typical fiber is a vector space, and the cocycle rules are such that the bundle charts are linear maps. This construction is done first by constructing some bundle charts

Definition 28.1. A *vector bundle chart* (U_α, ψ_α) to the vector space V is a fiber bundle chart

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times V \quad (28.1)$$

Then we say that for any two vector bundle charts $(U_1, \psi_1), (U_2, \psi_2)$ to the same vector space V that they are *compatible* if their fiber transition function $\psi_{12} = \psi_2|_{U_{12}} \circ \psi_1^{-1}|_{\psi_1^{-1}(U_{12})}$ is a fiber linear isomorphism, where the structure group is valued in $\text{GL}(V)$ and acts on the fiber V in the natural way :

$$\psi_{12}() \quad (28.2)$$

In other words, we want our fiber bundle to be of structure group $\text{GL}(V)$ with the standard representation on V .

Then to define the entire vector bundle, we simply need to consider an entire cover of our manifold by such vector bundle charts :

Definition 28.2. A *vector bundle atlas* is a collection of vector bundle charts

Just as in the case of manifolds, our vector bundle is then either define as a maximal vector bundle atlas on the manifold or simply by some vector bundle atlas in which case we define isomorphic vector bundles by the compability of their atlases.

Definition 28.3. A *vector bundle* $\pi : \mathcal{V} \rightarrow M$ is a fiber bundle such that the typical fiber $\pi^{-1}(p) \cong V$ has the structure of an F -vector space, and the

Example 28.4. The trivial bundle $\text{pr}_1 : M \times V \rightarrow M$ for some vector space V , with the vector bundle charts simply defined by restriction of the projection to subsets $U \subseteq M$, is a vector bundle with transition group $\{e\} \subset \text{GL}(V)$

Example 28.5. The Moebius strip is a real vector bundle $\pi : M \rightarrow S$ with typical fiber $\mathbb{R}$, with the two vector bundle charts [...] and structure group $\mathbb{Z}_2 \subset \text{GL}(\mathbb{R})$

28.1.1 Sections

Theorem 28.6. Every vector bundle admits a global section called the *zero section* denoted by 0

Sections form a module

Theorem 28.7. The sections of a C^k -vector bundle form a $C^k(M)$ -module

28.1.2 Morphisms of vector bundles

28.2 Affine bundle

An affine bundle is defined in much the same way as a vector bundle, with most of the definitions replacing vector spaces with affine spaces.

Definition 28.8. An *affine bundle chart* (U_α, ψ_α) is a bundle chart

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times A \quad (28.3)$$

Definition 28.9. An *affine bundle* $\pi : \mathcal{A} \rightarrow M$ is a fiber bundle such that the typical fiber $\pi^{-1}(p) \cong A$ has the structure of an F -affine space with structure group $\text{GA}(A)$

Just as for the case of vector bundles, the typical example of an affine bundle is the affine version of the tangent bundle

Definition 28.10. The *affine tangent bundle* ATM

28.2.1 Sections

28.2.2 Morphisms of affine bundles

28.3 Frame bundle

As every finite dimensional vector space can be equipped with a basis, so can a vector bundle be equipped with a basis at a given fiber. There is some linear isomorphism

$$e : E_p \rightarrow \mathbb{R}^n \quad (28.4)$$

From this we can define a bundle attaching a frame to the vector bundle at every point. This is the *frame bundle* $F\mathcal{V}$ associated to the vector bundle $\mathcal{V}$

28.4 Operations on a vector bundle

Similarly to the operations we can perform on vector spaces, it is possible to perform various types of operations on vector bundles. In fact, by the following theorem, any operation on a vector space is possible to translate to an operation on a vector bundle [64] :

Theorem 28.11. Given a smooth functor $F : \mathbf{Vect}_F^k \rightarrow \mathbf{Vect}_F$ in k variables, and $\{\mathcal{V}_i\}$ a family of k F -vector bundles of typical fiber $\{V_i\}$, then the set

$$\mathcal{V} = F(\mathcal{V}_1, \dots, \mathcal{V}_k) \quad (28.5)$$

$$= \bigcup_{p \in M} F(\mathcal{V}_1|_p, \dots, \mathcal{V}_k|_p) \quad (28.6)$$

has a natural vector bundle structure with typical fiber $F(V_1, \dots, V_k)$.

Theorem on how you can map any vector space operation onto a vector bundle operation [Lee?]

Tensor product

$$\mathcal{V} \otimes \mathcal{W} \quad (28.7)$$

Whitney sum

$$\mathcal{V} \oplus \mathcal{W} \quad (28.8)$$

External tensor product

$$\mathcal{V} \boxtimes \mathcal{W} \quad (28.9)$$

Hom bundle

Dual bundle

K-theory stuff [65]?

Affine bundle : free and faithful translation action

$$E \times_M W \rightarrow E : (e, w) \mapsto e + w \quad (28.10)$$

28.5 The tangent bundle

The most common example of a vector bundle is that of the *tangent bundle*, that we have already seen as the space of 1-jets of curves of a manifold, $J_0^1(L, M)$.

Theorem 28.12. The tangent bundle $TM = J_0^1(L, M)$ is a vector bundle.

Proof. The projection of the bundle $\pi : TM \rightarrow M$ is simply the source morphism $\alpha : J_0^1(L, M) \rightarrow M$, which is a continuous injection. For a coordinate neighbourhood (ϕ_α, U_α) , we can define a bundle chart

$$\psi : U_\alpha \rightarrow U_\alpha \times \mathbb{R}^n \quad (28.11)$$

□

Theorem 28.13. The tangent bundle of a product manifold $M \times N$ is the Whitney sum of their tangent bundle :

$$T(M_1 \times M_2) = TM_1 \oplus TM_2 \quad (28.12)$$

Proof. Given a product of two manifolds with the product topology, we have two continuous projections

$$\pi_i : M_1 \times M_2 \rightarrow M_i \quad (28.13)$$

and a family of injections. For any $p_2 \in M_2$, we have an injection

$$\iota_{1,p_2} : M_1 \rightarrow M_1 \times M_2 \quad (28.14)$$

$$p_1 \mapsto (p_1, p_2) \quad (28.15)$$

□

28.6 Line bundles

Line bundles are a special class of vector bundles modelled on one-dimensional vector spaces, of either real or complex dimension (or some other field, but we will not consider those examples), so that they are of typical fiber isomorphic to $\mathbb{R}$ or $\mathbb{C}$, called real line bundles and complex line bundles.

Theorem 28.14. The set of C^k functions on the manifold $C^k(M)$ is isomorphic to C^k sections of the trivial line bundle $LM = L \times M$

Proof.

□

Frame bundle of the line bundle

Classification of line bundles (Chern-Weil stuff)

We will see a few more example of interesting line bundles throughout this book, such as the tautological line bundle $[X]$, the determinant line bundle $[X]$, twisted line bundles $[X]$ and Hermitian line bundles $[X]$.

28.7 Vector bundle operations

There exists a wide variety of operations one can perform on vector bundles, many of them based on existing operations on vector spaces.

The study of such operations is part of *K-theory* [65, 66]

Theorem 28.15. Given a continuous functor between two vector spaces $F_V : V \rightarrow W$, there exists a unique functor of vector bundles $F_{VB} : \mathcal{V} \rightarrow \mathcal{W}$ such that etc

Given two vector bundles $\mathcal{V}, \mathcal{W}$, of typical fiber V and W

This leaves us with a variety of constructions for our vector bundles from the ones on vector spaces.

First of all is the *dual vector bundle*, where for every vector bundle $\mathcal{V}$ modelled on V , there exist a vector bundle $\mathcal{V}^*$ modelled on its dual V^* .

28.7.1 Whitney sum

Definition 28.16. The *Whitney sum* of two vector bundles $\mathcal{V}$ and $\mathcal{W}$, $\mathcal{V} \oplus \mathcal{W}$, with identical base space M , is the fiber bundle over M defined as the pullback of the bundle

$\pi_{1,2} : E_1 \times E_2 \rightarrow M \times M$ by the diagonal map

$$\Delta : M \rightarrow M \times M \quad (28.16)$$

$$p \mapsto (p, p) \quad (28.17)$$

We can show that the typical fiber of this pullback is indeed a direct su

28.7.2 Tensor product bundle

By theorem 28.15, we have that there exists a tensor product in the category of vector bundles. More explicitly,

Definition 28.17. The *tensor product bundle* $\mathcal{V} \otimes \mathcal{W}$

The *tensor product* of two vector bundles $\mathcal{V} \otimes \mathcal{W}$, of typical fiber V and W ,

Symmetric tensor product

Skew tensor product

Exterior tensor product

For two vector bundles $\mathcal{V}_1, \mathcal{V}_2$ with typical fiber V_1, V_2

28.7.3 External tensor product

Definition 28.18. For two vector bundles $\pi_1 : \mathcal{V}_1 \rightarrow M_1$, $\pi_2 : \mathcal{V}_2 \rightarrow M_2$, their *external tensor product* $\pi : \mathcal{V}_1 \boxtimes \mathcal{V}_2 \rightarrow M_1 \times M_2$

A bitensor is a tensor field that is defined on more than one point [67,68]. It is some map from two points to a tensor value :

$$B : M_1 \times M_2 \rightarrow T^r_s V \quad (28.18)$$

A bitensor bundle is therefore given by the external tensor product of two tensor bundles $T^{r_1}_{s_1} M \boxtimes T^{r_2}_{s_2} M$

We are generally interested in bitensors over the same manifold, as they are useful to convey some non-local quantities, such as the distance between two points, the direction between two points, etc.

Limit to one point

28.8 Complex vector bundles and complexified vector bundles

Similarly to a real vector bundle, the complex vector bundle is simply a vector bundle of typical fiber $V \cong \mathbb{C}^n$ and structure group $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$.

Complex structure

holomorphic vector bundle

For any given real vector bundle, we define its complexification by the complexification of its typical fiber.

$$E \otimes \mathbb{C} \tag{28.19}$$

Fiber is $V \otimes \mathbb{C}$

Conjugate bundle $\overline{E}$

28.9 Projective bundle

Just as every vector space has an associated projective space, every vector bundle has an associated projective bundle.

As with vector spaces, we first have to remove the zero vector, which here will correspond to the zero section. The resulting bundle is called the slit vector bundle (most commonly used for the *slit tangent bundle*, usually noted T^0M or $\overset{\circ}{T}M$), which is a bundle where each fiber is diffeomorphic to $V \setminus \{0\}$. The projective bundle is then the quotient of that slit vector bundle by rescaling fiber-wise.

Definition 28.19. For a vector bundle $\pi : \mathcal{V} \rightarrow M$, its projective bundle $P^*\pi : P(E) \rightarrow M$ is the slit vector bundle $\overset{\circ}{\mathcal{V}} \cong \mathcal{E} \setminus \dots$ with its zero section removed, quotiented by the equivalence relation

$$v_1 \sim v_2 \leftrightarrow \pi(v_1) = \pi(v_2) \wedge \exists k \in \mathbb{R}_{\neq 0}, v_1 = kv_2 \tag{28.20}$$

So that $P(E) = E / \sim$ and $P^*\pi =$

Bundle projection

$$\pi : \tag{28.21}$$

The projective bundle of the vector bundle $\mathcal{V}$ with typical fiber V trivially has typical fiber $P(V)$ from this definition. Similarly, we can also define the oriented projective bundle $O\mathcal{V}$ with typical fiber OV by restricting the equivalence relation to only positive scale factors.

Blow up

We can obtain our slit vector bundle back from the projective bundle by considering its pullback along the projective projection $P(\pi)$:

[...]

This is the *tautological line bundle* of the projective bundle $P(E)$. line bundle

Projective affine bundle

28.9.1 The projective tangent bundle

A common example of a projective bundle of a vector bundle is the *projective tangent bundle* PTM .

curve reparametrization

Projectivized tangent bundle

Celestial sphere

28.9.2 Projective frame bundle

Given a projectivized vector bundle $P\mathcal{V}$, there exists a corresponding projectivization of the frame bundle, $PF\mathcal{V}$.

28.10 Characteristic classes

Vector bundles have a variety of topological invariants that we can define to measure various properties of the manifolds, called characteristic classes. [69]

Definition 28.20. A *characteristic class* is a functor from the category of vector bundles to the category of cohomology classes, such that for any vector bundle $\mathcal{V}$ and group G , a characteristic class $c(\mathcal{V})$ belongs to the singular cohomology class $H^*(M, G)$.

By functoriality, for $f : M_1 \rightarrow M_2$, $c(f^*\mathcal{V}) = f^*c(\mathcal{V}) \in H^*(M_2; G)$

Theorem 28.21. For any characteristic class, $c(\mathcal{V}) = 0$ for any trivial bundle $\mathcal{V}$.

Proof. A trivial vector bundle will be isomorphic to $M \times F^n$. If we consider the map $f : M \rightarrow \{*\}$ to the manifold of one point, $M \times F^n$ is the pullback of the vector space F^n over the point, $F^n \cong \{*\} \times F^n : f^*(F^n) \cong M \times F^n$. [...]

Definition 28.22. Given a characteristic class c , its *characteristic number*

Definition 28.23. For a real vector bundle $\pi : \mathcal{V} \rightarrow M$, its *Stiefel-Whitney class* $w(E)$ is a member of the singular cohomology ring

$$H^*(M, \mathbb{Z}_2) = \bigoplus_{i=0}^{\infty} H^i(M, \mathbb{Z}_2) \quad (28.22)$$

Chern class $c_i(\mathcal{Z}) \in H^{2i}(M, \mathbb{Z})$ for a complex vector bundle $\pi : \mathcal{Z} \rightarrow M$

Pontryagin class $p_i(V) \in H^{4i}(M, \mathbb{Z})$ for a real vector bundle

Euler class $e(\mathcal{V}) \in H^n(M, \mathbb{Z})$ for an oriented n -dimensional real vector bundle.

[Is every curve of preserving/reversing orientation of the same homotopy type?]

Chern–Weil homomorphism

28.11 Category of vector bundles

The category of vector bundles **VectBund** is composed of vector bundles as objects and vector bundle morphisms as morphisms.

Monoidal category : Cartesian monoidal structure is the direct sum of vector bundles, symmetric monoidal structure is the tensor product of vector bundles.

External tensor product functor :

$$\boxtimes : \mathbf{VectBund} \times \mathbf{VectBund} \rightarrow \mathbf{VectBund} \quad (28.23)$$

Chapter 29

Principal fiber bundles

As with the case of Lie groups, the fibers of a fiber bundles can possess a group structure, or at least something resembling it.

A *principal G -bundle* P , where G is a Lie group, is a fiber bundle equipped with a continuous right action from G :

$$R : P \times G \rightarrow P \quad (29.1)$$

$$(p, g) \mapsto R_g p = p \cdot g \quad (29.2)$$

such that R preserves the fiber structure, and acts freely and transitively :

$$\forall g, \pi(p) = \pi(R_g p) \quad (29.3)$$

$$p \cdot g = p \rightarrow g = e \quad (29.4)$$

$$\forall p_1, p_2 \in P, \exists g \in G, p_1 \cdot g = p_2 \quad (29.5)$$

Given that the group acts freely and transitively on the fiber, each fiber of a principal bundle is a G -torsor, and each fiber of the bundle is homeomorphic to the group G itself (as a manifold). As the orbits of a group action on a G -torsor are the entire space, the orbits of G at any point $p \in P$ are the fibers $\pi^{-1}(\pi(p))$.

"An equivalent definition of a principal G -bundle is as a G -bundle $\pi : P \rightarrow M$ with fiber G where the structure group acts on the fiber by left multiplication. Since right multiplication by G on the fiber commutes with the action of the structure group, there exists an invariant notion of right multiplication by G on P . The fibers of π then become right G -torsors for this action. "

Just as torsors can be mapped to a group by selecting a preferred point $t \in T$ to act as the identity, so can the fibers of a principal bundle be mapped to its corresponding Lie

group by selecting a point of every fiber at every point of M . In other words, a given section gives us the structure of a Lie group at every point. In this case, the right action R_g essentially acts as the group product on every fiber of P under the condition that $R_e \cdot \sigma = \sigma$, ie that we only consider the stabilizer of the section.

Lie groups themselves are in fact an example of a principal bundle. If we take some Lie group G , then a G -torsor is isomorphic to a principal G -bundle over the point, $\pi : G \rightarrow \{*\}$, and the Lie group itself can be considered as such a bundle with the section $s : * \mapsto e_G$, selecting the neutral element of that fiber.

Theorem 29.1. A principal bundle P is trivial, $P \cong G \times M$, if and only if it admits a global section.

Proof.

□

Given two sections $s_1, s_2 \in \Gamma(P)$, those two sections define a unique group action

Section, difference between two sections and action of one section on another

29.1 Tangent bundle

As a manifold itself, a G -principal bundle has a tangent bundle TP . Since all of its fibers $\pi^{-1}(p)$ are diffeomorphic to a Lie group, the vertical bundle of a G -principal bundle at a point, $V_p P$ is in fact isomorphic to a Lie algebra $\mathfrak{g}$.

Proof : isomorphism from the fiber of P to a Lie group etc

TP has the fibers $F \cong \mathbb{R}^n \times \mathfrak{g}$

Theorem 29.2. The vertical bundle VP is isomorphic to the trivial bundle $P \times \mathfrak{g} \rightarrow P$

Vector fields on TP , projectable vector fields, exponential map

29.2 Classifying space

Definition 29.3. A principal bundle is *universal* if its total space is contractible.

A universal principal bundle is denoted by $\pi : EG \rightarrow BG$, where BG is the *classifying space* of G .

Theorem 29.4. Every Lie group G has a universal principal bundle.

Proof. □

Example 29.5. A basic example of a classifying space is the $U(1)$ classifying space $BU(1)$

29.3 Associated bundle

As the notion of a principal bundle is supposed to carry the notion of a local symmetry, we also need the corresponding notion of some space that symmetry is applied to, which is what the associated bundle to a principal bundle is.

If we have some G -principal bundle, let's consider some manifold S that group G could act on. The Lie group G has some left group action ρ on S :

$$\rho : G \times S \rightarrow S \tag{29.6}$$

$$(g, s) \mapsto g \cdot s = \rho(g, s) \tag{29.7}$$

To get some field for which our G -principal bundle acts on, we need to define a bundle with typical fiber S

Theorem 29.6. The quotient space of $P \times S$ by G is a G -bundle.

Proof. □

Definition 29.7. For a G -principal bundle $\pi : P \rightarrow M$, the *associated bundle* is the

Associated fiber bundle also denoted AFB for short

A common type of associated fiber bundle we will consider is an *associated vector bundle* (AVB), where the typical fiber will be a vector space V and the group action will be a linear map.

Theorem 29.8. The quotient space of $P \times V$ by G is a vector G -bundle.

Proof. □

29.4 Reduction

Given a group G and a subgroup $H < G$, and a G -principal bundle and H -principal bundle,

Reduction isomorphic to the coset space G/H

Trivial reduction $G/\{e\}$ is just the sections of P

Theorem 29.9. For a principal G -bundle $\pi_G : P_G \rightarrow M$ with a connection Φ , and some arbitrary point $e \in G$, and the subset $P(e, \Phi)$ of every points connected to e by a horizontal curve, then $P(e, \Phi)$ is a reduced bundle with structure group $\text{Hol}(e)$ and the connection Φ is reducible to a connection in $P(e, \Phi)$.

[put this after the holonomy section]

Obstruction

[70]

29.4.1 Case of subgroups

One of the most common case of reductions of structure groups, which gives it its name, is the case where H is a subgroup of G , $H < G$.

Action of H on G

29.4.2 Case of lifts

29.5 Characteristic class

"The first Chern class of a bundle P is the class of its determinant line bundle $\det P$ "

Chern Weil theory

29.6 Contraction

As principal fiber bundles have the structure of a Lie group, we can see what happens if we try to perform some kind of contraction on the structure group G .

As the group contraction requires us to define a basis for the contraction to occur, we will similarly require a basis for our principal bundle, ie a section of the frame bundle FTP associated with TP .

Take the tangent bundle TP , define a basis in FTP , perform some contraction in that basis [some compatibility conditions?], integration of the new algebra [?]

Theorem 29.10. Given some principal G bundle, [contraction to H leads to an H principal bundle?]

Proof. For any trivializing neighbourhood $\psi : \pi^{-1}(U) \rightarrow U \times G$, consider the global sections of the frames $FP|_U$ and a set of families of continuous functions $\{f_{i,\lambda}\}$ which contract the Lie algebra via

$$[e_\mu, e_\nu] = f_{\mu\nu,\lambda} C^{\mu\nu\rho} e_\rho \quad (29.8)$$

Leads to a contracted principal H -bundle on the trivializing neighbourhood

What about the cocycle condition?

□

Is it a "reduction of structure group" but not a subgroup

What's the action of the original group on the contracted group and what's the homogeneous space

Chapter 30

Lie groupoids

[71, 72]

In the case of manifolds, it can be useful to generalize the notion of a Lie group to that of a Lie groupoid, where we are allowed to have a copy of the same Lie group at different point of a space, in some sense :

Definition 30.1. A *groupoid* $\mathcal{G}$ is a small category where every morphism is invertible, with the set of all objects $\text{Obj}(\mathcal{G})$ is denoted by $\mathcal{G}_0$ and the set of all morphisms $\text{Mor}(\mathcal{G})$ is denoted by $\mathcal{G}_1$. The groupoid itself is denoted by :

$$\mathcal{G} : G_1 \rightrightarrows G_0 \quad (30.1)$$

The category is equipped with the functions

- $s, t : \mathcal{G}_1 \rightarrow \mathcal{G}_0$, mapping each morphism to its *source* and its *target*. If we have a morphism $f : A \rightarrow B$, $s(f) = A$, $t(f) = B$.
- The unit map $u : \mathcal{G}_0 \rightarrow \mathcal{G}_1$, mapping objects A to their identity map $u(A) = \text{Id}_A$
- The multiplication map $m : \mathcal{G}_{1s} \times_t \mathcal{G}_1 \rightarrow \mathcal{G}_1$, sending two morphisms $f, g \in \mathcal{G}_1$ such that $t(g) = s(f)$ to their composition $g \circ f$
- The inversion map $\text{inv} : \mathcal{G}_1 \rightarrow \mathcal{G}_1$ sending a morphism to its inverse : $\text{inv}(f) = f^{-1}$.

From what we saw of the category of groups, a group is indeed itself a groupoid of a single object. The difference with a groupoid is that when a groupoid has more than one object, it will not make sense to compose any two endomorphisms f, g if they are not over the same object. If we wish to do so, we will need to compose it with some morphism $t(g) \rightarrow s(f)$ first.

A Lie groupoid is a groupoid in which the objects G_0 and G_1 are both smooth manifolds, s, t are surjective submersions,

The endomorphisms of an object are a Lie group

The basic example of a Lie groupoid is the *pair groupoid* $M \times M \rightrightarrows M$, which is composed of fundamental groupoid

30.1 Lie algebroids

Just as we have a correspondence between Lie groups and Lie algebras, we have correspondingly the Lie algebroid.

Definition 30.2. A *Lie algebroid* is a triple $(A, [-, -], \rho)$ of a vector bundle $\pi : A \rightarrow M$, a Lie bracket on the sections $\Gamma(A)$, and an anchor $\rho : A \rightarrow TM$ which is a vector bundle morphism, such that for any two sections X, Y and a smooth function $f \in C^\infty(M)$, the Lie bracket obeys the Leibniz rule

$$[X, fY] = \rho(X)[f]Y + f[X, Y] \quad (30.2)$$

Just as with Lie groups and Lie algebras, those are connected to Lie groupoids via a Lie functor. To derive it, first we show

[Groupoid anchor]

Definition 30.3. A Lie algebroid with a surjective anchor map is a *transitive Lie algebroid*.

One of the most common type of Lie algebroid is the Atiyah algebroid

Theorem 30.4. For a principal G -bundle P , the vector bundle $A = TP/G$ is a Lie algebroid.

Differential graded algebra

Chevalley-Eilenberg complex

Chapter 31

Connections

To define a derivative of some kind of object on our manifold (where that object will be some section of a fiber bundle), we need some way of defining how that object has changed with respect to some change in position. For instance, given the line bundle $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ for vector-valued functions on $\mathbb{R}$, we need to compare the values of that function at a point x and $x + dx$ so that we can meaningfully subtract them, but while all the fibers are diffeomorphic, there is no canonical identification of them which would allow us to compare a point $y_x \in \pi^{-1}(x)$ with $y'_{x+dx} \in \pi^{-1}(x + dx)$.

Example of a straight v. twisted such identification

To define such a derivative uniquely therefore, we need some way of associating nearby points of fibers to each other.

31.1 Jets of bundles

The way to define our derivatives will involve some mapping of points of the fiber bundle onto other points in some continuous manner. This will be done by considering our fiber bundle as a manifold and considering a special class of diffeomorphisms that maps points of a fiber to points of a different fiber in some way that makes it going along to some object of the manifold.

For instance consider such a motion along a curve, where we have some initial point $\gamma(0)$ corresponding to some value e in our bundle E and we would like to find corresponding points all along that curve

As we saw previously, the way to consider the infinitesimal action of a diffeomorphism is by considering its jets. To start with, we will consider the first jet of such diffeomorphisms, corresponding to the tangent bundle TE of our bundle E .

Given a smooth fiber bundle $\pi_E : E \rightarrow M$, as E is itself a smooth manifold, we can

consider its tangent bundle $\pi_{TE} : TE \rightarrow E$.

In terms of bundle coordinates, a point in TE can be expressed by

$$\psi(e) : TE \rightarrow \mathbb{R}^n \times \mathbb{R}^d \times F \times \mathbb{R}^n \quad (31.1)$$

$$e \mapsto (x^\mu, \phi^\alpha, y^\nu, \partial_\sigma \phi^\beta) \quad (31.2)$$

A vector in TE roughly has two kinds of components to it, a direction along the base manifold itself, given by vectors of the form $X^\mu \partial x_\mu$, and a direction along the fiber, given by $Y^\alpha \partial y_\alpha$. Our connection should give us something of the form telling us that given some motion from the curve $X^\mu \partial x_\mu$, the corresponding change in the bundle will be of the form

$$X^\mu \partial x_\mu + Y^\alpha \partial y_\alpha \quad (31.3)$$

More generally jets of the fiber bundle

Sections

The sections of those tangent bundles will as usual induce some 1-parameter families of diffeomorphism by integration

$$\text{Fl}_X^t = \phi_t^X = \exp(tX) \quad (31.4)$$

Most of those diffeomorphisms will break the structure of a fiber bundle : two points initially in the same fiber may end up belonging to two different fibers. This is because the diffeomorphisms generated by those vector fields are morphisms in the category of manifolds and not those in the category of fiber bundles. To avoid such problems, we need to only generate *fiber-preserving diffeomorphisms*, which are diffeomorphisms

Bundle morphisms

Are there any "large" bundle morphisms?

Types of vector fields

Projectable vector fields

For a fiber bundle $\pi : E \rightarrow M$, X is a *projectable vector field* if there is a vector field τ on M such that

$$\tau \circ \pi = T\pi \circ X \quad (31.5)$$

$$X = X^\alpha(x^\mu)\partial_\alpha + X^i(x^\mu, y^j)\partial_i \quad (31.6)$$

Internal transformations (along the fibers) v. geometric transformations

Vertical vector fields

Vertical and horizontal bundles

Connection as jets, connection as map on tangent bundle, connection as map on vertical bundle

Connection as flow of a vector field on TM giving rise to a projectable set of curves

31.2 Lifts

Once a connection is defined on a bundle, it is possible to perform *lift* is

31.3 Connection

[If connection integrable, flat?]

31.4 Transport

The connection of our fiber bundle gives us the ability to perform a *transport* on a curve. Consider some curve γ with the pullback bundle γ^*E , giving us the fiber structure along that curve. If we consider some initial point $\gamma(0)$ (we can choose any other point without loss of generality), with some initial point in the fiber $e \in \gamma^*E_{\gamma(0)}$, we can define some notion of the curve preserving this value as the parameter evolves. By analogy with the case where the value to be preserved is an actual direction (as if our bundle is TM), it is called a *parallel transport*.

We say that a curve parallel transports this value if the section defined on the pullback bundle, defining a lifted curve $\gamma^\uparrow$ in E itself, is such that every tangent of that curve is in the horizontal subspace

Horizontal lift

Transport operator

Transport equation

$$V(\lambda) = \text{tra}_{\nabla, \lambda_0, \lambda} V(\lambda_0) \quad (31.7)$$

Parallel transport operator

$$\frac{d}{d\lambda} \text{tra} = A \text{tra} \quad (31.8)$$

31.4.1 Iterated integrals

The transport equation cannot be simply integrated,

$$\text{tra} = \mathcal{P} \exp\left(-\int_{\lambda_0}^{\lambda} \Gamma \frac{dx}{ds} ds\right) \quad (31.9)$$

[73]

31.4.2 Holonomy

Once we have the parallel transport operator, one operation we can do is compare the parallel-transported values to their original ones. For this we can use the holonomy of the connection

Definition 31.1. The *holonomy* of a connection on a principal or vector bundle is the difference on

$$\text{Hol}_p(\nabla) = \{P_\gamma | \gamma \in \Omega M(p)\} \quad (31.10)$$

linear and nonlinear connections

31.5 Covariant derivatives

Now that we have some tool to identify nearby fibers, we can finally define a derivative on our manifold.

Covariant derivative as the limit of the transport operator

$$\nabla_X s = \lim_{h \rightarrow 0} \frac{P}{h} \quad (31.11)$$

Covariant derivative as the vertical projection of the exterior derivative

$$\nabla_X s = \pi_V(ds(X)) \quad (31.12)$$

Koszul derivative

The transport we have defined so far is specifically related to curves in some sense : the transport of an object is performed along some curve γ on M , lifted to a curve $\gamma^\uparrow$ on E , and our connection is defined on TE , the jet space of curves. But connections can be also be defined for higher dimensional objects

Use in string theory

show that a connection on a worldsheet isn't just reducible to a connection on a curve

31.6 Curvature

The curvature of a connection roughly corresponds to the behaviour of multiple applications of a connection :

If we consider the holonomy of our connection Hol_∇

Curvature as infinitesimal holonomy

$$\Omega(u, v) = d\omega(u, v) + \frac{1}{2} [\omega(u), \omega(v)] \quad (31.13)$$

31.7 Pullback connection

Given a connection ∇ on a vector bundle $\mathcal{V}$ over N and a smooth map $f : M \rightarrow N$, the pullback connection $f^*\nabla$ on $f^*\mathcal{V}$ over M is the connection defined by

$$(f^*\nabla)_X(f^*s) = f^*(\nabla_{df(X)}s) \quad (31.14)$$

The simplest example of a pullback connection is that of a curve, with the map $\gamma : L \rightarrow M$. In this case, the pullback connection

Submanifold connection

For a product of two manifolds $M_1 \times M_2$, there is a pullback connection of the projection map

$$\text{pr}_i : M_1 \times M_2 \rightarrow M_i \quad (31.15)$$

$$\text{pr}_i^* \nabla \quad (31.16)$$

31.8 Principal connection

If we have a principal bundle $\pi : P \rightarrow M$, a connection $\omega \in \Omega^1(P, \mathfrak{g}) \cong C^\infty(P, T^*P \otimes \mathfrak{g})$ on that bundle is called a *principal connection* if it obeys the following properties :

- Given an element $g \in G$, the connection obeys the property $\text{Ad}_g(R_g^* \omega) = \omega$
- For an element of the Lie algebra $X \in \mathfrak{g}$ and its fundamental vector field on P $X^\#$, $\omega(X^\#) = X$

It should be noted that a connection on a principal bundle may fail to be a principal connection. In which case it is sometimes referred specifically as a *non-principal connection* (this will be used for instance later on when we consider the connection associated with observer space).

Examples of a principal connection can be given by the circle connection $U(1)$ and the Yang-Mills connection $SU(n)$, with respective Lie algebras $\mathfrak{u}(1) \cong \mathbb{R}$ and $\mathfrak{su}(n)$. In this case, the

$$\omega() = \quad (31.17)$$

Given a section $s \in \Gamma(P)$, the pullback of the principal connection via this section $s^* \omega$

Gauge transformation : for another section $g \in \Gamma(P)$ acting on the right via $sg = s(p)g(p)$ (unique group action taking s to g ?)

$$(sg)^* \omega = \text{Ad}_g^{-1} s^* \omega + g^{-1} dg \quad (31.18)$$

31.9 Koszul connection

If we have a vector bundle $\pi : \mathcal{V} \rightarrow M$,

31.10 Stack of connections

The notion of a connection as a distribution on a bundle has the drawback of not allowing as much flexibility as one may have in physics. In the case of gauge theory for instance, our gauge connection may be on a trivial G -principal bundle, or a twisted one, such as in the case of electromagnetism where the twist in the bundle will correspond to magnetic monopoles. In some cases we even need to consider all those bundles simultaneously, such as having to sum over every possible configuration of the electromagnetic field.

[Relation to the universal bundle EG]

31.11 Atiyah algebroids

Another description of connection comes in the form of Atiyah algebroids. If we have a fiber bundle $\pi : E \rightarrow M$, we have the following short exact sequence

$$0 \rightarrow VE \xrightarrow{\iota} TE \xrightarrow{d\pi} \pi^*TM \rightarrow 0 \quad (31.19)$$

where ι is the inclusion of VE in TE , and as it is simply the kernel of $d\pi$, we have that $\text{Im}(\iota) = \text{Ker}(d\pi)$.

[74]

$$0 \longrightarrow VP \longrightarrow TP \longrightarrow \pi^*TM \longrightarrow 0 \quad (31.20)$$

$$1 \longrightarrow \Gamma_c(\text{Ad}(P)) \longrightarrow \text{Aut}_c(P) \longrightarrow \text{Diff}_c(M) \longrightarrow 1 \quad (31.21)$$

31.12 Contraction of principal connections

If we have a principal connection (P, ω) on a G -principal bundle P , given some section $s \in \Gamma(P)$ which defines a Lie group structure on the fibers of P , we can attempt to define a contraction of that Lie group and connection

At any given point $(p, g) \in P$, consider the vertical space $V_{(p,g)}P \cong \mathfrak{g}$ given a basis $\{t\}$. Our connection ω can be expressed as

$$\omega = \quad (31.22)$$

Chapter 32

Natural bundles and geometric objects

One key aspect of general relativity is that it is a *geometric* theory, a term which we will define in the following chapter.

[75–77]

Geometric objects (Veblen)

A *natural bundle* is a bundle which admits a lift of the diffeomorphism group to bundle automorphisms.

Natural functor :

Definition 32.1. A *natural bundle functor* F is a covariant functor

$$F : \mathbf{SmoothMan} \rightarrow \mathbf{FibMan} \quad (32.1)$$

such that

$$B \circ F = \text{Id}_{\mathbf{SmoothMan}} \quad (32.2)$$

for the base functor B , for $\iota : U \rightarrow M$ some inclusion of an open submanifold, $FU = p_M^{-1}(U)$ and $F\iota$ is the inclusion of $p_M^{-1}(U)$ into FM

etc...

Example of natural bundle functors : T the tangent functor, T^r_k the (r, k) velocity functor

Weil functors

Natural bundles

Define natural bundles

Geometric objects as sections of natural bundles

Lifts to natural bundle

[74, 76, 78–80]

32.1 Jets and velocity bundles

The jet spaces $J^k(M, N)$ that we previously saw can be shown to have a bundle structure, and in particular a natural bundle structure.

Theorem 32.2. For two smooth manifolds M_1, M_2 , of dimension n_1 and n_2 , the jet space $J^k(M_1, M_2)$ has the structure of a fiber bundle $(\alpha, \beta) : J^k(M_1, M_2) \rightarrow M_1 \times M_2$, with typical fiber diffeomorphic to $\text{Poly}^k(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$

Given some map $X : \mathbb{R}^k \rightarrow M$,

For the jet between k -manifolds

Affine or vector velocity bundles

For the jet space $J(N, M)$ consider the trivial bundle $\text{pr}_2 : M \times N \rightarrow N$

Source and target map

Jet functor?

Theorem 32.3. The jet space $J^k(E)$ is a fiber bundle on M with projection s

Proof.

□

Projections : $\pi_{k-r}^k : J^k(E)$

Jet coordinates

Contact ideal

Contact one forms

$$\{\theta^\mu\}$$

form a basis of the contact ideal

”If $\pi : U_0 \rightarrow U$ is a local coordinate neighbourhood for E and $\Xi : U_0 \rightarrow J^\infty(U_0)$ satisfies $\Xi^*\omega = 0$ for all $\omega \in \mathcal{C}$, then there exists a local section $s : U \rightarrow U_0$ such that $\Xi(x) = j^\infty(s)(x)$ for all $x \in U$.”

32.2 The frame bundle

One of the most important spacetime object is the *frame bundle*. The frame bundle in terms of velocity bundles is the subset of the first jet of the n -dimensional velocity bundle that is invertible, $\text{inv}T_n^1M$, the bundle of invertible first jets for maps $\mathbb{R}^n \rightarrow M$. This can in fact be shown to be equivalent to the first jet of the coordinate charts, and also the first jet of diffeomorphisms on the manifold.

$$FM \cong \text{inv}J_0^1(\mathbb{R}^n, M) \cong J^1\text{Diff}_p(M) \cong J_p^1\phi_{U_p} \quad (32.3)$$

Theorem 32.4. The frame bundle is a principal bundle of the group $\text{GL}(n)$.

Proof. Implicit function theorem? etc □

Associated bundles

Theorem 32.5. The k -velocity bundles are associated bundles to the frame bundle of order k .

Proof. □

Theorem 32.6. The frame bundle F^kM is the frame bundle of the tangent bundle : $FM \cong FTM$.

Proof. □

Generalize for F^kM and FT^kM .

Lift the coordinate line for every coordinate

Higher order frame bundle

We also have the affine version of the frame bundle given by removing the condition of our jets being of fixed source, giving us the *affine frame bundle* :

$$AFM \cong \text{inv}J^1(\mathbb{R}, M) \quad (32.4)$$

Projective frames?

Theorem 32.7. The projective frame bundle PF^kM is the projective frame bundle of the contact elements k

Reps of the general linear group

32.3 Lifting properties

For a natural bundle based on $J^k(\mathbb{R}^r, M)$, lifting of submanifolds of dim r ?

Lifting of curves to TM , T^*M , PTM , etc

Definition 32.8. A *lift* of a k -submanifold S to a velocity bundle $T_k^r M$, $S^\uparrow$

[81]

32.4 Double tangent bundle

As one of the important connection we will have to deal with in general relativity is the connection on the tangent bundle, we have to consider the *double tangent bundle*, TTM .

The double tangent bundle TTM is as the name implies the tangent bundle of the tangent bundle itself, with the usual tangent bundle projection, treating TM as a manifold. It is important to denote this projection as there is another bundle structure one could impose on the total space TTM , which is the pushforward of the projection π , $d\pi$:

$$\begin{array}{ccc} TTM & \xrightarrow{j} & TTM \\ \downarrow \pi_{TTM} & & \downarrow d\pi \\ TM & & TM \\ \downarrow \pi_{TM} & & \downarrow \pi' \\ M & & M \end{array}$$

Interpretation of TTM wrt π_{TM} and $d\pi$

The double tangent bundle can also be considered as the second non-holonomic jet extension of curves on our manifolds,

$$TTM = J_0^1(L, J_0^1(L, M)) \quad (32.5)$$

and furthermore, it is a subbundle of the second jet of surfaces, $J^2(\mathbb{R}^2, M)$, allowing a nice interpretation for sections of TTM as corresponding to families of curves.

"In a natural way, the second order tangent bundle $T(TM)$ is a quotient of $J_0^2(\mathbb{R}^2, M)$, the manifold of 2-jets of maps from $\mathbb{R}^2$ to M with source 0. We call this quotient map $\mathcal{Q} : J_0^2(\mathbb{R}^2, M) \rightarrow T(TM)$."

[82]

Definition 32.9. The *canonical flip* j

$$j\left(\frac{\partial\Gamma}{\partial s\partial t}\right) = \left(\frac{\partial\Gamma}{\partial t\partial s}\right) \quad (32.6)$$

Projection down to J^2M

How it relates to doubly ruled surfaces in J^2M

The vertical bundle VTM has a canonical section $\iota : TM \rightarrow VTM$, the radial vertical vector field, given by $\iota(u) = \iota_u(u)$, or

$$\iota(u) = \iota(u^\mu \partial_\mu|_{\pi(u)}) = u^i \dot{\partial}_\mu|_u \quad (32.7)$$

32.5 Something jets

In a given coordinate chart ϕ , we define the holonomic basis $\{\partial_\mu\}$.

32.6 Soldering

One fundamental aspect of a natural bundle is that they are directly related to the geometric objects of the manifold in some sense, which is done via the notion of soldering :

Definition 32.10. For a fiber bundle $\pi : E \rightarrow M$ of typical fiber F and structure group G , we say that E is *soldered* to M if

- G acts transitively on F
- $\dim(M) = \dim(F)$
- E admits a section s
- There's a

[83]

Theorem 32.11. A fiber bundle equipped with a soldering is a natural bundle [Is that true]

Find a generalization of soldering for any jet space

A more geometric way to consider a soldering is to consider the 1-parameter families of diffeomorphisms on M . As we've seen, such maps produce a curve given a point p on the manifold, via

$$M \times C^\infty(\mathbb{R}, \text{Diff}(M)) \rightarrow C^\infty(\mathbb{R}, M) \quad (32.8)$$

$$(p, \phi_t) \mapsto \gamma(t) = \phi_t(p) \quad (32.9)$$

This family of diffeomorphism maps to a vector field by the Lie functor

If we consider an associated vector bundle of fiber $\mathbb{R}^n$ to the frame bundle, with the canonical action of $\text{GL}(n)$ [what kind of mapping would produce a vector field given ϕ_n]

Is the soldering related to the lift of diffeomorphism viz. the making of curves from 1-parameter families of diffeomorphisms

We can also define the soldering directly as the bundle isomorphism to the tangent bundle

1-form $\theta \in \Gamma(T^*FM \otimes \mathbb{R}^n)$

$\mathbb{R}^n$ -valued 1-form θ

Connections as 2-solder forms

[84]

Soldering from TM to the bundle directly

Frame fields, tetrads

Using our soldering, we can map elements of our fake tangent bundle $\mathcal{T}M$ to the tangent bundle

$$e : \quad (32.10)$$

$$\theta \in \Gamma(T^*FM \otimes \mathbb{R}^n) \quad (32.11)$$

For $v \in \Gamma(T_p FM)$ such that $v = \pi_* w$,

$$\theta(w) = \theta^a|_p(w)e_a \quad (32.12)$$

$$= \Theta^a|_x(v)e_a \quad (32.13)$$

For $\{e_a\}$ the canonical basis of $\mathbb{R}^n$ and $\{\theta^a\}$ a basis of T^*FM , $\theta = \pi^*\Theta^a$, $\theta^a|_p(w) = \Theta^a|_x(v)$

[...]

For $e \in FM$, $p \in M$ such that $\pi(e) = p$, there exists a $\mathbb{R}^n$ -valued form θ

$$\theta_e(Y_e) = \rho(\pi_2(e)^{-1})(\pi_*Y_e) \quad (32.14)$$

Pullback under a section $s \in \Gamma(FM)$:

$$s^*\theta_p(Y_p) = \theta_{s(p)}(s_*Y_p) \quad (32.15)$$

$$= \rho(\pi_2(s(p))^{-1})(\pi_*s_*Y_p) \quad (32.16)$$

$$= \rho(\pi_2(s(p))^{-1})((\pi \circ s)_*Y_p) \quad (32.17)$$

$$= \rho(g)^{-1}(Y_p) \quad (32.18)$$

$\rho(g)$ is the matrix corresponding to the frame $(X_i(e))$, ρ is the fundamental rep. Coordinates of Y_e in the $(X_i(e))$ basis

32.7 Gauge-natural bundles

Natural bundles can be generalized to include in addition some gauge symmetries (beyond the geometric type), producing objects that will likely fit most models in physics. [85,86]

For $B : \mathbf{FibMan} \rightarrow \mathbf{SmoothMan}$ the base functor. Fix a Lie group G and consider the category of principal G -bundles over n -manifolds, $\mathbf{PrincBund}_n(G)$, morphisms are principal bundle morphisms $f : P \rightarrow \bar{P}$

Definition 32.12. A *gauge-natural bundle* is a functor $F : \mathbf{PrincBund}_n(G) \rightarrow \mathbf{FibMan}$ such that every principal bundle is transformed to a fiber bundle over the same manifold etc

For $G = \{e\}$, this is just natural bundles

Example : $U(1)$ gauge-natural bundle

32.8 Concomitant objects

Just as there are concomitant objects which can be expressed by rational integral expressions of each other in the case of vector spaces, there are also differential concomitant geometric objects,

Concomitant objects serve the purpose of defining a class of basic objects in gravitational theories, giving us for instance the set of all objects of a certain type that could be build using a metric tensor, a connection, etc.

[78, 87–89]

Chapter 33

Cartan geometries

The goal of a Cartan geometry is to in some sense locally replicate the structure of a Klein geometry (in neighbourhoods around every point) and then stitch together those structures to a global structure.

The general case of a Cartan connection [90] is given by a Klein geometry (G, H) , such that $\dim(G/H) = n$, and a principal H -bundle $\pi : P \rightarrow M$

Fundamental vector field : for $A \in \mathfrak{h}$, A^*

A Cartan connection is a $\mathfrak{g}$ -valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ obeying

1. For any $A \in \mathfrak{h}$, $\omega(A^*) = A$
2. For $h \in \mathfrak{H}$, $(R_h)^*\omega = \text{ad}(h^{-1})\omega$
3. For any non-zero vector field X on P , $\omega(X) \neq 0$

Curvature :

$$\Omega = d\omega + \frac{1}{2} [\omega, \omega] \quad (33.1)$$

General case : reduction $P(M, G) \rightarrow P'(M, G')$

Specific case where it relates to the tangent bundle / frame bundle?

Higher connection case

[91]

Definition 33.1. A *Cartan geometry* is a manifold equipped with a Cartan connection, G -principal connection, reduction of structure group $H \hookrightarrow G$, isomorphism

$$T_p M \xrightarrow{\nabla} \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h} \quad (33.2)$$

Cartan connection for a :

$$\varpi = \theta + \omega \quad (33.3)$$

Torsion of a connection

$$\Theta = d\theta + \omega \wedge \theta \quad (33.4)$$

33.1 Cartan curvature

Curvature as map $\bigwedge^2 M \rightarrow VP_{\mathfrak{h}}$

33.2 Development

For a curve γ , the *development*

Rolling sphere analogy

33.3 Generalized circles

Definition 33.2. A *generalized circle* is a curve $\gamma : I \rightarrow M$ that is the projection of an integral curve of an ω -constant vector field on P .

Generalized circles in E : circles and straight lines?

33.4 Parabolic geometries

If the group G is semisimple [Why?] and $H \subset G$ is a parabolic subgroup, parabolic Cartan connection

[92]

Flag curvature?

33.5 Contraction of Cartan geometries

If we have a Cartan geometry (G, H)

Chapter 34

G -structures

Somewhat related to Cartan geometries are G -structures, which are in some sense the polynomial approximation of Cartan geometries.

Given the k -th order frame bundle $F^k M$, a G -structure of Lie group G , G a subgroup of the jet group $\mathrm{GL}^k(n)$, is a principal bundle reduction from $F^k M$ to some G -principal bundle $P(M, G)$.

[93]

34.1 Group prolongation

Given a Lie subgroup of $\mathrm{GL}(n)$, $G < \mathrm{GL}(n)$, with its Lie algebra $\mathfrak{g}$, we consider the space of symmetric multilinear mappings of order k , $L_{\mathrm{Sym}}((\mathbb{R}^n)^k, \mathbb{R}^n)$, so that if $t \in \mathfrak{g}_k$, t

$$t : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (34.1)$$

t belongs to the k -th prolongation of the Lie algebra $\mathfrak{g}$, noted $\mathfrak{g}_k$, if for $k - 1$ fixed vectors $(v_i)_{1 \leq i \leq k-1}$, $v_i \in \mathbb{R}^n$, then the map

$$\mathbb{R}^n \rightarrow \mathbb{R}^n \quad (34.2)$$

$$v \mapsto t(v, v_1, \dots, v_{k-1}) \quad (34.3)$$

belongs to $\mathfrak{g}$.

For a given Lie algebra $\mathfrak{g}$, we say that the smallest value of k such that the prolongation $\mathfrak{g}_k$ is the trivial algebra $\mathfrak{g}_k = 0$ is the *order* of the algebra. If $\mathfrak{g}_k$ is never trivial for any value of k , $\mathfrak{g}$ is said to be of infinite order.

Definition 34.1. The k -th prolongation of the general linear group $\mathrm{GL}(n, k)$ is called the *jet group* of order k , noted $\mathrm{GL}_k(n, k)$

Theorem 34.2. The jet group of order k is the group of inverse jets $\mathrm{inv} J_0^k(\mathbb{R}, \mathbb{R}^n)$.

[90]

Prolongation of $\mathfrak{o}$

Prolongation of $\mathfrak{co}$

34.2 G -structure

Definition 34.3. Given the k -th order frame bundle $F^k M$ with structure group $\mathrm{GL}_k(n)$ over M , a k -th order G -structures is a reduction of that bundle to a G -principal bundle $\pi : P \rightarrow M$

[94]

Connection from reduction of 2-frames :

$$d\omega^i = -\omega_p^i \wedge \omega^p \quad (34.4)$$

34.3 Integrability

The notion of geometric structures being locally similar to their model space can be upgraded to a more general case by the notion of an integrable G -structure.

Definition 34.4. A G -structure is said to be *integrable* if there exists an open cover $\{U_\alpha\}$ of the manifold such that the G -structure on each U_α is expressed as the default G -structure

Obstruction

Torsion

Theorem 34.5. Only torsion-free G -structure are integrable.

Proof.

□

A more local notion is given by that of an order k infinitesimally integrable G -structure :

Definition 34.6. For any integer $k \in \mathbb{N}$, a G -structure

”For $k \in \mathbb{N}$, a G -structure $\mathbf{c}$ on a manifold X modeled on V (e.g. a smooth manifold or supermanifold) is called order- k infinitesimally integrable if at each point $x \in X$ its restriction to the order- k infinitesimal neighbourhood $\mathbb{D}_V^0 \cong \mathbb{D}_x^X \hookrightarrow X$ is equal to the default G -structure $\mathbf{c}_0$.”

[95, 96]

34.4 Geometric structures

The G -structures that will interest us are the ones involved with spacetime symmetries

Two easy choices we can make are that the structure group is the entire general linear group $\mathrm{GL}(n)$, or that it is the trivial group $\{e\}$. The first case corresponds to simply having no structure in the first order, and a general affine connection in the second. The second case defines a *parallelism*, where the reduction simply gives rise to a frame field defined on the manifold.

Another important structure is the *volume structure*, with the structure group $\mathrm{SL}(n)$, the special linear group. This corresponds to frames defining the same volume form.

The bundle reduction gives rise to the *scale bundle*, which we will study in more details in chapter X.

Metric structure, conformal structure, symplectic structure

34.5 Complex structure

Another type of structure that will be important later on is the *almost-complex structure*. An almost-complex structure is defined on any even-dimensional manifold by invariance under the group $\mathrm{GL}(n)$ out of $\mathrm{GL}(2n)$.

The structure preserved here is a rank $(1, 1)$ tensor J which, interpreted as a vector automorphism $J : TM \rightarrow TM$, is an involution : $J \circ J = \mathrm{Id}_{TM}$.

The obvious example of an almost-complex structure is the case of $\mathbb{R}^2$ with the matrix defined by

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{34.5}$$

in Cartesian coordinates.

34.6 Symplectic structure

Symplectic form ω

The symplectic form defines a musical isomorphism

More details at [X]

Chapter 35

Higher gauge

We have considered the case of connections mostly in the case of curves, but this notion can be extended for the case of a submanifold of arbitrary dimension.

A first approach of higher gauge theories is to consider the case of connections for higher dimensional objects. If we consider a k -dimensional submanifold S instead of a curve, our

Higher gauge theory considers the case where

35.1 Lie k -groups

Another way to consider higher gauge theory is to base the theory on a k -group rather than a group.

[97–99]

Chapter 36

Kinematic groups

A *kinematic group* is some Lie group to describe the motion (and geometry) of objects on the model space. Given some model space X , the kinematic group is some group $G \subset \text{Aut}(X)$ such that the isotropy subgroup H forms a Klein space with G , (G, H) with X as its model space. A *kinematic algebra* is the Lie algebra of that group.

A spacetime model will in general be based on some kinematic group as a Cartan geometry, as other types of geometries will in general be unfit to model such theories.

There can be quite a large number of groups one can define for such a space, but kinematic groups are generally defined with some properties in mind. There are n generators for translations, including one of a preferred direction for time. Those are the generators of time translation H and space translation P^i . There are also generators for rotations J and boosts K .

We can also consider the more general case that includes conformal transformations, if the space we wish to model has some conformal symmetry. To this we can add the dilation operator D and the generator of special conformal transformations ?

The usual conditions that we impose on kinematic groups are the following :

- The one-dimensional subgroups generated by each boost generator are non-compact
- The kinematic group is isotropic
- The kinematic group is invariant under parity and time-reversal transformations.

The non-compactness of the boosts can be considered in the following way : one of the defining point of kinematic groups in physics is that we need to have them define a local causal structure : there is a split between timelike vectors going to the future, to the past, and spacelike vectors. There is no way of transforming a timelike vector into a spacelike vector, or a timelike vector of another time orientation by using a transformation connected to the identity.

If we consider simply the type of our vectors by simply considering the oriented projective space OV , the different types of vectors form different regions of that space : a region for future timelike vectors, past timelike vectors, and the rest. Depending on the kinematic group, this can be spacelike and null vectors, or just the very thin region separating the past and future for Galilean and Aristotelian groups, or the timelike regions can be compressed to a single point in the Carrollian case. [Aristotelian case?]. In each case, the stabilizer subgroup of the kinematic group will cause vectors in each region to span the entire region : the orbit of a future timelike vector under $\text{Stab}(K)$ is the entire timelike region, and the same goes for spacelike, null and past timelike vectors.

If our stabilizer subgroup was entirely compact (as the rotation group always is), the orbit of any given vector would be

[is it for causality purpose, compact group leads to compact orbit leads to no separation between types of vectors]

Argument from going to the light cone from inside the light cone by subsequence of applications of boosts and compactness? Shouldn't orbits also be S ? Does it go outside from the light cone since it is also open? and have to circle all the way back to the original point?

How does it relate to the past light cone?

[100, 101]

The kinematic group is isotropic if its Lie algebra for J is that of $\mathfrak{so}(n-1)$, and it integrates to the subgroup $\text{SO}(n-1)$. Furthermore, the other generators behave as follow under J :

$$[J, H] = 0 \quad (36.1)$$

$$[J, J] = J \quad (36.2)$$

$$[J, P] = P \quad (36.3)$$

$$[J, K] = K \quad (36.4)$$

Parity and time reversal transformations

Every kinematic group comes with its homogeneous part comprised of the stabilizer subgroup for spacetime points, $H = \text{Stab}$

[102–105] [106–108] Eleven kinematic groups for eight algebras :

Poincaré Galilean Bargmann Carrollian de Sitter anti de Sitter Newton-Hook Para-Poincaré Para-Galilean Static (Euclidian) (Aristotelian) Static (Spherical) Static (Hyperbolic)

Conformal versions of those?

Classification of kinematic groups

Klein group of a kinematic group, does it always have the weird incidence structures, polarities, conics, etc, of Minkowski space

Universal enveloping algebra, Casimir invariants

Clifford algebras, Dirac operators

Dynamic reps of the kinematic groups

"A group G is a covariance group of a representation if (i) G maps [kinematically possible trajectories] of that representation into [kinematically possible trajectories]; (ii) the [kinematically possible trajectories] constitute "the basis of a faithful representation of G " (i.e., no two elements of G produce identical mappings of the [kinematically possible trajectories]); (iii) G maps [dynamically possible trajectories] into [dynamically possible trajectories]. (Thorne et al., 1973, p. 3567)"

Carrol : [109]

36.1 Hierarchy of the kinematic groups

Most of the kinematic groups can be arranged in a hierarchy where the different groups relate to each other via a contraction of one of the characteristic scale of that group.

36.2 The little group

If we consider the action of a kinematic group on a vector space V with representation ρ , given any vector $v \in V$, its *little group* is the stabilizer subgroup of that vector.

Wigner classification

Definition 36.1. Given an action $\rho : G \times X \rightarrow X$ of a kinematic group G on a space X , and some subset $S \subset X$, the *little group* is the stabilizer subgroup of that subset.

Typically in physics the little group mostly concerns vector spaces, their subspaces, and group representations on those subspaces.

Example : rep for the Lorentz group is classified by spin and mass sign

Every kinematic group can define an equivalent Cartan connection via G -structures

Group contractions

Connections

Contraction scheme for kinematic groups

speed-time contraction, speed-space contraction, space-time contraction

36.3 Representations

36.3.1 Irreps of the Poincaré group

[110]

Irreps of the Poincaré group

36.3.2 Irreps of the Galilean group

Irreps of the Galilean group

Chapter 37

Orientation

As we saw all the way back in the chapter on manifolds, a manifold's orientability is defined by the Jacobians of its atlas covers. If two charts have a transition function with a Jacobian of positive determinant, they are said to define the same orientation, and the transition function is orientation-preserving. If we consider a subatlas consisting of only charts of the same orientation, we have an *oriented atlas*.

There are two possible oriented atlases. We will denote them by $\mathcal{A}^+$ and $\mathcal{A}^-$, although the choice of which one is of positive and negative orientation is arbitrary, as we can switch them.

A manifold is *orientable* if there is two such oriented atlases, and *non-orientable* otherwise.

We can define the orientation of our manifold by an *orientation bundle*, $\pi : OM \rightarrow M$, a principal natural bundle with a typical fiber $\mathbb{Z}_2$. The orientation fiber is given by considering for the atlas $\{(U_\alpha, \phi_\alpha)\}$ of M a bundle atlas $\{(U_\alpha, \psi_\alpha)\}$ of OM with structure group $\mathbb{Z}_2$ obeying the cocycle condition

$$g_{\alpha\beta} \tag{37.1}$$

If the bundle is trivial, the manifold is orientable, and the two sections correspond to the two orientations of the manifold.

$$\phi \tag{37.2}$$

Natural bundle, action of the diffeomorphism group by considering the orientation preserving v. non-orientation preserving diffeomorphisms

We can also consider the orientation bundle as belonging to the G -structure stemming from the reduction of the general linear group $GL(n)$ to the oriented general linear group $GL^+(n)$. This will work in fact for the reduction of any unoriented subgroup of the

general linear group to their oriented version, such as the orthogonal group, special linear group, etc.

When we consider any subgroup of $GL(n)$ that are not subgroups of $GL^+(n)$, defined by the subgroup $G \subseteq GL(n)$ and $G^+ = G \cap GL^+(n)$, we can define the short exact

$$0 \rightarrow G^+ \hookrightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 0 \quad (37.3)$$

Orientation as the bundle reduction $GL(n) \rightarrow GL^+(n)$ or $O(n) \rightarrow SO(n)$

The orientation bundle can also be seen as the subbundle of the density bundle from the determinant rep of the general linear group defined by the sign of this determinant :

$$e \mapsto \text{sign}(\det(e)) \quad (37.4)$$

Orientation by the top n -form bundle

$O(p, q)$

If we consider subgroups of $GL(n)$ that further split the tangent space into different regions, like the indefinite conformal or orthogonal groups $CO(p, q)$ and $O(p, q)$, we obtain additional notions of orientation. In the case of the relativistic signature $(n-1, 1)$, these are called *time orientability* and *space orientability*.

37.1 Time orientability

In the case of Lorentzian manifolds, an additional notion of orientability occurs in the form of time orientability.

Similarly to the n -form definition for orientability, we can consider the case of

See also the bundle of cones

Bundle of cones : orbit of a timelike vector field under action from the Lorentz group?

37.2 Space orientability

Similarly to the concept of time orientability is that of *space orientability*. We will say that a given spacetime is space orientable if there is a nowhere vanishing spacelike $(n-1)$ -form $\omega_{(n-1)}$

37.3 Correlations of various orientabilities

Theorem 37.1. If a spacetime is any two of orientable, space orientable or time orientable, it is also the third. If it is any two of oriented (by an n -form ω_n), space oriented (by an $(n-1)$ -form $\omega_{(n-1)}$) or time oriented (by a 1-form τ), there is a third orientation defined by a choice

$$\omega_n = \pm \omega_{(n-1)} \wedge \tau \quad (37.5)$$

Proof.

□

37.4 Associated bundles to the orientation

37.5 Obstructions to orientability

We have already seen a few cases of all types of orientability failing. [111, 112] We will now look at some generic conditions for cases where orientability of various types cannot be defined.

On an algebraic level, orientability can be described by relative homology

Theorem 37.2. A manifold is orientable if and only if its first Stiefel–Whitney class vanishes : $w_1(M) = 0$.

Proof. If a manifold fails to be orientable, there exists some loop γ

□

Stiefel-Whitney classes, obstruction

[113]

Chapter 38

The metric tensor

The fundamental object of most descriptions of general relativity is the *metric tensor*. There are quite a number of ways to define the metric tensor, and we will review them.

The most common definition of the metric tensor in general relativity is that of a rank $(0, 2)$ symmetric tensor with signature $(1, n - 1)$, so that it is a map

$$g : TM \times TM \rightarrow \mathbb{R} \quad (38.1)$$

It is common to write out the appropriate tensor bundle as $S_{p,q}M$ for the bundle of symmetric rank $(0, 2)$ tensors of signature (p, q) , so that our metric tensor can be written as a section of that bundle : $g \in \Gamma(S_{1,(n-1)}M)$.

Via the tensor-hom adjunction, it can alternatively be considered as a map from vectors to dual vectors :

$$g : TM \rightarrow T^*M \quad (38.2)$$

The metric tensor is often defined as an *inner product*, but in the case of general relativity, it fails to be positive definite : $g(v, v)$ can both fail to be non-negative and it can also fail to be non-zero for non-zero vectors. It otherwise obeys the usual properties of an inner product, that is, to be symmetric and linear in both terms. This is referred to as a *symmetric bilinear form*.

$$\begin{aligned} g(X, Y) &= g(Y, X) \\ g(aX_1 + bX_2, cY_1 + dY_2) &= ac \, g(X_1, Y_1) + ad \, g(X_1, Y_2) \\ &\quad + bc \, g(X_2, Y_1) + db \, g(X_2, Y_2) \end{aligned}$$

In terms of coordinates, if we pick a coordinate chart with holonomic basis ∂_μ with dual basis dx^μ , the metric tensor components are

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) \quad (38.3)$$

So that we can write out our metric tensor as

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \quad (38.4)$$

For historical reason, the metric tensor in general relativity tends to be written in the *line element* notation. For a metric tensor g , we write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (38.5)$$

disregarding the tensor product of the dual basis (implicit). This comes from the notion of integrating the length of a curve, which component-wise can be written as

$$L_\gamma = \int ds \quad (38.6)$$

$$= \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}} ds \quad (38.7)$$

Using the infinitesimal notation very loosely, this implies

$$ds^2 = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} ds^2 \quad (38.8)$$

$$= g_{\mu\nu} dx^\mu dx^\nu \quad (38.9)$$

As the metric is somewhat of an inner product, we can define the associated "norm" of a vector :

$$\|X\| = \sqrt{g(X, X)} \quad (38.10)$$

$$= \sqrt{g_{\mu\nu} X^\mu X^\nu} \quad (38.11)$$

38.1 The metric as a G-structure

A metric tensor can be defined by an $O(p, q)$ -structure, where we define a reduction of the frame bundle FM to the bundle of orthonormal frames OM

The reduction $GL(n) \rightarrow O(p, q)$ is equivalent to a section of the

$$GL(n)/O(p, q) \cong \mathbb{R}^{\frac{p+q}{2}} \times \mathbb{R}P^p \quad (38.12)$$

[check quotient]

Diagram of $SO(p, q)$, $GL(n)$, $SO(p)$, etc

$$GL(n)$$

Those orthonormal frames can be used as maps to the fake tangent bundle to define the metric tensor :

$$g : T_p M \times T_p M \rightarrow \mathbb{R} \quad (38.13)$$

$$(X, Y) \mapsto g(X, Y) = \langle \theta_e(X), \theta_e(Y) \rangle \quad (38.14)$$

Theorem 38.1. The metric as a G -structure is isomorphic to the metric as a section of $S_{p,q}M$.

Proof.

□

38.2 The metric as a Cartan geometry

As a Cartan geometry, the metric tensor is the description associated with the Cartan geometry $(\text{Poinc}(n), \text{Lor}(n))$

Frames

Map between the fake tangent bundle and TM

As the fake tangent bundle has a canonical inner product $\langle \cdot, \cdot \rangle$ defined on it, this defines an inner product

$$g(X, Y) = \langle \theta(X), \theta(Y) \rangle \quad (38.15)$$

38.3 Musical isomorphisms and inverse metric

Using the tensor-hom adjunction given by our metric tensor, we can define a canonical map between vectors and tangent vectors : [Riesz operator]

$$g : TM \rightarrow T^*M \quad (38.16)$$

$$v \mapsto g(v, -) \quad (38.17)$$

which has the action on vector fields $g(X, -)(Y) = g(X, Y)$. This mapping is denoted by the *flat musical isomorphism*, $X^\flat = g(X, -)$. The class of this musical isomorphism is the same as that of the metric. If we have a C^k metric, the corresponding musical isomorphism is a C^k map, mapping smooth vector fields to C^k 1-forms, since we have for any smooth X and Y

$$X^\flat(Y) = g(X, Y) \quad (38.18)$$

which by definition of the regularity of the metric tensor is a C^k map from $\mathfrak{X}(M)$ to $\mathbb{R}$.

This first musical isomorphism allows us to define the *inverse metric* g^{-1} , which is a rank $(2, 0)$ tensor defined by

$$g^{-1} : T^*M \times T^*M \rightarrow \mathbb{R} \quad (38.19)$$

$$(\alpha, \beta) \mapsto g^{-1}(\alpha, \beta) \quad (38.20)$$

$$g(X, Y) = g^{-1}(X^\flat, Y^\flat) \quad (38.21)$$

In components :

$$g_{\mu\nu} X^\mu Y^\nu = g^{\mu\nu} g_{\sigma\mu} X^\sigma g_{\tau\nu} Y^\tau \quad (38.22)$$

This leads us to the defining relation of the inverse metric tensor,

$$\delta^\mu_\sigma = g^{\mu\nu} g_{\sigma\nu} \quad (38.23)$$

In terms of matrices, treating the metric tensor as the matrix $\mathbf{g}$, the inverse metric tensor is the inverse matrix $\mathbf{g}^{-1}$, as the name would imply :

$$\mathbf{g}\mathbf{g}^{-1} = \mathbf{g}^{-1}\mathbf{g} = \mathbf{I} \quad (38.24)$$

This inverse matrix always exists as our metric tensor's determinant never vanishes.

The inverse metric tensor itself defines the inverse musical isomorphism :

$$g^{-1} : T^*M \rightarrow TM \quad (38.25)$$

$$\alpha \mapsto g^{-1}(\alpha, -) \quad (38.26)$$

written as the *sharp musical isomorphism* $\alpha^\sharp$

$$\alpha^\sharp(\beta) = g^{-1}(\alpha, \beta)$$

The composition of our two isomorphisms is given by

$$(X^\flat)^\sharp = g(X, -)^\sharp \quad (38.27)$$

$$= g^{-1}(g(X, -), -) \quad (38.28)$$

The two musical isomorphisms are therefore inverse of each other, making them bijective.

The musical isomorphisms extend to tensors. For a tensor T of rank (r, s) ,

$$T \quad (38.29)$$

This notion can be extended to more general tensors by the use of the musical isomorphisms, as this is equivalent to

$$\|X\| = \sqrt{\quad} \quad (38.30)$$

Norm of a tensor :

$$\|T\| = \sqrt{T^\sharp(T)} \quad (38.31)$$

Trace of the metric tensor :

$$\text{Tr}(g) = \text{Tr}(\Lambda\eta\Lambda^{-1}) = (p - q) \quad (38.32)$$

(use cyclic property)

38.4 Contractions

Given our metric and the musical isomorphisms it induces, we can now define the *contraction* of a tensor field using that metric. As we can define a contraction for any tensor $(r, s) \rightarrow (r - 1, s - 1)$, the musical isomorphisms will also let us define a contraction $(r, s) \rightarrow (r - 2, s)$ and $(r, s) \rightarrow (r, s - 2)$.

For two integers α, β , $1 \leq \alpha \leq r$, $1 \leq \beta \leq s$, we define the contraction map of a rank (r, s) by

$$C_{\beta}^{\alpha} : T^r_s M \rightarrow T^{r-1}_{s-1} M \quad (38.33)$$

The contracted tensor acts in the following way on a set of

Contractions with symmetric and antisymmetric tensors

$$X^{(\mu\nu)}Y_{\mu\nu} = X^{(\mu\nu)}Y_{(\mu\nu)} \quad (38.34)$$

Trace of a tensor $\text{Tr}(T) = T^{\mu}_{\mu}$

Theorem 38.2. The trace of the metric tensor is equal to the number of dimension of the space : $\text{Tr}(g) = n$.

Proof.

$$\text{Tr}(g) = \quad (38.35)$$

□

38.5 Volume form

A metric structure defines a unique volume form (up to $O(n)$ rotation)

Definition 38.3. The *metric volume form* ω_g is a nowhere vanishing n -form

Reduction $\text{SL}(n) \rightarrow O(p, q)$?

38.6 Hodge star

Given a metric on the tangent bundle, we can now define the equivalent of the Hodge star for forms.

Gram determinant

Definition 38.4. The *Hodge star* $\star$ of a differential k -form $\alpha \in \Omega^k(M)$ is a map

$$\star : \Omega^k(M) \rightarrow \Omega^{(n-k)}(M) \quad (38.36)$$

$$\alpha \mapsto \star \alpha \quad (38.37)$$

Such that its wedge product with another k -form $\beta \in \Omega^k(M)$ is

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \omega \quad (38.38)$$

$$\det(\alpha \wedge \star \beta) = \langle \alpha, \beta \rangle \quad (38.39)$$

In terms of coordinates

A common application of the Hodge dual is the definition of the Levi-Civita tensor, which is the Hodge dual of 1 :

$$\varepsilon = \star 1 \quad (38.40)$$

The Hodge star allows us to define a variety of other operators, first among these being the codifferential :

Definition 38.5. The *codifferential* δ is a differential operator defined as the Hodge dual of the exterior derivative (up to a sign based on degree), ie for a given k -form α , we have

$$\delta \alpha = (-1)^k \star^{-1} d \star \alpha \quad (38.41)$$

While the exterior derivative maps

Laplace-de Rham operator

Identities

38.7 Classification of vectors and curves

The metric tensor gives us the ability to classify vectors as we did in Minkowski space 14.3, since every tangent space in the tangent bundle can be mapped to the fake tangent bundle which has a natural Minkowski structure.

Just like in the case of affine spaces, we classify vectors as being either timelike, spacelike or null (or zero, usually kept as distinct from null). A vector field that is everywhere of the same norm sign can then also be classified as any of those.

If a curve is C^1 and has a tangent vector everywhere of the same norm sign, we say that this curve is of the type associated with that sign. A curve everywhere of norm > 0 is spacelike, everywhere of norm 0 is null, and everywhere of norm < 0 is timelike.

If a curve is only piecewise C^1 , we say that it is of a given causal type if it is of that causal type on all C^1 regions, and also that both its half-derivatives[X] at each turning point is of that type as well. Furthermore, if the curve is causal, we will say that the curve is [...] if the two half-derivatives are in the same light cone at the turning point.

Curves of a lower degree of regularity can still be attempted to be classified (from Kriele)

38.8 Metric invariants

Cartan–Karlhede algorithm

Carminati–McLenaghan invariants

In addition to the local metric invariants, the metric structure also comes with a variety of global metric invariants, most prominently the

Integral invariants

38.9 Space of metrics

If we want to look at the space of all possible sections of a metric, we can look either at the sections of all the $O(n-1, 1)$ -structures, or equivalently, the sections of symmetric rank $(0, 2)$ -tensors of signature $(n-1, 1)$.

[94, 114–119]

The space of all sections of metric tensors corresponds to

$$\text{Lor}(M) = \Gamma(S_{1,,n-1}M) \tag{38.42}$$

[120]

38.9.1 Moduli space of metrics

However, even after quotienting by the $O(n-1, 1)$ gauge, we still have leftover symmetries in the configuration space of metrics, as we will consider that two diffeomorphic Lorentzian manifolds are considered equivalent, so that for two such manifolds (M, g) and (M', g') , we will consider them equivalent if there exists a diffeomorphism $\phi : M \rightarrow M'$ such that the two are isometric : $\phi^*g = g'$.

In terms of coordinates, we can for instance consider two spacetimes

[121]

Superspace, moduli space, Teichmuller space

Case of Riemannian surfaces

Stratified space [122]

Definition 38.6. A *stratified space* $(V, S, \mathcal{S})$ is a topological space V , a collection S of connected smooth manifolds called *strata* such that V is their disjoint union

$$V = \bigsqcup_{X \in S} X \quad (38.43)$$

...

- Each stratum X is a locally closed subset and the decomposition S is locally finite.
- If $X, Y \in S$ and $Y \cap \overline{X} \neq \emptyset$, then $Y \subseteq \overline{X}$.
- $X = \{v \in T_X | \rho_X(v) = 0\}$, ρ is the distance function from the stratum X .
- For $X, Y \in S$, $Y < X$, $(\pi_Y, \rho_Y) : T_Y \cap X \rightarrow Y \times (0, +\infty)$ is a submersion
- For $X, Y \in S$, $Y < X$, $\pi_Y \circ \pi_X = \pi_Y$ and $\rho_Y \circ \pi_X = \rho_Y$.

Theorem 38.7. The moduli space of metrics is a non-Hausdorff stratified space.

Proof.

□

[123]

There are less pathological versions of the moduli space of metrics of a manifold that could be considered, the discussion of which is deferred until chapter X.

38.9.2 The two dimensional case

The two dimensional Riemannian case, called the moduli space of Riemann surfaces, is a good example as it is usually fairly simple and finite dimensional.

38.9.3 Minisuperspaces

If we have some additional informations about the spacetime, it is common to consider a smaller subspace of the moduli space of metrics, called the *minisuperspace*.

For instance, a simple case is to consider the space of all metrics of constant curvatures, since up to diffeomorphisms they are all only classified by the scalar curvature R . The minisuperspace of such metrics is therefore a one-dimensional subspace of $\text{Lor}(M)$, a curve $\gamma : \mathbb{R} \rightarrow [g_R]$ of the metrics of constant scalar curvature R up to diffeomorphisms.

Chapter 39

The Levi-Civita connection

The connection we will usually consider in our spacetime theories is the *Levi-Civita connection*. This corresponds to the Cartan connection for the indefinite orthogonal group, $(\mathbb{R}^n \rtimes O(p, q), O(p, q))$ that is torsion-free. We have some reduction from the general linear group $GL(n)$ to $O(p, q)$, giving us a solder form θ

[...]

Reduction from GL_2 to $O(p, q)$

What people will generally call the Levi-Civita connection is the connection defined on the actual tangent bundle :

$$\nabla^{\text{LC}} : \quad (39.1)$$

The actual Cartan connection defined on the associated bundle is usually referred to as the *spin connection*, ω

$$\varpi = \omega + A \quad (39.2)$$

While having nothing to do with spinors by itself, we will later see that it is so-called due to the form the actual connection on spinors takes.

The Levi-Civita connection does not have any obvious gauge invariance (outside of diffeomorphism invariance) despite being a Cartan connection.

Mapped from a connection on the fake tangent bundle to the tangent bundle

Given a soldering θ

Given a holonomic basis $\{\partial_\mu\}$, we can define the *Christoffel symbols* of the Levi-Civita

connection

$$(\nabla_{\partial_\mu} \partial_\nu)(dx^\sigma) = \Gamma^\sigma_{\mu\nu} \quad (39.3)$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\tau} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \quad (39.4)$$

Uniqueness

Dressing field

Curvature

$$\text{Riem}(X, Y, Z) = \quad (39.5)$$

Ricci tensor and scalar

$$\text{Ric}(X, Y) \quad (39.6)$$

Derivative along a curve

$$D_t X = \gamma^* \nabla_{\dot{\gamma}} X \quad (39.7)$$

Ricci decomposition

Weyl tensor

Cotton tensor

Interpretation of the Riemann and Ricci tensors

Further derivatives :

Divergence operator

Laplace-Beltrami

39.1 The Levi-Civita connection as a derivative

Many general relativity books define the Levi-Civita connection as a derivative on the space of tensors obeying certain properties.

- Linearity : $\nabla(aT_1 + bT_2) = a\nabla T_1 + b\nabla T_2$
- Leibniz rule : For any tensor field multiplied by a scalar function fT , $\nabla(fT) = (df) \otimes T + f\nabla T$
- Commutes with contractions : $\nabla(C_j^i(T)) = C_j^i(\nabla T)$
- Reduces to partial derivatives for scalars
- Torsion free
- Metric compatible : $\nabla g = 0$

39.2 The geodesic equation

For the Levi-Civita connection, the geodesic equation is as usual given by the parallel transport of the tangent vector by itself,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \quad (39.8)$$

$$\nabla_{u^\mu \partial_\mu} X^\nu \partial_\nu = 0 \quad (39.9)$$

Pregeodesics

In terms of coordinates,

$$\frac{d^2}{d\lambda^2} x^\sigma(\lambda) + \Gamma^\sigma_{\mu\nu} \frac{d}{d\lambda} x^\mu(\lambda) \frac{d}{d\lambda} x^\nu(\lambda) = 0 \quad (39.10)$$

Uniqueness of results, related to the prolongation of O being trivial

Theorem 39.1. Given a C^1 metric tensor, for some initial conditions $(\gamma(\lambda_0), \dot{\gamma}(\lambda_0))$, the geodesic equation admits a unique solution in a neighbourhood of λ_0 .

Proof. Proof by Picard (cf. Choquet-Bruhat)

□

Sum of parallel-transported vectors is parallel-transported

Schild's ladder

39.3 Curvature

As with any connection, the curvature of the connection is given by the Cartan structure equation,

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \quad (39.11)$$

Commutative formula :

$$\text{Riem}(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad (39.12)$$

$$R \dots = \quad (39.13)$$

Interpretation as infinitesimal holonomy

$$\text{Riem}(X, Y)(Z) = \frac{d}{ds} \frac{d}{dt} \text{tra}_s^{-1} \text{tra}_t^{-1} \text{tra}_s X \text{tra}_t Y Z \Big|_{s=t=0} \quad (39.14)$$

Proposition 39.2. The Riemann tensor obeys the following properties :

- Skew symmetry : $\text{Riem}(X, Y) = -\text{Riem}(Y, X)$
- Skew symmetry : $\omega[\text{Riem}(X, Y)(Z)] = Z^\flat[\text{Riem}(X, Y)(\omega^\sharp)]$
- First Bianchi identity :
- Second Bianchi identity :
- Interchange symmetry :

Dual Riemann tensor

We can contract the Riemann tensor to obtain the *Ricci tensor*

$$\text{Ric}(X, Y) = \text{Tr}(Z \mapsto \text{Riem}(Z, X)(Y)) \quad (39.15)$$

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu} \quad (39.16)$$

Proposition 39.3. The Ricci tensor is symmetric : $\text{Ric}(X, Y) = \text{Ric}(Y, X)$

Definition 39.4. The *Ricci scalar* R is the trace of the Ricci tensor :

$$R = \text{Tr}(\text{Ric}) \quad (39.17)$$

$$= g^{\mu\nu} R_{\mu\nu} \quad (39.18)$$

There are a variety of additional scalars we can construct from the Riemann tensor. [cf. concomitant tensors]

The three scalars we can construct at second order are the *Kretschmann scalar* K_1 , the *Chern-Pontryagin scalar* K_2 , and the *Euler scalar* K_3

$$K_1 = \langle \text{Riem}, \text{Riem} \rangle \quad (39.19)$$

$$= R^{\tau}{}_{\mu\nu\sigma} R_{\tau}{}^{\mu\nu\sigma} \quad (39.20)$$

$$K_2 = \quad (39.21)$$

39.4 Flag curvature

As the orthogonal group is a parabolic subgroup of $\text{GL}(n)$,

Sectional curvature, Gauss curvature

For a plane subspace A_p at $T_p M$ spanned by two vectors X_p, Y_p , $A = \text{span}(X, Y)$, the sectional curvature is

$$K(A) = K(X \wedge Y) = \frac{\text{Riem}(X, Y, X, Y)}{g(X, X)g(Y, Y) - (g(X, Y))^2} \quad (39.22)$$

39.5 Ricci decomposition

[124]

Decomposition of the Ricci tensor into Ricci tensor, Ricci scalar and Weyl tensor

Relation to conformal group and volume group?

Find the normal coordinates for conformal metric and volume form

Kulkarni-Nomizu product and what it means [125]

$\text{map } \bigwedge^2 V \rightarrow \mathfrak{so}(n-1, 1) \text{ eq. to } \bigwedge^2 V \rightarrow \bigwedge^2 V$

39.6 Derived operators

Given our Levi-Civita connection ∇ , there are a few different differential operators that we can define

Definition 39.5. The *divergence* of a vector field X is the contraction of its covariant derivative :

$$\mathbf{div} X = C(\nabla X) \quad (39.23)$$

$$= \nabla_m u X^\mu \quad (39.24)$$

$$\mathbf{div}(X) = \mathcal{L}_X \omega = d(\iota_X \omega) \quad (39.25)$$

Formula for divergence with Γ_μ

Definition 39.6. The *Laplace-Beltrami operator* is the contraction of two Levi-Civita connections :

$$\square_{LB} T = \nabla \nabla T \quad (39.26)$$

$$= g^{\mu\nu} \nabla_\mu \nabla_\nu T \quad (39.27)$$

Theorem 39.7. For any k -form α , the Laplace-Beltrami operator of α is given by

$$\square_{LB} \alpha = (d\delta + \delta d)\alpha \quad (39.28)$$

Formula for scalar fields (since $\delta f = 0$):

$$\square_{LB} f = \delta df = \star^{-1} d \star df \quad (39.29)$$

In coordinates :

$$\square_{LB} f = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu f) \quad (39.30)$$

This is a very common formula as it corresponds to the notion of the wave equation on curved spacetime.

Hessian

Definition 39.8. The Hessian $\text{Hess}(f)$ of a scalar function is the second derivative :

$$\text{Hess}(f) = \nabla(\nabla f) = \nabla(df) \quad (39.31)$$

$$\text{Hess}(f)[X, Y] = \langle \nabla_X df, Y \rangle \quad (39.32)$$

$$\text{Hess}(f)[X, Y] = X[Y[f]] - df[\nabla_X Y] \quad (39.33)$$

[91, 124]

Schouten tensor, Weyl tensor, Cotton tensor

If we have a bitensor field $B \in \Gamma(T_s^r M \boxtimes T_{s'}^{r'} M)$, there exists a two connections defined on it acting as

$$\nabla^1 : T_s^r M \boxtimes T_{s'}^{r'} M \rightarrow T_s^{(r+1)} M \boxtimes T_{s'}^{r'} M \quad (39.34)$$

(check if correct rank)

Those are the connections naturally defined by the definition of the external tensor bundle, as the pullback connection

$$\nabla^i = \quad (39.35)$$

39.7 Concomitant objects

Given a metric structure g and a Levi-Civita connection ∇ on a manifold, what are the concomitant objects

39.8 Sparling form

[126]

Chapter 40

Torsion

While the connection of general relativity is generally assumed torsion-free, it can be useful to consider the general case with torsion, leading us to *Einstein-Cartan gravity*.

The torsion is, as we've seen, defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad (40.1)$$

Torsion of a spin connection as the derivative of the solder form

Torsion of the connection [127]

$$T(\partial_\mu, \partial_\nu) = \nabla_{\partial_\mu} \partial_\nu - \nabla_{\partial_\nu} \partial_\mu - [\partial_\mu, \partial_\nu] \quad (40.2)$$

$$= \Gamma^\sigma_{\mu\nu} \partial_\sigma - \Gamma^\sigma_{\nu\mu} \partial_\sigma \quad (40.3)$$

$$= \Gamma^\sigma_{[\mu\nu]} \partial_\sigma \quad (40.4)$$

The torsion tensor in terms of coordinates is therefore expressed by the antisymmetry of the Christoffel symbols.

As we can always split our Christoffel symbols between its symmetric and antisymmetric parts as

$$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{[\mu\nu]} + \Gamma^\sigma_{(\mu\nu)} \quad (40.5)$$

Difference of two connections $\nabla = \overline{\nabla} + \omega$

The *contorsion tensor*

Contorsion tensor

Parallel transport with torsion

As the torsion only affects the geometry in an antisymmetric manner, the geodesic equation is not affected by torsion,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \nabla_{\dot{\gamma}}^{\text{LC}} \dot{\gamma} + C(\dot{\gamma}, \dot{\gamma}) \quad (40.6)$$

Infinitesimal torsion

If we disregard non-metricity for now, there are four options that our manifolds can have regarding the interaction of a metric connection and torsion. It can have neither curvature nor torsion, the flat case, it can have only curvature, which is the Levi-Civita connection, it can have both curvature and torsion, which is the Cartan connection(?), and it can have no curvature and torsion, called the Weitzenböck connection.

40.1 Torsion on parallel transport

The main impact of torsion on a manifold is how it affects the parallel transport of vectors along a curve. If we have a curve γ in flat space along which we wish to parallel transport a given vector,

$$\nabla_X \dot{\gamma} = X^\mu \partial_\mu \dot{\gamma}^\nu + K^\sigma_{\mu\nu} \dots \quad (40.7)$$

[Cartan's staircase example]

Example 40.1. Cartan's spiral staircase is a manifold with zero curvature and a totally antisymmetric torsion $\theta = \omega\epsilon$, for ω some nowhere vanishing scalar function and ϵ the Levi-Civita tensor.

40.2 Torsion of conformal structures

Chapter 41

Non-metricity

If the affine connection is not demanded to be metric-compatible, we get that the covariant derivative of the metric tensor gives us some rank $(0, 3)$ tensor N ,

$$\nabla_Z g(X, Y) = N(X, Y, Z) \quad (41.1)$$

N is the *non-metricity tensor*.

In terms of coordinates,

$$N_{\mu\nu\rho} = -\partial_\rho g_{\mu\nu} + \dots \quad (41.2)$$

One fundamental property of the non-metricity tensor is that it describes the evolution of angles and norms by parallel transport along a geodesic. If we take some quantity defined from some bundle E along a curve γ , the quantity along the curve belongs to the pullback bundle γ^*E , the parallel transport of this quantity is defined by

$$\nabla_{\dot{\gamma}} s = 0 \quad (41.3)$$

Evolution of an inner product along the curve :

$$\gamma^* \nabla g(X, Y) = \quad (41.4)$$

There are two cases of interest for this evolution, which is the evolution of angles and the evolution of lengths. If we want to look at the evolution of lengths, this is the case where we consider

$$\gamma^* \nabla g(X, Y) = 0 \quad (41.5)$$

$$\gamma^* \nabla g(X, X) = \omega \quad (41.6)$$

So that $N(X, Y, -) = 0$ unless $X = Y$, in other words, this is the symmetric part of the nonmetricity tensor, $N_{(\mu\nu)\rho}$.

Conversely, the evolution of angles is given by the antisymmetrization of the tensor, which get rid of the term depending on the derivatives of the metric.

In terms of coordinates, we are considering the case where X and Y are different vectors and where they are the same vector.

For a general case, let's consider a full $\{e_\mu\}$ basis, for which any two elements are indeed different. Then the metric expressed in that basis will have its diagonal representing the norm of each of those basis vectors, while the off-diagonal terms are their angles. This remains true no matter which basis we pick.

So

$$\nabla g(e_i, e_j) \quad (41.7)$$

Schrödinger's connection, equiaffine connection, conformal connection

Chapter 42

Scale and integration

A few related issues on manifolds are that of the *scale* and *integration*. Before we can look into all of this however, we first need to define a reasonable enough measure on manifolds.

42.1 Measures and Lebesgue measure

[maybe move to math section]

A *measure space* is a triplet (X, Σ, μ) of a set X , a σ -algebra Σ , and a measure μ , mapping elements of the σ -algebra to the extended non-negative real line, $\mathbb{R}_{x \geq 0} \cup \{\infty\}$

The σ -algebra consists of all the measurable regions in that space, and is composed of subsets of X , $\Sigma \subseteq \mathcal{P}(X)$. The algebra has to obey the following properties :

1. X itself is part of the σ -algebra : $X \in \Sigma$
2. Σ is closed under complementation : if $A \in \Sigma$, then $X \setminus A \in \Sigma$
3. Σ is closed under countable unions : for a collection $\{A_i\}_{i \in I}$, for some finite indexing set I ,

$$\bigcup_{i \in I} A_i \in \Sigma \quad (42.1)$$

Those axioms imply that the σ -algebra is also closed under countable intersection (for $A, B \in \Sigma$, $A \cap B \in \Sigma$, by De Morgan's law), and the empty set (since $X \setminus X = \emptyset$) is also part of the σ -algebra.

The *measure* μ is some function mapping elements of the σ -algebra to the extended real line, $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$:

$$\mu : \Sigma \rightarrow \mathbb{R}^* \quad (42.2)$$

$$A \mapsto \mu(A) \quad (42.3)$$

The measure has the properties of being non-negative ($\mu(A) \geq 0$), having a null empty set ($\mu(\emptyset) = 0$), and of being countably additive : if we consider a collection of disjoint sets $\{A_i\}_{i \in I}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, then the measure of their union is the sum of their measure :

$$\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i) \quad (42.4)$$

If we drop the requirement that the measure has to be non-negative, this is what we call a *signed measure*.

As the name implies, all of these properties define something resembling a measure (in the more informal sense). If we take the example of trying to measure volumes, as was done for the original measure theory, we have that the volume of an empty region is zero and that the volume of two disjoint regions is equal to the sum of their individual volumes.

42.1.1 Product measure

Given two measure spaces (X_i, Σ_i, μ_i) , the *product measure space* of the two is defined by $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$, with $X_1 \times X_2$ simply the Cartesian product of the two, $\Sigma_1 \otimes \Sigma_2$ the tensor product σ -algebra

42.1.2 Borel measures

As we will mostly be dealing with measures on a topological space, we need to define the notion of a Borel measure, which is a measure compatible with a space's topological structure.

For a topological space X , the Borel sets of that space are a kind of σ -algebra, being the minimal σ -algebra such that every open set is part of the σ -algebra : for every subset belonging to the topology $U \in \tau$, this subset also belongs to the σ -algebra, $U \in \Sigma$, and from this basis, every other measurable set can be constructed by complementation and intersection. As closed sets are the complements of open sets, this implies in particular that all closed sets are also measurable.

A *Borel measure* is then simply a measure where the σ -algebra is the Borel sets of the topological space.

42.1.3 Regularity and finiteness

To define a reasonable enough measure, we need some notions of the measure being small on a small enough region. A measure is *locally finite* if for any point $p \in X$, there exists a neighbourhood of p , $U_p \ni p$, that is of finite measure : $\mu(U_p) < \infty$. This will avoid having an infinite volume for arbitrarily small regions.

A measure is σ -finite if the space X can be covered by at most countably many measurable regions with finite measure.

Borel regular, inner regular, outer regular

"Any Borel probability measure on a locally compact Hausdorff space with a countable base for its topology, or compact metric space, or Radon space, is regular."

42.1.4 Haar measures

If our measurable space is a G -set for some group G , we can also require that our measure be invariant under the group action, giving rise to the notion of a Haar measure :

Definition 42.1. Given a measurable space (X, Σ) and a group G acting on X via some action ρ_g , a *Haar measure* is a measure invariant under the group action :

$$\mu(\rho_g(x)) = \mu(x) \quad (42.5)$$

42.1.5 Lebesgue measure

The standard measure on $\mathbb{R}$ is the *Lebesgue measure*, constructed from the measure of intervals given by

$$\mu([a, b]) = \mu((a, b)) = \mu([a, b)) = \mu((a, b]) = b - a \quad (42.6)$$

Radon-Nykodim

Equivalent measures

Definition 42.2. Given two measurable spaces (X_1, Σ_1) and (X_2, Σ_2) , a measurable mapping $f : X_1 \rightarrow X_2$, and a measure on X_1 $\mu_1 : \Sigma_1 \rightarrow [0, +\infty]$, the *pushforward measure* $f_*\mu$ is

$$f_\mu(A) = \mu(f^{-1}(A)) \quad (42.7)$$

Pullback measure

42.2 Measures on manifolds

As fundamentally a topological space, the measure on a manifold should probably be a Borel measure, and as a locally Euclidian space, we should also require that the measures on a coordinate neighbourhood be homeomorphic to a measure on $\mathbb{R}^n$. If we require the measure to be σ -finite (for the reasons we explained earlier), and that it be an equivalent measure to the Lebesgue measure (in the sense of having the same sets of zero measure), we can apply the *Radon-Nikodym theorem*

Theorem 42.3. For a measurable space (X, Σ) and two σ -finite measures on that space, μ_1 and μ_2 , if $\mu_1 \ll \mu_2$ (absolutely continuous measure with respect to μ_2), then there exists a measurable function $f : X \rightarrow [0, \infty)$ such that, for $A \in \Sigma$,

$$\mu_1(A) = \int_A f d\mu_2 \quad (42.8)$$

The Radon-Nikodym derivative :

$$f = \frac{d\mu_1}{d\mu_2} \quad (42.9)$$

For the signed measure case, $f : X \rightarrow \mathbb{R}$

If furthermore, the two measures are equivalent, then $f : X \rightarrow \mathbb{R}_{\neq 0}$ (for signed measure) or $\mathbb{R}_{>0}$ (for measures)

Therefore every measure on $\mathbb{R}^n$ equivalent to the Lebesgue measure is going to be some multiplication by a nowhere zero scalar function.

On the restriction to a coordinate neighbourhood, there will be a pullback measure from the measures on $\mathbb{R}$ to U :

$$\mu(A \subset U) = \mu_\ell(\phi(A) \subset \mathbb{R}^n) \quad (42.10)$$

One criterion we have to ask of our measure is that it be invariant under diffeomorphism. If we have two coordinate neighbourhoods (U_α, ϕ_α) and (U_β, ϕ_β) , with overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we want the measure to be independent

$$\mu_{\phi_\alpha}(U) = \mu_{\tau_{\alpha\beta}} \quad (42.11)$$

[equivariance?]

Measure should be equivalent to Lebesgue measure on the plot

42.3 The scale bundle and Weyl weights

Given the space of all Radon measures on each coordinate neighbourhood of the manifold, we can show that the set of all such measures on the manifold defines a bundle structure, called the *scale bundle*.

The set of all Radon measure on a given coordinate neighbourhood U is given by the Lebesgue measurable functions on a given coordinate chart ϕ . As all continuous functions are measurable, we can restrict our attention to these.

C^k measures defined by measures related by some C^k function $f : U \rightarrow \mathbb{R}_{\neq 0}$, it is some kind of line bundle

Theorem 42.4. The set of all measures on a manifold is given by a bundle structure of some $\mathbb{R}_{\neq 0}$ bundle over M .

Proof. From the previous demonstration, we already have that for each coordinate neighbourhood, we already have a bundle structure, with

$$\psi : \dots \quad (42.12)$$

We also have that every coordinate neighbourhood form an open cover of the base manifold M . All we need to now do is to show the gluing structure. For any two coordinate neighbourhood U_α, U_β , □

Find out the trivialization and structure group

Show that the patchwork of radon measures on coordinate neighbourhoods that match with each other forms a bundle

This bundle is referred to as the *scale bundle*, and it is in fact just another form of quite a lot of different bundle that we have seen before.

Theorem 42.5. The scale bundle is isomorphic to the density bundle $\Lambda^1 M$

Proof. □

Theorem 42.6. If the manifold M is orientable, the scale bundle is isomorphic to the top form bundle $\Omega^n M$.

Proof. □

Theorem 42.7.

Also isomorphic to the $SL(n)$ -structure

Natural bundle

Related to $SL(n)$ being the symmetry group of the Lebesgue measure

A more general type of object that one can consider on a manifold is that of the s -density bundles. Unlike the more specific density bundle, those are bundles that are representations from the frame bundle under

$$g \rightarrow \det(g)^s \quad (42.13)$$

Relation to the representation theory of the general linear group

Products of s_1 and s_2 density fields $s_1 + s_2$

Correspondence with fields of scaling degrees, weyl weight

Relation to algebra of fields and also the physical dimension of things? Are dimensionful constants just constant sections of the s -scaling bundle

42.4 Integration on a manifold

Now that we have our measure on our manifold, we can work out the integration of a function. Given a measure μ defined by some section of the scale bundle

In a given coordinate neighbourhood (U_α, ϕ_α) ,

$$\int_{U_\alpha} f d\mu = \int_{\phi_\alpha(U_\alpha)} f(x) \omega(x) d\mu_\ell \quad (42.14)$$

Theorem 42.8. Given a manifold equipped with a measure (M, ω) and a map $N \rightarrow M$ a diffeomorphism with the pullback measure $(N, h^*\omega)$, then for any integrable function $f : M \rightarrow \mathbb{R}$,

$$\int_M f \omega = \int_N f \circ h h^* \omega \quad (42.15)$$

Proof.

□

Integration by parts

Theorem 42.9.

Behaviour under rescaling

42.4.1 Riemannian measure

The most common measure used in the context of general relativity is the Riemannian volume form

Definition 42.10. A *Riemannian volume form* ω is a measure on a manifold such that $\omega = \star 1$.

In terms of coordinates, we have that in a given coordinate system,

$$\omega = \sqrt{\det(g)} \varepsilon_{\mu\nu\dots} \quad (42.16)$$

42.5 Integration of differential forms

42.6 Cohomology of integration

The integration of differential forms gives us a nice parallel between the notion of integration

Integration as a mapping from the singular homology and de Rham cohomology

If

42.7 Transgression of differential forms

Theorem 42.11. Given a k -form ω and a d -dimensional submanifold Σ , there exists a canonical induced form $\tau_\Sigma \omega$ of degree $d - k$ on the mapping space $[\Sigma, X]$

$$\tau_\Sigma \omega|_{\Phi_{(-)}} = \int_\Sigma (\Phi_{(-)})^* \omega \quad (42.17)$$

$$\int_\Delta \omega \quad (42.18)$$

42.8 Stokes' theorem

Theorem 42.12. For a smooth $(n-1)$ -form of compact support on an oriented manifold with boundary M , with the induced orientation on ∂M ,

$$\int_M d\omega = \int_{\partial M} \omega \quad (42.19)$$

Integration by parts :

$$\int_U \langle \nabla f, X \rangle = - \int_U f \operatorname{div}(X) + \int_{\partial U} f \langle X, \nu \rangle \quad (42.20)$$

ν the normal vector to ∂U

42.9 Integration of vector-valued forms

For an algebra-valued form $\omega \in \Omega^n(M, \mathfrak{g})$, given a basis θ_i for the algebra,

$$\left(\int_U \theta_i(\omega) d\mu[g] \right) e_i \quad (42.21)$$

Chapter 43

Weyl geometry and conformal transformations

There are a few different topics that come up related to the conformal group

If we consider our Cartan geometry to be based on some version of the conformal group rather than the Lorentz group, we obtain what is called *conformal geometry*, or *Weyl geometry*.

There are a few ways that conformal geometry can be treated overall, and they should not be confused. First conformal geometry can be considered entirely on its own, in which case we do not have a metric tensor at all but only the field associated to the reduction to the conformal group, the *conformal metric* c . We can also

43.1 The different conformal groups

There is some difficulty in conformal geometry in that the use of the *conformal group* can be somewhat ambiguous.

43.1.1 The Weyl group

As a vector space, the conformal group $\mathcal{CO}(p, q)$ is the linear transformation preserving the metric tensor of signature (p, q) up to a rescaling,

$$\Lambda^\top \eta \Lambda = k \eta, \quad k \in \mathbb{R}_{\neq 0} \tag{43.1}$$

This corresponds to the direct product of the Lorentz group $O(p, q)$ by the group of dilations, $D \sim \mathbb{R}_{\neq 0}$. To differentiate it from other kinds of conformal groups, it is also

called the *Weyl group*. The affine version of that group is simply the Weyl group with translations, $\mathbb{R}^n \rtimes \text{CO}(p, q)$.

The Lie algebra of the Weyl group

The Weyl algebra admits a prolongation up to second order, $\text{CO}_2(p, q)$, which is the group

$$c \tag{43.2}$$

[128, 129]

The more general conformal group is usually noted by $\text{Conf}(p, q)$.

Local v. global conformal transformations

Rigidity of the conformal structure

There are two ways in which the conformal group is used in general relativity to not confuse. For some original semi-Riemannian manifold (M, g) , we apply a diffeomorphism to obtain the manifold (ϕ^*M, ϕ_*g) , which is the same object up to isomorphism. We say that ϕ is a *conformal transformation* if $g' = \phi_*g = kg$. A classic example of the conformal transformation is the dilation of Minkowski space, via $x \rightarrow \lambda x$, leading to the conformal transformation

$$\eta \rightarrow \lambda^{-2}\eta \tag{43.3}$$

These are the same manifolds as before, as there exists an isometry between those two manifolds (which is simply the inverse transformation ϕ^{-1}).

The *Weyl transform*, by contrast, is to pick two entirely unrelated manifolds (M, g) and (M', g') . If M and M' are diffeomorphic, and one such diffeomorphism is such that $\phi_*g = \omega g'$, for some nowhere vanishing smooth function $\omega \in C^\infty(M')$, then the two manifolds are related by a Weyl transform. There are very simple examples of manifolds related by a Weyl transforms which are not isometric, as we'll see, since they often differ in curvature invariants, making them impossible to be isometric.

From this definition we can see that trivially, every conformal transformation is a Weyl transform, for which $M' = \phi M$ and $g' =$

43.2 The Weyl transform

From the previous section, we have seen that the Weyl transform is a map from a pseudo-Riemannian manifold (M, g) to another (M', g') , such that the mapping of the metric

tensor is $\phi_*g = \omega g$, with some nowhere vanishing positive smooth scalar function ω , called the *scale factor*. The scale factor is also often denoted by $e^{2\omega}$ for any smooth scalar function ω , with the factor of 2 being there for simplifying some formulas down the line.

It is common for the mapping $\phi : M \rightarrow M'$ to simply be the identity, so that our metric relation is simply $g' = \omega g$, in which case we can easily compare the two.

$$g \mapsto \omega g \quad (43.4)$$

The Weyl transform impacts every quantities involving the metric tensor

The inverse metric has to still obey the relation $C_1^1(g \otimes g^{-1}) = \text{Id}$, or $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^\sigma$, so that the Weyl transform of the inverse metric will simply be

$$g^{-1} \mapsto \omega^{-1}g^{-1} \quad (43.5)$$

Determinant

$$\det(g) \quad (43.6)$$

The derivatives of the metric transform in the obvious way by the Leibniz rule, $g_{\mu\nu,\sigma} \rightarrow \omega_\sigma g_{\mu\nu} + \omega g_{\mu\nu,\sigma}$, so that the Levi-Civita connection transform as

$$\nabla X \rightarrow \quad (43.7)$$

impact on the inverse metric, connection, Riemann tensor, etc

43.3 Conformal structures

An interesting property of a spacetime is its *conformal structure*, which is the information which remains valid for any metric tensor up to Weyl transform. Rather than consider the pair of our manifold and its metric tensor (M, g) , we consider the manifold and the equivalence class of its metrics up to the Weyl transform, $(M, [g])$, with

$$[g] = \{g' | g' = \omega g\} \quad (43.8)$$

The starting metric g is then a class representative of the conformal structure. For specific purposes, it can be useful to have our class representatives to obey certain properties. Those properties do not in general narrow down the representative to a unique metric. Frequent conventions for this are to consider a class representative with unit determinant, $\sqrt{g} = 1$, or with a unit time-time component in some chosen coordinate system, $g(\partial_t, \partial_t) = g_{tt} = 1$.

There are several ways we can consider the conformal structure of a manifold. Outside of the representation as an equivalence class, we can also make the conformal metric $[g]$ into its own object, and for this we have several methods.

43.3.1 The ambient construction

The ambient construction is a bundle on which we can define explicitly what the equivalence class of a metric up to Weyl transform looks like.

[130]

Let's first look at the model space on which to base our ambient construction. Let's start with Minkowski space in $(1 + (n + 1))$ dimensions, $(\mathbb{R}^{n+1,1}, \eta)$, with our metric

$$\eta = -dt^2 + \sum_{i=1}^{n+1} dx_i^2 \quad (43.9)$$

The null cone as for any Minkowski space is just

$$\mathcal{N} = \{v \in \mathbb{R}^{n+1,1} \mid \|v\| = 0\} \quad (43.10)$$

Its projectivized null cone $P\mathcal{N}$, homeomorphic to the sphere S^n , bundle structure $\pi : \mathcal{N} \rightarrow S^n$

The conformal structure on S^n is from the restriction of η on $\mathcal{N}$: for $p \in \mathcal{N}$, $g_{T_p\mathcal{N}}$ is degenerate annihilated by the radial vector field $X = \sum x^i \partial_i$, determines an inner product on $T_p\mathcal{N}/\text{Span}(X) \cong T_{\pi(p)}S^n$

"As p varies over a line in $\mathcal{N}$, the inner product on $T_{\pi(p)}S^n$ varies only by scale and are the possible values at $\pi(p)$ for a metric in the conformal class on S^n ."

$O(n+1, 1)$ acts on $\mathbb{R}^{1,n+1}$ by isometries of g preserving $\mathcal{N}$. Induced action on N preserves the conformal class of metrics, it's the conformal group baby

If instead of N we pick a mass hyperboloid,

$$\mathcal{H} = \{p \in R^{n+2} | Q(x) = -m^2\} \quad (43.11)$$

It's the Poincaré metric associated to S^n , hyperbolic metric of constant sectional curvature -1

$$g_+ = 4(1 - |x|^2)^{-2} \sum_{i=1}^{n+1} (dx^i)^2 \quad (43.12)$$

Conformal metrics

A conformal class of metrics on M : metric bundle $\mathcal{G}$, $\mathbb{R}_+$ bundle over M

43.3.2 Tractor calculus

The conformal metric c can be interpreted in several ways in its action. When acting on vectors, it will map those vectors to sections of the scale bundle [of weight 2?] we got from the ambient construction,

$$c : TM \times TM \rightarrow \Lambda^1 M \quad (43.13)$$

We could also define it to act on its own type of object called *tractors* (named after its inventor, Tracy Thomas [131]), which are essentially jets of the scale bundle $J^k(L, \Lambda^1 M)$ instead of the manifold by itself, as we will see.

$$c : \mathcal{T}M \times \mathcal{T}M \rightarrow \mathbb{R} \quad (43.14)$$

Tractor bundle as jet of the scale bundle [132]

$$0 \rightarrow \mathcal{E}_a(1) \longrightarrow \mathcal{E}_A \longrightarrow \mathcal{E}(1) \rightarrow 0 \quad (43.15)$$

$$\begin{pmatrix} \sigma \\ \omega \\ \rho \end{pmatrix} \mapsto \quad (43.16)$$

Position tractor

velocity tractor

acceleration tractor

[133–137]

43.3.3 Conformal connections

Conformal connections classified by the Weyl vector ω

For $\omega = 0$, unique conformal connection

This applies only if we consider its action on g and not on c ?

43.3.4 Scale setting

43.3.5 Things

How various objects transform under conformal transformation

Hodge star is conformally invariant in $2n$ dimensions

43.4 Weyl spacetimes

We can replace all the usual metric structure of a pseudo-Riemannian manifold with that of a conformal structure, giving us a Weyl spacetime.

Definition 43.1. A *Weyl spacetime* (M, c, ω, ∇) is a manifold equipped with a conformal metric c and a conformal connection ∇ such that the derivative of the connection is $\nabla g = \omega \otimes g$, ω a 1-form, the *Weyl vector*.

What is the formulation in terms of the conformal metric c ?

Compatibility of the connection and conformal metric : zero Weyl vector, unique conformal connection

Related to the prolongation of the conformal group : $\mathfrak{co}(n) \rightarrow \mathfrak{co} \times \mathbb{R}^{n*}$

The unique conformal connection

Weyl spacetime as an equivalence class of Lorentzian manifolds

Length curvature

Relation of ω to scaling

Chapter 44

Volume structure

The equiaffine structure, also called the *volume structure*, is the case corresponding to structures related to the subgroup $SL^\pm(n) \subset GL(n)$. As the general linear group splits quite neatly as $GL(n) = \mathbb{R}^\times \times SL(n)$, the reduction of the volume structure gives us a bundle $\mathbb{R}^\times$. If we consider the subgroup $SL(n)$, this is the structure of oriented volumes.

The origin of the name of volume structure comes from this structure simply giving us the scale bundle that we saw earlier defining the volume measure on our manifold :

Theorem 44.1. The volume structure is isomorphic to the scale bundle.

Proof.

□

This corresponds to the fact that $SL(n)$ is the group preserving the volume of the Lebesgue measure [volume from the frame is invariant under SL etc]

The volume structure therefore determines a scale on our manifold. If furthermore our manifold is orientable, we can also consider the oriented volume structure given by $SL^+(n)$, which gives us the more commonly used volume form.

" SL structures are the sections of the bundle associated with LM and the left action of GL on GL/SL . This is the volume bundle. Furthermore, the sections of VM correspond to GL -equivariant functions f of LM to GL/SL . The equivariance condition is $f(la) = |\det(a)|^{-1/n} I_n \cdot f(l)$, $a \in GL$."

$$Vol \leftrightarrow \Gamma(VM) \leftrightarrow C_{eq}^\infty(LM, GL(n)/SL^\pm(n)) \quad (44.1)$$

To second order, the volume structure gives rise to a connection

$$x \quad (44.2)$$

Volume - equiaffine connection

$$\nabla^V \omega = 0 \tag{44.3}$$

Schrödinger connection?

Chapter 45

Affine structure

An affine structure is a second order G -structure from the second order general linear group to the first, $GL^2(n) \rightarrow GL^1(n)$.

Defines an affine connection

Select specific curves as preferred curves (geodesics)

Chapter 46

Projective structure

The *projective structure* of a spacetime is based on selecting a preferred class of paths among the set of all curves $\mathcal{C}(M)$, which are the geodesics of that spacetime. "Geodesics" here is to be understood in the sense of the submanifold defined by that path having a horizontal distribution, including both geodesics and pre-geodesics.

In terms of pseudo-Riemannian geometry, a projective structure is an equivalence class of connections such that curves remain geodesics. For a manifold with two different metrics giving rise to two different Levi-Civita connections, (M, g, ∇) and (M, g', ∇') , for a curve $\gamma : L \rightarrow M$, the two manifolds are *projectively related* if

$$\nabla\gamma = f\gamma, \quad \nabla'\gamma = f'\gamma \quad (46.1)$$

so that both curves are pre-geodesics.

Two projectively related connections :

$$\Gamma^\sigma_{\mu\nu} \rightarrow \Gamma^\sigma_{\mu\nu} + \Pi^\sigma_{\mu\nu} \quad (46.2)$$

The obvious candidates for projectively related metrics are metrics related by a constant conformal factor, $g = kg'$, $k \in \mathbb{R}_{>0}$.

Other non-trivial projectively related metrics

The Cartan geometry of a projective structure is given by the *projective linear group*,

$$\mathrm{PGL}(n) = \mathrm{GL}(n)/D^\pm \quad (46.3)$$

Projective linear group, $\mathrm{PGL}(n)$ Cartan connection, connection with values $\mathfrak{p}$ in the group GL

Projective semilinear group PL?

Projective tractor

G -structure on the 2-frame bundle, projective group $\mathcal{P}$

$$G = \mathrm{PGL}(n, \mathbb{R}) = \mathrm{SL}(n+1, \mathbb{R})/Z(\mathrm{SL}(n+1, \mathbb{R})) \quad (46.4)$$

$$H = \left\{ \begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix} \in \mathrm{SL}(n+1, \mathbb{R}) \right\} / Z(\dots) \quad (46.5)$$

$A \in \mathbb{GL}(n)$, ξ is an n -vector,

[15, 138, 138]

46.1 Projective tractors

Chapter 47

Absolute parallelism

An *absolute parallelism* is the simplest of all structures, as it is simply the reduction to the trivial group :

$$\mathrm{GL}(n) \rightarrow \{e\} \quad (47.1)$$

As this is a trivial reduction, it simply corresponds to a section of the frame bundle itself : an absolute parallelism is the existence of a frame field $e \in \Gamma(FM)$ defined on M . From this, we can already say that the obstruction to this structure is that the manifold must be parallelizable to admit an absolute parallelism on it.

On a geometric level, it corresponds to a rule for the translation of tangent vectors

[139–141]

Trivialization of the frame bundle

Defines a rule for translation of tangent vectors

Equivalence of absolute parallelisms, smooth trivializations of the frame bundle, smooth connections with holonomy being the identity

The DoF is full on n^2

Which manifolds are parallelizable

Theorem 47.1. Every manifold admits local absolute parallelisms in its bundle neighbourhoods

If we allow our absolute parallelism to have torsion, we have a *Weitzenböck connection*

Chapter 48

Finsler geometry

The Finsler structure of a manifold is one which assigns lengths to curves. If we consider some curve γ , we would like to assign some functional

$$L[\gamma] : C(L, M) \rightarrow \mathbb{R} \quad (48.1)$$

Most Finsler structures are done with the case of positive assignments of that type, fitting the usual concept of a length, but as we are concerned with the Lorentzian case where such curve lengths may be negative, we will use what are more generally known as pseudo-Finsler structures, where the length may be negative.

A Finsler

Finsler function [142, 143]

Definition 48.1. A *Lorentz-Finsler function* is a function on the tangent bundle $F : TM \rightarrow \mathbb{R}$

The length of a curve from the Finsler function is then

$$L[\gamma] = \int_I F d\lambda \quad (48.2)$$

[Does the Finsler function define a 1-form along a curve]

Therefore the measure of our curve is defined by the pullback

Measures on the pullback on curves

Randers manifold

Chapter 49

Incidence geometry

If we remove most structures from a geometry, one of the most barebone structure is the *incidence structure*.

An incidence geometry has collections of different types of objects (in our case submanifolds of different dimensions), and an incidence relation between two objects of different types, meaning that

49.1 Affine incidence structure

For an affine space A , the incidence structure is composed of all the k -dimensional subspaces. If we classify them by dimension, it is a sequence of subspaces $(W_i)_{0 \leq i \leq n}$

- For every $(n + 1)$ -tuple of points $(p_i) \in A$, there exists a unique n -dimensional subspace W such that every point is incident to it : $I(p_i; W)$

For every two points a curve through them

For every non-parallel curve a point is incident

Affine geometric group $AG(n)$

49.2 Projective incidence structure

Projective incidence structure

Incidence structure of a manifold

Projective geometry group

Conics

49.3 Path structure

One aspect of geometry that uses the incidence structure is that in Euclidian space, if we are given two points $p, q \in \text{Euc}$, there exists a unique line L containing those two points, but dually this line could be defined by the point p and an initial vector v (or 1-distribution Δ since we only care about direction) at p . [144, 145]

In a bundle description, the latter is described by a point of the projectivized tangent bundle PTM

Definition 49.1. A *path structure* is a choice of two line subbundles E_1, E_2 on TM such that $E_1 \cap E_2 = \{0\}$ and $E_1 \oplus E_2$ is a contact distribution.

[...]

The correspondence : the lift of Σ to P maps down to a line in M and the lift of M to P projects to a line in Σ [?]

dual path geometry

49.4 Twistors

[146, 147]

Double fibration, flags

Correspondence

Schubert cells

Chapter 50

Structuralism

[Structuralism from Bourbaki Putnam etc]

We can define a lot of different G -structures on a given geometry, corresponding to various properties of the geometry its objects may have, such as distances, congruences, crossratio, incidence, etc.

But a further property of such structures is that they will usually interact with each other. In the cases of interest to us, the structure defining lengths and the structure defining volumes will have to be somewhat compatible as we want at least on the local level that given some n -form of length ℓ vectors, the corresponding volume should be ℓ^n . We want our incidence structure regarding projective conics to correspond to our conformal structure in that vectors in a given conic will correspond to a certain conformal norm. If a curve is a geodesic in the sense of the affine structure, its path will also be a geodesic in the sense of the projective structure, and so on.

We can roughly split the important structures that we saw into three classes :

1. Structures relating to space itself : the volumes, frames
2. Structures relating to curves : all the connections
3. Structures concerning their interactions

We can of course define for instance the metric structure without ever referring to the tangent bundle, simply defining it abstractly as the structure of a subgroup $O(p, q)$ of $GL(n)$, but this will not capture much of the semantics involved in how we typically use it. The volume bundle and frame bundle both derive from mappings of regions of space to other objects, the connections

We will see two kinds of compatibility between G -structures here : the first is the case where one structure is defined by a subgroup of another, and the second is the case where two structure groups intersect to form a third structure group. [148]

50.1 Substructures

Some G -structures are substructures of another. If we have a G -structure of group G and one of group H such that $H < G$,

$$H_2 \subset H_1 \subset G \quad (50.1)$$

A common example of this is the conformal structure and the metric structure, $\mathcal{CO}(p, q) \rightarrow \mathcal{O}(p, q)$.

If we have a reduction from G to H_1 , a bundle R_1 , and a reduction from G to H_2 , a bundle R_2 , for our two structures to be compatible, we need R_2 to be a subbundle of R_1 , given by a

Given a G -structure with group G and a subgroup H with a monomorphism $\iota : H \rightarrow G$, we get from the

As a subbundle of the frame bundle, a substructure will itself be a subbundle of that subbundle, each

How the sections relate

A section s_2 of the H_2 -structure will be a representative of the H_1 -structure. For some s_1 section, there exists a section s_2 such that s_2 corresponds to the orbit space of s_1 under the action of H_1/H_2 [normal subgroup?] $[s_2] = s_1 \in \Gamma(\dots)$

Section s_1 has to be $[s_2] = H_1 \cdot s_2$

This is why it is common practice in general relativity to write objects of other structures as equivalence classes of metric tensors or Levi-Civita connections, although those other structures have their own associated objects.

Reduction $H_1 \rightarrow H_2$ selects the subset in s_1

Relates to metricity for F^2 to F^1 :

$$\nabla[e] = 0 \quad (50.2)$$

Example : structure $O \subset CO$

The section $g \in \Gamma(\text{Fr}M/O(n-1, 1))$ are also sections of the larger $c \in \Gamma(\text{Fr}M/CO(n-1, 1))$

Given a section $\rho \in \Gamma(COM/O)$, corresponding to a scaling, a given metric g corresponds to a conformal metric c up to rescaling :

$$g = \rho \cdot c \quad (50.3)$$

Example : structure $O \subset SL^\pm$

Reduction of $SL^\pm(n)$: also a scale $GL(n)/SL^\pm(n) \cong R^\times$ (volume structure)

Reduction of $SL^\pm$ to $O(n)$:

Example : reduction of O to a single frame $\{e\}$: frame has to be orthonormal according to the metric?

Example : reduction of the affine structure Γ to the orthogonal group

Reduction of Γ^2 to the prolongation O^2

50.2 Intersection of two structures

given a group G with two closed subgroups H_1, H_2 , such that every element of g can be expressed as a product of those two subgroups,

$$\forall g \in G, \exists h_1 \in H_1, h_2 \in H_2, h_1 h_2 = g \quad (50.4)$$

for two reductions R_1, R_2 with H_1, H_2 , $R_1 \cap R_2$ is a reduction of P with typical fiber $H_1 \cap H_2$

If we have the sections associated to those reductions,

$$s_i : M \rightarrow P/H_i \quad (50.5)$$

50.3 Structure of general relativity

Pseudo-Riemannian structure with LC connection : $GL^2(n) \rightarrow SO^2(n) \rightarrow SO(n)$

$$SO(n) = SL(n) \cap CO(n)$$

$$A(n) = \mathcal{P}(n) \cap$$

Intersection of conformal and volume structure and projective and affine

How structures relate to concomitant tensors

In addition to intersecting G -structures, the structures involved also have all the substructures that one could have in between their group and the corresponding jet group. [path structure and causal structure?]

Metric structure, conformal structure, causal structure

Chapter 51

Submanifolds II

Now that we have seen all the structures that we will add to our manifolds, we can continue our discussion of submanifolds from 22 to see how those two topics relate.

For any C^k embedded submanifold $\iota : S \hookrightarrow M$, the jet bundle of r -submanifolds ($r \leq \text{Dim}(S)$) of order k will form a subbundle of the one for M . If we take some submanifold S' of S , $\iota' : S' \hookrightarrow S$, it will also form a submanifold of M by the composition

$$\iota_{S',M} = \iota \circ \iota' \quad (51.1)$$

Show that it is embedded also

In the first order case, for the $(1,r)$ -velocity bundles, this will be a subspace of the $(1,r)$ -velocity bundle of M .

pushforward by the inclusion map

We can define the pullback bundle of TM along ι , ι^*TM , inside which we find the subbundle generated by TS . This bundle also exists in M itself, and TS is then some bundle existing in $TM|_{\iota(S)}$.

The normal bundle is ι^*TM/TS

If we have a metric tensor, we can define a specific complement to this subbundle of the tangent bundle, called the *normal bundle* to S :

Split of the tangent bundle into the normal bundle :

$$T_p M = T_p S + T_p^\perp S \quad (51.2)$$

Given some vector v , either in the pullback ι^*TM or on $TM|_{\iota(S)}$, we can now split it along those two subspaces. The tangential part of v , $\tan(v)$, $\tan : TM|_{\iota(S)} \rightarrow TS$,

The normal part $\text{nor}(v)$, $\tan : TM|_{\iota(S)} \rightarrow T^\perp S$

We will say that a vector is tangent to S if its normal part is zero, and similarly, a vector is normal if its tangent part is zero. In the one dimensional case, this usage coincide with what we have already seen about tangent vectors for curves.

51.1 Tubular neighbourhoods

Around any embedded submanifold S , we can define an open subset of M which includes S called its *tubular neighbourhood*, which is in some sense the thickening of S as a k -dimensional submanifold into an n -dimensional open set. These are commonly used for proving a variety of theorems about submanifolds requiring us to extend some quantities of S into M .

The exponential map $\exp : N(M/S) \rightarrow M$ is a local diffeomorphism at $T^0 S$

There is a neighbourhood of $T^0 S$ in $N(M/S)$ on which $\exp$ is a diffeomorphism to a neighbourhood U of S in M .

S has a neighbourhood U' in M such that $\forall p \in M$, there is a unique geodesic connecting p to S of length $\rho(p, S)$ meeting S orthogonally.

Theorem 51.1. For any C^1 embedded submanifold of M , there exists a tubular neighbourhood of S in M .

Proof.

□

Coordinates adapted to the normal bundle

51.2 Induced structures

Given some kind of structure on the manifold M , such as a vector field, connection, G -structure, etc, this will typically induce a similar structure on S , called an *induced structure*. This can be as simple as simply restricting the

51.2.1 Induced connection

If we have some connection originally defined on some bundle $\pi : E \rightarrow M$, we can define our induced bundle by the pullback along the inclusion, $\iota^* \pi : \iota^* E \rightarrow S$. The tangent bundle of $\iota^* E$ is then

Split of the frame bundle $GL(n - k) \times GL(k)$ (or something larger idk)

51.2.2 Induced metric

If we have a semi-Riemannian metric g on M , there exists an induced metric on S called the *first fundamental form*, $g_S = \iota_* g$, giving a metric tensor on TS vectors.

$$g_S(X, Y) = g(\iota_* X, \iota_* Y) \quad (51.3)$$

Metric on submanifold

For a Levi-Civita connection ∇_{LC} , the induced connection

$$\nabla_X^S Y = \tan \nabla_X^M Y \quad (51.4)$$

The *second fundamental form*

$$\Pi(X, Y) = \text{nor} \nabla_X^M Y \quad (51.5)$$

$$\nabla_X^M Y = \nabla_X^S Y + \Pi(X, Y) \quad (51.6)$$

51.2.3 Geodesics

For a curve $\gamma : I \rightarrow S \subset M$,

$$\gamma^* \nabla_{\dot{\gamma}}^M Y = \gamma^* \nabla_{\dot{\gamma}}^S Y + \Pi(\dot{\gamma}, Y) \quad (51.7)$$

In particular, if we consider the geodesic equation,

$$\gamma^* \nabla_{\dot{\gamma}}^M \dot{\gamma} = \gamma^* \nabla_{\dot{\gamma}}^S \dot{\gamma} + \Pi(\dot{\gamma}, \dot{\gamma}) \quad (51.8)$$

A curve in S is a geodesic in M if its acceleration is everywhere normal to S .

51.2.4 Gauss-Codazzi equations

For an embedded submanifold $\iota : S \hookrightarrow M$ in a pseudo-Riemannian manifold M ,

51.3 Induced structures on submanifolds

If we consider the submanifold M of a manifold M' , with the inclusion map $\iota : M \hookrightarrow M'$

A vector field X on M passed through the inclusion map is a M' vector field on M .

Tangent space $T_p M$ is a non-degenerate subspace of $T_p M'$

$$T_p M' = T_p M \oplus T_p M^\perp \quad (51.9)$$

Vectors of $T_p M^\perp$ are called normal vectors to M

Induced connection :

Induced metric

timelike, spacelike and null submanifolds

Gauss-Codazzi equation

Level set function

Frames

51.4 Hypersurfaces

A hypersurface is an $(n - 1)$ dimensional submanifold. For an $(n - 1)$ -manifold Σ and an embedding $\theta : \Sigma \rightarrow M$, then we say that $\theta(\Sigma)$ is a hypersurface in M .

induced metric $\theta^* g$ on Σ where $X, Y \in T_p M$, $\theta^* g(X, Y)|_p = g(\theta_* X, \theta_* Y)|_{\theta(p)}$

Theorem 51.2. The intersection of an n -dimensional timelike submanifold and a spacelike hypersurface is an $n - 1$ spacelike submanifold.

51.4.1 Hyperquadrics and algebraic varieties

In the flat manifold $\mathbb{R}^n$:

Take the function q in $\mathbb{R}^{p+q}$

$$q = - \sum_{i=1}^p (x^i)^2 + \sum_{i=p+1}^{n+1} (x^i)^2 \quad (51.10)$$

$q(x) = \langle x, x \rangle$, hence $\text{grad } q = 2x$

$$\langle \text{grad } q, X \rangle = X(q) = X\langle x, x \rangle = 2\langle D_X x, x \rangle = 2\langle X, x \rangle \quad (51.11)$$

51.5 Foliations

Definition 51.3. A *foliation* of a manifold M is the decomposition of M into a disjoint union of connected subsets L_α , called leaves, such that they cover the manifold

$$M = \bigcup_{\alpha} L_{\alpha} \quad (51.12)$$

and there is an atlas (U_β, ϕ_β) such that the intersection of a leaf L_α with U_β

Foliations of interest in GR : foliation by spacelike hypersurfaces, foliations by timelike curves.

Froebinius theorem

Chapter 52

Varieties and schemes

A more general way to consider subspaces of a topological space is to take the solutions of functions on that space

52.1 Affine varieties

The basic example of a variety is that of an affine variety, which is roughly the zero set of a polynomial map.

Definition 52.1. An *affine variety* V is a subset $V \subset A^n$ of the n -dimensional affine space A^n over a field F that is the zero locus of a set of polynomials on that affine space, $S \subset K[x_1, \dots, x_n]$:

$$V = Z(S) = \{x \in A^n | \forall f \in S, f(x) = 0\} \quad (52.1)$$

Example 52.2. Any point $p = (p_i)$ is an algebraic variety defined by the first order polynomial

$$f(x_i) = x_i - p_i \quad (52.2)$$

Example 52.3. A lot of curves are basic examples of varieties, such as the straight lines defined by first order polynomials :

$$f(x_i) = a^i x_i + p_i \quad (52.3)$$

or the

Example : lines, curves, conics, etc

Theorem 52.4. The union of two algebraic sets is an algebraic set : Given $V_1 = Z(f_1)$, $V_2 = Z(f_2)$, there exists a function f such that $V_1 \cup V_2 = Z(f)$.

Proof. If we consider the polynomial defined by the product $f_1 \cdot f_2$, its zeros occur whenever either $f_1 = 0$ or $f_2 = 0$, which is precisely the union of those two algebraic sets. $\square$

Theorem 52.5. The intersection of a family of algebraic sets is an algebraic set : Given $V_i = Z(f_i)$, there exists a function f such that $\bigcap_i V_i = Z(f)$.

Proof.

□

Given those properties, and the fact that every point is already an algebraic set, the algebraic sets define a family of closed sets on A^n , and therefore a topology, called the *Zariski topology*.

Theorem 52.6. The zero-set of polynomials is their ideal

Submanifolds as zero-sets of functions

52.2 Hyperquadrics

A good example of the application of varieties in general relativity is the notion of hyperquadrics, which are second order varieties defined in some affine space of signature (p, q)

Hyperquadrics through the exponential map

52.3 Schemes

A generalization of varieties is that of schemes

Definition 52.7. For a commutative ring R , its *scheme*

Chapter 53

Foliations II

Foliation as the cover of the manifold by disjoint sets

Definition 53.1. A *foliation* $\mathcal{F}$ of codimension k of a manifold of dimension n is given by an atlas $\mathcal{A}_{\mathcal{F}}$ such that the transition maps

$$\tau_{\alpha\beta} : \mathbb{R}^{n-k} \times \mathbb{R}^k \rightarrow \mathbb{R}^{n-k} \times \mathbb{R}^k \quad (53.1)$$

are such that the projection of the transition map on the second factor $\mathbb{R}^k$ depends only on the second

$$y' = \tau_{\alpha\beta}(y) \quad (53.2)$$

Show that the foliation does a cover of the manifold by submanifolds of dimension k

53.1 Complementary foliations

Transversal mapping

Definition 53.2. For two foliations $\mathcal{F}_1, \mathcal{F}_2$ of codimension k_1, k_2 on the same manifold M , we say that $\mathcal{F}_1$ is *transversal* to $\mathcal{F}_2$, denoted $\mathcal{F}_1 \pitchfork \mathcal{F}_2$, if $k_1 + k_2 \leq n$ and any two leaves L_1, L_2 intersect transversally if $L_1 \cap L_2 \neq \emptyset$, the tangent spaces of their leaves span the full tangent bundle :

$$T_p L_1 \oplus T_p L_2 = T M_p \quad (53.3)$$

Where L_1, L_2 are the leaves going through p .

Example of transversal foliations of the plane by curves, example of a non-transversal foliation

Definition 53.3. For a pseudo-Riemannian manifold, we say that two foliations $\mathcal{F}_1, \mathcal{F}_2$ are *orthogonal*, denoted by $\mathcal{F}_1 \perp \mathcal{F}_2$, if the subspaces generated by their tangent spaces are mutually orthogonal : $T_p L_1, T_p L_2 \subseteq T_p M, T_p L_1 = (T_p L_2)^\perp$

Orthogonal foliations are transverse

53.2 Integrability

Just like for any object with a global and local version, we can wonder about the integrability of foliations. While every foliation defines a field of distributions, the reverse problem of integrability is to find a foliation from a given distribution.

Definition 53.4. A k -distribution Δ is said to be *integrable* if there exists a k -foliation of the manifold M with the distribution Δ everywhere tangent to the leaves.

Frobenius theorem

Integrability by k -blades instead of k -distributions

Corrolary 53.1. Every 1-distribution is integrable.

Proof. This simply follows from every vector in Δ being colinear, therefore $[X_1, X_2] = 0$ for any two vectors, which is always in the distribution. $\square$

Gives a parametrization mb

53.3 Geodesic foliations

53.4 Spacetime foliations

Foliations of spacetimes can be specialized by their causal nature. Just as for many other objects in the theory, we can have timelike, spacelike and null distributions, which are defined in the same way as their counterparts in vector subspaces for Lorentzian vector spaces : a timelike, spacelike or null foliation is one such that its distribution field is everywhere a timelike, spacelike or null subspace of the tangent bundle, defining appropriately timelike, spacelike and null submanifolds.

Theorem 53.5. Every spacetime admits a foliation by timelike curves

Proof. As every 1-distribution is integrable, and every spacetime admits a timelike 1-distribution, this follows from Frobenius' theorem. $\square$

A special type of foliation that will be perennially important in general relativity is that of the foliation into spacelike hypersurfaces :

Definition 53.6. A foliation by *spacelike hypersurfaces* is a foliation of codimension 1 where all the leaves are spacelike submanifolds.

This leads to a notion of the space of a spacetime at a given point in time, and will be of fundamental importance when we need to split

Chapter 54

Petrov and Segre classification

54.1 Subspaces of the tangent plane

n -dimensional vector subspaces of $T_p\mathcal{M}$: S_n, T_n, N_n

The subspace S_n is a spacelike subspace : its generators are all spacelike

The subspace T_n is a timelike subspace : its orthogonal space $T_n^\perp$ is spacelike

S_1, T_1, N_1 : subspace generated by a tangent vector that is spacelike, resp. timelike, null.

Jordan normal form :

Theorem 54.1. For a finite-dimensional vector space V and any linear operator $A : V \rightarrow V$, there exists a decomposition of V into a direct sum of invariant subspaces of A

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_k \quad (54.1)$$

and a basis $\{e_j^{(i)}\}_{1 \leq j \leq n_i}$ of each V_i such that

$$\begin{aligned} (A - \lambda_i \text{Id})e_1^{(i)} &= 0 \\ (A - \lambda_i \text{Id})e_2^{(i)} &= e_1^{(i)} \\ &\dots \\ (A - \lambda_i \text{Id})e_{n_i}^{(i)} &= e_{n_i-1}^{(i)} \end{aligned}$$

Decompose a matrix into block diagonal matrix

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix} \quad (54.2)$$

with J_i upper diagonal matrices

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{matrix}} & & & \\ & \lambda_2 & & \\ & & \lambda_3 & 1 \\ & & 0 & \lambda_3 \\ & & & \ddots \end{pmatrix} \quad (54.3)$$

54.2 Petrov classification

As we've seen [X], the Weyl tensor can be understood as a linear map on the space of bivectors :

$$\text{Weyl} : \Lambda^2 TM \rightarrow \Lambda^2 TM \quad (54.4)$$

This interpretation allows us to define algebraically special classes of spacetimes based on the eigenvalue problem of this operator. If we have a bivector X , let's consider

$$\frac{1}{2} C_{\mu\nu\rho\sigma} X^{\rho\sigma} = \lambda X^{\mu\nu} \quad (54.5)$$

given a unit timelike vector, take $X^\mu = X^{\mu\nu} u_\nu$

Petrov types :

- Type I
- Type D
- Type II
- Type N
- Type III
- Type O

Type I spacetimes are called algebraically general, while others are called algebraically special. In particular, type O spacetimes are simply the case of a zero Weyl tensor, ie being conformally flat spacetimes.

Bel criteria

54.3 Segre classification

Classification of rank 2 symmetric tensors

Rank (1, 1) tensors define a map $T : T_p M \rightarrow T_p M$, $T(X) = Y$

Eigenvalue problem on the symmetric tensor S

$$S^\mu{}_\nu X^\nu = \lambda X^\mu \quad (54.6)$$

$$X^\nu (S^\mu{}_\nu - \lambda \delta^\mu_\nu) = 0 \quad (54.7)$$

Segré notation : between brackets, orders of different Jordan blocks of a matrix. If multiple Jordan blocks for the same eigenvalue, written between parenthesis. Complex conjugate eigenvalues : Z and $\bar{Z}$

For a matrix with k real eigenvalues and l pairs of complex conjugates eigenvalues :

$$\{(p_{11} \dots p_{1r_1}) \dots (p_{k1} \dots p_{kr_k}) Z_1 \bar{Z}_1 \dots Z_l \bar{Z}_l\} \quad (54.8)$$

Example :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad (54.9)$$

Type [1111]

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \quad (54.10)$$

Type $[31]$ if $\lambda_1 \neq \lambda_2$, $[(31)]$ otherwise.

Possible types : partitions of n + replacement of 2, 11 by $Z\bar{Z}$

$$\begin{aligned} 2 &\rightarrow [2], [11], [Z\bar{Z}] \\ 3 &\rightarrow [3], [21], [Z\bar{Z}1], [111] \\ 4 &\rightarrow [4], [31], [22], [2Z\bar{Z}], [Z\bar{Z}Z\bar{Z}], [211], [1111] \end{aligned}$$

Pebłański notation

54.3.1 Segre classification of the Ricci tensor

$$R_{\mu\nu}X^{\mu\nu} = \lambda X^{\mu\nu} \quad (54.11)$$

4 basic Segre types :

A_1 :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad (54.12)$$

A_2 :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & Z & 1 \\ 0 & 0 & 0 & \bar{Z} \end{pmatrix} \quad (54.13)$$

A_3 :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \quad (54.14)$$

B :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \quad (54.15)$$

Type	Segre notation	Plebański notation
A_1	$[111, 1]$	$[S_1 - S_2 - S_3 - T]_{(1111)}$
	$[11(1, 1)]$	$[S_1 - S_2 - 2T]_{(111)}$
	$[(11)1, 1]$	$[2S_1 - S_2 - T]_{(111)}$
	$[(11)(1, 1)]$	$[2S - 2T]_{(11)}$
	$[1(11, 1)]$	$[S - 3T]_{(11)}$
	$[(111), 1]$	$[3S - T]_{(11)}$
	$[(111, 1)]$	$[4T]_{(1)}$
A_2	$[11, Z\bar{Z}]$	$[S_1 - S_2 - Z - \bar{Z}]_{(1111)}$
	$[(11), Z\bar{Z}]$	$[2S - Z - \bar{Z}]_{(111)}$
A_3	$[11, 2]$	$[S_1 - S - 2N]_{(112)}$
	$[1(1, 2)]$	$[S - 3N]_{(12)}$
	$[(11), 2]$	$[2S - 2N]_{(12)}$
	$[(11, 2)]$	$[4N]_{(2)}$
B	$[1, 3]$	$[S - 3N]_{(13)}$
	$[(1, 3)]$	$[4N]_{(3)}$

Chapter 55

Clifford algebras and bundles

So far we have not seen how spinors are dealt with in the context of general relativity, which are important both within the context of fermion physics as well as being a generally useful mathematical tool. You may remember from quantum theory that spinors are described by some associated vector space to some representation of the rotation group. For instance, in the case of Euclidian space with the rotation group $SO(3)$, the group for spin is the unitary group $SU(2)$, and its spinors are represented by the canonical representation on $\mathbb{C}^2$

[150]

We previously constructed both the tensor algebra and the exterior algebra for the tangent space, and used those to construct the associated tensor bundle and exterior bundle. In addition to those algebras, we can construct a more general version of the exterior algebra called the Clifford algebra, which will be used to build the space for spinors.

If we consider some vector space V equipped with a quadratic form $q : V \rightarrow \mathbb{R}$

$$q(av) = a^2q(v) \tag{55.1}$$

The map $(v, w) \mapsto q(v + w) - q(v) - q(w)$ is linear in v and w

We can verify that the bundle metric on the tangent plane T_pM forms a quadratic form

$$q(v) = \langle v, v \rangle \tag{55.2}$$

Exterior algebra on the vector space

antisymmetrized product of two vectors :

$$v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v) \quad (55.3)$$

inner product defines musical isomorphisms between V and V^* , which extends to $\bigwedge V$ and $\bigwedge V^*$, a hodge duality between the exterior algebras and the Clifford algebra

55.1 Multivectors

From this, we can define multivectors, which are made from sums of members of the exterior algebras $\bigwedge^p$.

Definition 55.1. The set of *multivectors* is the direct sum of the exterior algebras of the vector space

$$\bigwedge V = \bigoplus_{p=0}^n \bigwedge^p V \quad (55.4)$$

We will note a multivector A as

$$A = \sum_{i=0}^n A_i = A_0 + A_1 + \dots + A_n \quad (55.5)$$

with $A_i \in \bigwedge^i V$. A Polyvector of definite order is called homogeneous.

Definition 55.2. The projector $\langle \cdot \rangle_r$ projects a multivector on its homogeneous part of grade r .

$$\langle A \rangle_r = A_r \quad (55.6)$$

Proposition 55.3. The projector has the following properties :

1. $\langle A + B \rangle_r = \langle A \rangle_r + \langle B \rangle_r$
2. $\langle \lambda A \rangle_r = \lambda \langle A \rangle_r$
3. $\langle \langle A \rangle_r \rangle_s = \langle A \rangle_r \delta_{rs}$

Proof. 1.

$$A + B = \sum_{i=0}^n (A_i + B_i) \rightarrow \langle A + B \rangle_r = A_r + B_r = \langle A \rangle_r + \langle B \rangle_r$$

2.

$$\langle \lambda A \rangle_r = \langle \sum_{i=0}^n \lambda A_i \rangle_r = \lambda A_r = \lambda \langle A \rangle_r$$

3. x

□

A multivector that only has non-zero components of grade 0 and 1 is called a *paravector*.
 $A = A^0 + A^i e_i$, $A^0 = \langle A \rangle_0$, $A^i e_i = \langle A \rangle_1$

If we note the inner product (v, w) as $v \cdot w$, we define the Clifford product of two vectors as

$$uv = u \cdot v + u \wedge v \quad (55.7)$$

This is a multivector $uv \in \bigwedge V$, with the sum of a scalar and bivector.

This has the property

$$\begin{aligned} u \cdot v &= \frac{1}{2}(uv + vu) \\ u \wedge v &= \frac{1}{2}(uv - vu) \end{aligned}$$

The wedge product is generalized easily enough, simply by using

$$A \wedge B = \sum_{i,j}^n \langle A \rangle_i \wedge \langle B \rangle_j \quad (55.8)$$

If we pick an orthonormal basis $\{e_i\}$ for V , then, as $e_i \cdot e_j = 0$, it is not too hard to show that

$$e_i e_j = e_i \wedge e_j \quad (55.9)$$

which will help to extend the Clifford product to all multivectors.

Proposition 55.4. The Clifford product is distributive and associative.

Proof.

$$\begin{aligned} (a + b)c &= (a + b) \cdot c + (a + b) \wedge c \\ &= a \cdot c + b \cdot c + a \wedge c + b \wedge c \\ &= ac + bc \end{aligned} \quad (55.10)$$

$$(ab)c = \tag{55.11}$$

□

55.2 The Clifford algebra

Clifford algebra $C\ell(V)$ is the set of multivectors $\bigwedge V$ equipped with the Clifford product.

Definition 55.5. The *Clifford algebra* $C\ell(V, q)$ of a vector space V with quadratic form q is an associative unital algebra such that, given the ideal $\mathcal{I}_q$

Definition 55.6. The Clifford algebra has the following properties, for $A, B, C \in C\ell(V)$:

1. $AB \in C\ell(V)$
2. $1A = A1 = A$
3. $A(BC) = (AB)C$
4. $A(B + C) = AB + AC$
5. $(B + C)A = BA + CA$
6. For $a \in V$

$C\ell^k(V)$ is the vector space of polyvectors of grade k

$$C\ell(V) = \bigoplus_{k=0}^n C\ell^k(V) \tag{55.12}$$

$$C\ell^0 = \mathbb{R}, C\ell^1(V) = V$$

$$\mathbb{R} \oplus V = C\ell^0(V) \oplus C\ell^1(V) \subset C\ell(V) \tag{55.13}$$

Even and odd Clifford algebras :

$$C\ell^{\text{even}}(V) = \bigoplus_{k \in 2\mathbb{N}} C\ell^k(V) \tag{55.14}$$

$$C\ell^{\text{odd}}(V) = \bigoplus_{k \in 2\mathbb{N}+1} C\ell^k(V) \quad (55.15)$$

Both with dimension 2^{n-1}

$C\ell^{\text{even}}(V)$ is a subalgebra of $C\ell(V)$

Pseudoscalars : n -multivector. Orientation operator :

$$\varepsilon = e_1 \wedge \dots \wedge e_n \quad (55.16)$$

$$\varepsilon^2 = (-1)^{\frac{n(n-1)}{2} + s} \quad (55.17)$$

Multiples of ε are pseudoscalars.

If n is odd, ε commutes with all multivectors. Otherwise, it commutes with even grade multivectors and anticommutes with odd graded ones

$$\varepsilon P_r = (-1)^{r(n-1)} P_r \varepsilon \quad (55.18)$$

The center (set of elements that commute with all) of $C\ell(V)$ is $C\ell^0(V)$ if n even, $C\ell^0(V) \oplus C\ell^n(V)$ if odd

Talk about Grassman algebras, even/odd Clifford algebras, Pin group

55.3 Automorphisms

Reversion : Transformation from $C\ell$ to $C\ell$, such that for all polyvectors, the factors are reversed

$$(v_1 \dots v_k)^T = v_k \dots v_1 \quad (55.19)$$

Main involution : acts on basis vectors as $e_i^* = -e_i$.

$$a \mapsto (-1)^n \varepsilon a \varepsilon^{-1} \quad (55.20)$$

Factor of 1 for even graded elements, -1 for odd graded.

Clifford conjugation : $\bar{R} = (R^*)^T$

Scalar product on multivector : $\langle A, B \rangle = A \cdot B = \langle A^T B \rangle_0$

$$A \cdot B = \langle A \rangle_0 \cdot \langle B \rangle_0 + \langle A \rangle_1 \cdot \langle B \rangle_1 + \dots + \langle A \rangle_n \cdot \langle B \rangle_n \quad (55.21)$$

Hodge duality :

$$\star : \Lambda^p \rightarrow \Lambda^{n-p} \quad (55.22)$$

$$A_p \mapsto \star A_p \quad (55.23)$$

Frame $\{e_i\}$ on V defines a frame on $\bigwedge V$ by the multivector $e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$, $I = \{i_1, \dots, i_k\}$, $e_\emptyset = e_0 = 1$. Multivectors e_I gives a basis to $\bigwedge V$, and then to $C\ell(V)$, I goes from 1 to 2^n .

$$A = A^I e_I = A_0 + A^i e_i + A^{ij} e_{\{ij\}} + \dots + A^{i_1 i_2 \dots i_n} e_{\{i_1, i_2, \dots, i_n\}} \quad (55.24)$$

For an orthonormal basis $e_i \cdot e_j = \eta_{ij} = \pm \delta_{ij}$, we can define $\eta_{IJ} = e_I \cdot e_J$, then

$$A \cdot B = \eta_{IJ} A^I B^J \quad (55.25)$$

Complexified Clifford algebra : $\mathbb{C}\ell(V) = \mathbb{C} \times C\ell(V)$

55.4 Clifford groups

The *Clifford group* $\Gamma(V)$ is the restriction of the Clifford algebra to elements that admit an inverse, that is, for $x \in \Gamma(V)$, there exists an $x^{-1} \in \Gamma(V)$ such that $x^{-1}x = xx^{-1} = 1$. The group action on an element of the Clifford algebra is

$$\begin{aligned} \Gamma(V) \times C\ell(V) &\rightarrow C\ell(V) \\ (x, a) &\mapsto -xax^{-1} \end{aligned} \quad (55.26)$$

We will also ask that this group action preserves the vector space, so that for $v \in V$,

$$-xvx^{-1} \in V \quad (55.27)$$

Those conditions define the Clifford group

$$\Gamma(V) = \{x \in C\ell(V) \mid \exists x^{-1}, x^{-1}x = xx^{-1} = 1, \forall v \in V, -xvx^{-1} \in V\} \quad (55.28)$$

The existence of an inverse and the fact that $x1 = 1x = x$ proves that it has a group structure, since the Clifford product is associative.

Representation of $\Gamma(V)$:

$$\begin{aligned} \rho : \Gamma(V) &\rightarrow \text{GL}(V) \\ x &\rightarrow \rho(x) \end{aligned}$$

such that $\rho(x)v = -xvx^{-1}$

Theorem 55.7. ρ is a surjective homomorphism from $\Gamma(V)$ to $\text{O}(p, q)$

Theorem 55.8. ρ is a surjective homomorphism from $\Gamma^{\text{even}}(V)$ to $\text{SO}(p, q)$

Even and odd subsets of the Clifford groups :

$$\begin{aligned} \Gamma^{\text{even}}(V) &= \Gamma(V) \cap C\ell^{\text{even}}(V) \\ \Gamma^{\text{odd}}(V) &= \Gamma(V) \cap C\ell^{\text{odd}}(V) \end{aligned}$$

Since the product of two even elements is even, $\Gamma^{\text{even}}(V)$ is a subgroup of $\Gamma(V)$. $\Gamma^{\text{odd}}(V)$ is not since the product of two odd elements is not odd.

if $x \in V$:

$$-xvx^{-1} = v - 2\frac{v \cdot x}{x \cdot x}x \quad (55.29)$$

Reflection of v with respect to the hyperplane orthogonal to x .

Rotation is the product of two reflections : a rotation will be

$$(xy)v(xy)^{-1} = xyvy^{-1}x^{-1} = RvR \quad (55.30)$$

(xy) is a product of two vectors : $A_0 + A_2$.

Rotation : $R \in \Gamma^{\text{even}}$ such that $RvR^{-1}, \bar{R}R = \pm 1$

Definition 55.9. The Pin group $\text{Pin}(p, q)$ is the subgroup of $\Gamma(V)$

$$\text{Pin}(p, q) = \{s \in C\ell(V) | s_i \in V, s = s_1 s_2 \dots s_k, \bar{s}s = \pm 1\} \quad (55.31)$$

Definition 55.10. The Spin group $\text{Spin}(p, q)$ is the subgroup of $\Gamma(V)$

$$\text{Spin}(p, q) = \{s \in C\ell(V) | s_i \in V, s = s_1 s_2 \dots s_{2k}, \bar{s}s = \pm 1\} \quad (55.32)$$

$\text{Spin}(p, q)$ is a subgroup of $\text{Pin}(p, q)$ and $C\ell^{\text{even}}(V)$

$$\text{Spin}(p, q) = \text{Pin}(p, q) \cap C\ell^{\text{even}}(V) \quad (55.33)$$

$$\begin{aligned} \text{O}(p, q) &= \text{Pin}(p, q)/\mathbb{Z}_2 \\ \text{SO}(p, q) &= \text{Spin}(p, q)/\mathbb{Z}_2 \\ \text{SO}^\uparrow(p, q) &= \text{Spin}^\uparrow(p, q)/\mathbb{Z}_2 \end{aligned}$$

$\text{Spin}(p, q) \cong \text{Spin}(q, p)$, not true for $\text{Pin}(p, q)$ and $\text{Pin}(q, p)$

55.5 Specific algebras

55.5.1 Algebra of real space

A few simple examples are Clifford algebras of signature $(p, 0)$ for low dimensions.

The simplest example of a Clifford algebra is $\mathbb{R}$ with the quadratic form $Q(a) = a^2$. The space of multivectors is then just $\mathbb{R} \oplus \mathbb{R}$, of scalars and vectors (which are also pseudoscalars in this case), with basis $\{1, e_1\}$. An arbitrary multivector is then written as $a1 + be_1$, with the product rule

$$(a + bdx)(a' + b'dx) = aa' + ab'e_1 + a'be_1 + bb'e_1e_1 \quad (55.34)$$

using the ideal, we have $e_1e_1 = -Q(e_1)1 = -1$, we get

$$(a + be_1)(a' + b'e_1) = (aa' - bb') + (ab' + a'b)e_1 \quad (55.35)$$

This is exactly the algebra of complex number, we have then the isomorphism $C\ell(\mathbb{R}) = \mathbb{C}$, and we redefine the basis $e_1 = i$, so that an element of the algebra can be written as

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	-1	e_{12}	$-e_2$
e_2	e_2	$-e_{12}$	-1	e_1
e_{12}	e_{12}	e_2	$-e_1$	-1

Table 55.1: Multiplication table of the basis

$z = a + bi$. The main involution and Clifford conjugation are identical in this context and are equivalent to the complex conjugation.

The associated Clifford group $\Gamma(\mathbb{R})$ is then the subset of $\mathbb{C}$ such that for $z, z^{-1} \in \Gamma(\mathbb{R})$, we have $zz^{-1} = 1$, which, for $\mathbb{C}$, is simply $z \in \mathbb{C} \setminus 0$. The condition that $-xvx^{-1} \in V$ then corresponds to $\Re[-z(ib')z^{-1}] = 0$, which is always true. The even and odd part of the Clifford group are then simply the real and imaginary part : $\mathbb{R} \setminus \{0\}$ and $i\mathbb{R} \setminus \{0\}$. The even Clifford group is the real multiplication group $\mathbb{R}^\times$.

The rotations in $\mathbb{R}$ are then the elements of Γ^{even} such that $\overline{R}R = 1$, which is just the group $\mathbb{Z}_2 = \{-1, 1\}$, which is indeed the rotation group of $\mathbb{R}$.

For the pin group, we need to look at the elements of $\mathbb{R}$

Spin group :

In 2 dimensions, the space of multivectors is $\mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}$, scalars, vectors and pseudoscalars, with the basis $\{1, e_1, e_2, e_1e_2\}$. The products of elements of the basis is given by

$$e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = e_{12} \quad (55.36)$$

$$e_1e_{12} = e_1e_1e_2 = -Q(e_1)e_2 = -e_2 \quad (55.37)$$

$$e_2e_{12} = e_2e_1e_2 = -e_1e_2^2 = e_1Q(e_2) = e_1 \quad (55.38)$$

$$e_1^2 = -Q(e_1) = -1 \quad (55.39)$$

$$e_2^2 = -Q(e_2) = -1 \quad (55.40)$$

$$e_{12}^2 = e_1e_2e_1e_2 = -e_1e_2e_2e_1 = e_1Q(e_2)e_1 = -Q(e_1) = -1 \quad (55.41)$$

The table for all basis products is

This is exactly the multiplication table for the quaternions. Under the map $\phi(1) = 1$, $\phi(e_1) = i$, $\phi(e_2) = j$ and $\phi(e_{12}) = k$, the two dimensional Clifford algebra is isomorphic to the quaternions : $Cl(\mathbb{R}^2) \cong \mathbb{H}$. The reversion leaves all elements invariant except for $e_{12}^T = -e_{12}$, the main involution will just invert the signs for e_1, e_2 , and therefore the Clifford conjugation is the quaternion conjugation

$$(a + ib + jc + kd) \rightarrow (a - ib - jc - kd) \quad (55.42)$$

If a quaternion is invertible, $z \in \mathbb{H} \setminus \{0\}$, it acts on a $\mathbb{R}^2$ vector as

$$-xvx^{-1} = -(a + ib + jc + kd)(v_x i + v_y j) \frac{(a - ib - jc - kd)}{a^2 + b^2 + c^2 + d^2} \quad (55.43)$$

$$= (-(bv_x + cv_y) + (av_x - dv_y)i + (dv_x + av_y)j + (bv_y - cv_x)k) \frac{(a - ib - jc - kd)}{\|x\|} \quad (55.44)$$

$$= \frac{1}{\|x\|} [i((a^2 + b^2 - c^2 - d^2)v_x + 2(bc - ad)) \quad (55.45)$$

$$+ j(a^2 - b^2 + c^2 - d^2)v_y + 2(bc + ad)v_x \quad (55.46)$$

$$+ 2k(bdv_x + cdv_y - acv_x + av_y)] \quad (55.47)$$

[REDO]

The Clifford group $\Gamma(\mathbb{R}^2)$ is the group of invertible quaternions, $z \in \mathbb{H}$ that leave this product in $\mathbb{R}^2$

Rotation group

$$\text{Spin}(2) = \text{U}(1) = \text{SO}(2)$$

(3, 0) case :

In the three-dimensional case (what is generally meant for "algebra of real space"), the multivector space is given by

$$\Lambda V = \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R} \quad (55.48)$$

composed of the scalars, vectors, bivectors (in 3 dimensions isomorphic to the pseudovectors), and pseudoscalars. A basis for it is given by $\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\}$

$$C\ell(\mathbb{R}^2) = \mathbb{H} \oplus \mathbb{H} \quad \text{Spin}(3) = \text{SU}(2)$$

55.5.2 The spacetime algebra and the Dirac algebra

The case that will interest us will be for the case where the vector space will be the tangent space of the manifold, in which case the vector space will be $\mathbb{R}^n$ and the inner product will be either the bundle metric $\langle \cdot, \cdot \rangle$ or the metric tensor g (we will see later that this will lead us to the same thing).

We will denote this as $C\ell(\mathbb{R}^{1,n})$, called the spacetime algebra. Its complexification will be $\mathbb{C}\ell(\mathbb{R}^{1,n})$, called the Dirac algebra.

In $1 + 1$ -dimensions, the exterior algebra is $\mathbb{R} \oplus \mathbb{R}^{1,1} \oplus \mathbb{R}$, the scalars, vectors and pseudoscalars.

$\mathbb{C}\ell(\mathbb{R}^{1,n})$ isometric to $Cl(\mathbb{R}^{2,n})$?

Dirac algebra : Clifford group $\mathbb{C}\ell(\mathbb{R}^{1,3}) \approx \text{SL}(2, \mathbb{C})$

Dirac algebra basis : $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5 \gamma^\mu, \gamma^5\}$

55.6 The Clifford bundle

55.6.1 Constructing the Clifford bundle

Clifford bundle of multivector fields :

$$Cl(TM) = \sqcup_p Cl(T_p \mathcal{M}) \quad (55.49)$$

Clifford bundle of differential multiforms :

$$Cl(T^* \mathcal{M}) = \sqcup_p Cl(T_p^* \mathcal{M}) \quad (55.50)$$

Both have typical fiber the spacetime algebra $Cl(\mathbb{R}^{1,n-1})$

Isomorphism between $Cl(TM)$ and $Cl(T^* \mathcal{M})$ by extension of the musical isomorphisms.

Action of the connection on the Clifford bundle

55.6.2 Associated bundles and pinor bundles

Pinor : vector spaces acted upon by irreducible representation of a Clifford algebra

Real representation of the algebra $Cl(V)$:

$$\rho : Cl(V) \rightarrow \text{End}_{\mathbb{R}}(W) \quad (55.51)$$

complex rep :

$$\rho : Cl(V) \rightarrow \text{End}_{\mathbb{C}}(W) \quad (55.52)$$

Endomorphisms on some $\mathbb{K}$ -vector space W .

Representations are equivalent if

$$\begin{array}{ccc} & A & \\ \swarrow \rho & & \searrow \rho' \\ \text{End}_{\mathbb{K}}(W) & \xrightarrow{\text{Ad}_f} & \text{End}_{\mathbb{K}}(W') \end{array}$$

$$\text{Ad}_f : \text{End}_{\mathbb{K}}(W) \rightarrow \text{End}_{\mathbb{K}}(W') \quad (55.53)$$

Definition 55.11. A pinor representation of $\text{Pin}(V)$ is the restriction of an irreducible representation of $Cl(V)$.

Spinor bundle $\pi_S : S \rightarrow \mathcal{M}$

Notation : components of a spinor

$$\xi = \xi^A e_A \quad (55.54)$$

dual spinors, in the cospinor bundle $\pi_{S^*} : S^* \rightarrow \mathcal{M}$

$$\psi(\xi) = \psi_A \xi^a \quad (55.55)$$

Complex conjugate : $\xi \rightarrow \bar{\xi} = \bar{\xi}^A e_A$

55.6.3 Solder form and spinor metric

Solder form

$$\gamma : \quad (55.56)$$

One benefit of using frame fields rather than the metric tensor is that it permits to define the usual tensor operations on spinor fields naturally.

$$\gamma^\mu = \gamma^a e^\mu_a \quad (55.57)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{ab}(e^\mu)_a(e^\nu)_b I = 2g^{\mu\nu} I \quad (55.58)$$

55.7 Projective Clifford

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An extra generalization of the Clifford algebra can be done by going to one of the quotients of the vector space, either the projective space of $P(V)$, its oriented version $O(M)$, (also direction fields $V/\mathbb{Z}_2$)?

$$1 \rightarrow D \rightarrow V \rightarrow P(V) \rightarrow 1 \tag{55.59}$$

Relation to the projective orthogonal group $\mathrm{PO}(p, q)$?

Chapter 56

Spin and spinors

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56.1 Orientability and spin structure

Definition 56.1. A manifold is said to admit a spin structure if it possesses a principal Spin bundle $\pi : P \rightarrow \mathcal{M}$ with the universal cover

$$F : P \rightarrow SOM \tag{56.1}$$

Theorem 56.2. If the oriented orthonormal frame bundle exists and is trivial, there exists a spin structure on the manifold.

Pin group $\text{Pin}(p, q)$: double cover of the orthogonal group $O(p, q)$

if manifold is space orientable but not time orientable : spinors

if manifold is spacetime orientable : spinors

Spinors : group $\text{SL}(2, \mathbb{C})$

Spin structure

Rokhlin theorem

Definition 56.3. A manifold is called a *spin manifold* if it admits a spin structure.

56.2 Representations of the spin groups

56.3 Associated vector bundles

Once we have our pin structure, sin structure or spin structure, we can define the associated vector bundles

Spinor bundle S : Complex vector bundle associated to the principal bundle of spin frames over M

If the manifold admits a spin structure, we can define

$$\nabla_a \psi^A = \partial_a \psi^A - (\Gamma_a)^A_B \psi^B \quad (56.2)$$

$$(\Gamma_a)^A_B = -\frac{1}{4} \omega_{abc} (\gamma^b)^A_C (\gamma^c)^C_B \quad (56.3)$$

56.4 Discrete symmetries

The discrete symmetries of the Poincaré group, along with the additional discrete symmetries of the spinor bundle itself, have

The two discrete symmetries of the Poincaré group are the involutions of spatial inversion P and time inversion T

In addition to these, as a complex vector bundle, the spinor bundle possesses a *charge conjugation* symmetry

Charge conjugation : map from the conjugate representation of a group

Rep of the group of all symmetries of a spinor bundle?

CPT symmetry

Clifford product

Even and odd Clifford algebras

Blades and geometric algebra

Pin groups

56.5 Spin bundles and spinors

Flag bundles

Projective spin

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56.6 Spin connection

56.7 Dirac operators

If there is a Clifford algebra, there exists a Dirac operator $\not{D}$

Dirac operator squares to the operator from the Casimir operator Ω

Derived from the Clifford algebra of the decomposition of the Lie algebra into parabolic subgroups [Relates to flag description?]

56.8 Whitehead tower

Chapter 57

Singularities

Singularities play an important role in the study of spacetime, as they are a common feature of many realistic spacetimes, as well as of common non-realistic but useful examples, so it may be of interest to have the proper definitions and classification for them.

On flat spacetime, field singularities are generally defined by divergences in quantities at defined points. For instance, the singularity of the electric field of a static charge in electromagnetism

$$\vec{E}(\vec{x}) = k_e \frac{q}{r^2} \vec{e}_r \quad (57.1)$$

which presents a singularity at $r = 0$, as the field is unbounded around this value. More generally, we can consider points where the values of the field fail to converge, for instance by considering the topologist's sine function

$$f(r) = \sin(1/r) \quad (57.2)$$

which does not diverge but still fails to converge at $r \rightarrow 0$.

One might think that the definition of a singularity in general relativity is similar (as it usually appear in similar forms in such famous examples such as the Schwarzschild metric), but unlike classical physics, singularities in general relativity are not trivial to define due to a few obstacles :

- Points can be (and for singularities, are) removed from the manifold, in which case if a quantity diverges at p on M , we need to define what this means on $M \setminus \{p\}$.
- For tensor components, the values of a field depend on the atlas we use, and this can lead to divergences in one coordinate system but not in another, such as the horizon in the Schwarzschild metric.

- For some choice of coordinates, a divergence at a given coordinate can correspond to a quantity which is "at infinity", and as such not really problematic.

A trivial example is to consider Minkowski space with the coordinate patch covering all space with $x > 0$, under the transition map $x \rightarrow x' = \ln(x)$ from Cartesian coordinates. The components of the metric tensor are then

$$ds^2 = -dt^2 + \frac{1}{x'^2} dx'^2 \quad (57.3)$$

The component g_{xx} diverges at $x = 0$, but obviously this is not an issue coming from. Similarly, if we consider the Poincaré disk model for the hyperbolic plane, defined on the disk $|z| < 1$ on the complex plane $\mathbb{C}$ with metric

$$ds^2 = \frac{dzd\bar{z}}{(1 - |z|^2)^2} \quad (57.4)$$

and we consider its Weyl transform

$$ds^2 = \frac{dzd\bar{z}}{(1 - |z|^2)^2} \quad (57.5)$$

If we considered some spherically symmetric spacetime with a curvature diverging at infinity, such as taking the metric

$$ds^2 = \quad (57.6)$$

Considering the behaviour of scalars can be a useful way to evaluate singular spacetimes, but here again we have to take some precautions : the scalar field defined by the inverse of the Ricci scalar R^{-1} will be divergent if we consider some spacetime on which R is nowhere 0 but converging to it, so that we have to be careful about the specific class of scalars we consider.

Throughout this chapter, an important guideline to keep in mind is that the prototypical example of a singularity-free spacetime is Minkowski space. This is important to keep in mind because general covariance makes it difficult to be sure that a given definition for a singularity does not apply to Minkowski space itself.

One particular example is that there are inextendible timelike curves of finite proper time in Minkowski space.

Theorem 57.1. There exists an inextendible timelike curve γ in Minkowski space (Mink, η) such that its total proper time is finite.

Proof. Consider the two-dimensional Minkowski space $\mathbb{R}^{1,1}$, with Cartesian coordinates (t, x) . $\square$

Our fundamental example of completeness is Minkowski space, so we will first look into what its properties are.

Theorem 57.2. Minkowski space is geodesically complete

Theorem 57.3. Proof from the bijection of the exponential map

Theorem 57.4. Minkowski space has no inextendible curves of bounded acceleration with finite

Proof. For any curve γ in Minkowski space, with bounded acceleration :

$$\|\nabla_{\dot{\gamma}}\dot{\gamma}\| < B \quad (57.7)$$

$\square$

[Include the foliation definition from that one paper] [154] [155]

57.1 Regular boundary points

The simplest singularities arise when a well-behaved point is removed from a manifold. While they may appear as singularities in all the following definitions, unlike other more pathological singularities, they do not necessarily form a boundary of the manifold. For instance, for the Minkowski half-space $\{(t, x) | t > 0\}$, there is a singularity formed by $(t, x) | t = 0$, but it is possible to extend (in a non-unique way) the spacetime.

To differentiate regular boundary points from more pathological ones, we first try to extend the spacetime in a way that remove them.

Definition 57.5. An extension M' of a spacetime is a spacetime M' such that there exists an isometric embedding $\mu : M \rightarrow M'$.

Definition 57.6. A boundary point is a *regular boundary point* if there exists an extension of the spacetime such that the boundary point is not present in the extension.

Such singularities, while often used for examples, are often considered unlikely to be physical.

57.2 m -completeness and g -completeness

If a manifold is equipped with a Riemannian metric, we can define the notion of completeness of the manifold by its status as a metric space. We then say that the manifold is m -complete if every Cauchy sequence converges to a point in the manifold, where the norm of the Cauchy sequence is the metric d defined by the Riemann tensor. That is, for a sequence p_n of points of the manifold

$$\exists p \in \mathcal{M}, \forall \varepsilon, \exists N, \forall n > N, \quad (57.8)$$

Unfortunately, Lorentzian metrics do not permit the definition of a metric on the manifold.

Another notion of completeness we can use is that of g -completeness : a manifold is said to be g -complete if all of its geodesics do not end abruptly, so to speak.

Definition 57.7. A spacetime is said to be g -complete if every inextendible half-geodesic is defined for arbitrarily large values of its affine parameter.

Hopf-Rinow theorem [156]

The definition is limited to geodesics due to the fact that any spacetime may admit curves of finite affine length by a proper choice of the parametrization

Theorem 57.8. Even benign spacetimes may admit half-curves of finite affine length.

Proof. Consider a curve in Minkowski space with an acceleration depending on its parameter.

$$\|\ddot{\gamma}(\tau)\| = \alpha(\tau) \quad (57.9)$$

with the usual parametrization for timelike curves

$$\|\dot{\gamma}(\tau)\| = -1 \quad (57.10)$$

Then, if we consider for instance two-dimensional Minkowski space, we get the following relations

$$\ddot{x}^2(\tau) - \ddot{t}(\tau) = \alpha(\tau) \quad (57.11)$$

$$\dot{x}(\tau) - \dot{t}(\tau) = -1 \quad (57.12)$$

or, switching to the null coordinates $v = (t + x)/2$ and $u = (t - x)/2$

$$\ddot{u}(\tau)\ddot{v}(\tau) = \alpha(\tau) \quad (57.13)$$

$$\dot{u}(\tau)\dot{v}(\tau) = -1 \quad (57.14)$$

After some manipulations, this gives us $\dot{u} = -\dot{v}^{-1}$, and

$$\left(\frac{\ddot{v}}{\dot{v}}\right)^2 = \alpha^2(\tau) \quad (57.15)$$

If we consider for instance the positive solutions, this becomes

$$\ddot{v} - \dot{v}\alpha(\tau) = 0 \quad (57.16)$$

with the simple solutions

$$\dot{v}(\tau) = Ae^{\int_1^\tau \alpha(\xi) d\xi} \quad (57.17)$$

$$\dot{u}(\tau) = A^{-1}e^{-\int_1^\tau \alpha(\xi) d\xi} \quad (57.18)$$

That curve is both inextendible and of finite affine length. $\square$

This corresponds to a curve of unbounded acceleration.

Unfortunately, the notion of g -completeness is not sufficient to fully capture the concept of singularities in the Lorentzian case.

Theorem 57.9. There exists geodesically incomplete compact spacetimes.

Proof. cf Misner 63, O'Neill p. 193 $\square$

The notion of g -boundary gives rise to three classes of singularities, depending on the type of curve considered. These are naturally timelike singularities, spacelike singularity and null singularity. Spacetimes with those types of singularities are qualified as timelike incomplete, spacelike incomplete or null incomplete, or g_t , g_s and g_n incomplete.

Unfortunately, there is no obvious relation between those types of incompleteness, as it can be shown that they can all occur independently of each other.

As regular boundary points are a very practical source of examples, it will be useful to show that they are indeed singular in those senses :

Theorem 57.10. If we remove any point p from a manifold M , $M \setminus \{p\}$ is g -incomplete and m -incomplete at p

Proof. If M originally contains a point p , there exists a normal neighbourhood U_p around it for which we can find some geodesics crossing p . A curve $\gamma : [-\varepsilon, \varepsilon)$ with $\gamma(0) = p$ will be split into two curves, $\gamma^\pm : (0, \pm\varepsilon \rightarrow U_p$, which are inextendible at $\lambda = 0$: if they were extendible, there would be another curve in which some sequence of points (λ_i) would converge to 0 and by continuity, to the original point p . [better proof] $\square$

57.3 Types of singular spacetimes by causality

There are three types of curves we should watch out for when we study singularities : timelike, null and spacelike curves, with in addition the type of acceleration the curve is under. As we will see, the singularities associated with each type of curves do not in general imply anything about the completeness of other types of curves.

57.3.1 Singularity-free spacetime which still admits incomplete curves

One thing we need to watch out for in the Lorentzian case is that not all curves will contribute to our analysis of its singularities.

We can indeed still find inextendible incomplete curves in the case of Minkowski space

57.3.2 Spacetime which is spacelike and null complete but timelike incomplete

2D Conformal Minkowski space $g = \Omega\eta$, $\Omega = 1$ if $|x| > 1$, conformal factor is space reflection symmetric : $\Omega(x, t) = \Omega(-x, t)$, Ω decays fast as $t \rightarrow \infty$ (for instance t^6). This conformal factor could be built easily enough with bump functions.

Geodesic on $t = 0$ is timelike and finite length. Every spacelike or null geodesic will leave the region $(-1, 1)$ and hence be complete.

57.3.3 Spacetime which is spacelike complete but timelike and null incomplete

Metric due to Kundt

$$ds^2 = -f^{-2}(x)dt^2 - dx^2 \quad (57.19)$$

for $x \in (0, \infty)$, $t \in \mathbb{R}$

Geodesics :

$$s - s_0 = \int (f^2(x) + \epsilon)^{-\frac{1}{2}} dx \quad (57.20)$$

ϵ being -1 for timelike geodesics, 1 for spacelike geodesics and 0 for null geodesics.

for $f(x) = x$

57.3.4 Spacetime which is timelike complete but null and spacelike incomplete

[cf Kundt]

57.3.5 Spacetime which is timelike, spacelike and null incomplete

57.3.6 Spacetime which is timelike and null complete but spacelike incomplete

57.3.7 Spacetime which is spacelike and timelike complete but null incomplete

[Beem example [157]]

57.3.8 Spacetimes which are complete for all geodesics but not for timelike curves of bounded acceleration

In addition to the various types of curves that can be complete or not, a further type of issue is that a spacetime can be complete for all geodesics, but incomplete in the cases of accelerated curves, even if we restrict them to curves of bounded acceleration. An accelerated observer could in principle simply disappear from spacetime in finite time.

The first example of such a spacetime is due to

Example of a spacetime which is geodesically complete but with non-terminating timelike curves

Covering space of two-dimensional anti de Sitter space

$$ds^2 = -(1+x^2)dt^2 + (1+x^2)^{-1}dx^2 \quad (57.21)$$

Construct a spacetime with "geodesic traps" ended by a singularity

Blocking sets

57.4 Directional singularities

As singularities as defined from equivalence class of curves, one aspect of this is that singularities can be directional [158]. Some curves may be able to go through a given point with a velocity going in a particular direction, but others may abruptly terminate.

Explanations wrt singularities as points in the completion of the frame bundle

The common example of this phenomenon is the Israel-Khan metric, which is given by a system of two singularities kept in equilibrium. It is given by a static axisymmetric metric

$$ds^2 = -e^{2U} dt^2 + e^{-2U} (e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2) \quad (57.22)$$

with U, γ two functions of (ρ, z) such that U is a harmonic function obeying

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \frac{\partial}{\partial \rho}) U + \frac{\partial^2}{\partial z^2} U = 0 \quad (57.23)$$

and γ satisfies

$$\frac{\partial \gamma}{\partial r} = r \left[\left(\frac{\partial U}{\partial r} \right)^2 - \left(\frac{\partial U}{\partial z} \right)^2 \right] \quad (57.24)$$

$$\frac{\partial \gamma}{\partial z} = 2r \frac{\partial U}{\partial r} \frac{\partial U}{\partial z} \quad (57.25)$$

Chazy-Curzon solution :

$$U = -\frac{m}{\sqrt{\rho^2 + z^2}} \quad (57.26)$$

$$\gamma = -\frac{m^2 \rho^2}{2(\rho^2 + z^2)^2} \quad (57.27)$$

[159]

57.5 Naked singularities

A particularly important class of singularities in general relativity is the naked singularity. The exact definition may vary, as the idea of a naked singularity is more about its pathological nature than a specific definition, but a common definition is this :

Definition 57.11. A spacetime is *nakedly singular* if there is a point $p \in \mathcal{M}$ and a future-incomplete timelike or null geodesic γ such that $\gamma \subset I^-(p)$

(definition extended to causal curves of bounded accelerations maybe)

By this definition, it is easy to see that a spacetime that is both timelike and null complete is never nakedly singular, and so spacetimes that are only spacelike singular do not admit any such naked singularities.

Other examples of definitions of naked singularities due to Penrose are [160] :

1. There exists a PIP containing a singular TIP
2. There exists a PIF containing a singular TIF
3. There exists a PIP containing a TIP
4. There exists a PIF containing a TIF
5. There exists a ∞ -TIP containing a singular TIP
6. There exists a ∞ -TIF containing a singular TIF
7. There exists a singular TIP containing another singular TIP There exists a singular TIF containing another singular TIF

Cosmic censorship

57.6 *b*-completeness

The notion of completeness that fulfills best the broadest notion of singularity is *b*-completeness, or bundle completeness, where we define a Riemannian metric on the bundle of linear frames $L\mathcal{M}$

An inextendible curve in $\mathcal{M}$ defines a point of the *b*-boundary $\partial\mathcal{M}$ if and only if its euclidian length measured in a parallelly propagated frame is finite.

$$g_b(X, Y) = \sum_i \theta^i(X) \theta^i(Y) + \sum_{i,j} \omega_k^i(X) \omega_k^i(Y) \quad (57.28)$$

b-boundary : Completion of the orthonormal bundle $\pi : O \rightarrow M$ with the metric e on the bundle. The *b*-boundary

57.7 Classification of singularities

The b -boundary gives us the boundary points ∂M of the spacetime. These singularities can then be classified by the behaviour of the Riemann tensor near them.

First, as we have defined previously, singularities in ∂M which are actual points of a manifold M' , where M' is an extension of M , are called regular boundary points. If for one of those extensions M' , the Riemann tensor is a C^k tensor, and k is the highest value of any of those extensions, we say that it is a C^k regular boundary point. If there is no extension where the Riemann tensor is C^k , we say that this point is a C^k singular point.

If the metric is itself not smooth, the Riemann tensor might be itself C^k itself at some points, even discontinuous or worse (if we allow for weak derivatives). Although not part of the b -boundary, this will be of a similar behaviour to C^k regular boundary points. We may then get points as badly behaved as C^0 regular boundary points, for instance in the thin shell formalism or with gravitational shockwaves.

(manifold + b -boundary is a topological space)

57.7.1 Regular boundary points

As we saw previously, some boundary points are simply the result of spacetime not being fully extended.

Theorem 57.12. If a spacetime admits an extension, it contains *regular boundary points*.

Proof. Consider a spacetime $\mathcal{M}$ with an injection map $\iota : M \hookrightarrow M'$ into an extension M' such that $\text{Im } \iota \subset M'$. The image of the inclusion will then have a non-empty boundary ∂M . As they are part of the manifold $\mathcal{M}'$, there exists Cauchy sequences $(e_i)_{i \in \mathbb{N}}$ in the bundle of linear frame LM' , $e_i \in LM$ for all i , such that

$$\pi\left(\lim_{i \rightarrow \infty} e_i\right) \in \partial \text{Im } \iota \quad (57.29)$$

As M does not contain any point in ∂M , those sequences of $LM'|_M$ do not converge in the bundle itself and are boundary points. $\square$

57.7.2 Quasi-regular singularities

A boundary point $p \in \partial M$ is a quasi-regular singularity if it is not possible but the Riemann tensor can be extended in a parallel frame [161–163]

Definition 57.13. A singular boundary point b is called a *quasi-regular singularity* if, for every curve γ that ends on the boundary point, the frame field $\{e_a\}$ and all its derivatives on those curves remains bounded.

Local extendibility

Example 57.14. The conical 2-manifold constructed by removing a slice $r \in [0, \infty]$, $\theta \in (2\pi - \alpha, 2\pi)$ and identifying its boundaries for $r > 0$ has a quasi-regular singularity at 0.

Proof. As we removed the point $r = 0$, we can consider any curve which originally crossed that point as an example, such as the line $x(\lambda) = (r = \lambda, \theta = \theta_0)$ (for some angle $\theta_0 < 2\pi - \alpha$), which is singular at $\lambda = 0$ and cannot be extended. $\square$

Take Minkowski space, express in cylindrical coordinates

$$ds^2 = -dt^2 + d\rho + \rho d\theta^2 + dz^2 \quad (57.30)$$

remove the timelike plane $\{(t, \rho, \theta, z) | \rho = 0\}$

Effects of quasi-regular singularities : affects the parallel transport of curves

Topological invariant relating to angular defects?

57.7.3 Curvature singularity

A curvature singularity is the type of singularity most commonly associated with the idea of a singularity, where the curvature fails to converge properly.

Curvature singularity : Riemann tensor cannot be extended in a parallel frame

Definition 57.15. A boundary point b is called a *curvature singularity* if, for every curve γ that ends on the boundary point, the frame field $\{e_a\}$ or one of its derivatives on those curves is unbounded.

The curvature singularity is the most common example of singularity. It is for instance the type we may find in the interior Schwarzschild solution $r < 2M$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

Non-scalar singularity : no curvature scalar is badly behaved

Scalar singularity : badly behaved curvature scalar

A few example of scalars that are commonly used to check for scalar singularities are the Ricci scalar R , and the Kretschmann scalar $K = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}$

Interior Schwarzschild metric :

$$R = \tag{57.31}$$

Chapter 58

Killing vectors and symmetries

The most obvious definition we would use in physics would be to consider whether a given object depends on a given coordinates. For instance consider some scalar function $f(x, y, z)$. If we have want to say that f is invariant with respect to translation along the z axis, f being independent of z would be a good criteria :

$$\partial_z f = 0 \tag{58.1}$$

This can be unfortunately a misleading statement in more general spaces. Consider for instance the spacetime given by Minkowski space minus the spatial origin, $\mathbb{R}^n \setminus \{p \mid x = y = z = 0\}$. Any quantity not containing a reference to the spatial variables will seem to be invariant under the translation

$$(x, y, z, t) \rightarrow (x + a, y + b, z + c, t) \tag{58.2}$$

But any point in the line $(-a, -b, -c, t)$ will in fact not be mapped anywhere. If we attempted such a transformation, that point would simply not be mapped to any point of the manifold.

Our definition of a symmetry will then have to take into account the global properties of manifolds.

A *symmetry* (what we will later call a *spacetime symmetry*, to differentiate from more general types of symmetries) is a diffeomorphism $\phi \in \text{Diff}(M)$ that preserves a structure on the spacetime, generally the metric tensor. A diffeomorphism preserving the metric tensor in particular is an *isometry* : $\phi^*g = g$.

The set of all such isometries is written as $\text{Iso}(M) \subset \text{Diff}(M)$.

$$\text{Iso}(M) = \{\phi \in \text{Diff}(M), \phi^*g = g\} \tag{58.3}$$

If a given spacetime has isometries, it means that the action of the diffeomorphism group on the metric tensor fails to be free.

Theorem 58.1. $\text{Iso}(M)$ forms a Lie group.

Proof. The identity diffeomorphism is an isometry as $\text{Id}_M^* = \text{Id}_{TM}$. For two isometries $f_1, f_2 \in \text{Iso}(M)$, their composition gives us $(f_1 \circ f_2)^*g = f_1^*(f_2^*g) = f_1^*g = g$. The inverse diffeomorphism f^{-1} of an isometry f is also an isometry as $g = (f^{-1} \circ f)^*g = f^{-1*}(f^*g) = f^{-1*}g$. $\square$

Theorem 58.2. A diffeomorphism $f \in \text{Diff}(M)$ is an isometry if the pushforward to the frame bundle FM leaves the orthogonal frame subbundle $F_O M$ invariant : $e \in F_O M \subset FM \rightarrow f^*e \in F_O M$

Theorem 58.3. A bundle automorphism f of $F_O M$ such that $f^*\theta = \theta$ is induced by an isometry of M .

If furthermore we have some 1-parameter families of such isometries, these will generate vector fields via the Lie map, called Killing vectors :

Definition 58.4. For any 1-parameter subgroup $f_t : (\mathbb{R}, +) \rightarrow \text{Iso}(M)$, the vector field $X = df$ is called a *Killing vector field*

We can also define Killing vectors the opposite way, by considering an infinitesimal transformation from a vector field leaving the metric invariant.

$$\mathcal{L}_X g = 0 \tag{58.4}$$

This definition is however not perfectly symmetric as a Killing vector field may not integrate to an isometry. As we've seen with the case of flat space with a point removed,

Killing vector field forms an isometry if global flow?

Since a Killing vector field is still a vector field, there exists an adapted coordinate system for each Killing vector field by using the flowbox theorem 20.2.1.

If we have a system of several Killing vector fields,

Adapted coordinates for several Killing vector fields

Zero vector fields are always Killing vectors

Killing vector field as applied to geodesic equations etc

Flowbox theorem

Types of symmetries : free group action, etc

Set of all Killing vector fields : $\mathfrak{iso}(M)$

All linear combinations of Killing vector fields are Killing vector fields

Bracket of Killing is Killing

Killing equation :

Given the identity

$$\nabla X^b = \frac{1}{2}(\mathcal{L}_X g - dX^b) \quad (58.5)$$

for Killing vectors :

$$\nabla K^b = -\frac{1}{2}dK^b \quad (58.6)$$

$$K_{\mu;\nu} + K_{\nu;\mu} = 0 \quad (58.7)$$

$$\nabla_X \nabla_Y K = R(X, Y)K \quad (58.8)$$

Contracted :

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu \quad (58.9)$$

[164, 165]

$\mathfrak{iso}(M)$ is a Lie algebra, subalgebra of $\mathfrak{X}(M)$

$\mathfrak{iso}(M)$ has dimension at most $n(n+1)/2$ (O'Neill)

On a complete semi-Riemannian manifold, every Killing vector field is complete

Maximally symmetric spaces

Maximal symmetry for g can only be the Poincaré group, any other symmetry is a subgroup of Poincaré

[63, 166, 167]

Theorem 58.5. For a vector field $\xi \in \mathfrak{X}(M)$, then the following are equivalent :

1. ξ is a Killing vector field

2. $\xi(g(X, Y)) = g([\xi, X], Y) + g(X, [\xi, Y])$
3. $g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$

Theorem 58.6. For two Killing vector fields X, Y , if there exist a point at which $X_p = Y_p$ and $(\nabla X)_p = (\nabla Y)_p$, then $X = Y$.

$$\dim(\text{Iso}(M)) \leq \frac{n(n+1)}{2} \quad (58.10)$$

For any point $p \in M$, the linear maps f on $T_p M$ such that $g_p(f(X_p), Y_p) = -g_p(X_p, f(Y_p))$
: linearity of g and f , bracket operation :

$$g_p([A, B]X, Y) = g_p(ABX, Y) - g_p(BAX, Y) \quad (58.11)$$

$$= -g_p(X, ABY) + g_p(X, BAY) \quad (58.12)$$

$$= -g_p(X, [A, B]Y) \quad (58.13)$$

This is the Lie algebra $\mathfrak{o}(T_p M)$ of skew symmetric operators. In an orthonormal basis,

$$g_p(AX, Y) = -g_p(X, AY) \Leftrightarrow (AX)^\top \eta Y = -X^\top A^\top \eta Y \Leftrightarrow X^\top A^\top \eta Y = -X^\top \eta AY \Leftrightarrow X^\top \eta = -\eta A \Leftrightarrow A^\top \eta = \eta X \quad (58.14)$$

It is therefore isometric to the orthogonal Lie algebra $\mathfrak{o}(n)$

For the map $E : \mathfrak{iso}(M, g) \rightarrow T_p M \times \mathfrak{o}(T_p M)$ defined by

$$\xi \mapsto (\xi_p, (\nabla \xi)_p) \quad (58.15)$$

E is bijective, therefore

$$\dim(\mathfrak{iso}(M, g)) = \dim(E(\mathfrak{iso}(M, g))) \leq \dim(T_p M) + \dim(\mathfrak{o}(n)) = n + n(n-1)/2 \quad (58.16)$$

Theorem 58.7. If K is a Killing vector field of constant norm, $X(g(K, K)) = 0$ for every $X \in \mathfrak{X}(M)$, then the integral curves of K are geodesics.

Proof. □

This means that in a spacetime where there exist a congruence of freely-falling observer which observe the same metric locally along their trajectories, there exist a timelike Killing vector field.

58.1 Adapted coordinates

Our original definition of a symmetry as being independent of a given coordinate can indeed be found in some form with the case of Killing vectors, as they will admit some coordinate system in which this is true.

For a single Killing vector k , we already have the flowbox theorem 20.2.1 : for a nowhere vanishing Killing vector k , there is a coordinate chart ϕ such that [...].

In these coordinates, the Killing condition becomes

$$\partial_x g = 0 \quad (58.17)$$

which is exactly the condition we started out with, at least in the one dimensional case. Therefore, if our metric is independent of a given coordinate, the metric has the Killing vector field defined by the holonomic basis element of that coordinate.

If there are more than one Killing vectors, we could perform the same analysis, however we do not have any guarantee that the coordinate chart of one Killing vector field can be the same as that of another, so that we cannot have by default that the metric is independent of all those coordinates at once.

Theorem 58.8. For a set of Killing vectors $\{k_i\}$ such that the set of vector is involutive (ideal generated by $\{k_i^\sharp\}$ is differential condition?), then there exists a coordinate chart ϕ with coordinates $(x_1, x_2, \dots, x_i, x_{i+1}, \dots, x_n)$ such that the metric can be expressed as

$$g = g_{ab}(x^\mu) dx^a dx^b + g_{AB}(x^C) dx^A dx^B \quad (58.18)$$

and

$$k_i = k_i^a(x^C) \partial_a \quad (58.19)$$

If the Killing fields also commute,

58.2 Conserved quantities

For geodesics on a manifold with some Killing vector field K , we have that the quantity $g(K, u)$

$$D_t g(K, \dot{\gamma}) = \nabla_{\dot{\gamma}} g(K, \dot{\gamma}) + g(\nabla_{\dot{\gamma}} K, \dot{\gamma}) + g(K, \nabla_{\dot{\gamma}} \dot{\gamma}) \quad (58.20)$$

$$= g(\nabla_{\dot{\gamma}} K, \dot{\gamma}) \quad (58.21)$$

58.3 Types of symmetries

58.3.1 Spacetimes with one Killing vector

The simplest case of symmetry is a spacetime equipped with a single Killing vector K . We will in particular be interested in the case where this Killing vector is everywhere of the same causal type, either everywhere timelike, spacelike or null.

If the spacetime is well-behaved enough, it is possible to express the metric of such a spacetime in appropriate adapted coordinates [168]. Let's consider our spacetime (M, g) with a single Killing vector field K everywhere timelike or everywhere spacelike (in particular, it is nowhere vanishing). Being equipped with a nowhere vanishing vector field, we can define a foliation of this spacetime by the flow of its Killing vector Fl_K .

Consider the orbit space $S = M/\text{Fl}_K$. For $\gamma \in S$, γ is an inextendible curve of tangent K .

Mapping $\psi : M \rightarrow S$: for $p \in M$, $\psi(p)$ is the trajectory of K passing through p .

Assumption : S has the structure of a differentiable 3-manifold such that ψ is a smooth mapping (avoid dense curves)

If K is hypersurface orthogonal, S is a hypersurface in M that is everywhere orthogonal to K

Non-orthogonal case, no natural way to introduce those surfaces

S is a quotient space of M and not a subspace

There is a bijective correspondence between tensor fields and tensors operations on S and some tensor fields and operations on M .

Bijection : Tensor fields T' on S and tensor fields T on M satisfying $C(K, T) = 0$ for every index and $\mathcal{L}_K T = 0$

Straumann 2.9

58.3.2 Stationary spacetime

A spacetime is *stationary* if it has a Killing vector field that is timelike everywhere. From the previous section, we know

58.3.3 Static spacetime

Stationary and irrotational

$$ds^2 = -\alpha(x)dt^2 + g \quad (58.22)$$

58.3.4 Ultrastatic spacetime

If in addition to being static, the timelike Killing vector is of constant norm along the spacelike hypersurfaces, ie for any Σ_t , the orthogonal Killing vector k_t does not depend on its position, the spacetime is called *ultrastatic*. By being of constant norm, the flow lines of the timelike Killing vectors are geodesics, corresponding to

Ultrastatic spacetimes simply have the structure of a direct product of Riemannian manifolds $(L, -\delta) \times (\Sigma, g)$

$$ds^2 = -dt^2 + h \quad (58.23)$$

This gives them a very nice structure to work with. Ultrastatic spacetimes have the structure of an $\mathbb{R}$ -principal bundle, with the timelike Killing vector being a

Connection

Curvature

58.3.5 Spherically symmetric spacetime

The definition of spherical symmetry can be slightly tricky in the case of general relativity. Here are a few properties that may be involved :

- The orbit space of the isometry is a spacelike submanifold diffeomorphic to a sphere
- The orbit space of the isometry is a spacelike submanifold of constant positive curvature
- The isometry group is isomorphic to $SO(3)$
-

Definition 58.9. A spacetime is *spherically symmetric* if it admits an isometry group acting on $n - 2$ -hypersurfaces of constant positive Gaussian curvature

$$ds^2 = +Y(r, t)d\Omega^2 \quad (58.24)$$

See Stephani

Wald p. 121, coordinate form and when it breaks down

58.3.6 Axisymmetric spacetimes

Definition 58.10. A spacetime is *axisymmetric* if it has a rotation symmetry $O(1)$

Lewis-Papapetrou line element [170, 171] [172]

$$ds^2 = -f dt^2 - 2\kappa dt d\phi - \ell d\phi^2 - e^\mu (d\rho + dz^2) \quad (58.25)$$

functions depend on z, ρ alone

58.3.7 Cylindrically symmetric spacetimes

Axisymmetric and translation symmetric along the axis of rotation

Definition 58.11. A spacetime is said to be cylindrically symmetric if there exists a

58.3.8 Homogeneous spaces

Definition 58.12. A semi-Riemannian manifold (M, g) is said to be *homogeneous* if the group of isometries $\text{Iso}(M)$ acts transitively on it, so that for any two points $p, q \in M$, there exists an isometry $\phi \in \text{Iso}(M)$ such that $\phi(p) = \phi(q)$.

Homogeneity is quite a strong condition on the topology and geometry of a manifold, and given a dimension, there are only a handful of such manifolds to exist.

Theorem 58.13. If M is a homogeneous semi-Riemannian manifold, it is geodesically complete.

Theorem 58.14. A homogeneous space's isometry Lie algebra $\text{iso}(M)$ is the translation group $\mathbb{R}^n$.

58.3.9 Isotropic space

Definition 58.15. A semi-Riemannian manifold (M, g) is called an *isotropic* at a point p if for any two vectors at p , $v_p, w_p \in T_p M$, such that $|v_p| = |w_p|$, there exists an isometry $\phi \in \text{Iso}(M)$ such that

$$\phi_*(v_p) = w_p \quad (58.26)$$

The spacetime is further simply called isotropic if it is isotropic at every point.

For a Riemannian manifold in particular, this can be restated as the fact that the unit sphere of the tangent space at p , noted $S_p M$ (this is the set of all vectors of unit norm) obeys this property.

Isotropic equivalent to two-point homogeneity

Locally isotropic manifold

58.3.10 Maximally symmetric space

A semi-Riemannian manifold is *maximally symmetric* if it is both homogeneous and isotropic at every point, or alternatively, if it is two-point homogeneous.

Theorem 58.16. The Lie algebra $\mathfrak{iso}(M)$ of a maximally symmetric space is the Poincaré algebra.

Proof. For some point $p \in M$

□

The Killing vectors and their Lie bracket form the full Poincaré group

Riemann tensor's form

$$\text{Riem} = \frac{\alpha}{n(n-1)} g \otimes g \quad (58.27)$$

$$R_{\mu\nu\lambda\rho} = \frac{R}{n(n-1)} (g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho}) \quad (58.28)$$

58.4 Classification of symmetries by dimension

58.4.1 Symmetries in (1+1)-dimensions

Translation symmetries (time, space, null)

Boost ($\text{SO}(1, 1)$)

58.4.2 Symmetries in (1+2)-dimensions

58.4.3 Symmetries in (1+2)-dimensions

That theorem about how the max number of Killing vectors is 7 before maximally symmetric

58.5 Conformal Killing vectors

The notion of Killing vectors easily generalizes to whatever geometric structure that one can imagine.

Definition 58.17. A *conformal Killing vector field* is a vector field that preserves the conformal metric c of a conformal manifold under the Lie derivative :

$$\mathcal{L}_X c = 0 \quad (58.29)$$

Alternatively, the conformal Killing vector will preserve the metric tensor of a pseudo-Riemannian manifold up to scale :

$$\mathcal{L}_X g = \lambda g \quad (58.30)$$

Other possibilities :

$$\mathcal{L}_K \omega = 0 \quad (58.31)$$

58.6 More general symmetries

Backlund symmetries??? Q group??? Killing spinors?

[173]

Chapter 59

The exponential map and the world function

A big difference with geometric theories such as general relativity from other gauge theories is the fact that they select a special class of curves from their structure. There is no equivalent of geodesics in other kinds of gauge theories, and while we could try to find some equivalent with the notion of the trajectory charges follow in an electromagnetic field, this lacks the universality of the geometric case : curves are not only classified by their original position in $T^k M$, but by their charge-to-mass ratio. Two curves originally at the same point and the same velocity can diverge simply from their difference in charge.

This special class of curves allow us to map out points of the velocity bundle to the manifold itself, by considering the curve starting at a given point p with some initial velocity $v \in T_p M$ and find out which curve ends up there after a given length of time.

59.1 The exponential map

[174]

We have had a lot of exponential maps in our manifold so far, first relating to diffeomorphisms and vector fields, as

$$\exp : \mathfrak{X}(M) \rightarrow \text{Diff}(M) \quad (59.1)$$

via the flow of a vector field $\text{Fl}(X)$. We also saw the exponential map as it relates to Lie groups, where it is in many cases quite literally an exponential map

$$\exp : \mathfrak{g} \rightarrow G \quad (59.2)$$

Relating also in some way to the previous definition, since every point of a Lie group also defines a diffeomorphism on the Lie group by Lie group action,

$$\exp : \Gamma(\mathfrak{g}) \longrightarrow \text{Diff}(G) \quad (59.3)$$

and in the other direction, we can consider the space of vector fields to be a Lie algebroid and the diffeomorphism group its Lie groupoid.

If we have an Ehresmann connection on the tangent bundle, ie some connection $\Phi : TTM \rightarrow VTM$

[Correspondence between connections and semisprays]

Given this correspondence between the two, such connections also have an associated exponential map on the tangent bundle TM . If we integrate the semispray ξ of our connection $\Phi \in TTM$, as a simple vector field on TM , it will give rise to a flow that foliate TM [Is it true globally? Locally?]. As this is simply a vector field, we have exactly one curve going through each point of TM . Because of the [?] nature of the vector field, each curve is transversal to the fiber :

$$x \quad (59.4)$$

For any point (p, v) in TM (pair of a position and initial velocity), we have therefore have a unique curve going through that point. The projection of this curve down to the base manifold gives us a curve going through p with velocity v at p , with one such curve for every vector $v \in T_pM$.

The *exponential map* $\exp_{\nabla}$ of an affine connection ∇ is the map of this process, by considering the foliation of TM into those curves [geodesic flow?], and associating to a base point $p \in M$ and a given vector at that point $v \in T_pM$ the unique curve $\gamma : [0, 1] \rightarrow TM$ of initial point $\gamma(0) = v$, $\pi(v) = p$, and on this curve, the point $\pi(\gamma(1))$.

$$\exp_{\nabla} : TM \rightarrow M \quad (59.5)$$

$$v \mapsto \pi(\text{Fl}_1^{\Phi}(v)) \quad (59.6)$$

This map is obviously not well defined for the entire range of TM , as the flow of the connection may not be complete in general[IS IT?]. Furthermore, this map is not necessarily a diffeomorphism

Curves from the flow of the connection are first order : one curve flows per point (0-connection)

How does it relate to the connection being vertical or horizontal (all curves are non-fiber-wise I think)

There is one curve for every direction for every point of the tangent bundle mapping to the same point

diffeomorphism v. flow curve

The exponential map of an affine connection (works only for an affine connection because it relies on the scaling of vectors, projective connection will be weirder idk)

Existence of the exponential map for affine connections

Proof of existence via the geodesic equation [175]

Exp map for the integration of a Lie algebroid

The exponential map has the important property of preserving the local properties of the model space up to first order.

Derivatives of the exponential map

$$\frac{d}{dt}(\exp_p(tX)) \quad (59.7)$$

59.2 The cut locus

The boundary of the region for which the exponential map is a diffeomorphism is called the *cut locus*

Definition 59.1. The *injectivity radius* of the connection at a point p , $\text{inj}(p)$, is the supremum of geodesic length starting at p in the domain of the exponential map.

The injectivity radius tells us in some sense the distance from a given point where the manifold can be considered to be roughly Euclidian

Lorentzian injectivity radius

Derivatives of the exponential map

59.3 Riemann normal coordinates

The exponential map defines a natural set of coordinates. Those coordinates can either be given by mapping each point of the Riemann normal neighbourhood to its original vector at the origin, giving us some map to $\mathbb{R}^n$ after giving $T_p M$ a frame, or by taking the

vector field defined by the tangent vector of the geodesics at each point and obtaining a coordinate system by applying the flow box theorem.

Those types of coordinates are called *Riemann normal coordinates*, and they are in some sense what is generally meant when it is said that general relativity is locally Euclidian.

If we consider our semi-Riemannian geometry as a G -structure, the Riemann normal coordinates also correspond to the integration of the G -structure up to first order.

$$g = \tag{59.8}$$

Gauss' lemma

Theorem 59.2. Given a ball $B_\varepsilon(0)$ in the tangent space $T_p M$, of a radius ε small enough to fit within the domain of the exponential map, for $v, w \in B_{0,\varepsilon}$ and $q = \exp_p(v)$, then

$$\langle d \exp_p(v), d \exp_p(w) \rangle_q = \langle v, w \rangle_p \tag{59.9}$$

Define the derivatives more properly

$$d \exp_p(v) = \frac{d}{dt}(\exp_p(vt))|_{t=1} \tag{59.10}$$

idk

59.4 The world function

The metric structure of spacetime gives us the notion of the length of a curve via the length functional

$$L[\gamma] = \int_I d\mu[\gamma^* g] \tag{59.11}$$

$$= \int_{\lambda_1}^{\lambda_2} \sqrt{g(\dot{X}, \dot{X})} d\lambda \tag{59.12}$$

$$= \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu}(X(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{59.13}$$

$$\tag{59.14}$$

For any two points in the manifold, there exists curves connecting them, as spacetimes are path-connected, but there is no obvious preferred curve connecting them. However, if

we consider a totally geodesic neighbourhood U , for any two points $p, q \in U$, there exists a unique geodesic entirely in U connecting them. We can therefore define a map in this neighbourhood to give us the Lorentzian distance between any two points.

[Show that the exponential map defines the notion of distance for the length functional for geodesics]

Definition 59.3. The *world function* σ is a bitensor giving

$$\sigma_\gamma(p, q) = \int_{\gamma^{-1}} d\mu[\gamma^*g] \quad (59.15)$$

$$\sigma_\gamma(p, q) = \int_a^b \sqrt{g_{\mu\nu}(x(\lambda))\dot{x}(\lambda)\dot{x}(\lambda)} d\lambda \quad (59.16)$$

$$\sigma_\gamma(p, q) = \int_{\lambda_1}^{\lambda_2} \|\dot{\gamma}\| d\lambda \quad (59.17)$$

$$\Omega_\gamma(p, q) = \frac{1}{2}(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} g(\dot{\gamma}, \dot{\gamma}) d\lambda \quad (59.18)$$

In terms of the exponential map

$$\Omega(p, q) = \frac{1}{2}g(\exp_p^{-1}(q), \exp_p^{-1}(q)) \quad (59.19)$$

$$= \frac{1}{2}(1 - 0)g(u, u) \quad (59.20)$$

$$= \frac{1}{2}(1 - 0)g(u, u) \quad (59.21)$$

$$= \frac{1}{2} \int_0^1 g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda)) d\lambda \quad (59.22)$$

Derivatives of the world function

$$\nabla^1 \sigma(p, q) = \quad (59.23)$$

$$\nabla^2 \sigma(p, q) = \quad (59.24)$$

Derivative with respect to a curve :

$$\frac{d}{d\lambda} \sigma(\gamma(\lambda), p) = \quad (59.25)$$

59.5 The geodetic interval

The geodetic interval, also called the world function, assigns a positive semi-definite length to geodesics connecting two points. That is, for a geodesic γ connecting p and q , consider the quantity s_γ defined by

- γ timelike :

$$s_\gamma(p, q) = -l_\gamma$$

- γ null :

$$s_\gamma(p, q) = 0$$

- γ spacelike :

$$s_\gamma(p, q) = l_\gamma$$

The geodetic interval is then defined by

$$\begin{aligned} \sigma_\gamma(p, q) &= \pm \frac{1}{2} [s_\gamma(p, q)]^2 \\ &= \frac{1}{2} (\lambda_q - \lambda_p) \int_{\lambda_p}^{\lambda_q} g(u(\lambda), u(\lambda)) d\lambda \end{aligned} \quad (59.26)$$

If geodesics between p and q are unique, it can be considered as a bitensor field of rank $(0, 0)$.

Value independent of the parametrization

$g(u(\lambda), u(\lambda))$ is constant along γ so we can write

$$\sigma_\gamma(p, q) = \frac{1}{2} (\lambda_q - \lambda_p)^2 g(\gamma'(\lambda), \gamma'(\lambda)) \quad (59.27)$$

If parametrized so that $\gamma(0) = p$, $\gamma(1) = q$,

$$\Omega_\gamma(p, q) = \frac{1}{2} g(\gamma'(\lambda), \gamma'(\lambda)) \quad (59.28)$$

$$\Omega_\gamma(p, q) = \frac{1}{2} \varepsilon \left(\int_p^q ds \right)^2 \quad (59.29)$$

$\varepsilon = \pm 1, 0$ if the curve is spacelike, timelike or null.

For γ a geodesic, we define the geodetic distance as the world function on γ normalized, that is, for a timelike geodesic,

$$s_\gamma(p, q) = \int_p^q d\tau \quad (59.30)$$

for a spacelike geodesic

$$s_\gamma(p, q) = \int_p^q ds \quad (59.31)$$

and of course, for a null curve, geodesic or not, we always have $s_\gamma(p, q) = 0$.

Unlike Riemannian manifolds, the minimal

If the geodesic isn't timelike,

$$\nabla_\mu^x \sigma_\gamma(x, y) = \pm s_\gamma(x, y) \nabla_\mu^x s_\gamma(x, y) \quad (59.32)$$

We define the unit tangent vector of our geodesic from y to x by

$$t^\mu(x, \gamma, x \leftarrow y) = \pm g^{\mu\nu} \nabla_\nu^x s_\gamma(x, y) \quad (59.33)$$

$$\nabla_\mu^x \sigma_\gamma(x, y) = s_\gamma(x, y) t^\mu(x, \gamma, x \leftarrow y) \quad (59.34)$$

If the geodesic is null, $s_\gamma(x, y) = 0$, so that if we use the previous expression, it will be 0. The covariant derivative in this case will be proportional to some null vector k

$$\nabla_\mu^x \sigma_\gamma(x, y) = \zeta_\gamma(x, y) k_\mu(x, \gamma, x \leftarrow y) \quad (59.35)$$

We then have to introduce a canonical observer by a timelike vector ξ^μ at x that we will parallel transport

$$(g^{\mu\nu} \nabla_\mu \sigma_\gamma) \nabla_\nu \xi^\rho = 0 \quad (59.36)$$

Or, in other words,

$$l^\mu(x, \gamma, x \leftarrow y) \nabla_\mu \xi^\rho = 0 \quad (59.37)$$

If we define it to be of the same time orientation as l ,

$$l^\mu(x, \gamma, x \leftarrow y) \xi_\mu = -1 \quad (59.38)$$

$$\zeta_\gamma(x, y) = -V^\mu \nabla_\mu^x \sigma_\gamma(x, y) \quad (59.39)$$

ζ_γ : distance along the null geodesic as measured by the canonical observer.

We also define $m = 2\xi - l$

$$\begin{aligned} m_\mu \xi^\mu &= -1 \\ l_\mu m^\mu &= -2 \end{aligned}$$

$$\nabla_\mu^x \zeta_\gamma(x, y) = -\frac{1}{2} l_\mu(x, \gamma, x \leftarrow y) \quad (59.40)$$

Ex from Visser : prove ζ is a parametrization of γ , show $l\nabla l = 0$, hint : $l\nabla l = fl$, $l\nabla \xi = 0$, $l\xi = -1$, compute f , show $l\nabla m = 0$ show $l\nabla \zeta = 0$, $m\nabla \zeta = 0$

59.5.1 The van Vleck determinant

In the case where geodesics between two points are unique, such as in a totally geodesic neighbourhood U , there are two ways to specify a geodesic : by either specifying an initial point p and tangent vector u , or by specifying two points p and q . This is an application of the path structure we saw in 49.3. We can therefore specify them as a point in $U \times U$ or TU .

In terms of coordinates

Some equivalence between some bitensors $B : U \times U \rightarrow T_s^r U \boxtimes T_{s'}^{r'} U$ and functions on TU or something

The Jacobian associated with this change of variables is the *van Vleck determinant*, $\Delta(p, q)$.

[176]

[Rewrite in terms of path structure]

$$\Delta_\gamma(x, y) = (-1)^n \frac{\det(\nabla_\mu^x \nabla_\nu^y \sigma_\gamma(x, y))}{\sqrt{g(x)g(y)}} \quad (59.41)$$

Motivation : In the case where geodesics between two points are unique, there

$$J_{\gamma}(x, y) = \frac{1}{\sqrt{g(x)g(y)}} \det\left(\frac{\partial(x, u)}{\partial(x, y)}\right) \quad (59.42)$$

$$= \frac{1}{\sqrt{g(x)g(y)}} \det\left(\frac{\partial u}{\partial y}\right) \quad (59.43)$$

Chapter 60

Deviations and congruences

A common scenario in relativity is to compare the deviation of two curves. This is for instance the most standard case of all, that of free fall, where we compare the motion of the Earth and that of an object in freefall, or the case of the collapse of a gas cloud in an astrophysical context.

We need for this to be able to compare the motion of multiple curves. In the simplest case where we simply compare the motion of two curves, we will consider a family of curves

Definition 60.1. A *family of curves* $\Gamma : \mathbb{R} \times \mathbb{R} \rightarrow M$, also called a *doubly-ruled surface*, is a continuous set of curves

$$\Gamma(s, \lambda) = \gamma_s(\lambda) \tag{60.1}$$

For the case of two curves, they will correspond to two values of the parameter s , with the intermediary values useful for us to make some kind of analytical argument from it.

Families of curves defined in this sense are simply a special case of 2-dimensional submanifolds, with the double-ruling corresponding to some parametrization of the surface. However we won't require that this double-ruling be bijective, as many of the interesting cases for families of curve occur in cases where the family of curves collapse to a point.

As we saw in the definition of the double tangent bundle, families of curves naturally lift to families of curves in the tangent bundle via the derivative with respect to its first parameter :

$$\partial_s \Gamma : \mathbb{R} \times \mathbb{R} \rightarrow TM \tag{60.2}$$

and their second derivatives lift them furthermore to TTM , and J^2M

Families of geodesics

Definition 60.2. For a smooth family of curves Γ , its *Jacobi vector field* is a section of the pullback bundle Γ^*TM

$$J = \partial_s \Gamma(s, \lambda) \Big|_{s=0} \quad (60.3)$$

In terms of coordinates,

Theorem 60.3. A Jacobi vector field obeys the relation

$$D_\lambda^2 J + \iota_{\dot{\gamma}} \text{Riem}(J, \dot{\gamma}) = 0 \quad (60.4)$$

$$= 0 \quad (60.5)$$

Proof.

□

Tidal tensor

Geodesic deviation equation

A family of curves can be augmented to have any number of parameters,

Raychaudhuri equation

The effects of gravity from the curvature of spacetime can be best understood by looking at the effect it has on the motion of geodesics and curves, as this will be kind of effect we can most readily observe physically.

60.1 Geodesic deviation

Geodesic deviation considers the separation of a family of geodesics. This will be helpful to understand tidal forces in general relativity. Two or more points initially separated by a distance evolve on the spacetime and we analyse the evolution of their distance as time goes on.

To represent this situation, we consider a one-parameter family of geodesic curves, $\gamma_s(\lambda)$. A choice of n value of s will give us the geodesic representing the trajectories to look at.

As usual, the tangent vector is

$$U^\mu = \frac{d}{d\lambda} x_s^\mu(\lambda) \quad (60.6)$$

But we will also be interested by the deviation vector :

$$S^\mu = \frac{d}{ds} x_s^\mu(\lambda) \quad (60.7)$$

Relative velocity of geodesics :

$$V^\mu = U^\rho \nabla_\rho S^\mu \quad (60.8)$$

Relative acceleration :

$$A^\mu = U^\rho \nabla_\rho V^\mu \quad (60.9)$$

$$A^\mu = R^\mu_{\nu\sigma\tau} T^\nu T^\rho S^\sigma \quad (60.10)$$

60.2 The Raychaudhuri equation

A more general method to consider tidal forces is using a family of curves going through every point of space. If all the curves are timelike for instance, this could be considered like a collection of particles in a fluid.

Open subset $U \subset \mathcal{M}$, a is a family of curves such that for every point $p \in U$, there is one curve in the congruence passing through p . Tangents of a congruence defines a vector field in U and the integral curves of a vector field in U defines a congruence. Smooth congruence if the vector field is smooth.

Timelike congruence of geodesics, parametrized by proper time τ so that the tangents ξ^μ are normalized $\xi^\mu \xi_\mu = -1$. The tensor field $B_{\mu\nu}$

$$B = \nabla \xi^\flat \quad (60.11)$$

$$B_{\mu\nu} = \nabla_\nu \xi_\mu \quad (60.12)$$

is "spatial", ie

$$B(\xi, -) = 0 \quad (60.13)$$

$$B_{\mu\nu} \xi^\mu = 0 \quad (60.14)$$

For a one-parameter subfamily $\gamma_s(\lambda)$ of the geodesic congruence, with S^μ the deviation vector, then

$$\mathcal{L}_\xi S^\mu = 0 \quad (60.15)$$

Hence,

$$\xi^\mu \nabla_\mu S^\nu = S^\mu \nabla_\mu \xi^\nu = B_\mu^\nu S^\mu \quad (60.16)$$

B is the failure of S to be parallelly transported. It measures the spread, twist and compression of the congruence of geodesics along their path.

Projection of the metric

$$h = g + \xi \otimes \xi \quad (60.17)$$

$$h_{\mu\nu} = g_{\mu\nu} + \xi_\mu \xi_\nu \quad (60.18)$$

Expansion :

$$\theta = \langle B, h \rangle \quad (60.19)$$

$$\theta = B^{\mu\nu} h_{\mu\nu} \quad (60.20)$$

Shear :

$$\sigma = S(B) - \frac{1}{3} \theta h \quad (60.21)$$

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3} \theta h_{\mu\nu} \quad (60.22)$$

Twist :

$$\omega_{\mu\nu} = B_{[\mu\nu]} \quad (60.23)$$

$$B_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \quad (60.24)$$

By Frobenius theorem and B being purely spatial, the congruence is hypersurface orthogonal iff $\omega_{\mu\nu} = 0$

Chapter 61

The causal relations

A Lorentzian metric naturally define a causal structure on the manifold, corresponding to the regions that a causal curve can access, or physically, the spacetime events that can influence each others. This will be expressed by the causal relations between each points of the manifold.

For most of this chapter, the spacetimes will be assumed to be time-orientable, as most of the important concepts require the existence of a well-defined past and future direction for causal curves. If the spacetime is not time-orientable, some of these results can be salvaged by going to the time-orientable double cover.

61.1 Definition

Once we have a manifold equipped with a metric and a time orientation, it becomes possible to define some ordering relations on the manifold.

The fundamental relations on a spacetime are the chronological, strictly causal, causal and horismos relations.

- p chronologically precedes q if there is a future-directed timelike curve connecting p to q , noted $p \ll q$.
- p strictly causally precedes q if there is a future-directed causal curve connecting p to q , $p \neq q$, noted $p < q$.
- p causally precedes q if there is a future-directed causal curve connecting p to q , with possibly $p = q$, noted $p \leq q$.
- p horismos q if $p \leq q$ and $p \not\ll q$, noted $p \nearrow q$.

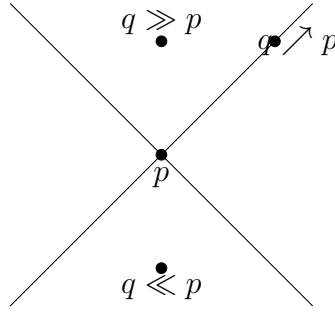


Figure 61.1: Some points on the causal structure of Minkowski space

Those causal relations will form the basis of the causal structure. We can also define their negation in the following way

Definition 61.1. A point p is *causally independent*, noted $p \parallel q$, if there is no causal curve connecting p and q .

Definition 61.2. A point p is *chronologically independent*, noted $p | q$, if there is no timelike curve connecting p and q .

As with relations in general, those can also be defined on subsets of M . If a

[177]

Unlike the special relativistic case, general relativistic theories can have rather pathological causal structures, which is exemplified in the notion of a vicious point

Definition 61.3. A point p is *vicious* if $p \leq p$. A point that isn't vicious is said to be a *virtuous point*.

A point being vicious implies the existence of a *closed causal curve*, which is a continuous mapping of S^1 to the manifold that is everywhere causal. If it is additionally everywhere timelike, we say that it is a *closed timelike curve*.

As the degenerate curve of length 0 is not timelike, and neither is the constant curve $\forall \lambda, \gamma(\lambda) = p$, $p \ll p$ and $p < p$ implies the existence of a second point on this curve, which implies that there exists a point $q = \gamma(\lambda_q)$, $\lambda_q \neq \lambda_p$, such that $p \ll q$ and $q \ll p$. The same applies to strict causality. This will be of use for further proofs.

61.2 Properties of causal relations

To prove a lot of properties of causal relations, it will be practical to approximate them by a series of piecewise geodesic curves, in the following way :

Definition 61.4. A trip from p to q is a curve piecewise composed of future-oriented geodesics such that its past endpoint is p and future endpoint is q . There is a series of

points (x_i) , $i = 1, \dots, n$, such that $x_0 = p$ and $x_n = q$, and such that x_i and x_{i+1} are linked by a timelike geodesic. The tangent vectors at endpoints are required to have the same time orientation : $g(u^+(x_i), - (x_i)) < 0$

We can then prove the following [41] :

Theorem 61.5. $p \ll q$ is equivalent to the existence of a trip going from p to q .

Proof. Cover the timelike geodesic by a finite number of convex normal neighbourhood N_i (this is possible due to the curve between two points being compact and hence having a finite subcover). The first point of the trip is $x_0 = p$, in the convex normal neighbourhood N_{i_0} . x_1 is the future endpoint of $\gamma \cap \bar{N}_{i_0}$. Since it is a convex normal neighbourhood, there exists a future-oriented timelike geodesic connecting x_0 and x_1 . If $x_1 = q$, it is proven. Otherwise, $x_1 \in N_{i_1}$, and the argument repeats.

Converse : Let α be the trip from p to q . μ and λ are two consecutive geodesic segments of α , with r the future endpoint of λ and the past endpoint of μ . Consider some convex normal neighbourhood of r . For some basis in $T_r\mathcal{M}$, $\exp_r^{-1}(\mu)$ has coordinates of the form $(\tau, \tau \tan(\chi), 0, 0)$, while $\exp_r^{-1}(\lambda)$ has coordinates of the form $(-\tau, \tau \tan(\chi), 0, 0)$ ($\tau > 0$, $\chi \in [0, \pi/4)$). This corresponds to coordinates for a future-oriented and past-oriented geodesic starting at r .

Connect the two segments $(\tau_0, \tau_0 \tan(\chi), 0, 0)$ and $(-\tau_0, \tau_0 \tan(\chi), 0, 0)$, for some small enough value of τ_0 , by a smooth curve η in $T_r\mathcal{M}$ that is everywhere timelike. For instance, by switching the plane (t, x) to polar coordinates

$$F(R, \theta) = R \cos\left(\frac{\theta\pi}{\pi - 2\chi}\right) - \exp(R^2 \sin^2\left(\frac{\theta\pi}{\pi - 2\chi}\right) - 1)^{-1} = 0 \quad (61.1)$$

This is a deformation of the standard bump function.

[INTERSECTION OF SMOOTHING AND CURVES : $\theta = \pi/(2 - \chi)$???

[Something something composition of the bump function in polar coordinates + coordinate change to $\theta \rightarrow \theta \frac{\pi}{\pi - 2\chi}$, smooth transformation so smooth joint]

Tangent vector of this curve in $T(TM)$:

$$\eta' = (\partial_R F, \partial_\theta F) \quad (61.2)$$

The slope of η never reaches the null cone. For a small enough neighbourhood (small enough τ_0), $\exp_r(\eta)$ is timelike in $\mathcal{M}$

□

Theorem 61.6. If γ_1 is a null geodesic from p to q and γ_2 is a null geodesic from q to r , then either $p \ll r$ or $\gamma_1 \cup \gamma_2$ is a null geodesic from p to r .

Proof. If $\gamma_1 \cup \gamma_2$ is not a single null geodesic, this is due to a discontinuity in the tangent vector at q (a "joint"). As previously seen, we can take $\square$

Corrolary 61.1. $p \leq q$ is equivalent to the existence of a causal trip between p and q . Also, if

Equivalence of chronological/causal/strictly causal/horismos relation with piecewise geodesic curves : cf Penrose

Horismos equivalent to piecewise null geodesic proof : zig zags also "this always works because the convex normal neighborhoods form an open cover"

With the equivalence to causal trips, we can then prove the following properties :

Proposition 61.7.

1. $p \ll q$ implies $p \leq q$.
2. $p \ll q$ and $q \ll r$ implies $p \ll r$.
3. $p \leq q$ and $q \leq r$ implies $p \leq r$.
4. If $p \ll q$ and $q \leq r$, $p \ll r$.
5. If $p \leq q$ and $q \ll r$, $p \ll r$.

Proof.

1. By definition.
2. Identifying the future endpoint of the trip $p \ll q$ with the past endpoint of $q \ll r$.
3. Identifying the future endpoint of the causal trip $p < q$ with the past endpoint of $q < r$.
4. Let α be a trip from a to b , γ a causal trip from b to c . Cover the trip γ from q to r with a finite number of convex normal neighbourhoods $N_1, \dots, N_r$ (because causal trips are compact). We can assume no loops in γ because the redundant parts can be deleted. $x_0 = q \in N_{i_0}$. x_1 the future endpoint of the connected component of $\gamma \cap \bar{N}_{i_0}$. $y_1 \in N_{i_0}$ on the final segment of α , $y_1 \neq x_0$.

$\square$

Property 1 implies that the chronological order relation is a subset of the causal order relation, $\ll \subset \leq$. This is also true for the horismos order relation, $\nearrow \subset \leq$.

Proposition 61.8. If $p \leq q$, then we have either $p \ll q$, $p \nearrow q$, $p = q$, or both $p = q$ and either $p \ll q$ or $p \nearrow q$.

61.2.1 Inverse relations

From the basic causal relations we can easily define their inverse. We define the inverse relations $p \gg q$, $p \geq q$, $p \nearrow q$ to signify the existence of a past-directed timelike, causal, causal but not timelike curve connecting p to q .

Proposition 61.9.

1. If $p \ll q$, then $q \gg p$.
2. If $p \leq q$, then $q \geq p$.
3. If $p \nearrow q$, then $q \nwarrow p$.

Proof. This can easily be constructed by taking the curve γ connecting p to q and then taking the curve $\gamma(-\lambda)$. $\square$

61.3 Restricted causal relations

We can define causal relations on subsets of the manifold. Given a subset $U \subset \mathcal{M}$, we say that the restriction of $\ll, \leq, \nearrow$ to U , noted $\ll_U, \leq_U, \nearrow_U$, is defined by

Definition 61.10. We write that $p \ll_U q$, $p \leq_U q$, $p \nearrow_U q$ if there exists a future-directed timelike, causal, causal but not timelike curve connecting p and q lying entirely within U .

We can in fact write every causal relation in this form, since pRq is simply equivalent to $pR_M q$.

61.3.1 Properties

Proposition 61.11. For two subsets U, V of $\mathcal{M}$ such that $U \subset V$, we have

- $\ll_U \subset \ll_V$
- $\gg_U \subset \gg_V$
- $\leq_U \subset \leq_V$
- $\geq_U \subset \geq_V$

Proposition 61.12. $\ll_U \subset \leq_U$ and $\gg_U \subset \geq_U$ remains true in a restricted relation.

Proposition 61.13. If $S \subset U$, then $S \leq_U S$ and $S \geq_U S$

61.4 Causality and maps

For a map f between two spacetimes (M_1, g_1) and (M_2, g_2) , we say that it is

- chronal preserving if $p \ll q$ implies $f(p) \ll f(q)$
- causal preserving if $p \leq q$ implies $f(p) \leq f(q)$
- a chronal isomorphism if f is bijective and f, f^{-1} are chronal preserving
- a causal isomorphism if f is bijective and f, f^{-1} are causal preserving

It can be shown that these relations hold under a Weyl transformation.

Proposition 61.14. If we consider the sets of relations on a spacetime (M, g) , the conformal map

$$f : (M, g) \rightarrow (M, \Omega g) \quad (61.3)$$

is a causal isomorphism.

Proof. if $p \ll q$ in the spacetime (M, g) , there exists a curve connecting p and q such that $g(\gamma', \gamma') < 0$. Under a Weyl transform, this curve will have the inner product $\Omega g(\gamma', \gamma')$. As $\Omega > 0$, this curve will always be timelike. The same applies to all other relations. $\square$

Thanks to this, we can define the causal structure as follows

Definition 61.15. The *causal structure* of a spacetime (M, g) is the set of all conformally equivalent spacetimes.

$$\text{coset}(M, g) = \{(M, \tilde{g}) | \tilde{g} = \Omega g, \forall p, \Omega(p) > 0\} \quad (61.4)$$

This will allow us to study the causal structure of the spacetime without taking into consideration more complicated structures such as the distances between points.

[Conformal structure]

Chapter 62

Causal sets

From those causal relations, several causal sets can be defined for regions of spacetime.

- The chronological future of a point p is the set of all points that can be reached by a future-directed timelike curve from p . It is noted $I^+(p) = \{q \in \mathcal{M} \mid p \ll q\}$. The same definition applies for the chronological past $I^-(p)$ with past-directed curves.
- The causal future of a point p is the set of all points that can be reached by a future-directed causal curve from p . It is noted $J^+(p) = \{q \in \mathcal{M} \mid p \leq q\}$. The same definition applies for the causal past $J^-(p)$ with past-directed curves. Unlike the chronological future and past, $\forall p, p \in J^\pm(p)$
- The future and past horismos of a point p is the subtraction of the chronological past/future from the causal past/future, the set of points that can only be reached by a null curve. It can be defined as $E^+(p) = \{q \in \mathcal{M} \mid p \nearrow q\}$, or $E^+(p) = J^+(p) \setminus I^+(p)$, with similar definitions for $E^-(p)$

We will here mostly only talk about I^+ , J^+ and E^+ , the generalization to I^- , J^- and E^- being trivial in most cases.

Those causal sets can be generalized to be defined on a set of spacetime points by

$$I^+(S) = \bigcup_{p \in S} I^+(p) \quad (62.1)$$

and identically for $J^+(S)$ and $E^+(S)$. This leads to the following useful identity by commutativity of the union

$$I^\pm(A \cup B) = I^\pm(A) \cup I^\pm(B) \quad (62.2)$$

The intersection of $I^+(p)$ and $I^-(q)$, also called the causal diamond of p and q , is useful enough to merit its own notation. It will be noted $\langle p, q \rangle$. The intersection of $J^+(p)$ and $J^-(q)$ will be noted as $\langle\langle p, q \rangle\rangle$.

Definition 62.1. The *chronology violating set* (resp. *causality violating region*) V is the set of all points $p \in \mathcal{M}$ such that $p \in I(p)$ (resp. there exists a non-trivial causal curve from p to itself).

If we replace the causal relations in the definition by their restriction to an open set U , we have their equivalent, noted

$$\begin{aligned} I^+(S, U) &= \bigcup_{p \in S} \{q \in U \mid p \underset{U}{\ll} q\} \\ J^+(S, U) &= \bigcup_{p \in S} \{q \in U \mid p \underset{U}{\leq} q\} \\ E^+(S, U) &= \bigcup_{p \in S} \{q \in U \mid p \underset{U}{\nearrow} q\} \end{aligned} \tag{62.3}$$

As with restricted causal relations, we have immediatly that $I^+(S, M) = I^+(S)$, and identically for J and E .

Some immediate consequences of those definitions and the properties of the causal relations are

Proposition 62.2.

- $I^+(S, U) \subset J^+(S, U)$
- $I^+(I^+(S, U), U) = I^+(S, U)$
- $J^+(J^+(S, U), U) = J^+(S, U)$
- $J^+(I^+(S, U), U) = I^+(J^+(S, U), U) = I^+(S, U)$

Proposition 62.3. The restricted causal sets on a geodesically convex set U are the intersection of those sets with U .

Proof. NOT SURE IF TRUE □

62.0.1 Past and future sets

Past and future sets P, F are sets that will contain all their own past and future points, that is, if $p \in P$, all points $q \ll p$ belong to P , and similarly with $p \ll q$ with F . This is defined by

Definition 62.4. A *future set* F obeys the property $I^+(F) = F$. Similarly, a *past set* P obeys $I^-(P) = P$.

62.0.2 Common past and future

Definition 62.5. The *chronological common past* and *future* of a set $U \subset M$ is the set of points

$$\begin{aligned}\downarrow U &= I^-(\{p \in M \mid \forall q \in U, p \ll q\}) \\ \uparrow U &= I^+(\{p \in M \mid \forall q \in U, p \gg q\})\end{aligned}\tag{62.4}$$

This means that first we consider the set of points

Set of all past and future sets : $\mathcal{P}, \mathcal{F}$

Definition 62.6. A *hull pair* is an element $(P, F) \in \mathcal{P} \times \mathcal{F}$ such that $P = \downarrow F$ and $F = \uparrow P$.

62.1 Minkowski space

As a lot of proofs will rely on the causal behaviour of objects in the tangent space $T_p\mathcal{M}$, it will be interesting to first study the causal structure of Minkowski space.

We consider Minkowski space as the spacetime $(\mathbb{R}^n, \eta)$, with the time orientation $\tau = (1, \vec{0})$.

In Minkowski space, the geodesic equation being simply $\ddot{x}(\lambda) = 0$, a geodesic between the points a and b can be written as

$$\vec{ab} = a + (b - a)\lambda\tag{62.5}$$

for $\lambda \in [0, 1]$. That geodesic is timelike/null if $|\vec{ab}| > 0, = 0$. Those are future/past oriented if $(b_t - a_t) > 0$.

Theorem 62.7. If $a \ll b$, then $b \not\ll a$, and if $a \leq b$, $b \not\leq a$.

Lemma 62.1. The metric tensor η has range $\mathbb{R}$.

Proof. Consider the timelike vector $(t, 0, 0, \dots)$ and the spacelike vector $(0, x, 0, \dots)$. Their norm will be respectively $-t^2$ and x^2 . Along with any null vector, this will span all of $\mathbb{R}$. $\square$

Theorem 62.8. $I^+(p)$ is open in Minkowski space, while $J^+(p)$ is closed.

Proof. $I^+(p)$ is the pre-image of the open interval $(-\infty, 0)$ with g , while $J^+(p)$ is the pre-image of the closed interval $(-\infty, 0]$. g being continuous, those pre-images are respectively open and closed. $\square$

62.2 Properties in a convex neighbourhood

The causal sets possess a lot of interesting properties in convex neighbourhood, which as we will happen to be the causal properties of Minkowski space. For this section, we will consider a point p in $\mathcal{M}$, with a convex neighbourhood U around p , and a point q such that $q \in U$ and $p \neq q$.

Theorem 62.9. $q \in I^+(p, U)$ (resp. $J^+(p, U)$, $E^+(p, U)$) is equivalent to the geodesic between p and q being timelike (resp causal, null) and future pointing.

Proof. Since $q \in I^+(p, U)$, there exists a timelike curve γ connecting p with q . If we consider the pullback of that curve in the tangent space $T_p\mathcal{M}$ via the exponential map $\exp^{-1}(\gamma)$, since it is a timelike curve by theorem X, it will lie entirely within the future light cone. Since D_p is convex, it is possible to define a segment from $\exp_p^{-1}(p)$ to $\exp_p^{-1}(q)$, lying in the future light cone. The image of this segment is a future pointing timelike geodesic. $\square$

Theorem 62.10. The image of $I^+(0) \cap D_p$ (resp. $J^+(0) \cap D_p$, $E^+(0) \cap D_p$) in the tangent space at p by the exponential map is $I^+(p, U)$, resp. $J^+(p, U)$, $E^+(p, U)$.

Proof. $\square$

Theorem 62.11. $I^+(p, U)$ is open in U and M , while $J^+(p, U)$ is closed.

Proof. Since $I^+(0)$ is open in Minkowski space and the domain of the exponential map is itself open, $I^+(0) \cap \exp_p^{-1}(U)$ is open. $\exp_p$ being a homeomorphism, the image of that set will be open, which is $I^+(p, U)$. $\square$

Theorem 62.12. $J^+(p, U)$ is the closure in U of $I^+(p, U)$

Proof. Since $\exp_p$ is a homeomorphism over the convex normal neighbourhood, $J^+(p, U) = \exp_p(J^+(0)) = \exp_p(\bar{I}^+(0)) = \overline{\exp_p(I^+(0))} = \bar{I}^+(p)$ $\square$

Theorem 62.13. Causal curves in a compact subset of U have two endpoints.

Proof. $\square$

t

62.3 Global properties

Proposition 62.14. The causal and chronological future of any set obeys the following properties

1. $\text{int} J^+(S) = I^+(S)$.
2. $J^+(S) \subset \overline{I^+(S)}$, and $J^+(S) = \overline{I^+(S)}$ if $J^+(S)$ is closed.
3. $\bar{J}^+ = \bar{I}^+$.

Proof.

1. Since $I^+(S)$ is open and $I^+ \subset J^+$, $I^+(S) \subset \text{int}(J^+(S))$ follow. If $p \in \text{int}(J^+(S))$, since $\text{int}(J^+(S))$ is open, there exists a $q \in I^-(p) \cap J^+(S)$. So $p \in I^+(q) \subset I^+(J^+(S)) = I^+(S)$. Since this is true for every point p , $\text{int}(J^+(S)) \subset I^+(S)$, and the equality is verified.
2. For any $q \in J^+(S)$, there will be a causal curve γ from $p \in S$ to q ...
3. $I^+ \subset J^+$, so $\bar{I}^+ \subset \bar{J}^+$. [USE PROP 1]

□

Let $(p, q) \in \bar{J}^+$, U and V neighbourhoods of p and q .

U, V open subsets of $\mathcal{M}$, $V \subset U$. V is causally convex in U if any causal curve in U with endpoints in V is entirely contained in V

if $U = \mathcal{M}$, V is called causally convex

$\leq_U$ is the causal relation when U is treated as a spacetime. If V is causally convex in U , then $\leq_U$ restricted to V is identical to $\leq_V$

Converse is false : example : $U = \mathcal{L}^2$, $V = \{(t, x) \in \mathbb{R} \mid |t|, |x| < 1\}$

Totally vicious spacetimes are such that only $V = \mathcal{M}$ is causally convex in $\mathcal{M}$

If $V \subset W \subset U$, V is causally convex in W

Proposition 62.15. $I^+(p)$ is open.

Proof. If $q \in I^+(p)$, then there is a trip α from p to q . For some convex neighbourhood V of q , let's pick a point q^- in between p and q , and p^+ in a convex neighbourhood U of p such that p^+ is between p and q . □

Proposition 62.16. The chronological future of every point forms a cover of the manifold : $\bigcup_{p \in \mathcal{M}} I^+(p) = \mathcal{M}$.

Proof. For any point $p \in \mathcal{M}$, we have a convex normal neighbourhood U_p . Since D_p is an open set around 0 in $T_p \mathcal{M}$, we have a past-directed geodesic to some point q in $I^-(p, U_p)$. The point p itself will be in $I^+(q, U_q) \subset I^+(q)$. Hence every point is in the chronological future of some point of the manifold. □

Proposition 62.17. $J^+(p)$ is not necessarily closed.

Proof. The standard counterexample for this is Minkowski space with a point removed. $\square$

Proposition 62.18. $J^+(p)$ is closed for all p if and only if the spacetime contains no naked singularities (???)

Proof. If the spacetime contains a naked singularity, there exists a point q and a future-directed timelike or null incomplete geodesic $\gamma \subset I^-(p)$ converging to p in the b -boundary of the spacetime. Since for all $p_i \in \gamma$, $p_i \leq p$ and $p_i \leq q$, $p \leq q$, so that $p \in E^-$

If $J^+(p)$ is closed, any sequence of points $\{r_i\}_{i \in \mathbb{N}}$ such that for all i , $r_i \in J^+(p)$.

Consider a point $q \in J^+(p)$. If there is a naked singularity, there will be a pair of points p, q such that there is a

Pick a point to the future of all q_n ??? No causal incomplete curve - All sequences converge??? $\square$

Proposition 62.19. For $i = \{1, 2, \dots\}$, the chronology violating set (resp. causal) corresponds to connected components of the form $V_i = I^+(p_i) \cap I^-(p_i)$ (resp. $J^+(p_i) \cap J^-(p_i)$). (cf Kriele)

Proof. For a connected component $V_i \subset V$, since V_i is connected, for every $p, q \in V_i$ there exists a continuous path $\gamma \subset V_i$ connecting them. If we consider for all $z \in \gamma$, $I^+(z)$ is a neighbourhood of z , since $z \in I^+(z)$ and $I^+(z)$ open. Then since γ is compact, we can cover it by $I^+(z_j)$, for a finite collection $z_j \in \gamma$, meaning that there is a future-directed timelike curve from p to q , and the same argument applies with $I^-(p)$, so that $V_i = I^+(p) \cap I^-(p)$. $\square$

62.4 Causal ordering

[177]

Definition 62.20. $g_1 < g_2$: the causal cone of g_1 is in the timelike cone of g_2

n -degree causal relations :

$$\begin{aligned} \overset{1}{<} \mathcal{S} &= \dots \\ (\mathcal{S} \overset{1}{>} &= \dots \end{aligned} \tag{62.6}$$

theorem : $p \in \mathcal{M}$, neighbourhood $U \ni p$. $\exists V \ni p, V \subset U$ such that there are two metrics on V , g^+ and g^- , such that $g^- < g < g^+$

62.5 Causally convex sets

Definition 62.21. An open set U is causally convex if and only if every causal curve γ that intersects it is such that $\gamma \cap U$ is connected.

$\mathcal{M}$ and $\emptyset$ are causally convex sets.

Causal diamond $\langle p, q \rangle$ is causally convex.

Definition 62.22. A point $p \in \mathcal{M}$ is strongly causal if for any neighbourhood $U \ni p$, there is a causally convex subset $V \subset U$.

62.6 Achronal sets

If we wish to partition our spacetime into synchronized surfaces in a meaningful way, foliating it into spacelike hypersurfaces may not be enough.

Definition 62.23. A subset S of $\mathcal{M}$ is achronal if for every point $p, q \in S$, there is no points such that $q \in I^+(p)$, or $I^+(S) \cap S = \emptyset$

Definition 62.24. The edge of an achronal set S is the set of all the points $p \in \bar{S}$ such that every neighbourhood U of p contains a timelike curve from $I^-(p, U)$ to $I^+(p, U)$ that does not intersect S . [WARNING DIFFERENT DEFINITION IN PENROSE AND O'NEILL, CHECK IT]

Definition 62.25. For a closed, achronal set S , its past (resp. future) domain of dependence $D^+(S)$ (resp. $D^-(S)$) is the set of points p such that every future (resp. past) inextendible causal curve going through p intersects S . The total domain of dependence $D(S)$ of dependence is the union of the two.

Definition 62.26. The future (resp. past) Cauchy horizon $H^+(S)$ (resp. $H^-(S)$) of an achronal closed set S is the set of points $p \in D^+(S)$ (resp. $D^-(S)$) such that the chronological future (resp. past) does not intersect with $D^+(S)$ (resp. $D^-(S)$). The total Cauchy horizon $H(S)$ is the union of the two.

$$H^+(S) = \{p | p \in D^+(S), I^+(p) \cap D^+(S) = \emptyset\} \quad (62.7)$$

$$H^-(S) = \{p | p \in D^-(S), I^-(p) \cap D^-(S) = \emptyset\} \quad (62.8)$$

$$H(S) = H^+(S) \cup H^-(S) \quad (62.9)$$

$$H^\pm(S) = D^\pm \setminus I^\mp[D^\pm(S)] \quad (62.10)$$

Properties :

for $S \subset \mathcal{M}$ a closed achronal set :

1. $D^+(S)$ is closed
2. $H^+(S)$ is achronal and closed
3. $S \subset D^+(S)$
4. $p \in D^+(S)$ implies $I^-(p) \cap J^+[S] \subset D^+(S)$
5. $\partial D^+(S) = H^+(S) \cup S$
6. $\partial D(S) = H(S)$
7. $I^+[H^+(S)] = I^+(S) \setminus D^+(S)$
8. $\text{int} D^+(S) = I^+[S] \cap I^-[D^+(S)]$

(proof as exercise in Penrose, gotta prove it)

Definition 62.27. A future Cauchy horizon is compactly generated if its null generators remain in a compact region K for $\lambda < \lambda_0$

Definition 62.28. The base set $\mathcal{B}$ of a compactly generated future Cauchy horizon is the set of all points $p \in H^+(S)$ such that there is a null generator γ such that p is a past terminal accumulation point of γ .

Either a closed timelike curve going through $p \in \mathcal{B}$

Theorem 62.29. For a compactly generated Cauchy horizon $H^+(S)$, the *base set* $\mathcal{B} \subset K$ is always non-empty. All null generators of $H^+(S)$ approach $\mathcal{B}$ asymptotically : for every past-directed generator and every open neighbourhood U of $\mathcal{B}$, there exists $\lambda_0 \in I$ such that for every $\lambda > \lambda_0$, $\gamma(\lambda) \in U$ [178].

Proof. Consider a past directed null geodesic γ in $H^+(S)$ and a monotone increasing sequence $\{\lambda_i\}$ without limit in I . There are infinitely many points $\gamma(\lambda_i)$ in K [show it]. Then there must be an accumulation point p of $\{\gamma(\lambda_i)\}$ that is within $H^+(S)$, since it is a closed subset of $\mathcal{M}$. Then $p \in \mathcal{B}$, and all of those accumulation points are in K by compactness (by contradicton something something cf Kay). $\square$

Theorem 62.30. $\mathcal{B}$ is composed of null geodesic generators entirely contained within $\mathcal{B}$ and are past and future inextendible.

Proof. Riemannian metric h on $\mathcal{M}$, parametrize all curves on $\mathcal{M}$ by arc length s with respect to this metric. For $p \in \mathcal{B}$, there is a null generator of $H^+(S)$ such that p is a past accumulation point. Parametrize γ so that s is increasing in the past direction. Since γ is past inextendible and K compact, s extends to infinite values even if λ doesn't. So there is a sequence $\{s_i\}$ diverging to infinity such that $\gamma(s_i)$ converges to p . $(k_i)^\alpha$ the tangent of γ at s_i in the arc length parametrization, with unit norm wrt h . Since the subset of TM such that $\pi(TM) = K$ and with vectors of unit norm is compact, there must exists a tangent vector k^α at p such that $\{(\gamma(s_i), (k_i)^\alpha)\}$ converges to (p, k^α) . Since each $(k_i)^\alpha$ is null, by continuity so is k^α .

For γ a maximally extended null curve with initial condition (p, k^α) , parametrized by arc length, $\gamma(0) = p$. For $q \in \gamma$ and $\gamma(s) = q$. Since $\{s_i\}$ diverges, for large i $(s + s_i)$ will be in the interval of γ . Since $\{(\gamma(s_i), (k_i)^\alpha)\}$ converges to (p, k^α) , by continuity of $\exp$ and s (wrt g and h), $\{\gamma(s + s_i)\}$ converges to q . Thus $q \in \mathcal{B}$ $\square$

Closed null geodesics of $\mathcal{B}$: fountains

There are cases where the null geodesics aren't closed (non-strongly causal spacetimes)

Theorem 62.31. Strong causality is violated at every $p \in \mathcal{B}$.

Theorem 62.32. $H^+(S)$, $\mathcal{B}$ its base set. For $p \in \mathcal{B}$ and U a globally hyperbolic neighbourhood of p , there exists $q, r \in U \cap \text{int} D^+(S)$ such that q and r are connected by a null geodesic, but not a causal curve in U .

62.7 Cauchy hypersurface

Cauchy hypersurfaces will be the model on which the notion of a specific moment in time is built for a spacetime.

Definition 62.33. A *Cauchy hypersurface* is a subset $S \subset \mathcal{M}$ that intersects every inextendible timelike curves exactly once.

Theorem 62.34. A Cauchy hypersurface S is a closed achronal hypersurface.

Proof. Since every inextendible timelike curve intersects S and every point in $\mathcal{M}$ contains a timelike curve, we have that $M = I^+(S) \cup S \cup I^-(S)$. $\square$

Example of a closed boundaryless achronal hypersurface that isn't a Cauchy hypersurface (Hyperbole in Minkowski space)

Theorem 62.35. A Cauchy hypersurface S intersects every inextendible causal curve.

Proof. $\square$

Theorem 62.36. A closed achronal set Σ is a Cauchy surface if and only if $D(\Sigma) = \mathcal{M}$

Theorem 62.37. A closed achronal set Σ is a Cauchy surface if and only if $H(\Sigma) = \emptyset$.

Theorem 62.38. If Σ is a closed, achronal and edgeless subset of the spacetime, then it is a Cauchy surface if and only if every inextendible null geodesic intersects Σ , enters $I^+(\Sigma)$ and $I^-(\Sigma)$.

Proof. x $\square$

Theorem 62.39. A nakedly singular spacetime does not admit a Cauchy surface (Geroch and Horowitz 1979)

Definition 62.40. A partial Cauchy surface is a closed boundaryless achronal surface such that every causal curve intersects it at most once.

As not all spacetimes admit a Cauchy surface, but may be somewhat well-behaved up to a point, we can also define the notion of a partial Cauchy surface.

Definition 62.41. A *partial Cauchy surface* is a closed achronal hypersurface that intersects every causal curve at most once.

From this definition, we can immediately see that any Cauchy hypersurface is also a partial Cauchy hypersurface.

Proposition 62.42. If a spacetime contains a partial Cauchy hypersurface, there is a submanifold containing it where it is a Cauchy hypersurface.

62.8 Causal diagrams

We will rely a lot on diagrams for specific examples of causality on given spacetimes, so that it will be useful to define a bit precisely what is going on in such diagrams.

We have already seen how such diagrams work in Minkowski space, as we have defined the synthetic rules of how Minkowski space work. Those will still be of use, but in the case of causality violating spacetimes, we will need a lot more specific cases, so that we need a few more rules to specify :

First, we need to consider the rules for a quotient of Minkowski space.

Chapter 63

Causal properties

LOOK UP THAT BUSINESS IN FRANKEL'S THE GEOMETRY OF PHYSICS ABOUT NON INTEGRABLE DISTRIBUTIONS LEADING TO CTCs

Caratheodory theorem

63.1 Time separation

For $p, q \in \mathcal{M}$, with $\mathcal{C}_t^+(p, q)$ the set of all future directed timelike curves connecting p to q . The time separation $d : M \times M \rightarrow [0, +\infty]$ is the longest proper time connecting p to q

$$d(p, q) = \begin{cases} 0 & \text{if } \mathcal{C}_t^+(p, q) = \emptyset \\ \sup[L(\alpha), \alpha \in \mathcal{C}_t^+(p, q)] & \end{cases} \quad (63.1)$$

Properties :

1. $d(p, q) > 0$ if and only if $p \in I^-(q)$
2. If there's a closed timelike curve through p , then $d(p, p) = \infty$, otherwise $d(p, p) = 0$
3. $d(p, q) \in (0, +\infty)$ implies $d(q, p) = 0$
4. Triangle inequality : $p \leq q \leq r$ implies $d(p, q) + d(p, r) \leq d(p, r)$

Theorem 63.1. d is lower semi-continuous. For $p_m, q_m \in \mathcal{M}$, with $\lim p_m = p$, $\lim q_m = q$, we have

$$\liminf_{m \rightarrow \infty} d(p_m, q_m) \geq d(p, q) \quad (63.2)$$

Example where d is not upper semi-continuous : remove line from spacetime, q on the light cone of that line's boundary, p_n just to the right of that line, converges to just under the line's edge. Then $\limsup_{n \rightarrow \infty} d(p_n, q) = 1$ while $d(p, q) = 0$

Proposition 63.2. If $p \ll p$, $d(p, p) = \infty$

Proof. Since $p \ll p$, there is a closed timelike curve γ , $\gamma(0) = \gamma(1) = p$. The length of that curve is positive. It is then possible to construct a piecewise timelike curve by making identical copies of this curve and connecting their endpoints, using $\gamma(k) = p$, $k \in \mathbb{Z}$. For n copies, the length of this curve will be n time the original length. As this process does not converge, $d(p, p) = \infty$. $\square$

Something about timelike singularities and convergence???

63.2 Time functions

A time function on a spacetime is a map assigning a time for each event of the manifold

$$t : M \rightarrow \mathbb{R} \quad (63.3)$$

Properties

Borel measure m on the spacetime such that

- m is finite : $m(\mathcal{M}) < \infty$
- For a non-empty open subset U , $m(U) > 0$

For an orientable spacetime, pick an orientation and ε be the oriented volume element associated to the metric. Take an open covering of $\mathcal{M}$ such that every open set has measure by ε inferior to 1. Take a partition of unity $\{\rho_n\}$ subordinate to this covering.

The measure m is the one associated to the volume element

$$\varepsilon_m = \sum_{n=1}^{\infty} 2^{-n} \rho_n \varepsilon \quad (63.4)$$

If the spacetime isn't orientable, go to the double cover and pullback, take half the measure

Definition 63.3. The future and past volume functions, t^+ and t^- are functions $t^\pm : M \rightarrow \mathbb{R}$ such that

$$t^\pm(p) = \mp m(I^\pm(p)) \quad (63.5)$$

Proposition 63.4. $t^\pm$ is non-decreasing on any future-directed causal curve.

Proof. If $p \leq q$, then $I^+(q) \subset I^+(p)$ and $I^-(p) \subset I^-(q)$. Since m is a measure, $-m(I^+(p)) < -m(I^+(q))$ and $m(I^-(p)) < m(I^-(q))$. $\square$

Definition 63.5. A generalized time function is a function $t : M \rightarrow \mathbb{R}$ that is strictly increasing on all future-directed causal curves. It is called a time function if it is continuous as well, and a temporal function if it is smooth with a past-directed timelike gradient ∇t .

Proposition 63.6. A spacetime with closed timelike curves admits no generalized time function. A spacetime with closed null curves admits no time function.

Proof. If a function is strictly increasing on a closed timelike curve, we have $f(p) > f(p)$. Hence there can be no such function. If the spacetime includes a closed null curve but no closed timelike curve, we can represent this curve as the limit of a sequence of future-directed timelike curves $(\gamma_n)_{n \in \mathbb{N}}$. If a generalized time function exists, then for every γ_n , $f(\gamma_n(0)) < f(\gamma_n(1))$, meaning that if f is continuous, the limit should be that $f(p) < f(p)$. $\square$

63.3 Spacetime topologies

Beyond the manifold topology of the spacetime itself, we may define topologies on it by its causal structure.

63.3.1 The path topology

[179]

Definition 63.7. The *path topology*, or Zeeman topology, is the topology in which a subset $U \subset \mathcal{M}$ is open if for every timelike curve γ , there is a subset $V \subset \mathcal{M}$ in the manifold topology such that

$$U \cap \gamma = V \cap \gamma \quad (63.6)$$

Strictly finer than the manifold topology since trivially true for any open set U in the manifold topology.

Prove that it's a topology :

Since it's finer than the manifold topology, $\emptyset$ and $\mathcal{M}$ are in it

Intersection of two open sets $U_1 \cap U_2$: if $U_1 \cap \gamma = V_1 \cap \gamma$ and $U_2 \cap \gamma = V_2 \cap \gamma$, then

$$(U_1 \cap U_2) \cap \gamma = U_2 \cap (V_1 \cap \gamma) = V_1 \cap (U_2 \cap \gamma) = (V_1 \cap V_2) \quad (63.7)$$

Union of a collection of open sets $\bigcup U_i$: if for all i , $U_i \cap \gamma = V_i \cap \gamma$, then

$$\begin{aligned} \left(\bigcup_i U_i\right) \cap \gamma &= \bigcup_i (U_i \cap \gamma) \\ &= \bigcup_i V_i \end{aligned} \quad (63.8)$$

Basis for the topology : light cones $I^+(p, U) \cup p \cup I^-(p, U)$ for a convex normal neighbourhood U .

63.3.2 The Alexandrov topology

In general, the Alexandrov topology is a topology defined by a partial order on a set

Definition 63.8. The *Alexandrov topology* on a set X is a topology where the intersection of arbitrarily many open sets is open.

This is in contrast with the usual definition of topology where only a finite intersection is required to be open.

[SHOULD ACTUALLY BE INTERVAL TOPOLOGY]

The topology of interest in general relativity is the one generated by the basis $\langle p, q \rangle$, for all p and q .

Proposition 63.9. The basis $\langle p, q \rangle$ generates an Alexandrov topology.

Since this will be the only Alexandrov topology of interest, it will usually be referred to as the Alexandrov topology of a spacetime.

Theorem 63.10. The Alexandrov topology is a topology

Proof. If we consider a normal neighbourhood of any point p , we can find two points p_p, p_f to its past and future so that p is within the causal diamond of p_f and p_p , so that the base indeed form a cover of the set.

Intersection of two causal diamonds is a causal diamond.

Union of causal diamonds???

□

Show that the Alexandrov topology is a topology : any point p is in a causal diamond, so M is a union of the base. Empty set is the union of no basis element.

Proposition 63.11. The Alexandrov topology is at least as coarse as the manifold topology.

Proof. This only requires us to prove that any open set of the Alexandrov topology is an open set of the manifold topology, which is trivially true since we have shown that $I^\pm$ is always an open set in the manifold topology. $\square$

The Alexandrov topology isn't necessarily a good one, as can be seen by considering the timelike cylinder, as the past and future lightcones of any point p in it are the entire manifold, meaning that for any two points $p, q \in M$,

$$I^+(p) \cap I^-(q) = M \quad (63.9)$$

The basis has only M in it, so that the topology is only composed of the total space and empty space, making it the trivial topology. We will see later on under which conditions the Alexandrov topology does coincide with the manifold topology.

63.4 Timelike homotopies

Similarly to the general notion of curve homotopies if there exists a continuous map mapping one curve to another, it can be useful to have a notion of homotopies specifically for timelike curves.

There are several ways to deal with this. The first one is due to Smith [180].

Definition 63.12. A q -loop based at p , $q \in \mathbb{N}$, is a piecewise closed timelike curve with exactly q corners.

The timelike homotopies of q -loops of arbitrary q would be too large, and indeed reduce to the usual homotopy group (we can approximate any loop with some q -loop zig-zagging in a very small neighbourhood of the original loop), but on the other hand, the product of q -loops of fixed q does not form a group, as the trivial curve is not a q -loop, and if we have a timelike curve γ , the product $\gamma * \gamma^{-1}$ will have $(2q + 2)$ corners. To prevent this, we define, in addition to q -loops, stings.

Definition 63.13. A *sting* based at p is a curve composed from an arbitrary finite curve γ such that the sting will be the path $\gamma * \gamma^{-1}$, with $\gamma(0) = p$.

And then we define the set of generalized q -loops.

Definition 63.14. The p -factorization of a curve γ , with $p \in \gamma$, is the decomposition $\gamma = \gamma_+ * \gamma_-$, where $\gamma_+(1) = p$.

Definition 63.15. For a curve ζ starting at p , the curve

$$\gamma^* = \gamma_+ * \zeta * \zeta^{-1} * \gamma_- \quad (63.10)$$

is said to be obtained from γ by the insertion of a sting. Likewise, γ is obtained from γ^* by the deletion of a sting.

Definition 63.16. A *generalized q -loop* is a curve made of a q -loop by a finite number of insertions and deletions of stings. The constant curve e_p will also be considered a generalized q -loop based at p .

Proposition 63.17. Any q -loop can be turned into a q' -loop, $q' < q$, by the insertion of a sting.

Proof. By inserting a purely spacelike sting at singular points, we can remove any singular point, as one of the left or right tangent vector will fail to be causal at that point. $\square$

Definition 63.18. equality modulo stings

Corrolary 63.1. Given a q -loop γ , there is a q -loop equivalent modulo stings to $\gamma * \gamma^{-1}$

Proposition 63.19. The set of generalized q -loops with the path product $*$ and equivalence relation modulo stings $\doteq$ forms a group. .

Proof. If we consider the $\square$

The group of q -loops and stings at p will be noted $\tau_q(L, x)$ under the operation

$$\gamma_1(\lambda) * \gamma_2(\lambda) = \gamma_{12} \quad (63.11)$$

Proposition 63.20. If a spacetime isn't time-orientable, all of its odd τ_q groups will be trivial.

Theorem 63.21. All $\tau_q(L, x)$ groups are trivial for Minkowski space.

Proof. $\square$

Universal coverings

Def : Covering space of a topological space X is a space C with a continuous surjective map $p : C \rightarrow X$, $\forall x \in X$, there exists a neighbourhood $U \ni x$ such that $p^{-1}(U)$ is a union of disjoint open sets in C which are mapped homeomorphically onto U by p .

Universal cover : simply connected cover

Ideas : check if all contractible CTCs $\rightarrow$ causal cover

If CTC is contractible $\rightarrow$ quasiregular singularities ("unwrapping CTCs") Otherwise no problem

Definition : timelike connected, timelike contractible.

Chapter 64

Causal hierarchy

It is possible to classify spacetimes by their causal pathologies. There are many possible schemes and many possible classes of such spacetimes, but the principal classification is the causal ladder, which classifies spacetimes, from most to least pathological, as

- Any spacetime
- Non-totally vicious
- Chronological
- Causal
- Distinguishing
- Strongly causal
- Stably causal
- Causally continuous
- Causally simple
- Globally hyperbolic

Each rung of the causal ladder implies the one above it. That is, if a spacetime is distinguishing, then it is also causal, chronological and non-totally vicious.

64.1 Totally vicious spacetime

While not directly part of the causal ladder, the lowest rung on it will be defined in opposition to this type of spacetime, which has the worst causal pathologies. A totally vicious spacetime is a spacetime for which every point is vicious.

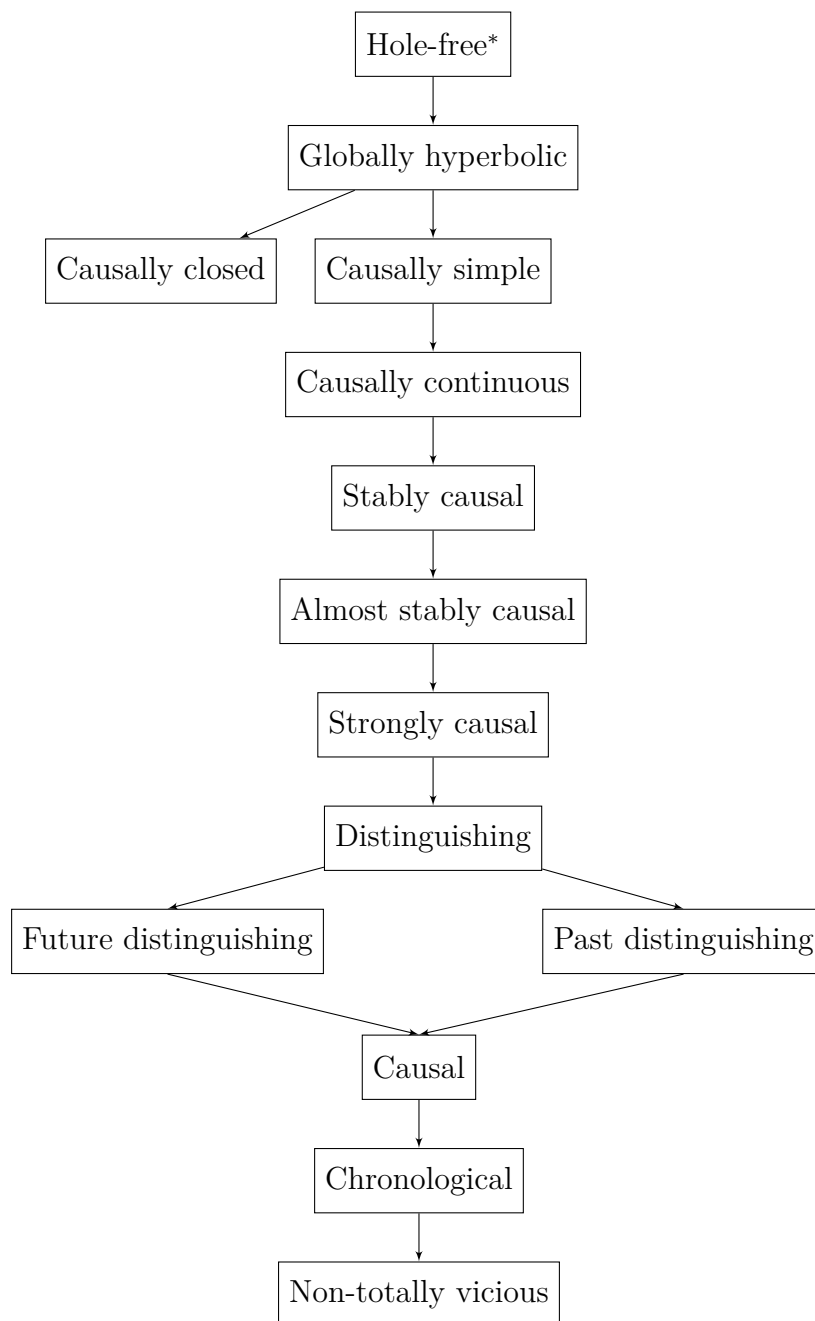


Figure 64.1: The causal ladder

Definition 64.1. A *totally vicious spacetime* is a spacetime in which all $p \in M$ obeys $p \ll p$.

Equivalently, the chronology violating region V is the entire spacetime. The typical example of a totally vicious spacetime is the Minkowski time cylinder, $S_t^1 \times \mathbb{R}^{n-1}$, as can be seen by simply taking any timelike geodesic oriented along the closed time coordinate. Other examples include the Gödel metric [...]

64.1.1 Properties

Proposition 64.2. A spacetime is totally vicious if and only if for all $p, q \in \mathcal{M}$, $d(p, q) = \infty$.

Proof. By the triangle inequality, $d(p, q) \geq d(p, p) + d(p, q)$. Since $d(p, p) = \infty$ if $p \ll p$, $d(p, q) = \infty$. Conversely, if $d(p, q) = \infty$ for all points, then $d(p, p) = \infty > 0$, meaning there is always a future-directed timelike curve between them, so $p \ll p$. $\square$

Proposition 64.3. A spacetime is totally vicious if and only if $I^+(p) = I^-(p) = \mathcal{M}$.

Proof. Since $p \in \mathcal{M}$, if $I^\pm(p) = \mathcal{M}$, $p \in I^\pm(p)$, or $p \ll p$.

Conversely, if we take $d(p, q) = \infty$ for all points per the previous property, for any point r , $d(p, r) > 0$ and $d(r, p) > 0$, meaning that for all points $p, r \in \mathcal{M}$, $r \in I^\pm(p)$, so $I^\pm(p) = \mathcal{M}$. $\square$

Proposition 64.4. A totally vicious spacetime has trivial Alexandrov topology, $\tau = \{\emptyset, M\}$.

Proof. For every basis set of the topology $\langle p, q \rangle$,

$$I^+(p) \cap I^-(q) = M \cap M = M \quad (64.1)$$

Their intersections and unions will only be $\mathcal{M}$, with $\emptyset$ being the empty union of no member of the basis. $\square$

Proposition 64.5. Totally vicious spacetime admits no generalized time function, and its future and past volume functions are constants.

Proof. Since it contains closed timelike curves, there is no generalized time function. Its future and past volume functions are

$$t^\pm(p) = \mp m(I^\pm(p)) = \mp m(\mathcal{M}) \quad (64.2)$$

$\square$

Theorem 64.6. All compact spacetimes with a timelike killing vector are totally vicious.

This relies on a few other theorems

Proof. For a compact spacetime with timelike Killing vector K^μ , if it is static, K is parallel for the conformal metric

$$\hat{g} = \frac{1}{g(K, K)}g \quad (64.3)$$

the 1-form $\omega = -\hat{g}(K, -)$ is closed so if $\mathcal{M}$ is compact it cannot be simply connected.
[TO FINISH] □

Timelike loop at every $p \in \mathcal{M}$ equivalent to timelike vector field with periodic integral curves

64.1.2 Examples

A few examples of totally vicious spacetimes include the Gödel spacetime, variations on the timelike cylinder $S_t^1 \times \Sigma$ or the van Stockum dust.

64.2 Non-totally vicious spacetime

As the name implies, this is a spacetime that just fails to be totally vicious.

Definition 64.7. A spacetime is *non-totally vicious* if it is not totally vicious, ie, there is a point p such that $p \not\ll p$.

Causal respecting set

This is a pretty mild causality condition

64.3 Chronological spacetime

A spacetime is chronological if it contains no closed timelike curves.

Definition 64.8. A spacetime is *chronological* if it obeys any of these equivalent conditions :

1. There is no point $p \in \mathcal{M}$ such that $p \ll p$.
2. Its chronology violation region is empty : $V = \emptyset$.

By definition, if a spacetime is chronological, it is also non-totally vicious.

64.3.1 Properties

Proposition 64.9. For all $p \in \mathcal{M}$, $d(p, p) = 0$.

Theorem 64.10. A spacetime is chronological if and only if $t^\pm$ is strictly increasing on any future-directed timelike curves.

Proof. □

Theorem 64.11. No compact spacetime is chronological.

Proof. Since $I^+(p)$ for all p forms an open cover of the manifold, if the manifold is compact, then this cover has a finite subcover, of the form $\{I^+(p_1), I^+(p_2), \dots, I^+(p_N)\}$. If there were no closed causal curves, p_1 would not be in $I^+(p_1)$, and hence must belong to $I^+(p_i)$, $i \neq 1$. But if $p_1 \in I^+(p_i)$, then $I^+(p_1) \subset I^+(p_i)$, and then the cover can be reduced to $\{I^+(p_2), \dots, I^+(p_N)\}$. The process can be repeated by induction until only one chronological future forms the entire cover of the manifold, meaning that p has to be within $I^+(p)$. □

Remark 64.12. The original proof of this theorem was a little more involved,

Original proof by Markus in "Remark on cosmological models"

A spacetime being chronological trivially implies that it is not totally vicious, but the converse is not true, simply by considering the case of a spacetime where the chronology violating region is not the entire manifold. A simple example of this is Misner space.

Proposition 64.13. A non-totally vicious spacetime may not be chronological.

Proof. Example : Toroidal spacetime, Killing vector field ∂_t , everywhere timelike except two points □

Proposition 64.14. A spacetime is chronological if and only if $t^\pm$ is strictly increasing on any future-directed timelike curve.

Proof. Since $t^\pm$ is constant on any closed timelike curve, being strictly increasing means that there is no point such that $p \ll p$. Conversely, if $p \ll q$ but $t^-(p) = t^-(q)$ □

64.3.2 Examples

64.4 Causal spacetime

Definition 64.15. A spacetime is *causal* if there is no point $p \in \mathcal{M}$ such that $p < p$.

By definition, if a spacetime is causal, it is chronological. The converse is however not true, which can be shown simply by finding a spacetime which has no closed timelike curves but does have a closed null curve.

Proposition 64.16. Not all chronological spacetimes are causal [181].

Proof. Consider the $(1 + 1)$ -dimensional spacetime modelled on a cylinder $\mathbb{R} \times S$, in coordinates (t, θ) , such that the spacetime is flat outside a range $(-T, T)$, and inside that range, its lightcone tips continuously with t until one of the null rays is colinear with ∂_θ . If we pick some bump function $\rho(T)$ that is 1 at $T = 0$ and 0 outside of $[-T, T]$, we can pick for instance

$$ds^2 = () \quad (64.4)$$

□

Theorem 64.17. A chronological but non-causal spacetime has a closed null geodesic.

Proof. If the spacetime isn't causal, then there's a point p such that $p < p$. Since it is chronological, that curve cannot be a timelike curve. So by theorem X, it must be a geodesic. □

64.5 Distinguishing spacetime

While a spacetime may lack closed timelike curves, another pathology spacetimes may suffer from is that two different point may have the exact same chronological future (or past), ie for two points p, q , $I^+(p) = I^+(q)$.

Definition 64.18. A spacetime is future (resp. past) *distinguishing* if the chronological future (resp. past) of two different point is not the same. If it is both future and past distinguishing, we say it is distinguishing.

$$I^\pm(p) = I^\pm(q) \rightarrow p = q \quad (64.5)$$

It is a stronger condition than causality :

Theorem 64.19. If a spacetime is either past or future distinguishing, then it is causal.

Proof. If a spacetime is non-causal, then there is a closed causal curve, meaning that we have two points $p, q \in \mathcal{M}$ with the property $p \ll q \ll p$. Which means that we have both $I^+(p) \subset I^+(q)$ and $I^+(q) \subset I^+(p)$, in other words, $I^+(p) = I^+(q)$. Hence it cannot be future distinguishing. The same proof applies for past distinguishing spacetimes. $\square$

Theorem 64.20. For two spacetimes $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$, $(\mathcal{M}_1, g_1)$ distinguishing, and $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ a diffeomorphism such that $p \leq q \rightarrow f(p) \leq f(q)$, then $(\mathcal{M}_2, g_2)$ is distinguishing and $g_1 = \Omega f^* g_2$.

Proof. $\square$

Related to rigidity of the conformal structure?

Proposition 64.21. A causal spacetime may not be distinguishing.

Proof. If we take the cylinder from the example 64.16 and remove a single point at $z = 0$, the spacetime loses all closed null curves, but its chronological future is still the entire upper cylinder. $\square$

Proposition 64.22. A past-distinguishing spacetime may not be future-distinguishing, and vice-versa.

Proof. Punctured cylinder - a strip example $\square$

64.6 Strongly causal spacetime

Another issue that could arise for a given spacetime is that while a timelike curve may not be closed, it could still be arbitrarily close to being so, such as having a timelike curve converging asymptotically towards itself. While not technically a causality violation, it is generally considered pathological and makes the spacetime much less well-behaved in a variety of ways.

Definition 64.23. A spacetime is *strongly causal* if for every neighbourhood U of every point $p \in \mathcal{M}$, there is a strongly causal neighbourhood $V \subset U$.

Definition from causal convexity and the other (Minguzzi)

Proposition 64.24. Strongly causal spacetimes are non-imprisoning.

Proposition 64.25. A spacetime is strongly causal if and only if the Alexandrov topology is equivalent to the manifold topology.

Example of a distinguishing spacetime that is not strongly causal : Carter spacetime

$\mathbb{R}^3$ with the identifications

$$\begin{aligned}(t, x, y) &\sim (t, x, y + 1) \\ (t, x, y) &\sim (t, x + 1, y + z)\end{aligned}\tag{64.6}$$

With $a \in (0, 1) \cap (\mathbb{R} \setminus \mathbb{Q})$

$$ds^2 = -c(t)(dt^2 - dx^2) - 2dtdy + dz^2\tag{64.7}$$

64.7 Almost stably causal spacetime

[paper by E. Howard Causal Stability Conditions for General Relativistic Spacetimes]

- There's at least one open set Q_g such that under small variations of g , large enough to introduce closed causal curves on $\mathcal{M}$, stable causality is still holding outside of Q_g
- The spacetime is globally causally stable and it's possible to widen the light cones on an open region in order to produce closed causal curves.
- Light cones can be open within the causal stable spacetime at a local and macroscopic level

64.8 Reflecting spacetime

Definition 64.26. A spacetime is past (resp. future) reflecting if $I^-(q) \subset I^-(p)$ implies $I^+(p) \subset I^-(q)$ (resp. $I^+(q) \subset I^+(p)$ implies $I^+(p) \subset I^+(q)$). It is reflecting if it is both past and future reflecting.

Proposition 64.27. In a reflecting spacetime, $(I^-(p), I^+(p))$ is a hull pair.

64.9 Stably causal spacetime

Definition 64.28. A spacetime is stably causal if there exists a metric $g' \in \text{Con}(\mathcal{M})$ such that $g < g'$ and g' is causal.

equivalent to : There exists a timelike vector field ξ^μ such that $(\mathcal{M}, g + \xi^\flat \otimes \xi^\flat)$ is chronological

Spacetime is stably causal iff it admits a global time function.

64.10 Causally continuous

$$I^+(p) \subset I^+(q) \equiv I^-(q) \subset I^-(p)$$

close points have roughly the same past and future

Stably causal spacetime that isn't causally continuous : Minkowski space minus a non-empty closed set

64.11 Causally simple

Definition 64.29. A spacetime is causally simple if it is causal and $J^\pm(p)$ is closed for every point p .

Theorem 64.30. A causally simple spacetime is distinguishing.

Proof.

□

64.12 Globally hyperbolic spacetime

A globally hyperbolic spacetime is the most well-behaved category of spacetimes regarding causality, and most reasonable spacetimes are assumed to be globally hyperbolic. It is roughly defined as having no closed timelike curves or naked singularities (although not all spacetimes with such characteristics are globally hyperbolic). It has a few equivalent definitions :

Definition 64.31. A spacetime is globally hyperbolic if it is causal and $\forall p, q \in \mathcal{M}$, $J^+(p) \cap J^-(q)$ is compact.

$J^+(p)$ and $J^-(p)$ are closed for all p

They can also be defined with respect to a Cauchy surface,

Theorem 64.32. A spacetime is globally hyperbolic if and only if it admits a Cauchy hypersurface Σ .

Proof. □

Theorem 64.33. A globally hyperbolic spacetime admits a Cauchy time function.

Proof. □

Geroch's theorem

Theorem 64.34. A globally hyperbolic manifold has the topology $M \approx \mathbb{R} \times \Sigma$, where Σ is the topology of one of its Cauchy surface.

Proof. □

Theorem 64.35. In a globally hyperbolic spacetime, any points $p, q \in \mathcal{M}$ such that $p \leq q$ can be joined by a causal geodesic with length equal to their time-separation.

64.12.1 Hole-freeness

While global hyperbolicity is sometimes defined informally as corresponding to the absence of naked singularities and closed causal curves, this is not quite the case. First because a spacetime can obey those conditions and still fail to be globally hyperbolic (as we will see, de Sitter space is the most common example), but also because there exists globally hyperbolic spacetimes that do not quite correspond to the intuitive notion of

Definition 64.36. A spacetime is called hole-free if for any spacelike hypersurface S and for any metric-preserving embedding $\theta : D(S) \rightarrow \mathcal{M}'$ into another spacetime $(\mathcal{M}', g')$, $\theta(D(S)) = D(\theta(S))$

64.12.2 Causal closedness

Definition 64.37. A spacetime is causally closed if for every $p \in \mathcal{M}$, $J^\pm(p)$ is closed.

Doesn't imply even chronology : Gödel is causally closed

Chapter 65

Abstract causal spaces

Causal structures?

It is possible to describe the causality of a spacetime without any reference to any manifold structure or topology. Those are called causal spaces. There is a number of different models for them.

65.1 Etiological space

Definition 65.1. An *etiological space* is a topological space X equipped with two binary relations $<$ and $\ll$, obeying the axioms that for $p, q \in X$,

1. $p < p$.
2. $<$ and $\ll$ are transitive.
3. if $p \ll q$, then $p < q$.
4. if $p \ll p$, then there exists a q such that $p \neq q$ and $p \ll q \ll p$.
5. The topology of X is a refinement of the Alexandrov topology from $\ll$.

Proposition 65.2. Any time-orientable spacetime is an etiological space.

Proof. If a spacetime is time-orientable, the relations $<$ and $\ll$ are well-defined on it. Then axioms 1, 2, 3 are from the properties of the causal and chronological ordering of points. Axiom 4 stems from $p \ll p$ implying a closed timelike curve, on which will lie at least one point. Axiom 5 is from property X. $\square$

Definition 65.3. A *causal space* is a quadruple $(X, <, \ll, \nearrow)$ of a set X and three relations on that set, with the axioms that for three points $p, q, r \in X$,

1. $p < p$.
2. if $p < q$ and $q < r$, then $p < r$.
3. if $p \ll q$, then $p < q$.
4. if $p < q$ and $q \ll r$, then $p \ll r$.
5. if $p \ll q$ and $q < r$, then $p \ll r$.
6. $p \nearrow q$ is equivalent to $p < q$ and $p \not\ll q$.

Those are precisely the rules that we derived previously for $<$ and $\ll$. We can then study their properties abstractly in this setting without worrying about the manifold structure behind it.

In addition, we may also add the axioms of causality for specific causal spaces

1. if $p < q$ and $q < p$, then $p = q$.
2. $p \not\ll p$

A few immediate consequences of the axioms of a causal space are

- $<$ is a partial order
- $\ll$ is anti-reflexive and transitive

Proposition 65.4. If $p < q < r$ and $p \nearrow r$, then $p \nearrow q \nearrow r$.

Define inverses $>, \gg, \nwarrow$

Relations on subsets $Y \subset X$, $<_Y, \ll_Y, \nearrow_Y$.

Definition 65.5. For a finite chain of N points $G = (p_i)_{1 \leq i \leq N}$ ordered by $<$ ($g_i < g_{i+1}$), $N \geq 3$, G is called a girder if for every i , $p_i \nearrow p_{i+2}$.

Girders, hypergirders, etc

65.2 Reinterpretation on causal anomalies

A tempting procedure to do if a spacetime has closed causal curves is to unwrap them : what if there was a cover of that spacetime where the closed causal curves are all sent to their own covering space of a straight line?

This idea goes all the way back to Reichenbach's investigation of causality on spacetimes [182]

Reichenbach on causal chains, similar vs identical events (p. 141)

If CTCs are multiply connected, going to covering space

Otherwise, what happens with unwrapping

Going to double cover for non-time orientable

Going from GR to Pauli-Fierz $\rightarrow$ exchanging CTCs for tachyonic effects

65.3 Theorems on causality

Lemma 65.1. For a closed curve γ in $\mathbb{R}^n$, for every component U^μ of the tangent vector in a coordinate basis, there exists a point $\gamma(\lambda)$ such that $U^\mu = 0$.

Proof. This is a fairly trivial application of Rolle's theorem. If we pick the trivial chart of $\mathbb{R}^n$, then the coordinates of the loop for a component μ just a map $\phi^\mu \circ \gamma : I \rightarrow \mathbb{R}$, with $\phi^\mu \circ \gamma(0) = \phi^\mu \circ \gamma(1)$. By Rolle's theorem, there exists a $\lambda \in I$ such that

$$U^\mu = \frac{d}{d\lambda}(\phi^\mu \circ \gamma(\lambda)) = 0 \quad (65.1)$$

□

Theorem 65.6. A static spacetime with trivial topology $M = \mathbb{R}^n$ is always causal.

Proof. Since the metric of a static spacetime can be expressed as

$$ds^2 = -\alpha(x)dt^2 + g_{ij}(x)dx^i dx^j \quad (65.2)$$

with g_{ij} a Riemannian metric and the coordinate t such that ∂_t is the Killing vector, then by the previous lemma, there exists a point in any closed curve where $U^t = 0$, meaning that at that point, the tangent vector will be

$$g(U, U) = g_{ij}U^i U^j \geq 0 \quad (65.3)$$

Since the Riemannian metric is only 0 for $U = 0$, any non-degenerate curve will have spacelike tangent vectors. □

Corrolary 65.1. Minkowski space is causal.

A related theorem is

Theorem 65.7. A spacetime with trivial topology $M = \mathbb{R}^n$ and a diagonal metric is always causal.

Proof. Using a similar proof, we have

$$ds^2 = -\alpha(x)dt^2 + \sum_{i=1}^N g_{ii}(x)(dx^i)^2 \quad (65.4)$$

where $g_{ii} \geq 0$ to preserve the signature of the metric. Then again for loops, there exists a point where

$$g(U, U) = \sum_{i=1}^N g_{ii}(U^i)^2 \geq 0 \quad (65.5)$$

which again implies that no loop can be timelike at every point. $\square$

Lemma 65.2. Any null-homotopic curve γ can be included within a single chart homeomorphic to $\mathbb{R}^2$.

Proof. I STILL DON'T KNOW $\square$

Theorem 65.8. A static spacetime only has non-null homotopic closed causal curves.

Find a proof first that contractible closed timelike curves have to be contained within a single simply connected region. $\square$

Corrolary 65.2. All static spacetimes have a causal cover.

Theorem 65.9. If a compact spacetime admits a covering space with a compact Cauchy surface, then it contains a closed timelike geodesic. (Tipler)

"Some spacetimes, such as Gödel spacetime, do not admit any global time slices. This is a consequence of three features: it is time orientable; a CTC passes through each point; and it is simply connected. The edge of an achronal surface S is the set of points p such that every open neighborhood $p \in O$ includes points in $I^+(p)$ and $I^-(p)$ that can be connected by a timelike curve that does not cross S ."

Proof : if the space is simply connected, then a boundaryless spacelike hypersurface Σ would split the spacetime into two disconnected pieces $\mathcal{M} \setminus \Sigma$ (Jordan Brouwer theorem, generalized). Components are D_+ (chronological future of Σ and D_- (chronological past).

$\mathcal{M}$ is time oriented : timelike vector field everywhere continuous $\rightarrow$ timelike vector field everywhere oriented the same way on the surface wrt the normal vector

If $p \in \Sigma$ and γ a future-oriented CTC, then at p , $\dot{\gamma}$ points to D_+ . When γ intersects with Σ again, $\dot{\gamma}$ points the way opposite to the time orientation vector field, which contradicts that the spacetime is time oriented.

Theorem 65.10. For $n > 2$, two points of an n -dimensional Lorentz manifold can always be connected by a spacelike curve.

Proof. Helicoidal curve between the two points in the tangent space

□

Chapter 66

Spacetime boundaries and asymptotic behaviours

We have already seen a few different ways to consider the boundaries and compactifications of various spaces. In this chapter, we will see how to construct such boundaries in a way that preserve some of the spacetime structures, and the information we can gather from the behaviour of our physics on those boundaries (points which are in some sense "at infinity").

[Why asymptotic behaviour doesn't work in coordinates bc it is coordinate dependent cf. Witten et al]

"Given two nowhere equal vector fields on a manifold and a curve on the manifold which goes to infinity in the sense that $\gamma : L \rightarrow M$ has $\gamma^{-1}(C)$ compact for all compact C , then there exists a chart with respect to which the components of the vector fields approach each other in the limit along the curve and also a chart with respect to which they do not."

66.1 Conformal compactification

[183]

The main method by which spacetimes are put into correspondence with a compact space is via the method of *conformal compactification*, where we attempt to embed a non-compact spacetime as a dense open subset of a given compact space, such that the embedding is a conformal map, thereby preserving all of the causal properties of this spacetime.

Formally speaking, if we have a spacetime (M, g) , it has a conformal compactification if there exists a larger spacetime $(\hat{M}, \hat{g})$ such that there exists a conformal embedding

$$\iota : (M, g) \mapsto (\iota M, \iota^* g) \subseteq (\hat{M}, \hat{g}) \quad (66.1)$$

so that $\iota^* g = \Omega g$, where ιM is an open submanifold of $\hat{M}$, with a boundary $\mathcal{S} = \partial \iota(M)$, with the conformal factor admitting a smooth extension in $\hat{M}$ with $\Omega(\mathcal{S}) = 0$, $d\Omega(\mathcal{S}) \neq 0$,

The last condition is there so as to [specify that the boundary is far away in g metric? Falls to 0 as $1/r$? Is that only important for asymptotics for spherically symmetric spacetimes]

”One might hope that, if the conformal factor Ω were chosen to approach zero asymptotically at an appropriate rate, then the resulting ”scaling down of the metric” might ”bring infinity in (with respect to g) to a finite region.”

In other words, a conformal compactification can be done if there exists a conformal extension of a given spacetime (M, g) , with conformal structure $(M, [g])$, to a larger conformal space (M', c') , in which the closure of the original spacetime $\iota(M)$ is compact.

If null asymptote : $g(d\Omega, d\Omega) = 0$

66.1.1 Conformal structure of Minkowski space

The basic example of this is to consider n -dimensional Minkowski space ($n \geq 2$). The easiest way to get its structure at infinity is to switch to spherical coordinates, which will allow us to consider the limit $r, t \rightarrow \infty$.

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad (66.2)$$

We can then switch to the radial null coordinates, $u = t - r$ and $v = t + r$

$$ds^2 = -dudv + \left(\frac{v-u}{2}\right)^2 d\Omega^2 \quad (66.3)$$

To compactify this space, we will need to look at it in some coordinate system where the points at infinity are within finite range of those coordinates. [Null coordinates to preserve conformal structure idk]

Take the conformal metric

$$\hat{g} = \frac{g}{(1+u^2)(1+v^2)} \quad (66.4)$$

Coordinate transformation $u = \tan(p)$, $v = \tan(q)$, $p, q \in [-\pi/2, \pi/2]$

$$ds^2 = -dudv + \left(\frac{\sin(p-q)}{2}\right)^2 d\Omega \quad (66.5)$$

Then finally $T = (p+q)/2$ and $\rho = q-p$, $\rho \in [-\pi, \pi]$

$$d\hat{s} = -dT^2 + \frac{1}{4}[d\rho^2 + \sin^{n-2}(\rho)d\Omega] \quad (66.6)$$

If we maximally extend this spacetime, we get $\mathbb{R}_t \times S^3$.

By suppressing the angular coordinates, we can represent this spacetime as a diagram. This is called a *Penrose diagram* or *conformal diagram*.

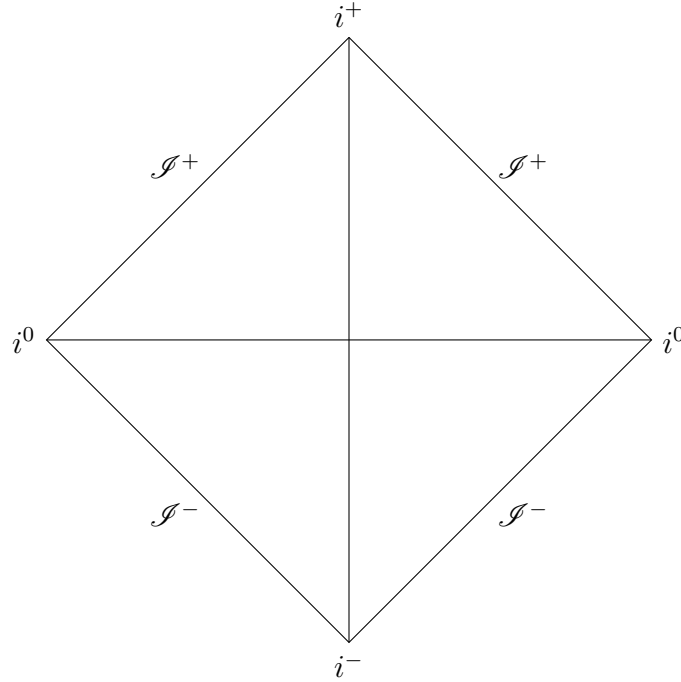


Figure 66.1: Conformal diagram of Minkowski space

The boundary of this conformal compactification is typically split into different components. i^0 , i^+ and i^- represent spatial, future temporal and past temporal infinity, $\mathscr{I}^+$ and $\mathscr{I}^-$ are future and past null infinity. We will see in the following section what those formally correspond to.

66.2 Spacetime boundaries

If we perform a conformal compactification on a non-compact spacetime, we'll end up with a few different points at infinity in the boundary $\mathcal{I}$. We may also wish to include the points corresponding to singularities in the boundary, as we did with the g -boundary and b -boundary. All those points will be referred to as *ideal points*.

There are many different method to classify ideal points and give them a topology, each with their own benefits.

66.2.1 The GKP construction

The Geroch-Kronheimer-Penrose boundary construction is performed by identifying points of spacetime with causal sets, which form a larger set than the spacetime itself, the remaining points forming the boundary. The basis for this construction is to associate each point with its chronological future and past, which will only be well-defined if the spacetime is distinguishing, so that if $p \neq q$, then $I^\pm(p) \neq I^\pm(q)$.

The relevant properties to extend this are that

- $I^\pm(p)$ is open.
- $I^\pm(p)$ is a past/future set 62.0.1.
- It cannot be expressed by the union of two proper subsets with the same properties, since for any proper open subset $S \subset I^\pm(p)$, $I^\pm(S) \subset I^\pm(p)$.

We then define the generalized sets with those properties. As usual, any proof relating to past versions of sets will generalize to the future versions by switching time orientation.

Definition 66.1. A future (resp. past) set is called *irreducible* if it is not empty and is not the union of two proper subsets which are themselves future (resp. past) sets. They are noted respectively IF and IP.

The set of all IP on $\mathcal{M}$ will be noted $\hat{\mathcal{M}}$, while the set of all IF will be noted $\check{\mathcal{M}}$.

Proposition 66.2. A subset $S \subset \mathcal{M}$ is an IP if and only if there exists a future-directed timelike curve γ such that $S = I^-(\gamma)$.

Proof. If we have a timelike curve γ , suppose that $I^-(\gamma) = A \cup B$, with A, B past sets. $I^-(\gamma)$ will be an IP if for all A, B , either $A \subset B$ or $B \subset A$, otherwise we will have both points $q \in A \setminus B$ and $r \in B \setminus A$. Since $q, r \in I^-(\gamma)$, there are points $q', r' \in \gamma$ such that $q \in I^-(q')$ and $r \in I^-(r')$. Since our spacetime is distinguishing and γ is a timelike curve, we must have either $q' \ll r'$, $r' \ll q'$ or $q' = r'$. Which means that if A or B contains the futuremost point of q' and r' , it will also contain both q and r . By contradiction, $I^-(\gamma)$ is an IP.

Conversely, for P an irreducible past set, consider $p \in P$. Since P is a past set, we have

$$P = I^-(P \cap I^+(p)) \cup I^-(P \setminus I^+(p)) \quad (66.7)$$

Since it is irreducible, either $P = I^-(P \cap I^+(p))$ or $P = I^-(P \setminus I^+(p))$, but since $p \notin I^-(P \setminus I^+(p))$,

$$P = I^-(P \cap I^+(p)) \quad (66.8)$$

meaning that if we pick a point $q \in P$, then $q \in I^-(P \cap I^+(p))$, so that there exists a point $r \in P \cap I^+(p)$ such that $q \ll r$, which cannot be p or q since our spacetime is distinguishing. As we have $q \ll r$ and $p \ll r$, this means that for any two points in P , there is a point which lies to the future of both.

If we pick a countably dense family of points $p_i \in P$, and $q_0 \in P$ such that $p_0 \ll q_0$. We can then pick a point q_1 to the future of both q_0 and p_1 , and then iteratively a point q_i to the future of both q_{i-1} and p_i . As $\{q_i\}$ is ordered by $\ll$, we can form a timelike curve γ with appropriate smoothing. The set $P \cap I^+(p_i)$ will be open in P , and since they are dense, there exists some $k \in \mathbb{N}$ such that $p_k \in P \cap I^+(p_i)$. $p_k \in I^-(q_k)$, so $p_i \in I^-(\gamma)$. Which means that $P \in I^-(\gamma)$. $\square$

Once we have irreducible sets, we can classify them between the ones that will correspond to spacetime points and the ones that will correspond to ideal points.

Definition 66.3. Future (resp. past) sets are classified as either *proper irreducible future* (resp. past) sets if they can be expressed by $I^\pm(p)$, or *terminal irreducible future* (resp. past) sets if they cannot. They are noted as PIF, PIP, TIF and TIP.

Proposition 66.4. A set S is a TIP if and only if there is a future-inextendible timelike curve γ such that $I^-(\gamma) = S$. It is a PIP if and only if there is a future timelike curve γ with endpoint p such that $S = I^-(\gamma)$.

Proof. If there is a curve γ such that $I^-(\gamma) = S$ $\square$

In Minkowski space :

Proposition 66.5. Minkowski space has two classes of TIP/TIF : the ones generated by inextendible curves of bounded acceleration and the ones generated by inextendible curves of unbounded acceleration.

Proof. Show that curves of bounded accelerations have $I^\pm(\gamma) = \mathcal{M}$, but this is not true for unbounded accelerations. $\square$

The set of TIP/TIF of curves of constant acceleration will be denoted $\mathcal{I}^-/\mathcal{I}^+$, called past null infinity and future null infinity, while the TIP/TIF composed of the entire manifold will be denoted i^-/i^+ , called past (timelike) infinity and future (timelike) infinity.

The chronological past and future form an injection from the spacetime to the set of past and future irreducible sets.

$$\begin{aligned} I^+ &: M \rightarrow \check{\mathcal{M}} \\ I^- &: M \rightarrow \hat{\mathcal{M}} \end{aligned} \tag{66.9}$$

the set of PIF is $I^+(\mathcal{M})$, the set of PIP is $I^-(\mathcal{M})$.

With irreducible sets, we can form the completion of the manifold $\bar{\mathcal{M}}$, such that $M \subset \bar{\mathcal{M}}$.

The basis of this completion is to consider the set $\check{\mathcal{M}} \cup \hat{\mathcal{M}}$. Since each element of M corresponds to two elements of $\check{\mathcal{M}} \cup \hat{\mathcal{M}}$, $I^+(p)$ and $I^-(p)$, we have to perform some identifications. The most obvious one is that $\check{p} \sim \hat{p}$ if $\check{p} = I^-(p)$ and $\hat{p} = I^+(p)$, but things are more complicated for terminal sets, as

$$\mathcal{M}^\natural = (\check{\mathcal{M}} \cup \hat{\mathcal{M}}) / \sim$$

Example where two terminal sets correspond to the same ideal point

If $\mathcal{M}$ is in addition strongly causal, the manifold topology is the Alexandrov topology, so that we can extend the Alexandrov topology

Definition 66.6. The *extended Alexandrov topology* is the coarsest topology such that the following sets are always open for $A \in \check{\mathcal{M}}$, $B \in \hat{\mathcal{M}}$:

$$\begin{aligned} A^{\text{int}} &= \{P^* \in M^\natural \mid P \in \hat{\mathcal{M}}, P \cap A \neq \emptyset\} \\ A^{\text{ext}} &= \{P^* \in M^\natural \mid P \in \hat{\mathcal{M}}, \forall S \subset \mathcal{M}, P = I^-(S) \rightarrow I^+(S) \not\subset A\} \\ B^{\text{int}} &= \{P^* \in M^\natural \mid P \in \check{\mathcal{M}}, P \cap B \neq \emptyset\} \\ B^{\text{ext}} &= \{P^* \in M^\natural \mid P \in \check{\mathcal{M}}, \forall S \subset \mathcal{M}, P = I^+(S) \rightarrow I^-(S) \not\subset B\} \end{aligned} \tag{66.10}$$

$\mathcal{M}^\natural$ with the extended Alexandrov topology has a topology that isn't necessarily Hausdorff. Equivalence relation R_H by the intersection of all equivalence relations $\subset \mathcal{M}^\natural \times \mathcal{M}^\natural$ such that $\mathcal{M}^\natural / R$ is Hausdorff. Then we have

$$\bar{M} = M^\natural / R \tag{66.11}$$

Proposition 66.7. If $\mathcal{M}$ is strongly causal, the image of $p \in \mathcal{M}$ in $\mathcal{M}^\natural$ will never be identified with the image of another point $q \in \mathcal{M}$.

c -boundary : $\bar{M} \setminus \mathcal{M}$

66.2.1.1 Budic and Sachs construction

Since for the set $\check{\mathcal{M}} \cup \hat{\mathcal{M}}$, we want to identify at least $I^+(p)$ with $I^-(p)$, it will be interesting to consider reflecting spacetimes, for which the pair $(I^-(p), I^+(p))$ forms a hull set

66.3 Asymptotic flatness

Once we have the boundaries of our spacetime, it is possible to define asymptotic behaviours by observing how different objects behave on the boundary.

A spacetime is *asymptotically simple* if every null geodesic in M has future and past endpoints on the boundary of its conformal compactification $\hat{M}$.

A spacetime is *weakly asymptotically simple* if there is an open set $U \subset M$ isometric to a neighbourhood of the boundary of the conformal compactification $\hat{M}$ of an asymptotically simple spacetime.

A spacetime is asymptotically flat if it is weakly asymptotically simple and its Ricci tensor vanishes in some neighbourhood of the boundary of $\hat{M}$.

Differences between asymptotic flatness at null infinity, spatial infinity or both

Coordinate version :

A globally hyperbolic spacetime is asymptotically flat if there is a contractible compact set C

outside of C , $g = \eta + h$

$$\begin{aligned}\lim_{r \rightarrow \infty} h_{\mu\nu} &= \mathcal{O}(r^{-1}) \\ \lim_{r \rightarrow \infty} h_{\mu\nu,\rho} &= \mathcal{O}(r^{-2}) \\ \lim_{r \rightarrow \infty} h_{\mu\nu,\rho\sigma} &= \mathcal{O}(r^{-3})\end{aligned}$$

66.4 More general asymptotic behaviour

Since quite a wide class of physically important spacetimes are not asymptotically flat, and the question of isolated objects within them is important, we broaden the definition of asymptotic behaviours to include more asymptotic spacetimes.

Asymptotically anti de Sitter

Asymptotically FRW

66.5 Asymptotic symmetry groups

"The group of all diffeomorphisms on the manifold of points at infinity which leave invariant the universal fields."

Ex : Poincaré for Minkowski space "with a distorted translation group"

Chapter 67

Horizons

The general notion of a horizon, as defined by Rindler [185], is a frontier between observables and unobservable things.

Horizons are generally defined with respect to observers with future-directed timelike curves γ , to correspond to the notion of a physical observer. This means that horizons are not entirely defined with respect to the spacetime itself, and even the most benign spacetimes such as Minkowski space possess families of observers with horizons. For such a curve, we define the horizon to be

Apparent horizons, trapped surfaces, event horizons, particle horizons, Hubble horizon, marginally trapped tubes, trapping horizons, future outer trapping horizons, dynamical horizons, etc

67.1 Event horizons

GKP : Event horizons are $\approx \partial TIP$ while the particle horizon is $\approx \partial TIF$.

In a time-oriented spacetime, the (future) event horizon of an observer with timelike curve γ is the boundary beyond which any point will never be observed by that observer. This means that no causal curve from any point q beyond the horizon will be able to reach any point of γ .

Hence to define the horizon, we will first consider the set of all points that can be observed, all points q such that $J^+(q) \cap \gamma \neq \emptyset$, or in other words, $J^-(\gamma)$. The horizon will then be the boundary of that set,

$$\text{EvtHor}^+(\gamma) = \partial J^-(\gamma) \tag{67.1}$$

In a similar manner, we will define the past event horizon, comprised of all points that

can never be influenced by the observer γ . It can be easily shown to be

$$\text{EvtHor}^-(\gamma) = \partial J^+(\gamma) \quad (67.2)$$

The set of future and past event horizons may be disconnected, in which case we say that every connected component of that set is an individual horizon.

Proposition 67.1. If for two timelike curves γ_1, γ_2 , we have that for any $p \in \gamma_1$ and $q \in \gamma_2$, $I^+(p) \cap \gamma_2 \neq \emptyset$ and $I^+(q) \cap \gamma_1 \neq \emptyset$, then

$$\text{EvtHor}(\gamma_1) = \text{EvtHor}(\gamma_2) \quad (67.3)$$

Proof. It suffices to show that the two sets defined by $\{q | J^+(q) \cap \gamma_{1,2} \neq \emptyset\}$ are identical. If this is true for the first observer, then there is a future-directed causal curve from q that intersects γ_1 . If we call this intersection point p , by our assumptions, we have that there exists a future-directed timelike curve from p to a point r of γ_2 . From the properties of timelike curves,

$$q \leq p, p \ll r \rightarrow p \leq r \quad (67.4)$$

Hence the two observers share the same event horizon. $\square$

This will allow us to define the event horizon of entire subsets of the spacetime for some given class of observers.

Example 67.2. Rindler observers in Minkowski space possess an event horizon, called the Rindler horizon.

Proposition 67.3. In an asymptotically flat spacetime, the horizon for an observer going from i^- to i^+ is $\partial I^-(\mathcal{I}^+)$

show that if a particle passes a future horizon, it never leaves it, and if it passes a past horizon, it has never been outside of it.

67.2 Black holes

The exact definition of a black hole can be complicated, but in

Definition 67.4. In an asymptotically flat spacetime, a *black hole* is the remainder of the spacetime minus the past of future null infinity

$$M \setminus I^-(\mathcal{I}^+) \quad (67.5)$$

A black hole corresponds to a region where no particle can escape to infinity.

We have similarly the definition of a white hole

Definition 67.5. In an asymptotically flat spacetime, a *white hole* is the remainder of the spacetime minus the future of past null infinity

$$M \setminus I^+(\mathcal{I}^-) \quad (67.6)$$

Boundary of black and white holes are horizons???

67.3 Particle horizon

67.4 Non-local horizons

Particle horizon : $E^+(p)$ for an observer at p

Chapter 68

Trapped surfaces

While it is quite common to talk about black holes in terms of their horizon, this is not a very easy notion to deal with. Horizons tend to rely on global properties of the manifold, making them a thorny issue to determine in most cases. If we do not know the global spacetime topology, can we talk about a region being a black hole?

Example of the retrocausality of horizons

This is why it can be useful to define the notion of a trapped surface, which is a purely local notion and therefore much easier to test for.

[186]

Trapped surfaces are generally spacelike surface of co-dimension two obeying certain properties making them resemble our usual notion of a black hole's event horizon. Let's consider some dimension $n - 2$ submanifold $\Phi : S \rightarrow M$, expressed in local coordinates as:

$$x^\mu = \Phi^\mu(y^a) \tag{68.1}$$

x coordinates on M , y coordinates on S .

Pushforward of $\partial_a \in TS$ to vectors of $T\mathcal{M}$

$$e_a = \Phi_*(\partial_a) \rightarrow e^\mu{}_a = \frac{\partial \Phi^\mu}{\partial \lambda^a} \tag{68.2}$$

First fundamental form γ :

$$\gamma = \Phi^*g \rightarrow \gamma_{ab}(\lambda) = g|_S(e_A, e_B) = g_{\mu\nu}(\Phi(\lambda))e^\mu{}_a e^\nu{}_b \tag{68.3}$$

S spacelike : γ Riemannian.

$$T_p M = T_p S \oplus T_p S^\perp \quad (68.4)$$

Definition 68.1. A future (resp. past) closed trapped surface is a closed spacelike surface with an area that decreases locally along any possible future (resp. past) direction.

Causal orientation of $\vec{H}$:

$\downarrow$: past-pointing timelike $\searrow$ or $\swarrow$: past-pointing null ($\vec{k}^\pm$) $\leftarrow$ or $\rightarrow$: spacelike $\cdot$: vanishes $\nearrow$ or $\nwarrow$: future-pointing null $\uparrow$: future-pointing timelike

Chapter 69

Distributions on manifolds

Functions on manifolds may not be sufficient for a variety of purpose, such as the computation of Green's functions, point-like sources, and other somewhat singular behaviours.

Distributions are dual function spaces, so that given a specific function space $F(M)$ on M , the corresponding space of distributions is the space of maps

$$\phi : F(M) \rightarrow F \tag{69.1}$$

for some field F .

Distributions are *functionals*. From what we saw earlier, another example of functionals is the integration

$$\int_U (-) d\mu : C^\infty(M) \rightarrow \mathbb{R} \tag{69.2}$$

On a manifold, the easiest function space to deal with without restrictions is the space of smooth functions of compact support, noted $C_c^\infty(M)$.

We can define the space of *test functions*

Schwartz distributions

Compact support distributions

Dirac distribution

Definition 69.1. The distribution where for any test function ϕ associates the value of ϕ at a point $p \in M$ is the *Dirac distribution* δ_p :

$$\delta_p[\phi] = \phi(p) \tag{69.3}$$

This notion generalizes for scalar functions to tensor fields by considering *test tensor fields* [187], which are compactly supported smooth rank (r, s) tensor fields. For non-singular distribution, this can be seen to correspond to

Distributional tensors

Another way to consider distributions is as the limit of certain sequences of functions. We consider some sequence of continuous functions $(f_i)_{i \in \mathbb{N}} \in C(M, \mathbb{R})^{\mathbb{N}}$, on which we define the product defined by

$$\langle f_i, \phi \rangle = \int_M f_i \phi \quad (69.4)$$

Definition 69.2. A sequence of integrable functions $(f_i)_{i \in \mathbb{N}}$ is *weakly convergent* if for any bump function $\phi \in C_c^\infty(M)$, their integral converge :

$$\lim_{i \rightarrow \infty} \langle f_i, \phi \rangle = \lim_{i \rightarrow \infty} \int_M f_i \phi d\mu \quad (69.5)$$

Distributions as sequences of functions and how it relates to contour integration

Principal value

$$\text{PV} \quad (69.6)$$

69.1 Currents

[188]

Integration over a submanifold S is a current:

$$[[S]](\alpha) = \int_S \alpha \quad (69.7)$$

Stokes theorem for currents

69.2 Distribution algebras

Despite their usefulness for generalizing functions, distributions come with a rather large issue : it is extremely difficult to define a meaningful product on them. This is expressed in the *Schwartz impossibility result* [189] :

Theorem 69.3. There exists no differential algebras of distributions such that the identity distribution Id is the algebra's unit, the distribution's derivative is the differential algebra's derivative, the restriction of distributions to continuous functions reduces to their pointwise product, and the Leibniz rule $D(fg) = (Df)g + f(Dg)$ holds.

Proof. □

This will get in the way of quite a lot of use of distributions, from their use in non-linear theories to the Green's function on a distributional source.

Because of this, if we hope to define an algebra on distributions, we need to circumvent this theorem by weakening one of its assumptions. This can range from defining the algebra on a subset of the distributions, to making it fail to be closed on $\mathcal{D}'$, to make it fail the Leibniz rule on some products.

[190]

69.2.1 Wavefront set

One way we can try to define an algebra for distributions is to first figure out exactly what subset of distributions can form an algebra without any help. As we've seen already, the image of continuous functions $C(M)$ into $D'(M)$ naturally forms an algebra the usual way real functions do, while products of singular distributions such as δ_p will not converge to an appropriate distribution.

This is not the largest such subsets we could use however, as we could check by considering the product of two distributions with a disjoint singular support. If we have $p \neq q$ for instance, let's try to consider the product $\delta_p \delta_q$ as a weakly converging sequence

$$(\delta_p \delta_q)[\phi] = \lim_{\varepsilon_1} \lim_{\varepsilon_2} \int_M \frac{1}{|\varepsilon_1 \varepsilon_2| \sqrt{\pi}} e^{-((\frac{x-p}{\varepsilon_1})^2 + (\frac{x-q}{\varepsilon_2})^2)} \phi(x) dx \quad (69.8)$$

$$= \quad (69.9)$$

Singular fiber

Definition 69.4. For a distribution $\phi \in D'(M)$, its *wavefront set* $\text{WF}(\phi)$

[191]

69.3 Green's functions

For a linear differential operator L

Green's functions on $\mathbb{R}^n$

Example : Greens function of the Laplacian

$$\Delta\phi(p, q) = \delta(p, q) \quad (69.10)$$

69.4 Renormalization

One issue we will run into quite frequently in classical general relativity and then a lot more often in quantum field theory is that we will run into Schwartz' impossibility result. A frequent example in classical settings is to consider some linear PDE

$$L\Phi = f \quad (69.11)$$

where we wish to calculate the Green's function of L to solve :

$$LG = \delta \quad (69.12)$$

and then obtain the solution

$$\Phi(x) = \int G(x, s)f(s)ds \quad (69.13)$$

What can we do if the source term f is itself a distribution, as is very common in physics? Consider for instance the case of a point particle going along a curve γ :

$$f(s) = \delta(s - x(\lambda)) \quad (69.14)$$

EM field of such a point particle :

$$A = \int G(x, y)q\delta(s - x(\lambda)) \quad (69.15)$$

Example of the field energy for EM, gravitation

Find some GR examples

69.4.1 Expansion into sequences

One way to look into the phenomenon first is to look at the sequence representation of distributions. For any distribution $f \in \mathcal{D}'$, there exists a weakly convergent sequence (f_n) such that

$$f[\phi] = \lim_{n \rightarrow \infty} \int_{\text{Supp}(\phi)} f_n \phi d\mu[g] \quad (69.16)$$

If we attempt this in the case of a charged point particle, we can look for instance at the most common representation of the Dirac distribution,

$$\delta_p[\phi] = \lim_{n \rightarrow \infty} \int_M \quad (69.17)$$

Chapter 70

Distributional metrics, thin-shells and other

If we have tensor distributions, we can have in particular distributional geometric objects such as metrics and other such objects. The most common use of this is to consider C^0 metrics that are only piecewise C^2 , so that we can deal with such things as sources of matter that vanish discontinuously outside of a region, or singular sources like shells, strings and points.

From what we have seen on algebras of distributions, different types of metrics will lead to different generalizations

What happens to the connection and curvature

If we wish our Riemann tensor to be a distribution, we need to consider the *Geroch-Traschen class* of metrics [187]. For the Riemann tensor to be a distribution, as it is composed of the product of elements of the connection and its derivatives, we will ask that the connection itself be a distribution, and some condition for its products to also be a distribution. In the Geroch-Traschen case, this is for the connection to be locally integrable.

Significance of having a distributional curvature

70.1 Geodesics

The uniqueness of geodesics in the normal neighbourhood that we saw in X was derived under the condition that the metric be of regularity C^{1-} (?), so that with our new metrics of fairly low regularity, we may have to start from scratch [...].

In terms of the geodesic spray, the geodesic equation can be seen as

$$(x^\mu, p_\mu) \rightarrow (g^{\mu\nu} p_\nu, \Gamma^\mu_{\alpha\beta}(x^\mu) p^\alpha p^\beta) \quad (70.1)$$

This type of differential equation cannot be analysed under most formalisms for them, as the connection Γ is discontinuous in x . To deal with this, we will have to use *Filipov differential inclusion* [192, 193].

In the context of the Filipov theory, we do not consider differential equations as equalities, but rather as

[194, 195] [196]

Israel junction condition [197]

Expression of the metric in adapted coordinate over the surface :

Discontinuity of a tensor

Definition 70.1. For a tensor distribution T , its *discontinuity* $[T]$

Discontinuity over a hypersurface is entirely normal :

Theorem 70.2. The discontinuity of the derivative of the metric $[\partial_\sigma g_{\mu\nu}]$ over some hypersurface Σ with normal vector n in the normal bundle obeys

$$[\partial_\sigma g_{\mu\nu}] = n_\sigma \gamma_{\mu\nu} \quad (70.2)$$

for some tensor γ .

Proof.

□

70.2 Cut and paste procedure

A natural source of distributional metrics is the cut and paste procedure. If we have some (not necessarily connected) manifold with boundary M with two connected boundary elements $\partial S_1, \partial S_2$ which are diffeomorphic by some diffeomorphism $h : \partial S_1 \rightarrow \partial S_2$, on which some fiber bundle E is defined,

Theorem 70.3. For a pseudo-Riemannian manifold with boundaries such that two connected components of that boundary admit an isometry $h : \partial S_1 \rightarrow \partial S_2$, $h_* g_{\partial S_1} = g_{\partial S_2}$, the adjunction space by that isometry is a C^0 pseudo-Riemannian manifold.

Proof. As we've already shown, the gluing of such a manifold along its boundaries is a smooth manifold [X], so that we now only need to prove that it is also an appropriate pseudo-Riemannian manifold.

If we have our sections σ_α defined on our original manifold's atlas, we now only need to show that those sections extend to the atlas of the glued manifold and that those sections are compatible. $\square$

Form of the coordinates on the boundary patch

[198]

Chapter 71

Trigonometry

71.1 Trigonometry in Minkowski space

In the Minkowski affine space, geometry can be defined in very much the same way as in Euclidian space, by replacing the inner product by the Minkowski bilinear form,

$$\overline{AB} \cdot \overline{AC} = \langle \overline{AB}, \overline{AC} \rangle \quad (71.1)$$

which allows us to define the congruence between two segments. If the lines are timelike, this will correspond to congruence of their proper time, and if they are spacelike, the congruence of their length. Every null line will be congruent in this case.

For two incident non-null segments $\overline{AB}$, $\overline{AC}$, the angle between them at A is

$$\widehat{BAC} = \frac{\langle \overline{AB}, \overline{AC} \rangle}{\|\overline{AB}\| \|\overline{AC}\|} \quad (71.2)$$

Just like in Euclidian geometry, this angle is invariant under the prolongation of the segments. If B' is a point colinear with $\overline{AB}$ such that B is between A and B' , then

$$\widehat{B'AC} = \widehat{BAC} \quad (71.3)$$

Definition 71.1. Given two non-null vectors $X_p, Y_p \in T_p M$, their angle is given by

$$\widehat{XY} = \frac{g(X, Y)}{\|X\| \|Y\|} \quad (71.4)$$

Gauss-Bonnet theorem

71.2 Trigonometry in curved space

In a totally geodesic neighbourhood U , we can consider segments between any two points $p, q \in U$ by taking the unique geodesic $\overline{pq}$ connecting them. A geodesic triangle is composed of three points connected by geodesics between any two pairs of points, so that a triangle is the complex defined by

$$\Delta = (p, q, r, \overline{pq}, \overline{qr}, \overline{pr}) \quad (71.5)$$

Family of geodesics between pq and pr , between qp and qr and rq and rp

$$\Gamma_{\overline{pq}, \overline{qr}}(s, t) = \exp \dots \quad (71.6)$$

Pythagorean theorem [199]

Gauss-Bonnet

Photon bounce

Chapter 72

Observers

An *observer* in general relativity is some abstraction of a physical object that is capable of serving as a device to perform experiments (the validity of such abstractions are delayed until chapter 215). Due to this pretty broad definition it can have quite a lot of definitions, but for the most part, observers are in general modeled by timelike curves γ (or paths $[\gamma]$) with extra structures defined on it.

Observers typically have some orientation defined on them, if the curve γ admits a consistent orientation. If they do, and the underlying spacetime is time-oriented, they are future-oriented curves.

An observer will generally carry a clock h , which is a map $h : I \rightarrow \mathbb{R}$. In other words the clock will be some parametrization of the curve. If the observer is simply described by a path this will just be a parametrization, while if it is a parametrized curve, the clock will be some reparametrization. The parametrization of the curve we consider does not need to be identical to that of the clock, useful parametrizations such as proper time can be useful for calculations while a clock is not guaranteed to have such properties. This clock in fact corresponds to a section of the frame bundle of the curve itself, FL , such that the parametrization h has that section as a holonomic basis.

An observer usually carries a frame, a section of the pullback frame bundle γ^*FM . It is generally assumed that whatever frame we have, one of its basis vector will be co-linear (and with the same orientation if oriented) as the tangent of the curve, $e_0 = \dot{\gamma}$. If given a clock, it will furthermore usually have the norm associated with that parametrization.

As a curve, we can lift it to any of the pullbacks of the velocity bundles, $\gamma^*T^k_1$, to obtain its derivatives $j^k\gamma = (\gamma, \dot{\gamma}, \ddot{\gamma}, \dots)$, although as most of our theories will focus on second derivatives at most, we rarely consider its properties beyond $k = 2$, so that we only consider its position $\gamma(\lambda)$, its velocity $\dot{\gamma}(\lambda)$, and its acceleration,

$$a = \nabla_{\dot{\gamma}}\dot{\gamma} \tag{72.1}$$

As an observer is typically assumed to be an open curve (either modelled on L , L^+ or I), and therefore a contractible space, the pullback of the frame bundle is trivial short of a self-intersection on the curve. Therefore observers will always have a globally definable local frame short of forming closed timelike curves.

Since an observer defines a preferred direction, it is natural to consider a structure on it. If we wish to only consider frame rotations that will preserve the direction of the curve as a basis vector, the observer's frame bundle splits to the subgroup $GL(n-1) \times GL(1) \times \mathbb{R}^{n-1}$ to preserve the time direction of its velocity, or a subgroup of this if we wish to also preserve its direction or norm.

For the case of orthonormal frames, this leaves us with an $O(n-1)$ bundle, the group of rotations for spatial frames.

Splitting on the observer using the time direction and spatial frame

[200]

Given a frame at some initial time $\tau = 0$, even considering a class of frame fields over the observer where the timelike basis vector is given by the onboard clock and direction of the observer, there are still quite a large number of possible sections. However, there exist a "natural" time evolution for a frame

Rotation of the frame

Fermi-Walker transport

72.1 Observer coordinates

For practical purpose, it is desirable to have adapted coordinates for observers, as these will typically be the type of coordinates that we will get (or that can easily be reconstructed from) experimental observations.

If the observer's curve is an embedded submanifold, it admits a tubular neighbourhood around it

72.1.1 Fermi coordinates

One common type of adapted coordinates on a curve (including observers) is the *Fermi coordinates*.

Fermi-Walker transport of the frame

Frenet frames

72.1.2 Märkze-Wheeler coordinates

[201]

72.1.3 Radar coordinates

A drawback of Fermi coordinates is that they rely on the use of spatial hyperslices without much indication of how one is supposed to derive them. One operational method to measure distances in Minkowski space is the *radar method*, which relies on the assumption that light has a constant speed, so that if the exchange of light signals between two observers takes a time $\delta\tau$ as measured by proper time, this will correspond to a distance of exactly

$$d = \frac{c}{2\delta\tau} \quad (72.2)$$

This is the *Einstein synchronization* convention. This convention can in fact be extended to the curved spacetime case within a certain distance of a given observer.

Radar coordinates

Märkze-Wheeler coordinates

Are radar coordinates always Fermi coordinates?

72.2 Observer fields

An observer field, also called a *mollusk* [202], is a nowhere vanishing timelike distribution $\mathcal{O}$, leading to a foliation of the manifold by timelike curves. Every point of the spacetime is therefore associated to an observer in this manner.

This foliation is of course not unique, but in particular, the topology of the leaves is not even necessarily fixed. We can take as an example the cylinder spacetime $L \times (S \times L)$, which admits foliation(????)

[Example with that Earman case(?)]

rotation

Radar neighbourhood

Foliation for observer fields

Chapter 73

Time-space splits and ADM formalism

Most of physics uses some splitting of time and space for practical calculations, to use Hamiltonian mechanics, or in the approximation where we consider classical mechanics, which has a canonical split. As we've seen in the chapters on causality, such splits of spacetime into time and space are not trivial, and in many cases impossible to do perfectly. We will therefore have to consider a variety of possible case, with various degrees of what we could consider being a proper time-space split as in the classical case.

73.1 Splitting structures and foliations

The splitting of spacetime into time and space can be considered at three levels : a point-wise structure, where we perform the split at a single point $p \in M$, a local structure, where the split holds in some region $U \subset M$, and as a global structure, where the split holds for the entire spacetime M .

At any given spacetime point, we can always split our tangent bundle by considering any 1-dimensional timelike subspace T and some complementary $(n - 1)$ -dimensional spacelike subspace S , often taken to be orthogonal to $S : S = T^\perp$. A further structure we can add is a pair of a timelike 1-form and spacelike $(n - 1)$ -form (τ, ϵ) that will define orientations and scales at that point, where any vector can be compared via its projection onto the subspace defined by those forms. Given a vector v , its projection on those pair of subspaces will be

$$x \tag{73.1}$$

For the pair of subspaces, the class of transformation of the vector space leaving those two structures unchanged are the intersection of the stabilizer subgroups of $GL(n)$, $P(1)$

and $P(n-1)$

$$\mathrm{GL}(n) \rightarrow \mathrm{GL}(n-1) \times \mathrm{GL}(1) \quad (73.2)$$

we can consider

$$\mathrm{O}(n-1, 1) \rightarrow \mathrm{O}(n-1) \times \mathcal{O}(1) \quad (73.3)$$

Splitting structure : timelike vector field

Similarly, the local and global structure of the split can take two different forms : we can consider a foliation by timelike curves [mollusk], giving in some sense the "time" structure, and we can consider a foliation by spacelike hypersurfaces, giving the "spatial" structure. Ideally we should be able to have the two at once, but as we will see, having a completely well-behaved spacetime split will not be a trivial matter.

Spacetimes always admit a foliation by timelike curve, as the presence of a Lorentz metric indicates the existence of a 1-distribution in the timelike direction. As 1-distributions are always involutive, by the Frobenius theorem, there is always such a foliation. This foliation doesn't necessarily have diffeomorphic leaves. Since we are in the one dimensional case, the leaves can only be lines or circles. A basic example is to consider the Deutsch-Politzer spacetime

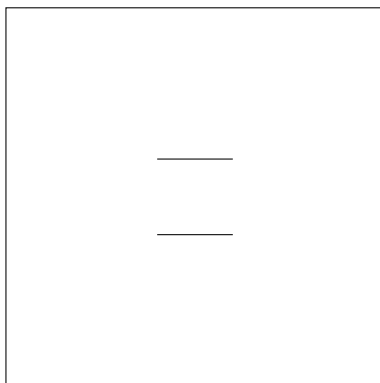


Figure 73.1: Foliation of the Deutsch-Politzer spacetime

That foliation is furthermore not unique [Time structure example from Earman]

Leaf space of the foliation

If our spacetime is furthermore time-orientable, we can furthermore consider a foliation by oriented timelike curves, which are more commonly what is assumed to be meant by an observer.

The spatial foliation is, as we saw previously, a lot more difficult to guarantee. We will always have some local such

If our timelike foliation has diffeomorphic leaves, we can consider endowing it with a bundle structure.

$$\pi : M \rightarrow \mathbb{R} \quad (73.4)$$

If the integral lines of the observer field forms a bundle

Type of bundle formed by the observer field

Non-principal connection on the observer bundle v. principal connection

[203]

There are therefore a gradation of such splits that we can do :

- On the infinitesimal level, the split can always be performed at any point.
- On a local level, the split can also always be performed
- On a global level, we can always define a global time direction
- If the spacetime is time-orientable, we can define a global oriented time direction, defined by a timelike vector field
-

Gradation :

Global section to the line bundle of timelike curves Global flow of this global section (obstruction : singularities?) Global slice to the observer field O (obstruction : non-integrable distribution) Unique global complementary sub-bundle Σ Global transverse foliation to O

Bott connection

Bel decomposition

[204]

Splitting map [205] :

$$S : \chi^k(P) \rightarrow \chi^k(X, G) \times \chi^{(k-1)}(X; G; \mathfrak{g}) \quad (73.5)$$

$$v \mapsto (k, \ell) = (\Pi v, \Pi \iota_\omega v) \quad (73.6)$$

Inverse map :

$$S^{-1} : (k, \ell) \mapsto v = \Sigma k + w \wedge \Sigma \ell \quad (73.7)$$

Splitting map of differential k -forms :

$$S^{-*} : \Omega^k(P) \rightarrow \Omega^k(X, G) \times \Omega^{(k-1)}(X; G; \mathfrak{g}^*) \quad (73.8)$$

$$\theta \mapsto (\alpha, \beta) = (\Sigma^* \theta, \Sigma \iota_w \theta) \quad (73.9)$$

Inverse map :

$$S^* : (\alpha, \beta) \mapsto \theta = \Pi^* k + \omega \wedge \Pi \beta \quad (73.10)$$

73.2 Split spacetime

Projection of the metric tensor on the split [206]

$${}^{(n-1)}g = \Sigma^* g \quad (73.11)$$

Lapse function

$$\alpha = O_* g \quad (73.12)$$

Shift vector field

$$\beta \quad (73.13)$$

Chapter 74

Morse theory

A way in which the global topology affects the local physics on a spacetime is given by the study of critical points of its functions, that is, the set of points on which a function vanishes. This topic constitutes what is referred to as *Morse theory*.

Critical points, critical locus

Chapter 75

Topology change and Lorentz cobordism

Topology change corresponds to the notion of the topology of space changing between two moments. The basic idea being that we have a topology change if we have two boundaryless achronal spacelike hypersurfaces that are not diffeomorphic. To make this rigorous, we will have to use the notion of cobordisms.

75.1 Cobordisms

A cobordism is an n -dimensional manifold with boundaries with two $(n - 1)$ -dimensional manifolds as boundaries Σ_1 and Σ_2 .

Lorentz cobordism : $(\mathcal{M}, v)$ is a Lorentzian manifold with boundaries $\mathcal{M}$ and a vector field v such that $\partial M = \Sigma_1 \sqcup \Sigma_2$ and v is interior normal on Σ_1 and exterior normal on Σ_2 .

[207–210]

M_1, M_2 are spacelike hypersurfaces

$\mathcal{M}$ is a "cutout" of a larger spacetime along Σ_1 and Σ_2 . We can extend $\mathcal{M}$ into a manifold without boundaries

Example of non-trivial Lorentz cobordisms :

Trouser spacetime : topology $S^2 \setminus (D^2 \sqcup D^2 \sqcup D^2)$, cobordism $M_1 = S^1, M_2 = S^1 \sqcup S^2$

Does topology change have some consequence of the "shape of space" (leaf space of M foliated by observers)

75.2 Cobordisms and causality

Theorem on cobordisms, singularities, time orientability and causality

Chapter 76

Non-Hausdorff spacetimes

[211]

While the Hausdorff condition is quite useful for a variety of theorems, and makes the resulting manifolds more intuitive as locally Euclidian spaces, we have so far not really given any strong reason to exclude them. Unlike paracompactness, there is no fundamental reason for us to reject them out of hand for spacetime models. Non-Hausdorff spaces do admit Lorentz metrics, connections and other such structures. So for this chapter, we will look into the properties and applications of non-Hausdorff spacetimes.

The Hausdorff condition as we've seen in 4.21 is the separation property that for any two points p, q , there should exist two neighbourhoods of those points U_p, U_q

Examples of non-Hausdorff manifolds

Example 76.1. The manifold given by the gluing of n copies of the real line $\mathbb{R}$ over the set $(-\infty, 0)$ is a non-Hausdorff manifold called the *branching real line*.

Example 76.2. The manifold given by the gluing of n copies of the real line $\mathbb{R}$ over the set $(-\infty, 0) \cup (0, +\infty)$ is a non-Hausdorff manifold called the *line with n -origins*.

76.1 Analysis

76.2 Non-Hausdorff spacetimes

The notion of non-Hausdorff spacetime is generally considered in some way to preserve locality, in the sense that the location of a spacetime's adjacent points cannot involve some arbitrary boundary. For this, it is generally assumed that any non-Hausdorff spacetime will split along the horismos of a given set.

76.3 Modality

One of the motivation behind the use of non-Hausdorff spacetimes is the use of *modality*, the notion that certain things are *possible*, corresponding in some sense to a particular branch of spacetime that one could find themselves in. This is a notion that is more commonly found in the domain of philosophy than physics, but those models exist nonetheless.

Use in QM

Definition 76.3. Given a proposition p , the necessity modality of p , $\Box p$, means that it is necessarily true, while the possibility modality $\Diamond p$, means that it is possibly true.

Modal axioms :

$$p \tag{76.1}$$

Chapter 77

Exotic spacetimes

As mentionned in chapter 16, a manifold with a given C^0 atlas $\mathcal{A}$ may admit more than one incompatible smooth structure $\mathcal{A}_S$. In other words, there may be two such smooth atlases such that their union is not compatible.

Every topological Hausdorff manifold up to dimension 3 admits a unique smooth structure (there are non-Hausdorff examples of multiple smooth structures as soon as dimension 1 though), but starting with dimension 4, coincidentally the dimension of the typical spacetime model for our universe, manifolds as trivial as $\mathbb{R}^4$ admit multiple smooth structures, and in fact infinitely many smooth structures in the case of $\mathbb{R}^4$.

Those manifolds are referred to as *exotic* $\mathbb{R}^4$, denoted by $\mathbb{R}_\Theta^4$.

small and large exotic $\mathbb{R}^4$

non-topological product $\mathbb{R}^3 \times \mathbb{R}^1$

[212]

Donaldson sphere

77.1 The 7-sphere

One of the most famous construction of an exotic manifold is Milnor's constructions of exotic 7-spheres [213]. Fortunately enough for us, the odd dimension coincides with the lack of obstruction for spacetime structures on a sphere, so that we can if we wish to build a spacetime out of it.

77.1.1 The Hopf fibrations

Before we can look into the 7-sphere, we first need to look into a specific construction of the 7-sphere. In the general case, this is the *Hopf fibration*.

The Hopf fibration allows us to give certain spheres a bundle structure, where a sphere S^n is a fibration of S^k by S^r , such that $k + r = n$. This construction only works for four different cases, based on the only four normed division algebras (by Hurwitz's theorem), the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

H-spaces, spheres, etc

If we have some normed division algebra A , we can consider the space of elements $a \in A$ of unit norm, $|a| = 1$, which is a sphere of dimension $\dim(A) - 1$, the space of pairs of elements of that algebra $(z_1, z_2) \in A \times A$ with squared norms summing up to 1, $|z_1|^2 + |z_2|^2 = 1$, of dimension $2\dim(A) - 1$, and the compactification of that algebra, AP , of dimension $\dim(A)$. Using homogeneous coordinates, AP can be written as the pair of coordinates (a, x) for $x \in \mathbb{R}$

Then for our pair of elements, we have the projection to AP given by

$$\pi(z_1, z_2) = (2z_1 z_2^*, |z_1|^2 - |z_2|^2) \quad (77.1)$$

Find the pre-image

For $\mathbb{R}$, the space of all points with norm 1 is simply $\mathbb{Z}_2 = \{-1, 1\}$, which is also the zero sphere S^0 , the compactification is the projective line $\mathbb{R}P^1 \cong S^1$, and the pair of points of summed norms 1 is the circle S^1 in $\mathbb{R}^2 : x^2 + y^2 = 1$. The resulting fibration is the real Hopf fibration

$$S^0 \hookrightarrow S^1 \rightarrow S^1 \quad (77.2)$$

This is simply a double cover of the circle over itself. The corresponding "spacetime" would be the one-dimensional spacetime $(S^1, -d\theta^2)$

In the complex case, the space of all points with norm 1 is the unit circle of the complex plane, $\mathcal{U}(1) \cong S^1$, the compactification is the Riemann sphere $\mathcal{C}P \cong S^2$, and the pair of points of summed norm 1 is the 3-sphere S^3 :

$$z_1 z_1^* + z_2 z_2^* = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 \quad (77.3)$$

The resulting fibration is the complex Hopf fibration (the original fibration of Hopf) :

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad (77.4)$$

The case of our interest is the quaternionic case $\mathbb{H}$. In this case, the space of all points of norm 1 is the space of unit quaternions, which is isomorphic to S^3 and $SU(2)$. Its compactification is the quaternionic projective space, $\mathbb{H}P$, homeomorphic to the 4-sphere S^4 , and the pair of points of summed norm 1 is the 7-sphere S^7 :

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad (77.5)$$

Let's consider two charts over S^4 , the usual standard charts U_N covering S^4 without the south pole and U_S without the north pole.

$$\phi_{N/S} : U_{N/S} \rightarrow \mathbb{R}^4 \quad (77.6)$$

In terms of quaternions, this maps projective quaternions to quaternions. The two charts correspond to projective quaternions without infinity (with homogeneous coordinates $[z; 1]$), and projective quaternions without zero (with homogeneous coordinates $[1, z]$)

$$\phi_N : [z; 1] \mapsto z \quad (77.7)$$

$$\phi_S : [1; z] \mapsto z \quad (77.8)$$

The transition map is given by the inversion map

$$\phi_S \circ \phi_N^{-1} : \mathcal{H}_{\neq 0} \rightarrow \mathcal{H}_{\neq 0} \quad (77.9)$$

$$z \mapsto z^{-1} \quad (77.10)$$

Standard Hopf fibration : total space S^7 is

$$\{((z_1, z_2); [z, w]) \in \mathbb{H}^2 \times \mathbb{H}P \mid z_1 w = z z_2, z, w \neq 0\} \quad (77.11)$$

Do the Milne spheres $M_{l,h}$, prove they are topological spheres, prove they are not diffeomorphic

Show what the tangent bundle looks like

Example Lorentzian metric on it with vector field along the S^1 , plus some Riemannian metric on S^7

77.2 Exotic $\mathbb{R}^4$

An important type of exotic manifold in the case of general relativity is the exotic $\mathbb{R}^4$, which is composed of manifolds homeomorphic to the canonical $\mathbb{R}^4$ with the identity map, but with an exotic smooth structure.

Theorem 77.1. There are infinitely many exotic $\mathbb{R}^4$.

Relation to $\mathbb{R} \times \mathbb{R}^3$

Singular structure

Chapter 78

Continuous metrics

We have so far mostly considered smooth metrics, metrics that are at least C^3 , or metrics which are at least piecewise regular. If we are willing to drop some of the machinery of differential geometry, we can consider the case of metrics that are only C^0 .

C^0 metrics will not admit connections or curvatures even in a distributional sense if irregular enough, so that we will only have the metric itself for our geometry. We can however still say a variety of theorems concerning the causal structure of such spacetimes.

[214]

Chapter 79

Discrete spacetimes

Either as a simplification or as a model in itself, we can want to have a spacetime based on a discrete structure, in the sense that our spacetime is composed of a countable number of points.

79.1 Regge calculus

Definition 79.1. A *simplicial manifold*

”The simplicial de Rham complex generalizes the de Rham complex from ordinary manifolds to simplicial manifolds.”

79.2 Rational spacetimes

We can replace the field in our definition of spacetimes from $\mathbb{R}$ to $\mathbb{Q}$. This method is not commonly used in general relativity, but it has been used in the description of special relativity, such as in attempts to define special relativity as a first order logical theory [215], as real numbers cannot be properly constructed as a first order theory.

One feature of such theories is that we will be typically unable to use any element of usual relativistic theories that rely on functions such as the square root etc. While this can make some typical theorems of relativity inaccessible, there are however still quite a lot we can say about them.

An example of such a theory is the **Basax** axiomatization of special relativity,

$$\mathbf{Basax} = (B, F, G, \text{Obs}, \text{Ph}, \text{Ib}, +, \cdot, \leq, E, W) \quad (79.1)$$

79.3 Finite field manifolds

Another angle we could take to talk about discrete spacetimes is to consider instead of being modeled on $\mathbb{R}^n$, which is an infinite complete field, by a finite field instead, as we defined in 6.3.

Finite vector spaces

[216]

Finite fields also have a natural structure of incidence spaces, where fields of order p^n are n -dimensional incidence spaces

Addition : translation along a line

Multiplication, inversion

Groups on $\mathbb{F}_{p^n}$

Finite Lorentz group

+

Chapter 80

Supergeometry

For some extensions of physics, as well as some formalisms that we will see later in this book such as the BRST formalism, we need to consider a generalization of manifolds to allow anticommutations of certain objects. This is the context of *supergeometry*.

80.1 Superalgebras

If we wish to make our supermanifolds in a manner similar to manifolds, built from charts to our model space, we need to consider a new space for it called the *supernumbers*.

The construction of supernumbers starts with a $\mathbb{Z}_2$ -graded vector space, called a *super-vector space*

$$V = V_0 \oplus V_1 \tag{80.1}$$

with the degree map

$$\deg : V \rightarrow \mathbb{Z}_2 \tag{80.2}$$

A vector of degree 0 is said to be even, while a vector of degree 1 is said to be odd, with other vectors being of mixed types.

Braiding operator :

$$\tau : v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v \tag{80.3}$$

Corresponds to the anticommutation of odd elements

$$(v + \theta_1) \otimes (w + \theta_1) = \quad (80.4)$$

Every $\mathbb{Z}$ -graded algebra can be converted into a superalgebra

Example 80.1. The exterior algebra $\bigwedge V$ is a superalgebra.

Proof. The exterior algebra $\bigwedge V$ is already a $\mathbb{Z}$ -graded vector space, so that we can project it down to a $\mathbb{Z}_2$ -graded vector space $\square$

Example 80.2. The Chevalley-Eilenberg $\text{CE}(\mathfrak{g})$ is a superalgebra.

80.1.1 Supernumbers

Consider the Grassmann algebra of N -generators, $\{\zeta^a\}$, denoted Λ_N . If there is a countably infinite set of generators, Λ_∞ set of supernumbers

$$z = z_B + z_S \quad (80.5)$$

z_B is the *body* of a supernumber, while z_S is the *soul* of a supernumber.

Topology and analysis on supernumbers

80.1.2 Supervector spaces

Supertrace operator

Berezinian

Definition 80.3. The *Berezinian* $\text{Ber}(A)$ of a supermatrix A is the unique operator obeying the properties

$$\text{Ber}(AB) = \text{Ber}(A)\text{Ber}(B) \quad (80.6)$$

$$\text{Ber}(e^A) = e^{\text{str}(A)} \quad (80.7)$$

80.2 Super Lie groups

Just as the general linear group is the space of all automorphisms on a vector space, we can define the *super general linear group* $\text{GL}(n|r)$ to be the automorphisms on a given supervector space

General linear Lie superalgebra $\mathfrak{gl}(n|r)$

Special linear Lie superalgebra $\mathfrak{sl}(n|r)$

orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2r)$

Super Poincaré group $\mathfrak{poinc}(n|r)$

80.3 Supermanifolds

Supermanifolds are the equivalent of manifolds which are in some sense modelled on a super vector space. There are several ways to do this, being roughly split into the two original definitions of a manifold that we saw, either as a topological space equipped with an atlas or as a locally ringed space

80.3.1 Supercharts

[217] The most common definition of a manifold is unfortunately quite difficult to turn into a basis for supermanifolds, as changing the space on which manifolds are modelled will have quite drastic consequences and quite a lot of basic mathematics will have to be changed to fit this formalism, such as the algebra and analysis defined on $\mathbb{R}^{p|q}$.

The model that we use for the anticommuting version of real numbers is the *Grassmann algebra* with k generators Λ^k . In the case where the number of generators is countably infinite, it is denoted by Λ^∞ .

80.3.2 Locally ringed spaces

The definition of supermanifolds from supercharts, while having the benefit of being close to the more common definition of manifolds in physics, tends to be a bit too heavy of a machinery for its use. In the context of supermanifolds, it is much more common to consider instead its structure as a locally ringed space.

While common manifolds are defined as the locally ringed space of functions $C^\infty(\mathbb{R}^n)$, we can expand this function space to the space of $C^\infty(\mathbb{R}^n) \times \Lambda^\bullet$

80.3.3 Supermanifold structures

Equipped with those definitions of supermanifolds, we can now

80.4 Berezin integrals

As we completely changed the fields that we are considering, we can no longer safely rely on the Lebesgue integral as the basis for integration on a supermanifold. Instead we have to look into the *Berezin integral* [218]

Berezin integral : graded linear functional of superfunctions, translation invariant

$$\int d\theta f(\theta + \theta_0) = \int d\theta f(\theta) \quad (80.8)$$

80.5 NQP manifolds

N -manifold : lifting of a supermanifold from $\mathbb{Z}_2$ grading to $\mathbb{Z}$ grading via the even-odd map $\mathbb{Z} \rightarrow \mathbb{Z}_2$

Q manifold : supermanifold with odd-graded homological Q vector field : $[Q, Q] = 0$

P -manifold : Graded symplectic structure

Shifted tangent bundle $TM[k]$, ΠTM

80.6 The odd Fourier transform

Like the Fourier transform mapping $C^\infty(TM)$ to $C^\infty(T^*M)$, the odd Fourier transform is a map

$$\mathbb{SF} : C^\infty(T[1]M) \rightarrow C^\infty(T^*[-1]M) \quad (80.9)$$

Chapter 81

Specificities of dimensions

Now that we have roughly seen all the geometric properties a spacetime can have in general, we will focus on the properties they can have for a specific number of dimensions, as the behaviour of spacetimes of different dimensions can be radically different.

The properties we will review here are all purely geometric/kinematic, the dynamical properties of spacetimes of specific dimensions are reviewed in [X].

81.1 Zero-dimensional spacetimes

It is a bit of a stretch to call the single point zero-dimensional manifold $\{*\}$ a spacetime, as there is no non-trivial metric that can be defined on it. However, it is a very useful toy model to have at our disposal due to its very simple properties, and we will quite often use it.

As we saw in [X], there is only one connected (non-empty) zero-dimensional manifold, the one composed of a single point.

[...]

One of the main benefit of the zero-dimensional spacetime is that its space of sections of fiber bundles are all finite-dimensional, as any fiber bundle over it are the manifold of the fiber itself. For any fiber F , we have that all fiber bundles over $\{*\}$ have a total space isomorphic to the fiber F itself [proof?], and the set of all sections, which is just the set of functions $C(\{*\}, F)$, is in fact isomorphic to F .

The zero-dimensional orthogonal group is slightly complicated by the definition of the determinant in zero dimensions. There is only one automorphism for a zero-dimensional vector space, the identity map, which is a 0×0 "matrix".

81.2 One-dimensional spacetimes

81.3 Two-dimensional spacetimes

Two dimensions is typically considered the lowest dimension in which it makes sense to talk of a spacetime, as we can properly differentiate time and space in such a case.

Theorem 81.1. Existence of isothermal coordinates

Isothermal coordinates, Conformal group, Classification, uniformization theorem

One particularity of two-dimensional spacetimes is that Minkowski space in two dimensions has a specific type of transformation which is the inversion of time and space

81.4 Three-dimensional spacetimes and manifolds

Theorem 81.2. Every orientable 3-manifold is parallelizable

Proof.

□

Parallelizability, Weyl tensor, etc

81.5 Four-dimensional spacetimes

As our universe is suspected to be at least up to some approximations four dimensional, the case of $(3 + 1)$ dimensions is of particular importance here.

Lemma 81.1. For any given finitely presented group $\langle R, S \rangle$, a 4-dimensional smooth manifold can be constructed with that group as its fundamental group.

Proof. For a group $\langle g_1, \dots, g_n; r_1, \dots, r_m \rangle$, consider a wedge sum of n circles $\bigwedge^n S^1$

□

Theorem 81.3. There is no decidable method to classify 4-manifolds.

Proof. As every finitely presentable group can be put in correspondence with a manifold, this would mean that a general method to classify 4-manifolds would involve a method to classify finitely presentable groups, ie, if we could figure out whether two manifolds M_1, M_2 deriving from such a group, those manifolds being homeomorphic would imply their fundamental groups being isomorphic. However, the problem of finding isomorphisms between two finitely presentable groups is known to be uncomputable

□

Classification of simply connected 4-manifolds

Smooth structures

Black holes in $3 + 1$

Chapter 82

Infinite dimensional manifolds

[35]

A lot of physics relies on the properties of infinite dimensional manifolds (and more generally infinite dimensional spaces) to give for instance geometric structure to function spaces

We have already seen the quite general setting of diffeological spaces (and briefly touched on Cartesian sheaves), but for some more specific applications, we may need spaces with more structure to them.

Many of the infinite dimensional manifolds that we will need will be function spaces, and quite often those are some subset of the space of continuous functions between two given manifolds : $C(M, N)$.

Frechet, Banach, IHL,

82.1 Infinite dimensional diffeological spaces

Theorem 82.1. The space of sections $\Gamma(E)$ is a diffeological space.

82.2 Frechet manifolds

Definition 82.2. A *Frechet space* X is a topological vector space that is etc

Example 82.3. The countably infinite dimensional space $\mathbb{R}^\omega$ of real sequences is a Fréchet space.

Example 82.4. The space of smooth real functions $C^\infty(\mathbb{R}^n)$ is a Fréchet space

Definition 82.5. A *Fréchet manifold* M is a topological space with an atlas $\{(U_\alpha, \phi_\alpha)\}$ of charts to a Fréchet space X

$$\phi_\alpha : U_\alpha \rightarrow \mathcal{O} \subset X \quad (82.1)$$

...

Tangent bundle and cotangent bundle

$$T_f C^\infty(X, Y) = C^\infty(X, f^*TY) \quad (82.2)$$

82.3 Banach manifolds

82.4 Pro-manifolds

Given a family of manifolds $(M_i)_{i \in I}$, lying in a sequence

$$M_0 \xleftarrow{\pi_0} M_1 \xleftarrow{\pi_1} \dots \quad (82.3)$$

The projective limit

$$M = \varprojlim_{i \in I} M_i \quad (82.4)$$

is a *pro-manifold*.

A typical example of a pro-manifold is the bundle of infinite jets, $J^\infty E$.

Theorem 82.6. The infinite jet bundle $J^\infty E$ is a pro-manifold.

Part III

Dynamics

We have thus far only described spacetime geometry as a given object. We are given spacetime as a whole, with the informations of all of its structure for all time and space. Physics is however generally about the study of *dynamics*, the evolution of our objects of study. We would not expect to know in advance all the properties of the object we study, but at best only its property in the present and past, and hopefully extrapolate from there what it will be in the future.

Now that we are mostly concerned with more physical properties of our spacetime, it is time to fix what a spacetime actually is. From our previous chapter, we will take our spacetime to be, beyond being a set M ,

1. A topological space (M, τ)
2. A topological manifold $(M, \tau, \mathcal{A})$
3. A smooth manifold $(M, \tau, \mathcal{A}, \mathcal{A}_S)$
4. A Lorentzian manifold $(M, \tau, \mathcal{A}, \mathcal{A}_S, g)$
5. A Lorentzian manifold with a connection (usually adapted) $(M, \tau, \mathcal{A}, \mathcal{A}_S, g, \nabla)$

We can at will switch those definitions to other formalisms to describe those types of objects, replacing our atlas with a sheaf of rings of functions as in [X], our metric by frame fields or G -structure, and so forth, but this is on the whole the type of object we will be dealing with.

All those properties constrain it to be *second-countable*, *paracompact*, and admit some 1-distribution, constraining it further to be either noncompact or compact with an Euler characteristic of 0.

In addition, we will require

1. The spacetime should be *path-connected*, and therefore *connected*. There's no particular physical reasons to require it to be connected, but if we model any of our objects and observers as curves, we cannot perform any physics related to any disconnected part of our universe, so that considering only the connected part will simplify things.
2. The spacetime should be *Hausdorff*. While we saw that spacetimes can tolerate the dropping of the Hausdorff condition, in practical terms there are no benefits for it under current physical theories.

We will occasionally consider spacetimes with different Cartan geometries, and therefore different structures from the metric and Levi-Civita connection, but we will specify it if we do so.

It will also generally be the case that the spacetimes we consider are *time-orientable*, *space-orientable*, *orientable* and usually with some kind of causality condition, although a lot of this section will not necessarily demand it, but if those conditions are to be explicitly dropped, we will specify it. We will see in particular why it is generally assumed

Chapter 83

Physics and geometry

A lot of physics can be pinned down to the notion of a *configuration space*, which is the space of all the possible states a physical system can be in, some *boundary conditions*, usually in the form of *initial conditions*, where we describe how that system is in some region of spacetime that we are aware of, and then some *equations of motion*, describing how those objects will evolve in space and time.

83.1 Configuration space

One aspect of physics we will have to consider in some detail here is that of the *configuration space*, which corresponds to the space of all possible configurations a physical system can have. If we take the case of some field theory modeled by some vector bundle $\mathcal{V} \rightarrow M$, a first approximation of our configuration space will be the space of all smooth sections $\text{Conf} = \Gamma^\infty(E)$.

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As we saw in 17, Conf can be given the structure of a few different spaces, depending on the type of base manifold, fiber and regularity.

Convenient vector space, nuclear Fréchet space, diffeological space

While as a vector space, configuration spaces are typically infinite dimensional, we commonly refer to their dimension as the dimension of the typical fiber of its associated fiber bundle, if it has one. To avoid confusion, it is also commonly referred to as its *degrees of freedom*.

$$\text{d.o.f.}(\Gamma(E)) = \dim(F_E) \tag{83.1}$$

Consider its topology, differential structure (in particular jets and tangents), metric if it

exists

As any type of space that we consider will still be a diffeological space, we can define a tangent space to it, $T\text{Conf}$, as well as a cotangent space $T^*\text{Conf}$. If we have some one-parameter family of configurations ϕ_λ , a vector will be [...]

For the cotangent space, given any functional $F : \text{Conf} \rightarrow \mathbb{R}$, we have the Gateaux derivative [...]

Derivative of a path in Fréchet space, directional derivative in Fréchet space

Relation between the Fréchet space $\Gamma^\infty(E)$, its derivatives $T\Gamma^\infty(E)$ and like idk $J^\infty(E)$ or the automorphism group or something

Definition 83.1. A C^k -functional is a C^k scalar function (valued on some field F) on a function space, $C^k(X, F)$

Functionals on our configuration space are therefore members of $C^k(\text{Conf}, F)$.

"The dual of a Fréchet space that is not a Banach space is never a Fréchet space, therefore one cannot e.g. define both the tangent and the cotangent bundle as Fréchet manifolds."

Topological vector space : Conf can be identified with its tangent space $T_\phi\text{Conf}$

Functionals are roughly $C^\infty(\text{Conf})$, and their derivatives are $T^*\text{Conf}$

The full configuration space of all of physics is rarely considered as a single unit. For instance, if we had two scalar fields ϕ_1, ϕ_2 , belonging to two line bundles $L_1 \oplus L_2$, we could consider a single field $\phi = \phi_1 + \phi_2$ belonging to the configuration space of $L_1 \oplus L_2$. The full configuration space of physics could be considered by the Whitney sum of all individual fields into some extremely high dimensional bundle.

However, we generally consider some separation for clarity. We also do not try to separate them too much, such as separating each dimension for a vector field. To do this, we generally considered that a field worthy of study on its own is an irrep of the symmetry group of the theory.

If we consider the full symmetry group of our theory, we can separate each field by their irreps

[Can you separate everything entirely based on irreps, what about flavors]

The main symmetries that we are likely to deal with in a physical theory are the kinematic group, the diffeomorphism group, and the gauge group. If a field is not an irrep of the kinematic group etc

Configuration space is a sheaf

Example :

Case of complex fields with the $U(1)$ symmetry

Sigma models

Evolution of initial conditions

83.1.1 Finite dimensional models

A lot of the mathematics involved in this chapter is complicated by the fact that most configuration spaces are infinite dimensional. To simplify things somewhat, it can be useful to consider some finite dimensional toy models which will involve similar mathematical processes.

we have already seen what spacetime is like in zero dimensions $[X]$, and we can easily define simple models on it

Another type of mode we can consider is that of Morse theory, which is the equivalent of theories on the configuration space if our configuration space was finite dimensional.

Shell as intersection

Derived intersection

Chapter 84

Differential equations and sprays

Most of dynamics in physics is described (at least locally) by some system of differential equations. If we consider some configuration space with k degrees of freedom, with a set of values (ϕ^i) , $1 \leq i \leq k$, the dynamics will be something of the form

$$F^j(p, \phi^i(p), \partial_\mu \phi^i(p), \partial_\mu \partial_\nu \phi^i(p), \dots) = 0 \quad (84.1)$$

A set of functions mapping the derivatives of our fields to the real numbers, and with the solutions being the kernel of those functions.

We generally only consider equations of motion in physics to be local in the sense that the PDE will only involve the fields evaluated at the same point, as opposed to a non-local operator such as [integro-differential equation?] :

$$\phi'(x) + \phi'(x+a) + \phi'(x+b) + \dots + \phi(x) + \dots = 0 \quad (84.2)$$

An example of such non-local dynamics can be considered for instance by considering the motion of particles that will move in unison [220]. If we have some system of differential equations along with the additional equation

$$\phi(x) = \phi(x+a) \quad (84.3)$$

or the differential equation of the Fabius function

$$f'(x) = 2f(2x) \quad (84.4)$$

[...]

The geometric way of describing such local differential equations is to consider some map $f : J^\infty E \rightarrow F$, mapping the jets of the bundle E to the bundle F , typically (the line bundle over the base space?) $F = \mathbb{R}^n$ to have some system of differential equations.

A basic example of this can be considered by taking the first order differential equation of a scalar function over the line, $y : \mathbb{R} \rightarrow \mathbb{R}$,

$$y' = g(x, y) \quad (84.5)$$

The function y is given by a section on the trivial line bundle

$$\text{pr}_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad (84.6)$$

This trivially admits the bundle coordinates (x, y) on any trivialization map.

map from the line bundle over $\mathbb{R}$, with coordinates (x, y) , first jet is (x, y, y')

Our differential equation is then defined as a map from the first jet space to the line bundle, which in terms of coordinates would be

$$f(x, y, y') = g(x, y) - y' \quad (84.7)$$

Contact form of the jet bundle

Definition 84.1. A *system of differential equations* of order q $\mathcal{R}_q \rightarrow M$ is a subbundle of $J^q E \rightarrow M$. A solution is a section of E , $\sigma \in \Gamma(E)$, such that its prolongation $j^q \sigma$ belongs to $\mathcal{R}_q$.

[221]

If E and $\mathcal{R}_q$ are vector bundles, this is a linear PDE

Notation : $\mathcal{R}_q \subseteq J^q(E)$

Pick some bundle coordinates of $J^q(E)$:

$$(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots) \quad (84.8)$$

Subbundle : there exists a coordinate system of $J^q(E)$ (x, Φ) such that $\mathcal{R}_q$ is given by the level set functions $\Phi_i = 0$.

In the original coordinates :

$$\Phi_0(x, y, \dots) = 0 \quad (84.9)$$

$$\Phi_1(x, y, \dots) = 0 \quad (84.10)$$

$$\dots \quad (84.11)$$

$$\Phi_k(x, y, \dots) = 0 \quad (84.12)$$

For a section :

$$\Phi_0(x, \phi(x), \partial_\mu(x), \partial_\mu \partial_\nu \phi, \dots) = 0 \quad (84.13)$$

Lie pseudogroup $\Gamma \subset \text{Aut}(M)$

Berwald connection

Sprays, semisprays, evolutionary vector fields, projectable vector fields

Principal symbol of a linear differential operator : for $L \in TJ^k(E)$, the principal symbols of L is the linear map from E_p to F_p

Define elliptic, hyperbolic, etc operators

Relation to kinematic groups

Hessian stuff

84.1 Vessiot theory

Vessiot theory

[222–224]

84.2 Covariance of equations

The space of all possible differential equations is a bit too large for physics, we would like in addition that our equations be *covariant*.

The exact meaning of this is that if we consider that our geometry is based on some kinematic group K , with some representation $\rho_{\text{rep}, K}$. Our configuration spaces are typically

going to be some vector bundles E which are associated vector bundles to the Cartan principal bundle associated with K .

left invariant differential operator, how does it relate

Tangent of $\mathcal{R}$ is left invariant under some lift of the kinematic group? Ditto for kinematic-gauge group of a gauge-natural bundle

As a tangent of jet space of the field, probably some associated bundle to the GK group that has an action on it

Influence of the kinematic group :

I want to say that the equations are covariant if for some $\phi \in \text{Diff}(M)$, with rep on the field $\rho_E : \text{Diff}(M) \times E \rightarrow E$, and a rep on the PDE $\rho_{TJ^k E} : \text{Diff}(M) \times TJ^k E \rightarrow TJ^k E$, something good happens (invariance or something), and more specifically something wrt the action of the kinematic group at least pointwise.

Cf. Korte and Coleman somewhere maybe

Example : are $\square$ and Δ both appropriate PDEs for a Poincaré kinematic group?

At a given point p , in a gauge-natural bundle E , we have the space E_p

Lifting of ϕ to E_p , acting on the section $\sigma \in \Gamma(E)$ as $\phi_*\sigma$

Lifting of ϕ to the bundle of PDEs

See what happens to $\phi_*D(\phi_*\sigma)$, or whatever is supposed to relate to the kinematic group (restriction on ϕ ? such that $j^1\phi|_p \in K$?)

Are $\square A$ and $d * dA$ both appropriate PDEs for a vector representation of the Poincaré group for A ?

How does it work with the wave equation???

Consider some gauge-natural vector bundle $\pi : E \rightarrow M$, E is an associated bundle of the PFB P_K of the kinematic group K , with rep

$$\rho_{K,\text{rep}} : K \rightarrow V \quad (84.14)$$

$$E = P_K \times V / K \quad (84.15)$$

Consider the jet space $J^2 E$, find the equations that are invariant under the action of K on P_K , therefore of E

Consider the linear case, which is the space of all vector subbundles for $J^2(E)$

Does it mean that at a point p , there's a coordinate system for which the (linearized?) Rq is of an invariant form under K ?

Distribution for the R_q of the wave equation : $J^2E = (x, \partial_\mu \phi_i, \partial_\mu \partial_\nu \phi^i)$

LB operator : $\square_{LB}$, in some coordinate system at p (in terms of $(x, u_\mu, u_{\mu\nu})$):

$$\square = \eta^{\mu\nu} u_{\mu\nu} \quad (84.16)$$

Application of a section to this PDE :

Chapter 85

Lagrangian mechanics

Modern physics will usually assume that fundamental dynamics is described by an action principle. An *action* is a functional on the configuration space :

$$S : \text{Conf} \rightarrow \mathbb{R} \quad (85.1)$$

$$\phi \mapsto S[\phi] \quad (85.2)$$

(or possibly some other field F), and the dynamics of the theory is given by the variation of the action functional vanishing, $dS = 0$, which is usually written in physics as $\delta S = 0$. The variation of the action functional is generally simply given as some notion of a derivative defined on the topological vector space of the configuration space, such as a Gateaux derivative or Fréchet derivative.

Most of the physics that we consider involve that our action is *local*. The goal of the definition of locality was to have the standard type of action, where the action is the integral of a Lagrangian, and it was originally thought that merely requiring that it be local in the sense of partial additivity. If we have two configurations $\phi_1, \phi_2 \in \text{Conf}$, which are disjoint, ie $\text{supp}(\phi_1) \cap \text{supp}(\phi_2) = \emptyset$, then the action of their sum is the sum of their action :

$$S[\phi_1 + \phi_2] = S[\phi_1] + S[\phi_2] \quad (85.3)$$

In some sense making the dynamics of disjoint regions independent of each other, to avoid the cases mentioned earlier where the dynamic in a given region could depend on another far off region.

For an action to be local, we demand that it satisfies the additivity rule

$$S[\phi_1 + \phi_2 + \phi_3] = S[\phi_1 + \phi_2] + S[\phi_2 + \phi_3] - S[\phi_2] \quad (85.4)$$

for $\text{supp}(\phi_1) \cap \text{supp}(\phi_3) = \emptyset$

The relation of additivity and partial additivity is that if we take $\phi_2 = 0$,

$$S[\phi_1 + 0 + \phi_3] = S[\phi_1] + S[\phi_3] - S[0] \quad (85.5)$$

As the last term is a constant, it should not have an influence on the dynamics that we seek. [225] Example of violation? Interpretation as locality?

One important consequence of this is the

Another condition on the action is that it should not be too singular,

Condition on the wavefront set

$$\Delta^k(M) = \{(p, \dots, p) \in M^k\} \quad (85.6)$$

$$\text{WF}(F^{(k)}(\phi)) \perp T\Delta^k(M) \quad (85.7)$$

Localized actions can be written in the form

Theorem 85.1. Any local functional can be written as the integral of some map $L : J^\infty E \rightarrow \Lambda M$

$$S[U, \phi] = \int_U \mathcal{L} \quad (85.8)$$

Proof.

□

$\mathcal{L}$ is the *Lagrangian density* of the action. While not always the case, it is quite common for the Lagrangian density to be decomposed as a product of a scalar function L , the *Lagrangian*, and density ω , with ω usually being a volume density of the manifold. If the domain of integration U is orientable in addition, the density can be described by an n -form. In this formalism, we can describe the action functional as

$$S[\phi] = \int_U L \circ j^\infty(\phi) \omega \quad (85.9)$$

For a few examples throughout this chapter, let's consider some standard action functionals. The most common type will be something like the Klein-Gordon action, which is given by

$$S[\phi; U] = \int_U [\|d\phi\|^2 + m^2 \phi^2] d\mu[g] \quad (85.10)$$

and the action of a point particle,

$$S[X; [t_1, t_2]] = \int_{[t_1, t_2]} [\|dX\|^2 + m^2\phi^2] d\mu \quad (85.11)$$

85.1 The Lagrangian gerbe

Considering our Lagrangian as some integral over spacetime has the risk of diverging. For instance if we consider the Lagrangian of some scalar field $L = \|d\phi\|^2 + m^2\phi^2$ for the constant field $\phi = 1$ on Minkowski space, we will get

$$S[M; 1] = \int_{\mathbb{R}^n} m^2 d\mu[\eta] \quad (85.12)$$

$$= \int_0^\infty \int_{\Omega} m^2 r^2 dr d\Omega \quad (85.13)$$

$$= m^2 \text{Vol}[S^{n-1}] \int_0^\infty r^2 dr \quad (85.14)$$

$$= \lim_{R \rightarrow \infty} m^2 \text{Vol}[S^{n-1}] \frac{R^3}{3} \quad (85.15)$$

Which diverges. A common solution to this is to pick only fields that would not diverge in such contexts, such as Schwartz functions, but in some context that may constrain the theory a bit too much, such as cosmology, where the case of a metric of constant curvature will also diverge.

To avoid these issues, one choice that can be made is to only define the Lagrangian on compact subsets, on which any continuous function will remain bounded. If we wish to work out our problem over all space, we then need to work out how different Lagrangians on overlapping domains interact.

85.2 The variational bicomplex

Before we look more into detail the properties of the action, let's look at the Lagrangian first.

Our Lagrangians will always be a map from the jet bundle of our configuration space to the density bundle,

$$L : j^\infty E \rightarrow |\Lambda M|^1 \quad (85.16)$$

We will decompose our Lagrangian by first separating out the volume form ω from it, which will just be the usual pseudo-Riemannian volume form or at most some volume structure element.

$$L = \mathcal{L}\omega \tag{85.17}$$

$\mathcal{L}$ is the *Lagrangian density*, which is simply a scalar function on the jet bundle.

Lagrangian cohomology

Cartan form

Lepage form

Lepagean equivalent

Variation problem

Variational bicomplex

[226]

Boundaries

Global issues (Locally defined Lagrangian)

Euler-Lagrange equation

Something about how we need to map the fields of some rep to either $C^\infty(M)$ or $|\Lambda|^n M$, meaning that we generally need some polynomial map from the field to $\mathbb{R}$

Example : for a vector field, need some maps $V \rightarrow \mathbb{R}$, etc etc

Check if that has to go along with additivity of the action

Also has to be invariant under the kinematic group

85.3 The Euler-Lagrange equations

The fundamental principle for Lagrangian mechanics is the *principle of stationary action* : for any infinitesimal variation of the configuration ϕ , the variation of the action must vanish to first order. This is generally written as

$$\delta S = 0 \quad (85.18)$$

In terms of functional analysis, this is a Gâteaux derivative. If we take some "direction" in the configuration space h , the Gâteaux derivative of a functional is

$$\frac{\delta S}{\delta_h \phi} = \lim_{\varepsilon \rightarrow 0} \frac{S[\phi + \varepsilon h] - S[\phi]}{\varepsilon} \quad (85.19)$$

The set of all configurations at which the variation vanishes is called the *critical locus* of the action :

$$\text{Crit}(S) = \{\phi \in \text{Conf} \mid \frac{\delta S}{\delta_h \phi} \big|_{\phi} = 0\} \quad (85.20)$$

Topology of the function space?

Master equation :

$$\langle S, S \rangle = 0 \quad (85.21)$$

Euler-Lagrange equation as a section of the tangent of the jet bundle $T(J^2 E)$

Transgression of the EL operator is the de Rham derivative of the configuration space

[226]

85.4 Global stuff

We now need to connect all those elements together

Theorem 85.2. The zero locus of the action functional corresponds to

How boundaries work in this system

Chapter 86

Symplectic and polysymplectic geometry

Before we go onto looking at defining Hamiltonian mechanics after we define Lagrangian mechanics, let's first look at the geometric foundation of Hamiltonian mechanics, which is *symplectic geometry*, and its generalization, polysymplectic geometry.

Symplectic geometry is the geometry associated with the *phase space* of a physical system, so it is good to remember that we are dealing with essentially phase spaces, at least within the framework of this book (symplectic geometry also occurs in optics etc). The standard example here is that of classical mechanics for a point particle, where the phase space is the space of positions and momenta, which is typically represented in coordinate form as

$$\text{Phase} = (q_1, \dots, q_n, p_1, \dots, p_n) \quad (86.1)$$

A space that is classically $\mathbb{R}^{2n}$. The equation of motion on this phase space is given by the *Hamiltonian*, with Hamilton's equations,

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \quad (86.2)$$

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \quad (86.3)$$

To consider this in a geometric manner, we need to consider what kind of structure on our phase space will allow us to write this down in a coordinate invariant manner, and this is where the notion of a symplectic space comes in.

Definition 86.1. A *symplectic space* is a vector space V equipped with a bilinear form ω that is non-degenerate (if $\omega(u, v) = 0$ for all v then $u = 0$), alternating ($\omega(v, v) = 0$), and skew-symmetric ($\omega(u, v) = -\omega(v, u)$).

As a non-degenerate bilinear form, the symplectic form ω defines musical isomorphisms, sending vectors to their dual, ie

$$\omega : V \rightarrow V^* \quad (86.4)$$

The standard example of a symplectic space is the $2n$ -dimensional real vector space V equipped with a symplectic form ω which can be expressed in some basis as

$$\omega_{ij} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (86.5)$$

Then in that basis, two vectors $u = (u_q, u_p)$ and $v = (v_q, v_p)$ have the symplectic product

$$\omega(u, v) = u_q v_p - u_p v_q \quad (86.6)$$

First we need to check with what generality we can write out a symplectic space in such a form.

Theorem 86.2. A finite dimensional symplectic vector space (V, ω) is even-dimensional.

Proof. □

Given this dimension, we can now show that every such symplectic space can in fact be expressed in a basis such that the symplectic form has this form.

Theorem 86.3. Any finite dimensional symplectic space can be given a basis for which the symplectic form comes in its canonical skew-symmetric form, called a *Darboux basis*.

Proof. Similarly to the Gram-Schmidt process, we start out with an arbitrary basis $\{v_i\}_{1 \leq i \leq 2n}$ □

86.1 Symplectic subspaces

As in the case of Euclidian and Lorentzian vector spaces, there are specific types of symplectic subspaces

As in the case of the other bilinear forms, for any symplectic subspace W , there exists a *symplectic complement*, which is the set of all vectors that are "orthogonal" via the symplectic form :

$$W^\perp = \{v \in V | \forall w \in W, \omega(v, w) = 0\} \quad (86.7)$$

Properties :

$$(W^\perp)^\perp = W \quad (86.8)$$

$$\text{Dim}(W) + \text{Dim}(W^\perp) = \text{Dim}(V) \quad (86.9)$$

Symplectic complements differ from orthogonal however in that the intersection $W \cap W^\perp$ needs not be the zero vector subspace $\{0\}$

Symplectic subspace : $W \cap W^\perp = \{0\}$

Subspaces (isotropic, lagrangian, etc)

86.2 Symplectic group

The stabilizer subgroup of $\text{GL}(2n, F)$ that leaves the symplectic form invariant is called the *symplectic group* $\text{Sp}(n, F)$.

$$\text{Sp}(n, F) = \{A^\top \omega A = \omega\} \quad (86.10)$$

In a basis around the canonical form

$$\omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (86.11)$$

we have

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (86.12)$$

obeying

$$-C^\top A + A^\top C = 0 \quad (86.13)$$

$$-C^\top B + A^\top D = I_n \quad (86.14)$$

$$-D^\top B + B^\top D = 0 \quad (86.15)$$

Symplectic matrices have determinant 1 : $\text{Sp}(n, F) \subset \text{SL}(n, F)$

Centers : I_{2n} and $-I_{2n}$, it is a simple Lie group

Symplectic algebra :

$$\mathfrak{sp}(2n, F) = \{X | \omega X + X^\top \omega = 0\} \quad (86.16)$$

In coordinates,

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (86.17)$$

obeying

$$A = -D^\top \quad (86.18)$$

$$B = B^\top \quad (86.19)$$

$$C = C^\top \quad (86.20)$$

Symplectomorphisms

Example : scaling, rotations, translations, Fourier transform, fractional Fourier transform, Fresnel transform, etc

86.3 Symplectic manifolds

Just as with every vector space structure, we can transfer it to a manifold structure by defining it on a vector bundle.

Definition 86.4. A *symplectic manifold* is a manifold with a non-degenerate closed smooth 2-form ω

Tautological one-form,

The standard example of a symplectic manifold is the cotangent bundle of any manifold. If we take some manifold Q , our symplectic manifold is formed by $M = T^*Q$ along with the

Specific example of the cotangent bundle of $\mathbb{R}^n$

Definition 86.5. Symplectic manifolds are $2n$ -dimensionals.

The symplectic manifold (M, ω) defines a symplectic vector space at every point by $(T_p M, \omega_p)$

Symplectic atlas

Symplectic form defines a musical isomorphism

G -structure with the group $\mathrm{Sp}(2n, k) \subset \mathrm{GL}(2n, k)$

Set of all diffeomorphisms with values in the symplectic group :

Definition 86.6. The subgroup of $\mathrm{Diff}(M)$ such that their first jet is the group of *canonical transformations* $\mathrm{Can}(M)$.

Hamiltonian vector fields

86.4 Poisson algebra

Definition 86.7. A *Poisson algebra* $\mathrm{Poiss} = (A, \cdot, \{-, -\})$ is an F -vector space A equipped with two algebraic operations $\cdot$ and $\{-, -\}$ obeying the following properties :

- $\cdot : A \times A \rightarrow A$ forms an associative F -algebra with A
- $\{-, -\}$ forms a Lie algebra with A
- $\{-, -\}$ acts as a derivation on $\cdot$ in the sense that it obeys the Leibniz rule :

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$$

Module A , commutative associative algebra $\cdot : A \otimes_k A \rightarrow A$, Lie bracket $\{-, -\} : A \otimes_k A \rightarrow A$

Universal enveloping algebra of a Lie algebra has a Poisson structure

isomorphisms

Central extension of the Hamiltonian vector field algebroid

86.5 Poisson manifold

As with all of our other structures, we can turn the structure of a Poisson algebra on a vector space to one on a manifold

Definition 86.8. The *Poisson bivector* $\pi \in \Omega^2(M)$ is a 2-form with a vanishing Schouten–Nijenhuis bracket : $[\pi, \pi] = 0$

Theorem 86.9. For any Poisson bivector π , there exists a coordinate system (U, ϕ) in which

$$\pi|_U = \pi^{\mu\nu} \partial_\mu \partial_\nu \quad (86.21)$$

in the holonomic basis of ϕ .

The correspondence between the Poisson bracket and the Poisson bivector is given by the map

$$\{-, -\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) \quad (86.22)$$

$$(f, g) \mapsto \pi(df \wedge dg) \quad (86.23)$$

Poisson maps (ichtyomorphisms)

Category of Poisson manifolds, monoidal

86.6 Dynamical groups

By themselves, symplectic spaces have nothing to do with kinematic groups, as they can represent quite a wide variety of things, but in most of the cases of interest to us, a kinematic group will obviously have some impact on it. If we pick our basic phase space $T^*\mathbb{R}^n$ with canonical coordinates, a change of coordinates will affect both the spatial part q in the usual way as well as the momentum p with the lift of that transformation to the cotangent bundle.

Definition 86.10. A Lie group G acting on a (pre)symplectic manifold V via a representation $\rho : G \rightarrow \text{Aut}(V)$ is a *dynamical group* if

$$\rho(G) \subset \text{Can}(V) \quad (86.24)$$

Rep of a kinematic group on (V, ω)

Central extension

86.7 Polysymplectic geometry

The case of symplectic geometry will in general not be enough for our purpose because as stated, it corresponds to the notion of phase space

A polysymplectic vector space is quite similar to the symplectic case, except in this case the polysymplectic form ω is a vector-valued 2-form, $\omega \in \Lambda^2(V, W) \cong \Lambda^2(V) \otimes W$.

[227]

While the typical example of a symplectic geometry is the phase space (p, q) , the polysymplectic geometry equivalent is the case of the phase space

Chapter 87

Hamiltonian mechanics

Pontryagin bundle $T^*Q \oplus TQ$

Hamiltonian mechanics in the general case of an arbitrary physical theory in a general spacetime can be quite complex, so that to keep the thread of the usual Hamiltonian mechanics, I will keep at hand the case of Minkowski space (M, η) and the physical theory of a point particle, with bundle $E = \mathbb{R} \times M \rightarrow \mathbb{R}$.

[228, 229]

87.1 Legendre transform and Legendre bundle

The first step for any Hamiltonian theory is to construct some version of the Legendre transform, where we map the jets of our configuration space to their dual.

[Check whether this transformation exists and is bijective and point to constraint chapter if not]

For a Lagrangian $L : J^\infty E \rightarrow \Lambda^n M$, its vertical tangent map $VL : VJ^\infty E \rightarrow V\Lambda^n M$ gives the linear morphism

$$J^1 E \times_M (T^* M \otimes_E V E) \rightarrow J^1 E \times_E \Lambda^n M \quad (87.1)$$

and the corresponding morphism

$$\hat{L} : J^1 E \rightarrow V^* E \otimes_Y \Lambda^n M \otimes_E TM \quad (87.2)$$

This is the *Legendre map* associated with a Lagrangian L . The Legendre bundle

$$\Pi = \bigwedge^n T^*X \otimes_Y TX \otimes_E V^*E \quad (87.3)$$

Infinite jet version

Bundle coordinates : (x, y, p)

$$p \circ \hat{L} = \pi$$

Multisymplectic form on the Legendre bundle $\Omega = dp \wedge dy \wedge \omega \otimes \partial$

[230]

87.2 Polysymplectic formalism

[230, 231] Vertical tangent morphism VL to L

$$VJ^1Y = J^1Y \times_Y (T^*M \otimes_Y VY) \quad (87.4)$$

$$\overline{VL} : J^1Y \times_Y (T^*M \otimes_Y VY) \xrightarrow{\quad} J^1Y \times_Y \bigwedge^n T^*M \quad (87.5)$$

Legendre bundle :

$$\Pi = \bigwedge^n T^*M \otimes_Y TM \otimes_Y V^*Y \quad (87.6)$$

Legendre morphism :

$$\hat{L} : J^1Y \xrightarrow[Y]{} \Pi \quad (87.7)$$

$$p_i^\mu \circ \hat{L} = \pi_i^\mu \quad (87.8)$$

Explain why multisymplectic doesn't work

87.3 Hamiltonian

[232]

Connection Θ on $\Pi \rightarrow M$ is Hamiltonian if $\Theta \lrcorner \Omega$ is closed

$$dH = \Theta \lrcorner \Omega \quad (87.9)$$

Symplectic manifold and polysymplectic

$d\omega = 0$ implies that the force doesn't depend on velocity?

Show the mapping of polysymplectic to symplectic by a choice of time coordinates

A Hamiltonian system describes a

Poisson algebra, Poisson manifold

Heisenberg group

Scalar field on symplectic manifold : dynamical variable

Dynamical group : Lie group with action on the symplectic manifold is symplectomorphisms

For a Lie algebra $\mathfrak{g}$, the dual algebra $\mathfrak{g}^*$ is a torsor

For a dynamical group G , a variable torsor μ is a *moment* of the G action if there exists a smooth map from (V, ω) to $\mathfrak{g}^*$ such that

$$\omega(Z_V) = -d(\mu \cdot Z) \quad (87.10)$$

For every constant $Z \in \mathfrak{g}$

"To every torsor there corresponds a field of 1-forms on G which is invariant under right translations and which takes the value μ when x is the identity element. These 1-forms are called Maurer*-Cartan forms. The systems of forces in elementary mechanics are the torsors of the group of Euclidian displacements."

For dynamical groups

$$\omega([Z, Z']_V) = -d(\omega(Z_V, Z'_V)) \quad (87.11)$$

Cohomology of a dynamical group

Moment map [233], cohomology

[234]

"One can show that every motion of the system can be described by the uniform rectilinear motion of the center of mass (the principle of Galilean relativity) and the motion of the system "around" its center of mass (Koenig's theorem)"

decomposable motion

"The cohomology of the Galilei group allows us to associate to each dynamical system a number m which is independent of the motion (mass)"

"It is cohomology that gives mass the status of a derived dimension $M = ATL^{-2}$ (A the dimension of action)

"Since the cohomology of the Poincaré group is zero, there is no longer a number which is characteristic for the system, such as the classical mass; the role of mass is played by the relativistic energy which no longer contains an arbitrary additive constant, but which depends on the motion. Furthermore, there is no barycentric decomposition because this decomposition was a consequence of the existence of a privileged subgroup of the Galilei group which doesnot have an equivalent in the Poincaré group"

What kind of causal structure allows a proper use of Hamiltonian mechanics from polysymplectic?

[235]

Chapter 88

Lagrangian mechanics and symplectic structures

One thing that could be asked regarding the use of symplectic geometry for Hamiltonian mechanics is the fundamental necessity of going to the Legendre transform of our configuration space to impose all the symplectic structures upon it. As the Legendre duals are, as the name implies, dual, is it not possible to simply use them on the Lagrangian directly?

It is in fact possible to do so. I will briefly review the method for this, although it should be noted that typically almost all physics is traditionally done on the Legendre dual instead.

Almost tangent structure

Chapter 89

Complex mechanics

It is quite common for structures in physics to have a natural complex structure to them. For instance, our main example of a symplectic manifold, the cotangent bundle of Minkowski space T^*M , admits a natural Kähler structure [236]. If we consider the tensor field on T^*M given by

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad (89.1)$$

Metric on the cotangent bundle :

Symplectic structure

If we take a vector field ξ on T^*M , $\xi = (dx, dp)$ splits along the vertical etc

$$J\xi = (-dp, dx) \quad (89.2)$$

$$\xi_1 + J\xi_2 = (dx_1 - dp_2, dp_1 + dx_2)$$

Complex structure is part of symplectomorphism

Kähler polarization

[237]

"Lagrangian is not holomorphic in ϕ so the derivative is real linear" "Every real linear maps on $\mathbb{C}$ decomposes into complex linear and complex antilinear parts"

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \quad (89.3)$$

independent $\bar{z}$, z , in $\mathbb{C}^2$, then go to the subspace where they are related?

89.1 Complex structure on the electromagnetic theory

A common example of a theory admitting a complex structure is that of electromagnetism.

$$E + iB$$

Chapter 90

Symmetries and Noether's theorems

In the context of physics, symmetries refer to the invariance of some object under a group action. In the context of Lagrangian mechanics, there are a few objects this could be referring to :

- Symmetries of the action functional under group action on the configuration space
- Symmetries of the Lagrangian under group action on the configuration space
- Symmetries of the equation of motion
- Symmetries of a given configuration
- Symmetries of the observables of a system

Symmetries on differential equations (Vessiot and Janet theory etc)

[238]

90.1 Actions and constraints

The Euler-Lagrange equation we obtain from the variational principle is not necessarily a well-posed problem, as can be seen from this quick demonstration. Consider the Euler-Lagrange equation of a given Lagrangian :

$$\text{EL}(L) = d_V L = 0 \tag{90.1}$$

In terms of coordinates,

$$\partial_\mu \partial_\nu \phi^\sigma \frac{\partial^2 L}{\partial(\partial_\mu \phi^\sigma) \partial(\partial_\nu \phi^\tau)} = \frac{\partial L}{\partial \phi^\tau} - \partial_\mu \phi^\sigma \frac{\partial^2 L}{\partial \phi^\sigma \partial(\partial_\mu \phi^\tau)} \quad (90.2)$$

Second jet in terms of the first (defines an acceleration field?), unique solution if, for a given τ , the matrix is invertible

$$\det\left(\frac{\partial^2 L}{\partial(\partial_\mu \phi^\sigma) \partial(\partial_\nu \phi^\tau)}\right) \quad (90.3)$$

Hessian

[239–242]

If we wish for the equation of motion to have unique solutions at least locally, we need the Hessian to be invertible [X].

Otherwise, non-unique solutions [Specify how this is a local issue unrelated to the global issue of the Cauchy problem]

Example 90.1. Consider the trivial Lagrangian $L(q, \dot{q}) = 1$ for any path of q . Its equation of motion is simply $0 = 0$, which is true for any path given any initial conditions (the locus of the action is the entire configuration space).

Example 90.2. The action of the free electromagnetic field in flat space

$$S[A] = \int dA \wedge \star(dA) \quad (90.4)$$

$$= \int (\partial_{[\mu} A_{\nu]}) d^n x \quad (90.5)$$

has the equation of motion

$$d \star (dA) = 0 \quad (90.6)$$

$$= 0 \quad (90.7)$$

with the Hessian for the Lagrangian

$$\text{Hess}(L) = \quad (90.8)$$

In geometric terms, the submanifold of Conf defined by the shell $dS = 0$ is not a single point

[Dimension of the shell v. the appropriate dimension of the gauge reduced shell in Conf v. the dimension as it relates to dof in the bundles?]

The simplest way theoretically to have a unique solution for the equation of motion's initial value problem is to simply select a submanifold from the shell that would

Submanifolds of solutions

Rank of the derivatives

To give the equations of motion a unique solution, we have to further constrain the solution space by the addition of some *constraints*, where we will pick a specific

$$\Phi(j^k\phi) = 0 \tag{90.9}$$

[243]

[238]

The constraints define a submanifold

Constraints

Dirac bracket

[244]

first v. second order constraint

primary v. secondary constraint

Constraint in Lagrangian v. Hamiltonian

Degrees of freedom, propagating degree of freedom, reduced phase space

First and second Noether theorem, global and local version

90.2 Constraints in Hamiltonian mechanics

The method of constraints in most textbooks is a more common feature of Hamiltonian mechanics than Lagrangian mechanics.

If we have a singular Lagrangian L , the corresponding Legendre transform to its polysymplectic version is

Legendre transform is presymplectic instead of symplectic

From Gotay :

"The following is a list of the types of presymplectic manifolds and the causes of degen-

eracy which are encountered in physics :

- Domain problems
- Constrained systems
- Time-dependent systems
- Parametrized and homogeneous formalisms
- A priori presymplectic systems

”

Domain problem : weakly nondegenerate symplectic submanifold of a strongly nondegenerate manifold (ex : $T_C Q \subset TQ$), inherits a presymplectic structure from the imbedding

Constrained systems : ex : particle constrained on a sphere, circle, etc, or internal consistency conditions (EM, gravity). $FL : TQ \rightarrow T^*Q$, $H \circ FL = E$, E is the energy of L , canonical momenta : $p_i \circ FL = \partial_{v^i} L$. Presymplectic if L is weakly regular or irregular : FL is no longer a local diffeomorphism but an injection, range of F is a submanifold of T^*Q . This submanifold is the primary constraint submanifold.

Our constraint terms in Lagrangian mechanics give rise to corresponding constraint momenta in

$$\hat{L} \tag{90.10}$$

90.3 Some more algebraic topology

[245]

Before we go on to look at symmetries in a more geometric way, we will first have to get some more algebraic topology done. What we want to fundamentally look at is the action of our symmetries on our configuration space. To do this, we will have to look at the theory of equivariant maps

Our configuration space is typically infinite dimensional, so that we will have to keep our theory rather general. As we've seen, those spaces still have the structure of a diffeological space.

Given some gauge transformation on our configuration space (which is given as a Lie groupoid(?)), the configuration gets acted on by this

Left G -space : group homomorphism $\mu : G \rightarrow \text{Homeo}^{\text{op}}(X)$

A G -equivariant space M

Theorem 90.3. The quotient of a G -manifold by a non-free action of G can fail to be Hausdorff

Proof. □

example of two different quotients with different group actions having the same quotient [246] Nice quotient is the case where $X \rightarrow X/G$ defined a principal G -bundle, no loss of information on the group action (true if G acts freely)

For $X \rightarrow X/G$, take the pullback the action is free iff the pullback is $G \times X$

Action groupoid $\text{Act}_G(X)$

For a space T , take the trivial groupoid object defined by identity arrows. $\text{Hom}(T, \text{Act}_G(X))$ is taken in the 2-category of groupoids, Hom is a groupoid.

$T \mapsto \text{Hom}(T, \text{Act}_G(X))$ defines a presheaf of groupoid on spaces

Canonical morphism $X \rightarrow \text{Hom}(-, \text{Act}_G(X))$ of presheaves of groupoids

Stackify $\text{Hom}(-, \text{Act}_G(X))$ into $X//G$

Equivariant cohomology [247], Homotopy quotient, stacks

For two derived Lie algebroids $\mathfrak{a}, \mathfrak{a}'$ with $f : \mathfrak{a} \rightarrow \mathfrak{a}'$,

$$\text{CE}(\mathfrak{a}) \quad \text{CE}(\mathfrak{a}')$$

$$C^\infty(X) \quad C^\infty(Y)$$

Canonical homomorphisms $p : X \rightarrow X/\mathfrak{g}$, $p^* :$

Define projective module

Projective resolution of an R -module M

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow M \tag{90.11}$$

Example for the ring $\mathbb{Z}$ and $\mathbb{Z}$ -module $\mathbb{Z}/\mathbb{Z}_2 :$

$M = \text{coker}(\phi_1)$, M is entirely defined by its projective resolution

90.4 Derived geometry

Derived geometry [248, 249]

The simplest non-trivial example of derived geometry we can look at is that of the cross space, which is defined as the variety $Z[f]$ for $f \in k[x, y]$ defined by $f(x, y) = xy$, whose zeroes are the two axis of $\mathbb{R}^2$ in the shape of a cross intersecting at $(0, 0)$.

Morse theory stuff

90.5 The BV-BRST formalism

[250–259]

(Becchi-Rouet-Stora-Tyutin)

Another way to consider the case of gauge theories is to look at the geometry of our configuration space.

In a geometric fashion, our overdetermined system is a too large subspace in the configuration space of on-shell solutions, $\text{Conf}_{EL=0}$, and the initial conditions do not reduce its dimensionality enough for the problem to be determined. The extra dimensionality will come from physically identical states of the system related by a gauge transformation. To get our gauge-fixed system, we need to perform some reduction of the solution locus $\text{Conf}_{EL=0}$ by the orbits of those symmetries.

The notion that we will need for this is that of an *action Lie groupoid*, where given some Lie group G , some smooth manifold M and a smooth action $\rho : G \times M \rightarrow M$, the action Lie groupoid is the homotopy quotient $M//_{\rho}G$

Action lie groupoid : objects are points in M , arrows are the pairs in the product $G \times M$, $s(g, p) = p$, $t(g, p) = g \cdot p$, multiplication : $(h, q) \cdot (g, p) = (hg, p)$ for $q = g \cdot p$.

$$(s, t) : G \times M \rightrightarrows M \tag{90.12}$$

In our case, the action Lie groupoid will be the group of symmetries on the action, and the manifold to be considered will be the on-shell configuration space $\text{Conf}_{EL=0}$ (which only has a manifold structure in the more general context of a Frechet manifold).

The [n-?]group[oid?] action that we are looking into is the one leaving our action on-shell. Consider our derivative of the action $\delta S \in T^*\text{Conf}$.

$$\mathrm{Ker}(dS) \tag{90.13}$$

Koszul-Tate resolution

We have already seen in X the type of spaces that we can obtain under direct quotient by some symmetry group, leaving us with very pathological stratified spaces. To avoid such issues we will instead use the *homotopy quotient*

Gauge parameter bundle $\mathcal{G}$

BRST

BV-BRST

[260–265]

Restriction to the stationary surface : Koszul-Tate differential δ Quotien by the gauge transformation : longitudinal operator γ

$BV - BRST$ differential operator :

$$s = \delta + \gamma + \dots \tag{90.14}$$

Extra terms to make s nilpotent ($s^2 = 0$)

ghost antighost field antifield

auxiliary fields

Nakanishi-Lautrup field

Gauge fixing [266]

L_∞ -algebra

Dressed fields method

Chapter 91

Observables

The observables of a physical system are, in the mathematical sense, the space of all maps from the configuration space to some field F

$$O : \text{Conf} \rightarrow F \tag{91.1}$$

Typically the field F is either $\mathbb{R}$ or $\mathbb{C}$. "Observables" also refer in a more experimental sense to the type of quantities we can hope to observe in the real world from the mathematical model, but we will see what this means and how this relates to this notion in 214. For now, we can just stick with the notion that they roughly correspond to quantities that we can measure in the real world.

As we saw in the previous chapter, gauge degrees of freedom lead to a lack of determinate evolution of the configuration. For this reason, under the assumption that any physical quantity that we may observe has a deterministic evolution, we assume that observables are a gauge invariant map :

$$\forall g \in \text{Gau}(\text{Conf}), O(\rho_g \phi) = O(\phi) \tag{91.2}$$

If our configuration space does not have a trivial gauge group, the space of observables is therefore the space of gauge invariant functionals on Conf :

$$\text{Obs}(\text{Conf}) = [\text{Conf}, F] / \text{Gau}(\text{Conf}) \tag{91.3}$$

[267, 268]

Lagrangian v. Hamiltonian methods

91.1 Field observables

The most obvious type of observable we can formulate is simply the value of our configuration at a point, mapped to some appropriate basis if they are vector quantities. Those are *field observables*. For a configuration space corresponding to a vector bundle $V \hookrightarrow \mathcal{V} \rightarrow M$, in some local basis, a field observable for $\phi^i \in \text{Conf}$ will be an element in that basis at a given point (it can also further be split into components if the field it is defined on is not the real numbers) :

$$\phi^i : M \rightarrow \mathbb{R} \quad (91.4)$$

$$p \mapsto \phi^i(p) \quad (91.5)$$

There is therefore one field observable per point of space, defined as

$$O_p(\phi) = \phi(p) \quad (91.6)$$

While the most direct type of observable we can get from the theory, these are not believed to actually be directly observable, partly due to the general lack of physical phenomenon that is likely to respond to the value of the field so directly, partly because such measurement would probably be point-wise in some sense, which is also not believed to be possible.

Example : For a point particle $X : I \rightarrow M$, represented by the fiber bundle $\pi : I \times M \rightarrow I$, the gauge-reduced configuration space is [...]

The field observables in this case are simply the positions of the point particle at a given time t , $X(t)$, given by the evaluation

$$\Gamma(I, M) \times I \rightarrow M \quad (91.7)$$

$$(X, t) \mapsto X(t) \quad (91.8)$$

If we want those positions to correspond to a real number, setting an appropriate coordinate patch on M ,

$$O_t^\mu(X) = \text{pr}_\mu \circ \phi \circ X(t) \quad (91.9)$$

With the local vector space structure of our configuration space, we can also construct *linear observables*,

$$L : M \rightarrow \mathbb{R} \quad (91.10)$$

$$p \mapsto a_i \phi^i(p) \quad (91.11)$$

and, if we further use the algebra defined on our configuration space [Poisson algebra? More general with terms like $g(v, v)$?], we can also define *polynomial observables*

$$P : M \rightarrow \mathbb{R} \quad (91.12)$$

$$p \mapsto a_0 + a_i \phi^i(p) + a_{ij} \phi^i(p) \phi^j(p) + \dots \quad (91.13)$$

While slightly better, it still remains an unrealistic form for actual observables. "Real" observables are generally not thought of as point-wise, but are instead *local observables*. An observable is local if it can be expressed in the integral form

$$L[\phi^i] = \int_M f(j^\infty \phi^i) b d\mu[g] \quad (91.14)$$

For some function f on the jet space of our configuration space and b some bump function on M .

Local observables can themselves be local field observables, local linear observables or local polynomial observables depending on their form.

microcausal observable : Polynomial local observable such that $a(k)$ has a wavefront set excluding points where k -vectors are in the closed future cone or all in the closed past cone.

field, local linear regular, microcausal, polynomial, general

Observables in classical physics

Poisson algebra?

Chapter 92

Free fields

A lot of physics is done using free fields as a basis, as they tend to be the simplest model theory for any type of field, and allow for the computation of more complex ones by perturbation around that free theory.

Defining a free theory can be complicated, as the equivalence of the theory to a linear one can be obscured. Consider for instance the point particle and the Nambu-Goto action

$$L = \left[g(\dot{X}, \dot{X}) \right]^{\frac{1}{2}} \quad (92.1)$$

Its equation of motion is the pregeodesic with some arbitrary parametrization

$$\dots \quad (92.2)$$

However, if we fix the gauge defined by the reparametrization invariance, we simply obtain the geodesic equation

$$D_\tau \gamma = 0 \quad (92.3)$$

which, due to the equivalence principle, can be cast at a point as

$$\frac{d^2}{d\tau^2} x^\mu = 0 \quad (92.4)$$

Both the gauge and the coordinate form can obscure such things

We will ask for a free field that there exist both a coordinate set and a choice of gauge around which

[Relation to BRST for gauge and diffeomorphisms?]

[269]

For a theory on $J^k E$, order $k + 1$? PDE that is covariant under the kinematic group

Linearization of the PDE subbundle?

Wave equation

n -th order, linear and sourceless : if

$$f_0 + f_1 j^0 \phi + f_2 j^2 \phi + \dots = 0 \quad (92.5)$$

Then $f_0 = 0$

The linear operator acting on it is a left invariant operator wrt the kinematic group's rep in the appropriate vector bundle (associated vector bundle to the kinematical group)

See what kind of classification of such equations that gives

Classification by little groups

Definition 92.1. A *free field* : finite rank $\mathbb{Z}$ -graded vector bundle E , vector bundle map $\langle -, - \rangle_{\text{loc}} : E \otimes E \rightarrow |\Lambda|M$ that is fiberwise nondegenerate, antisymmetric and of cohomological degree -1

$$\langle s_0, s_1 \rangle = \int_M \langle s_0, s_1 \rangle_{\text{loc}} \quad (92.6)$$

a differential operator $Q : \mathcal{E} \rightarrow \mathcal{E}$ of cohomological degree 1 such that $(\mathcal{E}, Q)$ is an elliptic complexe, Q is skew-self adjoint wrt the pairing :

$$\langle s_0, Qs_1 \rangle = -(-1)^{|s_0|} \langle Qs_0, s_1 \rangle \quad (92.7)$$

Action :

$$S[U; \phi] = \int_U \langle \phi, Q\phi \rangle \quad (92.8)$$

EoM :

$$QE = 0 \quad (92.9)$$

Influence of the kinematic group on Q and $\langle, \rangle$. Demand that E be a gauge-natural bundle and that it transforms properly under the kinematic group?

[270]

Source term $J\phi$

92.1 Classification of free fields

Given the various representations of the Poincaré group, we can fully classify their free fields.

[All the concomitant, invariant terms of the action]

92.1.1 Scalar fields

The most basic free field is the scalar field, with the Lorentz irrep $(0, 0)$.

92.1.2 Vector fields

92.1.3 Spinor fields

92.1.4 Rarita-Schwinger fields

Chapter 93

Mass

An important notion that has not particularly appeared so far is that of *mass*. In physics in general, what we call mass can be a variety of different properties :

1. Mass is a proportionality constant for acceleration under non-geometric forces
2. (squared) mass indicates the orientation of an object's momentum with respect to the light cone
3. Mass is a source for the stress energy tensor

For instance, consider the case of the free point particle. If the particle follows the curve γ , of norm tangent norm $\|\dot{\gamma}\|$, the particle has a squared mass > 0 if it is timelike, 0 if it is null, and < 0 if it is spacelike (we will show why that is so in [X]). There is however not much we can say about the magnitude of this mass since in the free case, by the equivalence principle, all objects will follow the same geodesic paths.

The magnitude of the mass will enter into the theory more properly once it is subject to non-geometric forces [271]. Let's consider the case of a point particle modelled by a curve moving according to some force it is submitted to.

Objects of mass zero moving on the light cone

Equivalence : inertial mass is equal to gravitational mass

Proportionality factor for the linearized EoM? (kinetic term)

What is mass? Action of the kinematic's group cohomology on the symplectic space, free particle case (2-homogeneous part of the Lagrangian)

Cohomology class of the kinematic group

How it relates to the conformal group, symmetry breaking of the conformal group

Reliance on the moment map associated with boosts? Classical : CoM, relativistic :

Time evolution on the symplectic space is a symplectomorphism, how does it relate to boost moment maps

Bare mass v. effective mass

[272, 273]

Chapter 94

The vacuum Einstein field equations

The physical theory we want to make for general relativity is that of a geometric theory [...]

94.1 The Einstein-Hilbert action

In general, our geometric gravitation theory (based on the Poincaré gauge) will be based on some local action functional of the Cartan connection for a Lorentzian geometry :

$$S[U; e, \omega] = \int_U (j^\infty(e, \omega))^* \mathcal{L} \quad (94.1)$$

From the general properties that we want for the theory, we want the resulting theory to be torsion free and that the connection be adapted to the frame.

While we could use just about any map from the Cartan connection to 1-densities, most of these would not give us necessarily interesting results. A basic example of this is that some types of terms would be identical, the basic example being [Gauss Bonnet term?]

To avoid this first issue, we will only consider the set of all 1-densities concomitant to the frame and connection.

As the volume element already fulfills any requirement as far as the Weyl weight goes, the terms of the Lagrangian simply have to be scalars in terms of the frame and connection. In

An additional property that we require of the gravitational action is that it is divergence free, that is, every equation of motion produced by the action is such that its divergence is zero :

$$\operatorname{div}(\operatorname{EL}(L_g)) = 0 \quad (94.2)$$

As we will see in more detail later on, this is a condition required for the resulting theory to have local conservation of energy.

Any invariant functional $\mathcal{L}[g]$ has vanishing covariant derivative, hence Lovelock theorem implies that a linear function of R is up to a total divergence the most general lagrangian density

The *Einstein-Hilbert action* is the most general action we can get from these requirements that will hold in $(3 + 1)$ -dimensions :

$$S[U; e] = \int_U R \, d\mu[g] \quad (94.3)$$

In terms of the metric tensor,

$$S[U; g] = \int_U (R + \Lambda) \, d\mu[g] \quad (94.4)$$

[116]

While this action fulfills the conditions that we set out for the general behaviour of the theory, we would also like the variation of our action to vanish on the boundary

Gibbons–Hawking–York boundary term

In term of frames :

$$*R = \Omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta \quad (94.5)$$

94.2 BRST terms

The gravitational action admits two obvious kinds of symmetries. First, if we consider it written out as frame fields and spin connections, the action is invariant under its gauge group of the Lorentz group :

$$(e, \omega) \rightarrow (\Lambda e, \dots) \quad (94.6)$$

Secondly, the action is invariant under diffeomorphism. There are many nuances to this statement (see 222), but here we consider "active" diffeomorphisms, where the action of the diffeomorphism group simply generates a new field over the same spacetime.

$$\mathcal{L}_\xi \quad (94.7)$$

BV-BRST

[135, 260, 274, 275]

constraints and coordinate choices

Harmonic coordinates

Definition 94.1. For a smooth manifold, *harmonic coordinates* are a type of coordinate chart $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ for which every function defined by the coordinate $x_\alpha^\mu = \text{pr}_\mu \circ \phi_\alpha$ is a harmonic function on this patch :

$$\square_{\text{LB}} x_\alpha^\mu = 0 \quad (94.8)$$

Theorem 94.2. A coordinate chart is harmonic if and only if its Christoffel symbols are such that $\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = 0$.

Proof.

□

Theorem 94.3. In a pseudo-Riemannian manifold, for any point $p \in M$, there exists an open neighbourhood of p for which harmonic coordinates exist.

in 2D : isothermal coordinates, $du = \star dv$

94.3 The equation of motion

Derivation of the EFEs :

Variation of the EH

$$\text{Ric}[g] + \frac{1}{2}Rg + \Lambda g = 0 \quad (94.9)$$

For brevity we will sometimes use the *Einstein tensor* instead, defined as

$$G = \text{Ric}[g] + \frac{1}{2}Rg \quad (94.10)$$

Properties of the Einstein tensor :

As we required the theory to be divergence free in our original statement, the equation of motion is obviously divergence free :

$$\nabla(G + \Lambda g) = 0 \quad (94.11)$$

Trace :

$$\text{Tr}(G) = R + \frac{1}{2}R\text{Tr}(g) = R + \frac{n}{2}R \quad (94.12)$$

Lovelock theorem [276, 277]

Einstein-Hilbert with cosmological constant term :

$$S[U; g] = \int_U (R + \Lambda) d\mu[g] \quad (94.13)$$

This leads to the *Einstein field equations* :

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (94.14)$$

94.4 Einstein manifolds

Given the Einstein-Hilbert action, the Lorentzian manifolds (M, g) which extremize the action are called *Einstein manifolds*, which are manifolds obeying the vacuum Einstein field equations :

$$\text{Ric} + \frac{1}{2}Sg = 0 \quad (94.15)$$

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = 0 \quad (94.16)$$

The most well-known of Einstein manifolds is Minkowski space itself, with constant metric

$$ds^2 = -dt^2 + \sum_{i=1}^n (dx_i)^2 \quad (94.17)$$

There are many other such Einstein manifolds

Schwarzschild, Kerr, gravitational waves spacetimes

”Every solution to the Yamabe problem on a closed Einstein manifold is Einstein.”

Chapter 95

The stress-energy tensor

Physics does not only consist of the gravitational field, and we typically want our physics to include a variety of other fields and objects which interact with it. In terms of the Lagrangian formalism, we perform a split between the gravitational Lagrangian and the matter Lagrangian :

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M \quad (95.1)$$

The gravitational Lagrangian contains only dynamical terms based on the geometry (metric, connection, solder form, frame field, etc), while the matter Lagrangian's dynamical terms all contain at least one non-geometric term. [Use locality to show which is which?]

$$S[U; g, \phi_i] = \int_U [\mathcal{L}_G + \mathcal{L}_M] d\mu[g] \quad (95.2)$$

The *stress-energy tensor* is the tensor obtained by the variation of the matter Lagrangian by the metric tensor (or equivalent object)

$$\delta S = \int_U [\mathcal{L}_G + \mathcal{L}_M] d\mu[g] \quad (95.3)$$

95.1 Classification of stress-energy tensors

Just as we classified Riemann tensors (and others) in chapter [X], we can perform the same analysis for the stress-energy tensor.

Hawking classification

Type I II III IV

Chapter 96

Mass in general relativity

The notion of mass that we get from classical field theory

Mass not generally defined in general relativity due to the lack of conservation of energy

Mass classically :

$$M = \int_D \rho(x) d^3x = -\frac{1}{4\pi G} \int_D \Delta\phi d^3x \quad (96.1)$$

If we try to apply it directly, most general case for some spacelike hypersurface Σ and two timelike vector fields ξ, χ :

$$M = \int_{\Sigma} T(\xi, \chi) d\mu^{(3)}g \quad (96.2)$$

If we assume a foliation by spacelike hypersurfaces :

$$M(t) = \int_{\Sigma_t} T(\xi, \chi) d\mu^{(3)}g \quad (96.3)$$

Conservation of mass :

$$\partial_t M(t) = \quad (96.4)$$

This is due to the general invalidity of Noether's theorem, as metrics will not generally be time-invariant.

96.1 Bondi mass

96.2 Komar mass

For a stationary asymptotically flat spacetime, with Killing vector ξ , the Komar mass is

$$M = \int_D (2T(u, \xi) - Tg(u, \xi)) d\mu[g] \quad (96.5)$$

As a surface integral

$$M = -\frac{1}{8\pi G} \int_S \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma \quad (96.6)$$

96.3 ADM mass

96.4 Positive mass theorem

Theorem 96.1. If the dominant energy condition holds, the total ADM mass of an asymptotically flat spacetime is non-negative. If its mass is zero, it can only be Minkowski space.

Paper on why a spacetime with the Casimir effect still has positive mass.

”Bondi mass cannot become negative in higher dimension”

Chapter 97

Symmetries, Noether currents in general relativity

[278]

[279]

For a given geometric theory of gravity, there are two possible symmetries to consider : the symmetry by their kinematic group, and the symmetry by the diffeomorphism group.

[...]

If we consider the dressed geometric theory, we are left only with the diffeomorphism group acting on g .

Gauge from symmetries example : change of coordinates that is the identity near the initial value

$$S[U; g] \rightarrow S[\phi(U); \phi^* g] \tag{97.1}$$

Configuration space is equivalent under K and $\text{Diff}(M)$

Harmonic coordinates, gauge fixing

Diffeomorphisms generating the stress energy current

Einstein tensor

Extended Lagrangian [258]

97.1 The Einstein pseudotensor

While we are in general unable to use Noether's first theorem for energy and momentum due to the lack of Killing vector fields, it can still be practical in some circumstances to have a version of energy and momentum conservation in general relativity.

Pullback of the Sparling form

Landau–Lifshitz pseudotensor

Einstein pseudotensor

The Weyl one

97.2 Degrees of freedom

As a gauge theory, the true number of degrees of freedom of the theory has to be

Degrees of freedom and propagating degrees of freedom

[280]

Chapter 98

Einstein algebras

[281]

Chapter 99

Polysymplectic general relativity

As spacetimes are very much not guaranteed to have a well-behaved time-space split on the outset, it is best to start out any phase space formulation in the polysymplectic formalism.

Legendre transformation :

jets of the metric tensor : $J^k S_{n-1,1}M$, with components $(x^\mu, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\tau})$

Phase space (g, π, π_2)

[230, 282]

Chapter 100

The ADM formalism

As we saw in chapter 73, we can decompose our spacetime into a spatial and temporal part, to various degrees, and those splits lead to some induced geometry on each part of the decomposition. This will allow us to transform the polysymplectic version of general relativity that we've seen into a Hamiltonian version, called the *ADM formalism*.

If we have a nowhere vanishing timelike vector field T ,

Interior product of the polysymplectic Hamiltonian with the time structure

ADM mass

100.1 Constraints

Diffeomorphisms, gauge, propagating degrees of freedom

Chapter 101

Peblanski formalism

[283]

Chapter 102

Ashtekar variables

In the context of the ADM formalism, the dynamical fields of general relativity can be recast in the form of the *Ashtekar variables*.

$$E + i\omega \tag{102.1}$$

Chapter 103

Gravitational observables

Just like any other type of physical theory, we can define the set of observables for a gravitational theory, although if we do, we encounter some difficulties. [284]

If we consider

Diffeomorphism invariant observables, pure gravity observables v. gravity + matter observables, Kuchar dust, etc

Chapter 104

Gravitational waves

There exists wavelike solutions in general relativity, most famously in the form of the pp-wave solutions :

$$g = g_R(x) + 2dudv + H(x, u)du^2 \quad (104.1)$$

Relates to the propagating degrees of freedom of the field

cf. maybe that bit in the global stability of Minkowski space

"The symbol σ_ξ when reduced to Q_ξ is seen to have zero null space whenever ξ is not null. Moreover, when ξ corresponds to a non-zero null vector, choosing a null conjugate $\bar{\xi}$ to ξ , that is, another null vector in the same component of the null cone at p such that $\langle \xi, \bar{\xi} \rangle = -2$, we can identify Q_ξ with the space of all $h \in S_2(M_p)$ such that $\iota_\xi h = 0$. Then $N(\sigma_\xi)$ is found to be $\hat{S}_2(\Pi)$, that is, the space of trace-free symmetric 2-covariant tensors on the 2-plane Π , defined to be the orthogonal complement of the plane spanned by ξ and $\bar{\xi}$. Therefore the 2-dimensional space $\hat{S}_2(\Pi)$ is the space of dynamical degrees of freedom of the gravitational field at a point."

How it relates to the Weyl tensor and propagation in vacuum

Chapter 105

Newman-Penrose formalism

Two non-colinear null vectors $l, \bar{l}$

Complementary subspace of the tangent bundle is given a complex structure (needs to be even dimensional)

Does this work for almost complex structure

Does this work for arbitrarily many dimensions

Chapter 106

Energy conditions

Without any input from the matter physics, general relativity allows a near complete freedom as far as the metric tensor is concerned, bar a few topological obstructions. But if we consider the actual constraints given by the matter fields' equations of motions and boundary conditions, the problem may quickly become intractable.

To somewhat constrain our expectations, it can be useful to add some broad constraints on the stress-energy tensor so as to avoid unphysical matter sources.

The energy conditions are usually devised with some local physical conditions on matter fields in mind, such as the flow of energy outside the light cone. As we will see though, it is not as trivial to impose them

A wide variety of energy conditions concern

Null energy condition : for a null vector k

$$T(k, k) = T_{\mu\nu} k^\mu k^\nu \geq 0 \quad (106.1)$$

Weak energy condition

Strong energy condition

Dominant energy condition

Chapter 107

Causality, topology and general relativity

As the matter content will define the metric on the manifold for general relativity and related theories, this means that the behaviour of matter will affect the causal structure and possibly the topology itself.

107.1 Chronology and the energy conditions

Violation of the chronology condition is generally considered to be unphysical. Because of this, many theorems were developed to study under which physical conditions we can be sure that it is preserved.

One of the first such theorem was Tipler's theorem, trying to show that for a physically reasonable matter content, the resulting spacetime was not achronal

Theorem 107.1. An asymptotically flat spacetime cannot be null geodesically complete if

1. $R_{ab}k^ak^b \geq 0$ for all null vectors k
2. The generic condition is satisfied
3. It possesses a partial Cauchy surface S
4. The chronology condition is violated on $J^+(S) \cap J^-(g^+)$

Proof.

□

Hawking's theorem [285] :

Theorem 107.2. A spacetime with a compactly generated Cauchy horizon will violate the weak energy condition if the partial Cauchy surface is non-compact. If it is compact, then at best $R_{ab}l^al^b = 0$.

Proof. □

Theorems on CTCs : Hawking on compactly generated CTCs, that one about compact Cauchy horizons (rigidity theorem)

”At first blush the suggested approach to identifying the operation of a time machine by means of the Potency Condition is threatened by a result of Krasnikov (2002) showing that every time oriented spacetime (M, g_{ab}) without CTCs has as an extension a maximal time oriented spacetime such that any CTC in the extension lies to the chronological past of the image of M in the extension.”

107.2 Singularities and general relativity

Singularity theorems [286]

Theorem 107.3. Spacetimes that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and there exists a compact slice are timelike or null geodesically incomplete.

Proof. □

Tipler?

Theorem 107.4. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and there exists a trapped surface are timelike or null geodesically incomplete.

Theorem 107.5. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and there exists a contracting region (cf Ellis 2007) are timelike or null geodesically incomplete.

Theorem 107.6. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and stable causality is not satisfied are timelike or null geodesically incomplete.

Cosmic censorship hypothesis

107.3 Topological restrictions

Energy conditions and topology change

107.3.1 Topological censorship

While not directly related to the restriction of topology, the topological censorship theorem gives some constraints on whether or not some topological features are measurable.

Theorem 107.7. Every causal curve extending from past null infinity to future null infinity can be continuously deformed to a curve near infinity.

Chapter 108

Some specificities of general relativity in n dimensions

108.1 Very low dimensions

In very low dimensions, general relativity is generally not considered at all, as any dimension lower than 2 will not really allow for anything resembling a metric tensor. But very low dimensional spacetimes are occasionally used as toy models, so let's very quickly look at the 0 and 1 dimensional cases :

In 0 dimensions, the only (non-empty) connected manifold is the point, $\{p\}$ up to homeomorphism. Due to the low cardinality, we can write out this homeomorphism explicitly : $f : \{p\} \rightarrow \{q\}$ is the function defined by $f(p) = q$, which is simply the pair (p, q) as a set, obviously bijective and continuous, as those single points are open sets.

The unique chart is the map $\phi : \{p\} \rightarrow \mathbb{R}^0$, such that $\phi(p) = 0$. As there is only a single chart, the set of transition function is simply the identity map.

The tangent bundle of the point is the point itself : any curve on the point is simply the trivial curve $\gamma(\lambda) = p$, so that the coordinate representation is simply $x : \lambda \rightarrow 0$, a constant function with only the zero vector as a derivative. The tangent bundle is simply the trivial bundle $T\{p\} = \mathbb{R}^0 \times \{p\}$, with the identity trivialization map.

There is only one linear map on this tangent space, which is the identity map. The theory of 0×0 matrices can be a bit tricky to use with the usual matrix theory, so that it is generally best to treat them directly

Orthogonal group : the identity map, which is of det 1

(det of a 0×0 matrix is 1)

One dimensional spacetimes :

There are only two possible topologies for a Hausdorff paracompact 1-manifold : the line L and the circle S^1 , both of which only admit one possible metric [show], the flat metric.

The "Lorentz group" in one dimension, $O(1, 0) \cong O(1)$, is simply composed of the group of time reversal, $\mathbb{Z}_2$, and the full group of isometries is the group $\mathbb{R} \times \mathbb{Z}_2$ of time translations and reversal.

The flat Minkowski metric is simply $\eta = -dt \otimes dt$. In terms of components, any metric can be written as the function

$$g = -f(t)dt \otimes dt \quad (108.1)$$

for f a nowhere vanishing positive function.

Theorem 108.1. There is only one metric in the moduli space $\text{Lor}(L)$ and $\text{Lor}(S^1)$

Proof. Taking the form of our metric, consider the coordinate change given by

$$t = \int_0^{t'} \frac{1}{f^{\frac{1}{2}}(s)} ds \quad (108.2)$$

As f is nowhere vanishing and smooth, there is no particular issue with this integral.

The Jacobian of this transformation is

$$\frac{dt}{dt'} = \frac{1}{f^{\frac{1}{2}}(t')} \quad (108.3)$$

Giving us the coordinate change on the metric

$$-\frac{dt}{dt'} \frac{dt}{dt'} f(t) = -1 \quad (108.4)$$

□

One dimensional spacetimes as spaces for point particles in the sigma model

108.2 General relativity in $(1 + 1)$ dimensions

Minkowski space structure in $1+1$ d

In $(1+1)$ dimension, the Lorentz group can be considered as the group leaving the matrix

$$(1) \tag{108.5}$$

space-time inversion symmetry in $1+1$ D : flip the signature $(+-)$ to $(-+)$ (NO group?)

Anti isometries in $\text{NO}(n)$

Definition 108.2. An *anti-isometry* on $\mathbb{R}^{p,q}$ is a linear transformation L such that $g(L(u), L(v)) = -g(u, v)$

The Lorentz group in $(1 + 1)$ -dimensions reduce to

Number of independent components of the Riemann tensor

Lorentz surfaces book

Classifications of surfaces

108.2.1 The Gauss-Bonnet theorem

[287, 288]

$$\int \tag{108.6}$$

108.3 General relativity in $(1 + 2)$ dimensions

In $(2 + 1)$ -dimensions, gravity ceases to be topological in character and starts to be dynamical again, although not quite as much as we would hope.

All (orientable) 3-manifolds are parallelizable

Riemann decomposition : zero Weyl tensor, no gravitational propagation

Counting of DoF

If our spacetime can be foliated by spacelike hypersurfaces, we can use the existing classification of 2-manifolds and the uniformization theorem to completely classify such spacetimes.

Uniformization theorem

108.4 General relativity in $(1 + 3)$ dimensions

Our universe

Since (orientable) 3 manifolds are parallelizable, globally hyperbolic manifolds are parallelizable

Lack of unique smooth structures

Chapter 109

Einstein-Cartan

If we drop the condition on the torsion of the connection, so that the spacetime is defined by (M, g, ∇) , with ∇ a metric connection, we obtain the *Einstein-Cartan theory* of gravity.

Connection with torsion

Configuration space can be expressed as (g, T)

Action :

$$S[U; g, T] = \int_U R[\nabla] \quad (109.1)$$

Spin tensor

why is the rotational part (R) linked to the translational part (T) and the translational part (torsion) linked to the rotational part

Chapter 110

Metric-affine gravity

If we remove all constraints on the connection, not requiring it to be either torsion-free nor metric, we end up with *metric-affine gravity*, where we treat the frame field and connection independently.

[289]

First order formalism, fully general connection

(g, T, N)

$$S[U; e, \nabla] = \int_U \Omega[\nabla] d\mu[e] \tag{110.1}$$

As an affine connection, the gauge group involved is the full general linear group, so that the extra degrees of freedom involved in the gauge are one for dilations and shears [squeeze mappings?]

dilation and shear currents

Chapter 111

Geometric theories of gravity

While the standard theory of general relativity is the one we have seen, of a $(\text{Poinc}(n), \text{O}(n-1, 1))$ Cartan geometry with a torsion-free connection, we can use just about any of the Cartan geometries for a similar geometric theory of gravity. This produces a whole family of alternative theories mostly classified by their gauge groups.

111.1 Order of geometric theories

We could presumably make infinitely many such theories, by taking various subgroups of the jet groups $\text{GL}_k(n)$, but there are generally physical reasons leading us to believe that we will in fact never go beyond theories of the second order. [290]

[...]

111.2 Teleparallel gravity

The simplest such alternative is that of *teleparallel gravity*, which is the theory associated with the smallest possible gauge group, the trivial group $\{e\}$. This selects a single frame field as preferred by the theory, giving us an absolute parallelism over the manifold [X].

Always flat, manifold must be subset of $\mathbb{R}^n$, torsion connection

Concomitant densities under $\text{GL}(n)$ and affine connections with torsion

111.3 Weyl gravity

The case of *conformal gravity* considers the Cartan connection with a conformal gauge group

As we saw in [X], there are in fact two types of conformal symmetries we can consider, the conformal group $\text{Conf}(p, q)$ and the Weyl group $\text{CO}(p, q)$

Conformal gravity (order 1 or 2), Weyl spacetime

Concomitant densities of the conformal metric

111.4 Whitehead gravity

Whitehead gravity was based on the idea that there was no fundamental metric or conformal structure to spacetime, and that only the projective structure was fundamental.

111.5 Unimodular gravity

Selection only of orthonormal frames of constant volume

111.6 MacDowell–Mansouri gravity

MacDowell–Mansouri gravity [291] is a theory of gravity based on either the de Sitter or anti-de Sitter kinematic group, $\text{SO}(4, 2)$ or $\text{SO}(3, 2)$, depending on the sign of the cosmological constant. As a theory of gravity based on a kinematic group very much similar to the Poincaré group, it does not suffer from the weirder issues of some of the previous theories which entirely lack some geometric aspect of the theory.

Part IV

Matter

Specific example of matter in general relativity

Chapter 112

Points particles

The most common abstraction of matter in classical mechanics is that of a point particle, which is some object of zero extent in space following some trajectory, with a few properties of its interactions with various forces, such as mass, charge, etc.

In general relativity (and in any geometric theory), point particles can be modelled by curves, usually causal curves unless we deal with tachyonic particles. This curve is simply described by a map from the line to M

$$X : L \rightarrow M \tag{112.1}$$

We can also alternatively describe point particles as *paths* rather than curves, which we will note by $[X]$. This notion may be more natural as there is not necessarily any natural parametrization one could offer, and the description should be reparametrization invariant anyway.

The basic description of a free particle is that of a geodesic. In the case of general relativity, a geodesic is also (locally) a curve of extremal length, so a natural starting point to describe the action of a point particle is to consider the length functional

$$L[X] = \int_I d\mu[X^*g] \tag{112.2}$$

$$= \int_{t_0}^{t_1} \left[g_{X(\tau)}(\dot{X}(\tau), \dot{X}(\tau)) \right] d\tau \tag{112.3}$$

or alternatively the energy functional

$$E[X] = \int_X \dots \tag{112.4}$$

The description from the length functional give us the *Nambu-Goto action*

$$S[U; X, g] = m \int_{X^{-1}(U)} d\mu[X^*g] \quad (112.5)$$

$$S[U; X, g] = \int_{X^{-1}(U)} \sqrt{g(\dot{X}, \dot{X})} \sqrt{\det(g)(X)} d^n x \quad (112.6)$$

while the definition from the energy functional gives us the *Polyakov action*

As a fiber bundle, we can describe the point particle as being given by the bundle $\pi : \mathbb{R} \times M \rightarrow \mathbb{R}$,

The solution from the Nambu-Goto action

$$\frac{\partial L}{\partial \dot{x}^\mu} = -m \frac{\sqrt{l}}{\partial l} \frac{\partial l}{\partial \dot{x}^\mu} \quad (112.7)$$

$$= -m \frac{\dot{x}_\mu}{2\|\dot{x}\|} \quad (112.8)$$

$$-\frac{m}{2} \frac{d}{d\tau} \left[\frac{\dot{x}_\mu}{\|\dot{x}\|} \right] = \quad (112.9)$$

Compute the Hessian

Definition using paths?

Double bundle?

Geodesic spray

The action obeys the appropriate rep from the Casimir invariant?

112.1 Constraints

As we've seen, the Nambu-Goto action is not a gauge free theory. The symmetry relating to this is the reparametrization symmetry of the curve, as there is

$\text{Diff}(L)$

Algebra of the vector field algebroid on L

[292, 293]

Polyakov

$$S[U; X, e] \quad (112.10)$$

112.2 Projective point particle

As we've seen, the point particle is in fact independent of its parametrization. The actual description of a point particle can therefore be given by its *path* instead of its curve. The formal description of an object using its path is done using contact elements and the spacetime's projective structure, along with the tractor formalism.

$$[X] \quad (112.11)$$

idk projective connection or something

$$S[U; [X]] = \int_{\Sigma} c([X], [X]) \quad (112.12)$$

(projective) tractor formulation of Polyakov

Relation with generic Polyakov

Setting of a scale to get it back

112.3 Interaction of point particles and fields

In the case of the kinematic group being the Poincaré group, it turns out that we cannot use the usual point-wise interactions for multiple particles that classical mechanics will usually allow, such as the Coulomb interaction $\propto \|x_2 - x_1\|^{-2}$ or the harmonic oscillator $\propto \|x_2 - x_1\|$.

[294]

Hubbard–Stratonovich transformation, Poincaré on interpretation of fields as particles [295]

One problem that we have with the description of the point particle so far is that its action happens on the pullback to the curve itself, and not the entire spacetime. If we are to include both extended objects and fields, the action will be written split in between two integrals on each space :

$$S[U; X, e, A] = \int_{X^{-1}(U)} (d\mu[X^*g] + qX^*A) + \int_U \text{Tr}(F \wedge *F) \quad (112.13)$$

$$S[U; X, e, A] = \int_{t_1}^{t_2} \left[g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu + e^2 M \right] e^{-1} d\lambda + \int_U \text{Tr}(F \wedge *F) \quad (112.14)$$

If we are to look at its variation,

$$\delta S = \int_{X^{-1}(U)} (d\mu[X^*g] + qX^*A) + \int_U \text{Tr}(F \wedge *F) \quad (112.15)$$

There are several methods to try to put the two entities on the same level as integral actions on M . The most commonly used one in physics is to define it as some distribution δ_X , which in some coordinate patch will be

$$\delta_X = \delta(x - X(\lambda)) \quad (112.16)$$

$$S[U; X, e, A] = \int_U \square \quad (112.17)$$

$\Sigma \rightarrow M \rightarrow F$, connect the bundles $\Sigma \times M \rightarrow \Sigma$ and $E \rightarrow M$

112.4 General form of free and accelerated motion

The equation of motion of a free particle is generally considered to be such that the curve described by a free particle is the same as that of a geodesic of the underlying spacetime.

The form of the motion of a free particle can be used to consider what kind of geometry a spacetime can admit given some general principles on the motion of such a particle.

If we take the equation of motion of a single particle, everything else being considered fixed, its motion can be simply considered by an acceleration map

Acceleration field

$$A : J^1 M \rightarrow J^2 M \quad (112.18)$$

Principle of inertia and constant mass [271]

112.5 Null mass

The action of a particle of null mass is a bit trickier to work out as if we try to write it out in the naive manner with the Nambu-Goto action, the action simply vanishes, $S = 0$, for which every point of the configuration space is a critical point.

If we use the Polyakov action, the zero mass version turns out to be

$$S[U; X, e] = \int_{\Sigma} \langle \dot{X}, \dot{X} \rangle d\mu[e] \quad (112.19)$$

Action for a point particle of null mass

112.6 Tachyonic irreps

If we consider the irreps of the Poincaré group classified by $P^2 > 0$, we are dealing with what are called *tachyons*, particles with spacelike momentum. An appropriate action for tachyons is given by

$$S[U; X, e] = \int \left[\|\dot{X}\|^2 - e^2 m^2 \right] e d\lambda \quad (112.20)$$

[FIX]

Constraint : $P^2 = -m^2$

Tachyons obviously do not have any rest frame in the sense of having a frame in which they have a velocity of 0, but historically people have tried to get around this issue with [complex coordinates]

[296]

Point particles on-shell for a tachyonic action are spanned by spacelike curves due to the constraint.

112.7 Braiding

For systems of multiple point particles, a further symmetry that can occur in the theory is that of *braiding*, or in simpler cases, also called *interchange symmetry*.

Consider a system of N identical point particles, with identical masses. The composite system is the direct sum

$$X : L \rightarrow \bigoplus_{i=1}^N M \quad (112.21)$$

[Does it have to be in some cobordism]

[297]

$$S[X_i] \quad (112.22)$$

Symmetry under the interchange of the targets and initial conditions

Symmetric group of degree n , S_n , group of all permutations of a set of cardinality n

$$\sigma \in S_n$$

Braid group

Symmetrized configuration space

The symmetrized space of N particles is called the *configuration space* (not to be confused with its more general meaning)

There are two types of configuration space we can consider. In the case of two particles being coincident at some point, $x_1 = x_2$, we can either consider that such configurations are forbidden, excluding them from the configuration space, or we can allow them. Allowing such configurations will remove the structure of a manifold to the configuration space and we will instead have to use an orbifold structure.

Orbifolds

Example :

If we take the case of two point particles in a one-dimensional target space $M = L$, the one-particle configuration space is simply $X_1 = \mathbb{R}$. For two particles, the configuration space before symmetrization is $\mathbb{R}^2$. Up to coincidence, the quotient by the equivalence relation $(x_1, x_2) \cong (x_2, x_1)$ will give us a space isomorphic to the upper right

$$X_2 = [0, \infty) \times \mathbb{R} \quad (112.23)$$

112.8 Super point particles

If we give our point particle and map it to a supermanifold of dimension $(p|q)$, its tangent vector will be valued in the

$$\dot{X} = \quad (112.24)$$

θ_i

Action of a super point particle

Equivalent to fermion line

Chapter 113

Extended objects

Beyond point particles, we can also consider the motion of any k -dimensional object through spacetime. There are quite a few applications for this, leading to a certain variety of terminology. This can be used first to simply model objects that can be abstracted to be k -dimensional manifolds, such as topological defects, or can be applied for string theory.

The terminology : extended object, k -brane, worldvolume (worldsheet or worldline for 2 and 1 dimensions)

This generalizes the action appropriately to a map $X : \Sigma \rightarrow M$, $\text{Conf} = [X, M]$

Bundle $\Sigma \times M \rightarrow \Sigma$

Nambu-Goto action

$$S[U; X] \tag{113.1}$$

Polyakov

As the action for a free extended object coincides with the volume functional of a submanifold, this means that free extended objects are also *minimal submanifolds*.

113.1 Branes

Unlike the case of point particles, one thing we have to consider for extended objects is the behaviour of their boundaries. The behaviour of the boundary of the worldline of a point particle is simply dictated by the boundary condition of the problem, as we ask that the boundaries of the line on the initial and final surface be equal to the initial and

final conditions, or that the initial jet $j^1\gamma = (x, v)$ entirely fixes the final boundary, but since there are boundaries even on spacelike hyperslices in the higher dimensional case, we still have to decide on their behaviour.

If we consider the totality of the boundary $\partial X(\Sigma)$ in the target space, we need to fix the boundary conditions for every connected component of this boundary. The boundary conditions that are of interest to us are the ones regarding the position of the jet space of appropriate order of our boundary, $j^k X(\Sigma)$, which needs to be fixed at a particular value.

On a theory depending on first jets like the Polyakov action, there are two types of boundary conditions we can consider : the *Dirichlet boundary conditions* and the *von Neumann boundary conditions*.

Definition 113.1. A *Dirichlet boundary condition* on the boundary $\partial X(\Sigma)$

Definition 113.2. If the extended object's boundary (outside of initial and final timeslices) is empty, we say that the object is closed.

von Neumann condition : $X'_\mu = 0$

Dirichlet condition : X_μ is on a fixed surface

The fixed surface is a D-brane

Branes and D-branes

Dirichlet boundary condition, von Neumann

topological defects

113.2 Constraints

The set of symmetries of any extended objects described by the way of a submanifold will always include at least the group of diffeomorphisms of the surface, $\text{Diff}(\Sigma)$, as it is quite simply a symmetry of the volume functional $\text{Vol}(X)$.

$$\text{Vol}(X) = \text{Vol}(\phi^* X) \quad (113.2)$$

internal symmetries : conformal symmetry for worldsheets [128] [Only for the Polyakov action?] Two dimensional case is also the conformal group

Constraint algebra?

Virasoro group

113.3 Strings

The most common example of extended objects outside of point particles are *strings*, which are the 2-dimensional case for 1-branes.

The two dimensional case has the peculiarity of its Polyakov action also having a conformal symmetry

The most common local coordinates used on the string's worldsheet are the usual time-space split (if we assume that the string is not entirely spacelike), $\sigma = (\sigma_t, \sigma_x)$, where the coordinate lines correspond to timelike and spacelike curves and the induced sheet metric is the $(1+1)$ -dimensional conformally flat metric, and the *light-cone coordinates*, $\sigma = (\sigma^-, \sigma^+)$

$$S = \int d\sigma \quad (113.3)$$

Poisson algebra :

$$[p_i, X^j] = -i\delta_i^j \quad (113.4)$$

$$[p^-, X^-] = -i \quad (113.5)$$

113.3.1 Open string

113.3.2 Closed string

113.4 Superstring

Just like the point particle, we can define a supersymmetric version of a string, called a *superstring*.

Chapter 114

Solid objects

[298, 299]

Solid objects form the majority of the objects used experimentally in general relativity. They include such diverse objects as pendulums, springs, rulers, planets, etc. Unfortunately they are also the most difficult to treat in that theory, particularly in a formal manner.

114.1 Rigidity

One of the fundamental problem of treating solid objects in general relativity is that the main simplifying assumption of such objects in non-relativistic mechanics, that of *rigidity*, will not work as is in the formalism, even in flat space.

114.1.1 Classical rigidity

In non-relativistic mechanics, an object is called rigid if the distance between any two points of that object remains constant at all times. If we have some difficulties to express this in a coordinate invariant manner by settling on what "a point on an object" mean, we can say instead that there exists some parametrization of that object where that property is true. Alternatively, we can say that there exists a 1-parameter family of isometries corresponding to the time evolution of the image of the object $X : S \rightarrow M$:

$$X(t; \sigma) = f_t^* X(0; \sigma) \tag{114.1}$$

The simplest example of this is to simply consider a rigid rod, defined by some segment $X : I \cong [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}^3$. If we give it the trivial parametrization where each point is labeled by $x \in [0, 1]$, we say that its motion is rigid if

$$\forall x, y \in [0, 1], \quad d(X(x, t_1), X(y, t_1)) = d(X(x, t_2), X(y, t_2)) \quad (114.2)$$

This will remain true if we use any other parametrization of the points on the object that is independent of time.

[Is there some way to connect extended body motion to the locality condition of the action?]

Isometric flow???

114.1.2 Rigidity in special relativity

As this example is already using a spatial distance, we can already see the issue coming : there are no coordinate invariant notion of a spatial distance in a relativistic spacetime. Therefore, if we simply try to use this definition, we will already run into trouble by picking another set of coordinates. Even in the simple case of Minkowski space, if we consider some basis (x, t) , defining a distance on the surfaces of constant t , and two free points particles moving rigidly in that frame, $X_1(\tau) = x_1$, $X_2 = x_2$, if we switch to some coordinate frame by a Lorentz boost, we obtain a new spacelike hypersurface

$$x \quad (114.3)$$

[diagram]

Ehrenfest paradox

[300–304]

114.2 Microscopic interpretation of solid objects

If we wish for some more accurate interpretation of a solid object in general relativity, we need to look at a simple model of one.

Simple example of a solid object as a series of bound point masses

[305]

The classic example of the impossibility of rigidity in special relativity is done by considering a pole of a given length l (the example is usually a pole one light-year long) to be pushed rigidly [306].

114.3 Elastic objects

[307]

114.4 Strings and springs

As we saw in X, there are no direct particle to particle interactions allowed in the relativistic context, the classical case being Galilean instantaneous fields being integrated out of the theory. This means that any such terms to use in our model with solid objects will have to either rely on fields or some approximation that fails to be Lorentz invariant.

In the non-relativistic case, the expression of the simple harmonic oscillator relies on the distance of the point considered to some given center, but in the relativistic case, there is no natural notion of spacelike distance that we can use, so that

Lagrangian of a SHO :

$$L = \sqrt{g(\dot{X}, \dot{X})} \tag{114.4}$$

String as a series of connected points

[308–312]

Chapter 115

Field theories

A *field* in physics will generally correspond to a physical system mapping the spacetime manifold to some bundle over it, as opposed to the case of extended objects where the bundle structure operates the other way around. Our configuration space is then simply the space of sections for that bundle

$$\text{Conf} = \Gamma(E) \tag{115.1}$$

Generally we consider fields individually, ie where the bundles correspond to irreps of the symmetries of the theory, given in general by some product of the kinematic group K and the gauge group G (this is due to the Coleman–Mandula theorem that we will introduce later on). Our fields will therefore be the associated vector bundles [gauge natural vector bundles?] that transform under some irrep of $K \times G$.

For instance if we consider the standard model, with gauge group $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$, we

Do I also have to consider some notion of concomitance to separate different gauge-natural bundles (primitive concomitant gauge-natural irrep vector bundle as fields)

Fields as cocycles? Does this only work for gauge fields

Quasinormal modes?

115.1 Classification of fields

Reps of the kinematic group, gauge group, flavors?

Maybe classifications of gauge-natural bundles (of a certain order?) overall

linearization + the rest (polynomial terms?)

Differential algebra from left-invariant diff. operators? [313, 314]

Spin 1 : $j^1 A = (A, \partial A)$

find the differential invariants of this

$$S_A = - \int_U [A \square A + (\operatorname{div}(A))^2] \quad (115.2)$$

Find the Pauli-Lubanski of this

115.2 Propagation

As we've seen from X, the fundamental free theory of a field will be the *wave equation* in the appropriate representation of the field. We can consider for instance the scalar case

$$\square \phi = 0 \quad (115.3)$$

[315–317]

The concept of the propagation of a field is a complex notion even in flat space

Let's consider for instance the flat space solution, given by the Green function

$$\phi(t, x^i) = \int \quad (115.4)$$

It is common in flat space to consider two notions for velocity : the group and phase velocity

flat space notions : group v. phase velocity

Relativistic notion : Front wave velocity

Propagation of discontinuities [317]

Green function, split for time-orientable manifolds into the future and past Green function [true for all kinematic groups?]

[316]

115.3 Complex fields

Fields can in addition be complex. In a geometric manner, we will say that a complex field is a field invariant under some $U(1)$ symmetry (corresponding to a phase rotation)

[318]

Chapter 116

Scalar field

If we consider the trivial representation of our kinematic group, such as the Poincaré group,

$$\rho_0 : \text{Poinc}(n-1, 1) \rightarrow \{e\} \quad (116.1)$$

The associated vector bundles are all scalar fields, with the only irreducible rep being that of a line bundle (of a given field such as $\mathbb{R}$ or $\mathbb{C}$).

Section of the real or complex line bundle, transforms under the $(0, 0)$ representation of the Lorentz group

The free fields of a real scalar field of order 2 are classified by mass

$$\mathcal{L}_{\text{Free}} = \frac{1}{2}g(\nabla\varphi, \nabla\varphi) - m^2\varphi^2 \quad (116.2)$$

$$\mathcal{L} = \frac{1}{2}g(\nabla\varphi, \nabla\varphi) - m^2\varphi^2 - \xi R\phi - V(\phi) \quad (116.3)$$

Complex :

$$\mathcal{L} = \frac{1}{2}g(\nabla\varphi, (\nabla\varphi)^*) - m^2|\varphi|^2 - \xi R|\phi| - V(|\phi|) \quad (116.4)$$

In coordinates :

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - m^2\varphi^2(x) - \xi R(x)\phi(x) - V(\phi(x)) \quad (116.5)$$

Klein-Gordon equation

$$(\square + m^2)\varphi + \xi R + V(\phi) = 0 \quad (116.6)$$

Canonical momentum for a :

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \xi^\mu \partial_\mu \phi \quad (116.7)$$

Hamiltonian density :

$$\mathcal{H} = \quad (116.8)$$

Important class of potentials :

$V(\phi) = 0$: free field $V(\phi) = g\phi^3$: cubic interaction $V(\phi) = g\phi^4$: quartic interaction
 $V(\phi) = g \sin(\phi)$: Sine-Gordon

Ghost and phantom fields

116.0.1 Complex scalar fields

A complex scalar field uses the complexification of the line bundle,

$$\pi : C = L \times \mathbb{C} \times M \rightarrow M \quad (116.9)$$

which is simply isomorphic to $\mathcal{C} \times M$. The complex line bundle C has a conjugate rep vector bundle $\overline{C}$.

E and $\bar{E}$ are isomorphic as real vector bundles

Dual bundle $E^* = \text{Hom}(E, \mathcal{O})$, $\mathcal{O}$ the trivial complex line bundle

Our complex scalar field's product will then be either represented by some hermitian metric $h(\phi_1, \phi_2)$, or alternatively, by the musical isomorphism $\phi^\flat =$

$$S[U; \phi] = \int_U \langle d\phi, d\phi \rangle_{\mathbb{C} \otimes TM} + m^2 \langle \phi, \phi \rangle_C \quad (116.10)$$

$$(\square + m^2)\phi = 0 \quad (116.11)$$

$$(\square + m^2)\bar{\phi} = 0 \quad (116.12)$$

The theory is still technically equivalent to the theory of two scalar fields $\phi = \phi_1 + i\phi_2$,

$$L = \langle d\phi_1, d\phi_1 \rangle + \dots \quad (116.13)$$

116.1 Twisted fields

Scalar field do not have to be modeled by the trivial line bundle. We can also consider non-trivial line bundles under the scalar rep.

"The first Stiefel–Whitney class classifies smooth real line bundles; in particular, the collection of (equivalence classes of) real line bundles are in correspondence with elements of the first cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients; this correspondence is in fact an isomorphism of abelian groups (the group operations being tensor product of line bundles and the usual addition on cohomology). Analogously, the first Chern class classifies smooth complex line bundles on a space, and the group of line bundles is isomorphic to the second cohomology class with integer coefficients. However, bundles can have equivalent smooth structures (and thus the same first Chern class) but different holomorphic structures. The Chern class statements are easily proven using the exponential sequence of sheaves on the manifold. "

Such non-trivial bundles are referred to as *twisted fields*.

The simplest model of a twisted field is to consider the "Moebius band" line bundle on a spacetime $M = L \times S^1$. If we give it the two coordinate patches $\mathbb{R} \times (-\varepsilon, \pi + \varepsilon)$ and $\mathbb{R} \times (\pi, 2\pi)$, the twisted line bundle is the one with the trivialization

$$\psi \quad (116.14)$$

$\mathbb{Z}_2$ structure group

Identification of the positive and negative values of the field

Chapter 117

Spinor fields

For any given manifold, we have a projective pinor bundle

[319]

Pinors, spinors, sinors

Section of the associated bundle to the Clifford bundle

If our manifold is equipped with a spin structure, we can define a spinor action on it.
The free Dirac action

$$S[U; \bar{\psi}, \psi] = \int_U [\sigma(\bar{\psi}, \nabla \psi) + \bar{\psi} \psi] d\mu[g] \quad (117.1)$$

Majorana, Weyl

Representations : $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$$(e_a^\mu \gamma^a D_\mu + m)\psi + V(\psi) = 0 \quad (117.2)$$

Potentials :

$V(\psi) = 0$: Free field $V(\psi) = (\bar{\psi}\psi)^2$: 4-point interaction $V(\psi) =$ (the axial current one)

Types of spinors

Weyl spinors

Majorana equation

Chapter 118

Electromagnetic fields

The *electromagnetic field* is the connection form associated to a $U(1)$ principal connection. While it can also be a $\mathbb{R}$ principal connection, as the algebras are identical, giving us locally the same type of object, it is generally considered to be $U(1)$ as this will be necessary later on for the quantization of this field.

As we are in the one-dimensional case, there exists only one possible Lie algebra for such a Lie group, the Lie algebra $\mathbb{R}^1$, as trivially $[a, b] = 0$ if the vectors are colinear, and therefore there can only be one Lie algebra defined by a single generator e_1 with $[e_1, e_1] = 0$. This Lie algebra has two possible integration, which are the only two 1-manifolds and the only two one-dimensional Lie groups (up to isomorphism), the real line $\mathbb{R}$ and the circle bundle $U(1) = SO(2) = S^1$.

Description as a gauge-natural bundle?

Cocycle description?

The Lie algebra of $U(1)$, $\mathfrak{u}(1)$,

Exponential map :

$$\exp : \mathfrak{u}(1) \rightarrow U(1) \quad (118.1)$$

$$\alpha \mapsto e^{i\alpha} \quad (118.2)$$

Classification of line bundles and circle bundles

Curvature 2-form $F = dA$

$$F = dA \quad (118.3)$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad (118.4)$$

Field	$c^\dagger$	$a^\dagger$	a	c
Degree	-2	-1	0	1

Table 118.1: Caption

$$\mathcal{L} = \frac{1}{4\mu_0} \text{Tr}(F \wedge F^*) = \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \quad (118.5)$$

Constraints :

BV-BRST :

Auxilliary field : $b \in \mathcal{G} \cong M \times L$

BRST extended Lagrangian :

$$\mathcal{L} = \quad (118.6)$$

Maxwell equation :

$$\nabla_\mu F^{\mu\nu} = j^\nu \quad (118.7)$$

$$\nabla_\mu (\nabla^\mu A^\nu - \nabla^\nu A^\mu) = j^\nu \quad (118.8)$$

$$\square A^\nu - \nabla_\mu \nabla^\nu A^\mu = j^\nu \quad (118.9)$$

Gauge invariance : invariant under the action of U(1) on the connection

For the Lorentz gauge :

$$\nabla_\mu A^\mu = 0 \quad (118.10)$$

$$\square A^\nu + [\nabla^\nu, \nabla_\mu] A^\mu - \nabla^\nu \nabla_\mu A^\mu = j^\nu \quad (118.11)$$

$$\square A^\nu - R^\nu{}_\rho A^\rho = j^\nu \quad (118.12)$$

Dual electromagnetic tensor

118.1 The Einstein-Maxwell equations

Combining the Einstein field equations and the Maxwell equations gives us a set of partial differential equations called the *Einstein-Maxwell equations*, which describe spacetimes with sourceless electromagnetic fields, also called electrovacuum spacetimes. The equations are then of course, on the gravitational side,

$$\text{Riem} - \frac{1}{2}gR = \quad (118.13)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa(F_{\mu\sigma}F_{\nu\rho}g^{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}) \quad (118.14)$$

$$d * F = 0, \quad dF = 0 \quad (118.15)$$

$$\nabla_\mu F^{\mu\nu} = 0, \quad \nabla_{[\mu} F_{\mu\nu]} = 0 \quad (118.16)$$

Properties

Rainich conditions :

$$T = \quad (118.17)$$

$$T^\mu{}_\mu = (F_{\mu\sigma}F^{\mu\sigma} - \frac{n}{4}F_{\sigma\rho}F^{\sigma\rho}) \quad (118.18)$$

For $n = 4$, $T = 0$, implying $R = 0$

Penrose-Newman

Redshift

Gravitational Hall effect

Speed of light in GR

118.2 Kaluza-Klein

Splitting structure on a manifold [320]

118.3 Non-trivial bundles

Most of our assumptions on electromagnetism so far has been under the assumption that our $U(1)$ bundle is trivial, but there are quite a lot of possible principal $U(1)$ -bundles over a given manifold.

Classified by $H^2(M, \mathbb{Z})$, isomorphism class of principal $U(1)$ -bundles is isomorphic to the homotopy classes of maps $M \rightarrow BU(1)$

$$[M, BU(1)] \cong [M, \mathbb{C}P^\infty] \cong H^2(M) \quad (118.19)$$

Euler class or equivalently first Chern class of a smooth complex line bundle (since $U(1)$ is homotopy equivalent to $\mathbb{C} \setminus \{0\}$)

If our spacetime is the contractible space $\mathbb{R}^n$, the first Chern class c_1 is trivial and therefore we only have the ordinary

A good example of a non-trivial electromagnetic bundle is the case of a punctured space, such as $\mathbb{R} \times (\mathbb{R}^{n-1} \setminus \{p\})$.

Classification of circle bundles over $M \setminus \{p\}$

Magnetic monopoles

EM fields in non-simply connected spacetimes

118.4 Higher circle connections

We can consider the electromagnetic theory of extended objects by the use of a higher order connection.

p -form electrodynamics

Kalb-Ramond

[321, 322]

118.5 Montonen–Olive duality

118.6 Splitting

If we do the spacetime splitting by a timelike vector u and a spacelike distribution ε ,

$$B = \iota_u F \tag{118.20}$$

$$E = \tag{118.21}$$

$$B = \epsilon F, E = tF$$

118.7 The geometric limit

It is very common to consider the electromagnetic field in the *geometric limit*, where we consider light to be represented by null geodesics.

This limit can be done by considering [323, 324]

Mariot-Robinson theorem

[325, 326]

Chapter 119

The Proca equation

If we drop the interpretation of electromagnetism as a gauge theory, we can investigate the massive case of a vector field, called the *Proca field*. The general free case for this is simply given

The free Proca action is simply given by the appropriate

$$S[U; A] = -\frac{1}{2} \int_U [(dA) \wedge *(dA) + m^2 g(A, A)] d\mu[g] \quad (119.1)$$

$$= \quad (119.2)$$

Massive EM

If we compare this action to that of the massless case of electromagnetism, the mass term breaks the gauge symmetry. Under the transformation $A \rightarrow A + df$,

$$m^2 g(A, A) \rightarrow m^2 [g(A, A) + 2g(df, A) + g(df, df)] \quad (119.3)$$

Solution of the Proca equation on flat spacetime :

(make it an appropriate nabla on the first index)

$$\nabla(dA) + m^2 A = 0 \quad (119.4)$$

$$\nabla_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0 \quad (119.5)$$

Effects of the mass term (cf. Feynman)

Dispersion of light, chromatic aberration

Chapter 120

Yang-Mills fields

[327]

The Yang-Mills field is a generalization of the electromagnetic field : rather than having a gauge group $U(1)$ and a field based on its gauge algebra $\mathfrak{u}(1)$, we consider the case

$\mathfrak{su}(n)$, with the G -principal bundle corresponding to an integration of that algebra (usually $SU(n)$), but occasionally quotiented by some of the group's center). The connection with $U(1)$ being that if we consider the $U(n)$ group, the unitary group, its relation to the special unitary group $SU(n)$ is given by the short exact sequence

$$SU(1) \rightarrow U(n) \rightarrow U(1) \quad (120.1)$$

Let's consider for now the case of $SU(n)$, and in particular, the two physical cases of $SU(2)$ and $SU(3)$, for the weak and strong interaction.

The standard basis used for the Lie algebra $\mathfrak{su}(2)$

Connection on the principal bundle $SU(N)$ (or $U(1)$ for $N = 1$).

Other possible bundles : bundles of the same algebra

We will only consider here $N > 1$ since the case of $U(1)$ was already done in the previous section.

Generators of $SU(N)$: $N - 1$ dimensions so $N - 1$ generators T_a of the algebra $\mathfrak{su}(N)$ of traceless hermitian complex $N \times N$ matrices

$$\{T_a, T_b\} = \frac{1}{N} \delta_{ab} I_n + \sum_{c=1}^{N^2-1} d_{abc} T_c \quad (120.2)$$

$$[T_a, T_b] = i \sum_{c=1}^{N^2-1} f_{abc} T_c \quad (120.3)$$

$$content... \quad (120.4)$$

While the Yang-Mills fields are rarely considered classically, as their quantum theory diverges drastically from their classical version, we can still look at the behaviour of the classical fields.

Chapter 121

Other fields

There exists other less commonly used representations of the Poincaré group

121.1 Rarita-Schwinger fields

After the $(0,0)$, $(1/2,0)$, $(0,1/2)$ and $(1/2,1/2)$ irreps of $\text{SO}(1, n-1)$, the next natural choice of field to consider is the spin $3/2$ field, usually called the *Rarita-Schwinger field*.

There are two spin $3/2$ irreps for the Lorentz group, $(1, 1/2)$ and $(1/2, 1)$

$(1, 1/2) \oplus (1/2, 1)$ rep

Causality issues

[328]

121.2 Continuous spin representation

The classification of little groups leaves one strange case out of all the rest. In the case of zero mass

$$[T^i, T^j] = im^2 S^{ij} \quad (121.1)$$

T^i an $n-2$ vector, S^{ij} an $\text{SO}(n-2)$ Lie algebra member (rotation on the light cone)

$T^i = 0$: massless finite dimensional rep case

$T^i \neq 0 : T^i$ are c -numbers components of a transverse vector

Massless

121.3 Spin 2 fields

What is generally called a spin 2 field is a field transforming under the irrep $(1, 1)$ of the Lorentz group, corresponding to a field h being a traceless symmetric tensor, $h \in \Gamma(ST_0^2 M)$.

$$h(v, w) = h(w, v), \text{Tr}(h) = 0 \quad (121.2)$$

$$h_{\mu\nu} = h_{\nu\mu}, g^{\mu\nu} h_{\mu\nu} = 0 \quad (121.3)$$

This type of field is most famously used for theories of linearized gravity

Pauli-Fierz equation

Are all the good spinor fields just gravity

121.4 Higher spin fields

As the irreps of the Lorentz group are graded on $\mathbb{Z}^2$, there are infinitely many spin representations that we could consider. Those do occur in the realm of composite particles,

[How composite particles work]

Coleman-Mandula theorem

Higher spin rep of fields

Forbidden due to QFT arguments

Chapter 122

Spontaneously broken symmetries

As we saw, symmetries can apply to a variety of objects in physics. A symmetry can be applied to the action S , it can apply to the Euler-Lagrange jet EL , and it can apply to a configuration ϕ .

If we consider the set of symmetries of the action, we say that a configuration exhibits *spontaneously broken symmetry* if it is not left invariant under every group element of the action's symmetries :

$$\exists g \in G(S), g(\phi) \neq \phi \tag{122.1}$$

Unbroken symmetry group : $G_\phi \subset G$

Mexican hat potential

Chapter 123

Supersymmetric fields

Just like the other fields we've seen so far, the supersymmetric fields are representations of the kinematic group, except that we are considering the larger kinematic groups that admit supersymmetries. Chiefly among these we will consider the super Poincaré group

Reps of the Super Poincaré group

Example of $\mathbb{R}^{n|2}$

The assignment of which Poincaré representations will get the commuting or anticommuting variables is somewhat arbitrary, as we can simply assign them as we please,

$$0 \oplus \mathbf{Scalar} \tag{123.1}$$

or

$$\mathbf{Spinor} \oplus 0 \tag{123.2}$$

However, for reasons that will become clear in the chapter on quantum field theory, we will usually consider only representations for which the commuting part of the algebra is for integer-spin representations, and the anticommuting part for half-spin representations.

Chapter 124

Topological defects

[329]

Vacuum topology

Homotopy group of the vacuum

Example : $\mathbb{Z}_2$ vacuum

$$\mathcal{L} = \frac{1}{2} \langle d\phi, d\phi \rangle + m\phi^2 + \phi^4 \quad (124.1)$$

Topological charge

Monopoles of dimension $n - k - 2$ if π_k is non-trivial

Domain walls : $\pi_0(G/H)$ is non-trivial Cosmic strings : $\pi_1(G/H)$ is non-trivial Monopoles : $\pi_2(G/H)$ is non-trivial

Part V

Statistical mechanics and thermodynamics

A lot of the actual objects by general relativity will be neither objects nor fields (or at least not in any helpful way), but very large collections of particles, such as the gas in stars.

Statistical mechanics in GR

There are a few ways we can consider such systems

Chapter 125

Probabilities

To work out the statistical methods of statistical mechanics, we first have to define the notion of probability. To define this, we fundamentally use the notion of *Kolmogorov probability*, which is defined by the following axioms :

Definition 125.1. Given a measure space (Ω, Σ, P) , this is a *probability space* if

1. For any $E \in \Sigma$, $P(E) \geq 0$
2. $P(\Omega) = 1$
3. The measure is σ -additive :

A more in depth interpretation of what probability means will have to be delayed until [X], but broadly speaking, Ω is the *sample space*, the space of all possible outcomes for whatever system we consider, Σ is the space of events, where we look at all the possible disjunction of events in Ω , and P is the *probability* of any given event.

The prototypical example of a probability space for instance is the space of fair coin tosses, where the sample space is the outcomes of a coin toss (heads or tails), the event space is all the possible conjunctions of those events, and the probability is the uniform probability given by

$$P(\emptyset) = 0 \tag{125.1}$$

$$P(H) = \frac{1}{2} \tag{125.2}$$

$$P(T) = \frac{1}{2} \tag{125.3}$$

$$P(H, T) = 1 \tag{125.4}$$

Few basic properties

Theorem 125.2. The empty set has probability zero : $P(\emptyset) = 0$

Proof. This stems from the empty set being disjoint from the sample space Ω :

$$P(\emptyset) = P(\emptyset) + P(\Omega) - P(\Omega) \quad (125.5)$$

$$= P(\emptyset \cup \Omega) - P(\Omega) \quad (125.6)$$

$$= P(\Omega) - P(\Omega) \quad (125.7)$$

$$= 0 \quad (125.8)$$

□

This simply means that the probability implies that at least one of the stated possible outcome should occurs.

Complement- rule :

$$P(A^c) = P(\Omega \setminus A) = 1 - P(A) \quad (125.9)$$

$$P(E) \in [0, 1] \quad (125.10)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (125.11)$$

125.1 Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (125.12)$$

As an axiom :

$$P(A \cap B) = P(A|B)P(B) \quad (125.13)$$

Bayes theorem :

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad (125.14)$$

125.2 Statistical variables

125.3 Probability distributions

125.4 Probability in mechanics

Chapter 126

Statistical mechanics

If we consider the case of a vast amount of point particles, the configuration space that we have to consider is some n -fold product of the manifold for our worldline :

$$Q = (\mathbb{R} \times M)^n \quad (126.1)$$

Exterior tensor product of the bundles (but not vector bundles)?

Section should be : $s = (X_1(t_1), X_2(t_2), X_3(t_3), \dots)$

$$S[U; X_i, e_i] = \sum_{i=1}^N S_{\text{Poly}}[X_i, e_i] + \sum_{i < j} \int V(X_i, X_j) \quad (126.2)$$

We will also have to possibly take the interchange symmetry of the action

Globally hyperbolic case, static case, etc

[330]

126.1 Probability measures

[331]

Chapter 127

Fluid

There are a few ways of defining fluids in general relativity. We can consider a fluid as some statistical ensemble of point particles obeying certain conditions, we can view them as a congruence of curves integrating some causal vector field, or we can consider it as some case of a scalar field.

127.1 Geodesic congruences

Let's consider some vector field $u \in \mathfrak{X}(M)$ with u everywhere causal. In general we consider them as either all timelike or all null, and in the timelike case we will only consider the case of unit norm $\|u\| = -1$, as the parametrization will not serve us here.

Let's consider some observer field $\mathcal{O}$, with some tangent vector v . projection operator $P_\nu^\mu = (\delta_\nu^\mu + v^\mu v_\nu)$

Split of the stress energy tensor into 3 terms, the energy density $\rho v \otimes v$, $\rho = T(v, v)$, the pressure $p_\sigma = -P_\sigma^\mu T_{\mu\nu} u^\nu$, and the stresses $S_{\mu\nu} = P_\mu^\sigma P_\nu^\tau T_{\sigma\tau}$

Equation of state

127.2 Statistical ensemble

Another way to consider a fluid is to take some statistical ensemble of test particles, possibly interacting with each other.

idk something Synge

Ensemble of point particles

Ensemble of distributions

Euler equations derive from ∇T

For a type I stress-energy tensor perfect fluid,

$$T = (p + \rho)u \otimes u + pg \quad (127.1)$$

$$T_{\mu\nu} = (p + \rho)u_\mu \otimes u_\nu + pg_{\mu\nu} \quad (127.2)$$

Dust $p = 0$

127.3 Perfect fluid

[332]

The perfect fluid is the case of a fluid without any internal stresses and an isotropic pressure

$$T = \rho u \otimes u + p(g + u \times u) \quad (127.3)$$

Conservation equations :

$$\nabla T = \nabla(\rho u \otimes u + p(g + u \times u)) \quad (127.4)$$

$$= 2(\rho + p)u \otimes \nabla(u) + u \otimes u \nabla(\rho + p) + g \nabla(p) \quad (127.5)$$

[IT'S A DIV]

By contraction with u :

127.4 Viscous fluids

Navier-Stokes etc (Choquet-Bruhat)

Chapter 128

Contact geometry

Equilibrium thermodynamics can be described in terms of contact geometry

[333–335]

For a contact manifold (M, η) , M $(2n + 1)$ -dimensional, $\eta \wedge (d\eta)^n \neq 0$

Reeb vector field R is unique $\iota_R \eta = 1$ and $\iota_R d\eta = 0$

Isomorphism

$$\flat : \mathfrak{X}(M) \rightarrow \Omega^1 M \quad (128.1)$$

$$X \mapsto X^\flat = \iota_X d\eta + \eta(X)\eta \quad (128.2)$$

$$\flat^{-1}(\eta) = R$$

Generalized Darboux : there exists some coordinates (q_i, p_i, S) such that

$$\eta = dS - p_i dq^i \quad (128.3)$$

$$R = \partial_S \quad (128.4)$$

Whitney decomposition of the tangent bundle : $TM = \ker(\eta) \oplus \langle R \rangle$, projectors are

$$P = \text{Id} - R \otimes \eta \quad (128.5)$$

$$Q = R \otimes \eta \quad (128.6)$$

Example : $T^*Q \times \mathbb{R}$, with

$$\eta_Q = \text{pr}_2^*(dS) - \text{pr}_1(\theta_Q) = dS - \theta_Q \quad (128.7)$$

θ_Q is the Liouville 1-form $\theta_Q(X_{\mu_q}) = \langle \mu_q, T_{\mu_Q} \pi_Q X_{\mu_Q} \rangle$, $X_{\mu_Q} \in T_{\mu_Q} T^*Q$.

In coordinates, $\eta = dS - p_i q^i$. Also $\omega = d\eta|_{\text{Ker}(\eta)}$ so $(\text{Ker}(\eta), \omega)$ is a symplectic vector bundle over $T^*Q \times \mathbb{R}$.

Unique vector field Δ_Q tangent to $\text{Ker}(\eta)$ such that $\iota_{\Delta_Q} \omega_Q|_{\text{Ker}(\eta)}$: the Liouville vector field, given by

$$\Delta_Q = p_i \partial_{p_i} \quad (128.8)$$

Thermogeometrodynamics

Gibbs form :

$$\Theta = d\Phi - \delta_{ab} I^a E^b \quad (128.9)$$

Space of thermodynamic equilibrium states $\mathcal{E}$ is the subspace of $\mathcal{T}$ given by $\phi : \mathcal{E} \rightarrow \mathcal{T}$ such that

$$\phi^*(\Theta_G) = 0 \quad (128.10)$$

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b \quad (128.11)$$

Derivative of entropy by the extensive quantities give the intensive ones, thermodynamic equilibrium condition

On $\mathcal{E}$, this implies the first law of thermodynamics :

$$dS - \delta_{ab} I^a E^b = 0 \quad (128.12)$$

Given our embedding $\phi : (E^a) \rightarrow (S, E^a, I^a)$

Second law of thermodynamics :

$$\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0 \quad (128.13)$$

Convexity condition

Statistical manifolds

Contact form

Chapter 129

Statistical mechanics and thermodynamics

[336, 337]

Most of the thermodynamics that is usually considered is what is called *equilibrium thermodynamics*

”In thermodynamic equilibrium, there are no net macroscopic flows of matter nor of energy within a system or between systems. In a system that is in its own state of internal thermodynamic equilibrium, no macroscopic change occurs. ”

Requirement for stationary or quasi-stationary spacetime?

Condition on length scale (short range interaction v. curvature length scale)

Gauss coordinates or Fermi coordinates around a point so that locally Euclidian

Statistical mechanics in a flat Klein space of a kinematic group :

Probability distributions on a spacetime : L^2 sections?

[338]

Chapter 130

Einstein-Vlasov

From Bruhat :

From the Hamiltonian bit : distribution is a positive function on the phase space, $C^\infty(\text{Ph})$

Interpreted as a density of particles

$$f : \text{Ph}_M \rightarrow \mathbb{R} \quad (130.1)$$

Restriction to good momenta : $\|p\| \geq 0$, and in a time oriented spacetime, $p^0 \geq 0$

For particles of the same mass, fibration of the phase space by the *mass hyperboloid*

$$\text{Ph}_{m,q} = T_q M \cap \{g_q(p, p) = -m^2, p^0 > 0\} \quad (130.2)$$

For $m > 0$, $p^0 > 0$. For $m = 0$, $p^0 = 0 \rightarrow p = 0$

Given the volume forms ω_g on M and ω_p on Ph_q (realistically on $T_q^* M$),

Volume $2n$ form on $T^* M$:

$$\theta = \omega_g \wedge \omega_p \quad (130.3)$$

$$\omega_g = \det^{\frac{1}{2}}(g) dt^2 \wedge dx_1 \wedge \dots \wedge dx_n \quad (130.4)$$

$$\omega_p = \det^{\frac{1}{2}}(g) dE^2 \wedge dp_1 \wedge \dots \wedge dp_n \quad (130.5)$$

Volume form on the mass hyperboloid $\omega_{m,p}$ is (Leray form) such that

$$d[\frac{1}{2}(g_{\mu\nu} - m^2)] \wedge \omega_{m,p} = \omega_p \quad (130.6)$$

$$\omega_p = \frac{1}{p_0} \det^{\frac{1}{2}}(g) dE^2 \wedge dp_1 \wedge \dots \wedge dp_n \quad (130.7)$$

Moments :

Moments of order 0 :

$$r_0(x) = \int_{\text{Ph}_x} f \omega_p = \int_{\text{Ph}_x} f \mu_p \quad (130.8)$$

Density of presence

For a $2n - 1$ submanifold Σ of Ph , mean number of particles crossing Σ is

$$N_\Sigma = \int_\Sigma f \iota_X \theta \quad (130.9)$$

$X = (p, Q)$ is a vector at $(x, p) \in \Sigma$ tangent to trajectories of particles crossing Σ at that point.

Moment of order 1 :

$$P^\alpha = \int p^\alpha f(x, p) \mu_p \quad (130.10)$$

Mass : $r^2 = -P^\alpha P_\alpha$

Macroscopic flow velocity : $u^\alpha = r^{-1} P^\alpha$

Second moment :

$$T^{\alpha\beta} = \int f(x, p) p^\alpha p^\beta \mu_p \quad (130.11)$$

Stress-energy tensor of f

Theorem 130.1. If f depends only on p via a scalar product $\langle V, p \rangle$ with V timelike, then the first two moments are the particle number-momentum and stress energy tensor of a perfect fluid. Unit flow colinear to V

Higher moments : Müller-Ruggeri thermodynamics

Vlasov equations : non-interacting particles following geodesics

$$p = \dot{\gamma} \quad (130.12)$$

$$\dot{p} = -\Gamma(p, p) \quad (130.13)$$

Geodesic flow

The scalar $g(p, p)$ is constant along an orbit of $X = (p, -\Gamma(p, p))$ in TM

Liouville theorem : θ is invariant under geodesic flow

$$\mathcal{L}_X \theta = 0 \quad (130.14)$$

Maxwell-Vlasov : Vlasov + some constant force

$$X = (p, Q + \Phi) \quad (130.15)$$

$$\Phi = e\iota_p F$$

Theorem 130.2. If f satisfies the Liouville-Vlasov equation, then its moments satisfy

$$\nabla_\sigma M^{\sigma abc\dots} = .. \quad (130.16)$$

Einstein-Vlasov system : Geodesic particles + SET from that distribution

Chapter 131

Recurrence time

It is a well-known theorem in classical statistical mechanics that for a wide class of statistical systems, given some initial configuration, the system will eventually return (close to) this configuration. This is expressed by the *Poincaré recurrence theorem* :

Theorem 131.1. For any second-countable, Hausdorff measure space (X, Σ, μ) with a one-parameter map T_t that preserves the measure on X , then for $A \in \Sigma$ with $\mu(A) > 0$, the set of points for there exists no $t \in \mathbb{R}$ such that $T_t(x)$ is not in an arbitrarily small neighbourhood of x has measure 0.

Proof. Consider a basis of open sets $\{U_n\}_{n \in \mathbb{N}}$ for X . For every $n \in \mathbb{N}$, we define

$$U'_n = \{x \in U_n, \forall n \geq 1, f^n(x) \notin U_n\} \quad (131.1)$$

□

Tipler's theorem [339]

Chapter 132

Black hole thermodynamics

If we consider the raw laws of thermodynamics in a generic spacetime, it is easy enough to soon enter into violations of the laws as we commonly consider them.

Thermodynamics near a black hole

Part VI

Evolution

The *Cauchy problem* is to consider the evolution of our system given known values on a hypersurface.

While the Cauchy problem could have been put in the part on dynamics, the topic in general relativity is large enough as to deserve its own section.

Well-posed problems [340]

Definition 132.1. For a manifold M with a bundle E , a section of jets $J^k E$ etc, a hypersurface $S \subset M$, and a value of $\Gamma(E)$ on S , we say that this is a *well-posed problem* if

1. There exists a solution
2. The solution is unique
3. Continuous behaviour wrt initial conditions

There exists many counterexamples to this idea. Consider for example the equation of the backward heat equation defined on the cylinder $L \times S^1$, with coordinates (t, x) ,

$$\partial_t f + \partial_x^2 f = 0 \tag{132.1}$$

with some arbitrary initial condition $f(0, x) = f_0(x)$ and the boundary conditions $f(t, 0) = f(t, \pi) = 0$

A famous physical example can be found in classical mechanics, with *Norton's dome*. Consider a dome shaped like the relation between the height h and radius r in cylindrical coordinates

$$h(r) = \frac{2}{3} \frac{b^2}{g} r^{\frac{3}{2}} \tag{132.2}$$

for $r \in [0, g^2/b^4]$.

$$\ddot{x}(t) = \sqrt{x(t)} \tag{132.3}$$

Infinitely many solutions for given initial conditions

Chapter 133

Differential equations and Sobolev spaces

133.1 Some generic results on PDE solutions

Peano theorem

The simplest case of first order ordinary differential equation is fairly well-known via the *Picard–Lindelöf theorem* :

Theorem 133.1. For a rectangle $D \subset \mathbb{R} \times \mathbb{R}^n$ in the space of coordinates (t, x) , with some point $(t_0, x_0) \in \text{Int}(D)$, and $f : D \rightarrow \mathbb{R}^n$ some function continuous in t and Lipschitz continuous in x , then there exists some $\varepsilon \in \mathbb{R}$ such that the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \quad (133.1)$$

has a unique solution on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$

Proof.

□

By considering the Legendre transform of a system, we can extend this theorem to the case of order k ordinary differential equations

Theorem 133.2. For a system of ordinary differential equations of order k of the form

$$y^{(n)}(t) = \quad (133.2)$$

$$y^{(n-1)}(t) = \quad (133.3)$$

Non-uniqueness example

We have already seen a few theorems make use of the Picard–Lindelöf theorem so far, such as the theorem for the uniqueness of geodesics and the Frobenius theorem. The

Frobenius theorem in particular can be used for manifolds to extend this theorem to the case of PDEs

Frobenius theorem as applied to solutions of PDEs

Caratheodory theorem

Relation to path geometry?

133.2 Sobolev spaces

Sobolev spaces are a commonly used type of function spaces that have a variety of useful properties

For a Riemannian manifold (M, g)

A function f or tensor field T belongs to the Sobolev space $W^{k,p}$ if its derivatives up to order k admit a finite L^p norm, the L^p norm being

$$\|f\|_{k,p} = \left(\int_M \sum_{i=0}^k |\nabla^{(i)} f(x)|^p dx \right)^{\frac{1}{p}} \quad (133.4)$$

with the component representation

$$\nabla^{(i)} f(x) = \nabla_\mu \nabla_\nu \dots \nabla_\sigma f(x) \quad (133.5)$$

and the tensor norm $|T| = T^*(T)$, with T^* the dual of T , or, in components

$$|T| = T_{\mu\nu\dots\sigma} T^{\mu\nu\dots\sigma} \quad (133.6)$$

Sobolev Banach space $W_0^{k,p} \subset W^{k,p}$: closure with respect to the norm of the space of smooth functions of compact support $\mathcal{D}$

Theorem 133.3. The $W_0^{k,p} = L^p(M)$

thm :

if M has a non-zero injectivity radius [Injectivity radius = inf of injectivity radius at every point, which is the largest radius for which the exponential map is a diffeomorphism](ie is complete), then $W_0^{1,p} = W^{1,p}$.

corrolary : true for compact manifolds

If M has a non-zero injectivity radius, and a Riemann tensor uniformly bounded, and its derivatives up to $k - 2$, then for $k \geq 2$, $W_0^{k,p} = W^{k,p}$

Chapter 134

Fields and particles

134.1 The classical case

From the Picard–Lindelöf theorem, we can already see how the case of a point particle will work out. If we have a non-relativistic point particle, with equation of motion

$$\dot{v} = F(x, v, t) \tag{134.1}$$

$$\dot{x} = v(t) \tag{134.2}$$

The Cauchy reduces to that of a system of ordinary differential equations of first order. Meaning that in the case where v is continuous in t (so simply a continuous curve) and F is everywhere Lipschitz continuous in x and v and continuous in t , then there is a unique solution given the initial data of some initial time t_i of t with initial position and velocity $x_i = x(t_i), v_i = v(t_i)$.

Already it isn't too hard to find cases which will not have a well defined Cauchy problem. If we consider a very poorly behaved force such as a divergent one, let's say $F(x) = \tan(x)$ on

$$\ddot{x} = \tan(x) \tag{134.3}$$

Space invader

However even a much milder behaviour can pose problems. The usual gravitational potential is already too ill-behaved to be a well-posed problem.

Painlevé non-collision singularities

If we consider a system of multiple point particles, it can be shown that there exist non-collision singularities [341]

Another such classical example which will have important consequences in the relativistic case is the *inverse cubic force*. If we pick a force

$$F(x) = \frac{\alpha}{\|x\|^3} \quad (134.4)$$

Field theory case

134.2 The Cauchy problem of classical fields and particles on a globally hyperbolic spacetime

Evolution of a set of fields (sections of vector bundles) with values known in a subset of spacetime to the entirety of spacetime, for a fixed spacetime $(\mathcal{M}, g)$

Initial conditions on hypersurface $S : (\phi_i|_S, \nabla\phi_i|_S)$

Second order linear PDEs :

$$A_{\mu\nu} \frac{\partial^2 u(t, x_i)}{\partial x_\mu \partial x_\nu} + B_\mu \frac{\partial u(t, x_i)}{\partial x_\mu} + C(t, x_i)u(t, x_i) = D(t, x_i) \quad (134.5)$$

If derivatives commute : $A_{\mu\nu}$ is symmetric

Theorem 134.1. The second order terms can be rewritten as $A_\mu u_{\mu\mu}$ with an appropriate change of coordinates.

Canonical form :

$$A_\mu u_{\mu\mu} + B_\mu u_\mu + Cu = D \quad (134.6)$$

Quasilinear hyperbolic equations

Definition 134.2. A PDE is said to be :

- *elliptic* if all the eigenvalues of $A_{\mu\nu}$ are positive or all are negative
- *hyperbolic* if none are zero and one has the opposite sign of the $(n - 1)$ others
- *parabolic* if one is zero and all others are of the same sign

134.2. THE CAUCHY PROBLEM OF CLASSICAL FIELDS AND PARTICLES ON A GLOBALLY H

- Ultrahyperbolic if none are zero and the same number are positive and negative.

In the canonical form :

Elliptic :

$$\sum_{i=1}^n u_{ii} \quad (134.7)$$

Hyperbolic :

$$-u_{tt} + \sum_{i=2}^n u_{ii} \quad (134.8)$$

Parabolic :

$$\sum_{i=2}^n u_{ii} \quad (134.9)$$

Ultra hyperbolic :

$$\sum_{i=1}^{n/2} u_{ii} - \sum_{i=n/2+1}^n u_{ii} \quad (134.10)$$

Constraint equations

134.2.1 Point particles

Cauchy problem for causal point particles $\rightarrow$ uniqueness of solutions for causal curves

Solution is unique for globally hyperbolic spacetimes

Show tachyons don't have a well defined Cauchy problem in general

Geroch article on why maybe tachyons are cool

134.2.2 Classical fields

That paper Valter Moretti linked on the wave equation in GR

Paper by Geroch on tachyons

134.3 The Cauchy problem of classical fields and particles on non-globally hyperbolic spacetime

Definition 134.3. An open set $U \subset \mathcal{M}$ is causally regular if for every $\varphi \in C(U)$ that satisfies $\square\varphi = 0$, there is a function $\varphi' \in C(\mathcal{M})$ that satisfies $\square\varphi'$ and $\varphi'|_U = \varphi$

Definition of Yurtsever :

Definition 134.4. An open set U is causally regular if its closure $\bar{U}$ contains an open neighbourhood U' such that for every $(n/2 - 1)$ -form $\omega \in \Gamma(\Lambda^{n/2-1}U')$ satisfying $\square\omega$ on U , there exists a smooth form ω' on $\mathcal{M}$ such that $\omega'|_U = \omega$ and $\square\omega' = 0$.

With this definition, $\mathcal{M}$ itself is obviously causally regular.

Causally regular : a local solution can be extended to a global solution

Definition 134.5. A point $p \in \mathcal{M}$ is causally regular if every neighbourhood U such that $p \in U$ contains a causally regular neighbourhood.

(Uniqueness of solution?)

(Does it apply for fields that are hyperbolic but not free scalars)

Definition 134.6. A spacetime is causally benign if for every open set U , there is an open set $U' \subset U$ which is causally regular.

Chapter 135

The Cauchy problem in general relativity

In a manner similar to the Cauchy problem for fields, the Cauchy problem for spacetime itself will involve a set of initial data for the spacetime to evolve.

Definition 135.1. A set of initial data for a spacetime is a triplet $(\Sigma, \bar{g}, K)$ such that Σ is a hypersurface of codimension 1, $\bar{g}$ is a Riemannian metric and K is a rank $(0, 2)$ tensor on Σ .

From given initial data find a solution (ideally unique) for future times.

Definition 135.2. A development of a set of initial data is a triplet $(\mathcal{M}, g, \sigma)$ where $(\mathcal{M}, g)$ is a Lorentz manifold and σ is a diffeomorphism $\sigma : \Sigma \rightarrow \mathcal{M}$

In other words, we have an initial spacelike hypersurface (which we can consider as some spacelike neighbourhood at a given time), with the values of the metric and its derivatives on it.

135.1 The Cauchy problem for globally hyperbolic vacuum solutions

The Cauchy problem for (lambda-)vacuum spacetimes is done by considering the Einstein field equations after some gauge fixing.

Uniqueness of equations : $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ + some gauge condition

Splitting via ADM formalism

135.2 The Cauchy problem for globally hyperbolic solutions with matter fields

cf Hawking

135.3 The Cauchy problem for non-globally hyperbolic solutions

Once we drop the requirement of global hyperbolicity, the situation becomes much more complicated, as can be seen simply by considering the examples of taking Minkowski space $(\mathbb{R}^n, \eta)$ and Minkowski space minus the timelike line $\{(t, \vec{x}) | t > 0, \vec{x} = 0\}$. Not only is the development of some surface at $t = 0$ non-unique (in fact, without any further conditions, there are infinitely many developments), but we are not even required to have the same topology of the spacelike hypersurface.

Non-unique development from topology (cf Krasnikov)

135.3.1 Cauchy problem and singularities

Boundary conditions on singularities

One simple example of a singular spacetime with an ill-defined Cauchy problem is the case of Minkowski space with a half-line removed.

Consider the Minkowski space in canonical coordinates (t, x) with the line $(t > 0, x = 0)$ removed, and let's try to solve the Klein-Gordon equation on it.

$$S[U; \phi] = \int_U \langle d\phi, d\phi \rangle \quad (135.1)$$

One common example of a nakedly singular spacetime is the extremal Kerr metric.

$$ds^2 = \quad (135.2)$$

135.3.2 Cauchy problem and closed causal curves

Another common type of non-globally hyperbolic spacetime is the

Example : Minkowski torus, Gödel metric, Deutsch-Politzer

Krasnikov on the non-uniqueness of development for non-causal spacetimes

Non-unique matter evolution

Benign closed timelike curves

The nonlinear case is much less explored, as exact solutions in the globally hyperbolic case is already hard enough, but there are a few elements we can consider.

That one with non-linear functions on a one-dimensional spacetime with branches

There exists a few solvable nonlinear field theories on Minkowski space which admit periodic solutions, allowing some analysis of

Sine-Gordon

Breather solutions

Chapter 136

Quantum evolution

Chapter 137

Chaotic behaviour in relativity

A less drastic issue of relativity than its well-posedness is that of its stability. While a system may be well-defined in the sense that its evolution depends continuously and uniquely on its initial data, that dependence may be such that minimal differences between initial conditions can lead to large differences in evolution.

Definition 137.1. In a dynamical system, the system is said to be *chaotic* if the

Example 137.2. The double pendulum is an example of a chaotic system

[342, 343]

Part VII

Practical formulas and numerical relativity

So far we have mostly seen exact formulas relating to general relativity, but most practical general relativity is not done with those. The Einstein field equations and other related formulas are, except for very symmetric or exceptional cases, usually intractable. They are very high dimensional and non-linear

Chapter 138

Linearized gravity

To make general relativity more tractable, the most obvious step is to linearize the theory. To linearize the theory around some background metric, we consider an arbitrarily small region around this metric in the moduli space of Lorentz metrics

$$g_\lambda \tag{138.1}$$

Cf. Besse, [344, 345]

[Related to the variation δg]

For g some metric on M , curve $g_t \in \text{Lor}(M)$, first jet $h = j_0^1 g_t$, linearized EFE :

$$E'_g(h) = \frac{d}{dt} E(g_t)|_{t=0} = 0 \tag{138.2}$$

$$2E'_g(h) = D^* D h - 2\delta^* \delta h - D d(\text{Tr}(h)) - 2\overset{\circ}{R} h \tag{138.3}$$

Theorem from Berger-Ebin [346] : A symmetric 2-tensor field h is an infinitesimal Einstein deformation of g iff

$$(D_g^* D_g - 2\overset{\circ}{R}_g) h = 0 \tag{138.4}$$

$$\delta_g h = 0 \tag{138.5}$$

$$\text{Tr}_g h = 0 \tag{138.6}$$

Space of infinitesimal Einstein deformations is finite dimensional

For a formal series

$$g_t = g + \sum_{k=1}^{\infty} \frac{t^k}{k!} h_k \quad (138.7)$$

$$E(g_t) = E(g) + \sum_{k=1}^{\infty} E_g^k(h_1, \dots, h_k) \quad (138.8)$$

A tangent vector $h_1 \in \text{Ker } E_g^1$ is formally integrable if there exists a formal series g_t such that $E(g_t) = 0$

An infinitesimal Einstein deformation integrates into a curve of Einstein metrics iff it is formally integrable

Linearization of any function depending on g : for some function/functional $f : \text{Met}(M) \rightarrow X$, the object in X will be [the part of the object in X such that its derivative wrt g is zero]

$$\frac{d}{d\lambda} f(g_\lambda) \quad (138.9)$$

$$g = g_0 + \lambda h \quad (138.10)$$

Perturbative inverse metric via the von Neumann expansion

$$(I + \varepsilon B A^{-1})^{-1} = \sum_{k=0}^{\infty} (-\varepsilon B A^{-1})^k \quad (138.11)$$

$$= I - \varepsilon B A^{-1} + \sum_{k=0}^{\infty} \varepsilon^{k+2} (-B A^{-1})^{k+2} \quad (138.12)$$

Determinant

Connection :

$$\Gamma^\sigma_{\mu\nu} = \quad (138.13)$$

Curvature

$$\text{Riem} \quad (138.14)$$

138.1 Linearized gravity as an independent theory

Do that problem of the Lagrangian thing from that presentation and the Feynman book

138.2 The spin 2 field

An alternative way to consider linearized gravity is to view it as a simple field operating on a fixed background.

As we've seen, h is a symmetric rank $(0, 2)$ tensor

Theorem 138.1. The field h is

It should be noted here that *the entire configuration space $\Gamma(T_2^0 M)$ cannot be used as the graviton's configuration space.* [CHECK IF THAT'S TRUE]

As we have used the assumption that h is the vector in some small neighbourhood around the background field, h will itself be in some small neighbourhood around the zero section [prove it?]

An easy way to see it is to just consider that some arbitrary field h is liable to change the signature of the overall metric

Check if that's true even for the metric $g_t = g + \sum \dots$

Impact of diffeomorphism invariance on h

As our gravitational field started out as a Cartan geometry with a lift of diffeomorphisms to field automorphism, we should look into what will happen to the field h under coordinate change.

For our field h in a trivializing neighbourhood U , and a diffeomorphism ϕ on U ,

$$\phi_* g_t \tag{138.15}$$

In coordinates

Tetrad version

138.3 Pauli-Fierz gravity

[270]

If the background metric we compute our linearized theory around is the Minkowski metric, in some topologically trivial neighbourhood (U, η) , then the linearized theory we obtain is the *Pauli-Fierz theory*.

$$S[U; h] = \int_U \square d\mu[\eta] \quad (138.16)$$

138.4 Gravitational waves

One benefit of linearized gravity is the ease with which one can derive gravitational wave solutions.

In the harmonic gauge :

$$\square \xi = -d(h - \frac{1}{2}\text{Tr}(h)\eta) \quad (138.17)$$

$$\square \bar{h} = 0 \quad (138.18)$$

Source term

Quadrupolar moment

Degrees of freedom (related to propagating DoF)

Relation to Weyl tensor

Energy carried by the wave (higher order expansion)

Coordinate artefact?

Feynman bead argument

There were some doubts about whether gravitational waves

Chapter 139

Perturbative gravity

While there are a lot of approximations we can do on general relativity, a question that is rarely considered is what is the error size of those approximations. A few of the different approximations that are commonly used are :

1. Using an approximation of the EFEs, such as linearized gravity around a background (or higher order approximations)
2. What we can reconstruct of the initial conditions
3. The approximation of the stress-energy tensor of an object by abstracting its structure or equation of state
4. The approximation of the stress-energy tensor by ignoring far-off sources

The error given by those approximations is usually given in terms of big- O notation, which tell us the behaviour of the error with respect to some parameters but fails to give us an actual size of this error. As we are not in a position to simply make the length scale arbitrarily smaller in experiments, it is of some use to try to work them out.

Error size on the approximation of a PDE

The error size of a function $f : X \rightarrow \mathbb{R}^n$ given by an approximation $\tilde{f}$ is the difference of the two,

$$R = f - \tilde{f} \tag{139.1}$$

Since knowing the actual error function is tantamount to actually solve the actual problem without error, and since we will generally speaking not be able to have such narrow estimates, what we typically want is to find some bound on the error size. This can take a variety of forms, such as a global bound on the error

$$|R| < M \quad (139.2)$$

with $M \geq \text{lub}(R)$, or it can take the form of some parametrized error size, such as observing the evolution of the error with time.

[347]

If we have a differential operator D [def. from jets?], and an approximated differential operator $\tilde{D}$ such that

$$D = \tilde{D} + \Delta \quad (139.3)$$

with $\tilde{D}\tilde{f} = 0$, then

$$D(f) \quad (139.4)$$

Possible influences on the error bound : upper and lower bounds on functions, bounded variation, etc

Sobolev functions?

Example : Linearized gravity

Error size on the initial condition $\phi_0 + \Delta\phi_0$, or maybe some neighbourhood in the function space

Far off sources : if DEC we can cut off outside the light cone, otherwise ???

Point 1 : Perturbative theory, error estimation

Einstein pseudo-tensor here?

Chapter 140

The classical limit

[348, 349]

A lot of the formulations of general relativity simplifies if we consider the classical limit, but what the classical limit is can be a bit difficult to define. There's a few (possibly related) elements that make up what is considered the classical limit :

- The kinematic group is contracted using the $c \rightarrow \infty$ limit
- There is an absolute time defined by a preferred time direction
- There is an absolute space defined by a preferred spatial distribution
- The spacetime is globally hyperbolic
- The theory is linearized
- The elements of the stress-energy tensor are of low value, $\|T\| \ll 1$
- In particular, the pressure is much lower than the energy density : $p/\rho \ll 1$
- The matter velocity is low : $v/c \ll 1$
- The spacetime is asymptotically flat

We'll see how all those requirements mesh together to give us various limits of general relativity, ending up on the classical limit.

Furthermore, the classical limit here can refer to different possible interpretations. We could consider the classical limit of the kinematic structure, where we look at the limit of (M, g) as a Lorentzian manifold, we could consider it for the dynamics, giving us the limit of the theory, we can consider the limit of a configuration of the metric, and we can look at what happens to the matter.

Topology of the manifold and integrability of the time to avoid CTCs $n \wedge dn = 0$ also $p/\rho \ll 1$, the matter velocity v resp. the newtonian coordinate system is $v/c \ll 1$

Find the relations between all of these and the flow diagram for all the approximations, GR, linearized gravity, Newton-Cartan, Newton, etc

140.1 Kinematic contraction

One important element of the classical limit is that the speed of light is considered to go to infinity, $c \rightarrow \infty$, leading to a contraction of the kinematic group to a simpler group.

As we've seen however^{25.11}, group contractions are not merely done by considering the limit of some parameter in the group, we have to pick a specific coordinate transformation. There can be a variety of such transformations involving the speed of light so that it is best to be careful about the exact meaning of the term, as a naive contraction of the metric tensor would for instance simply end on a Riemannian 3-manifold :

[matrix]

When we say that the speed of light is infinite, geometrically, we want that there should not be any causal vector in the tangent bundle with a velocity larger than that of light. This corresponds to the light cones closing in on each other and the disappearance of the spacelike region :

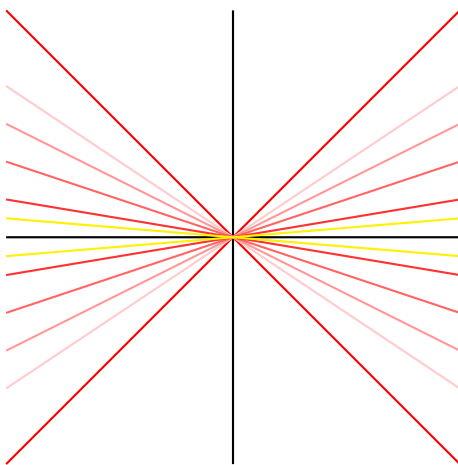


Figure 140.1: The $c \rightarrow \infty$ limit

For the purpose of this chapter, we do need to put the speed of light c back into our equations. Given an orthonormal basis of the tangent bundle made of timelike and spacelike vectors, (e_t, e_i) , the speed of light c converts the time dimension into a space dimension, giving us the basis (ce_t, e_i) .

In the usual case, a timelike vector v is given by the components,

$$v = cv^t e_u + v^i e_i \quad (140.1)$$

In this context, we have that for light, its velocity obeying $g(v, v) = 0$, means that v is in the plane formed by the fusion of the two light cones, in the basis formed by that plane and the timelike vector orthogonal to it, we have

$$u = 0e_t + v^i e_i \quad (140.2)$$

Its "speed" is given by $\|\vec{v}\|/v_0$, which in the limit of the two light cones closing, is indeed infinite.

For any timelike vector, it will simply be some value

$$u = v^t c e_t + v^i e_i \quad (140.3)$$

with its speed $\|\vec{v}\|/v_0$ a well-defined quantity.

This does not seem a basis invariant process, as we are picking a preferred time direction and a preferred spacelike plane to do our contraction on. We will see what the specific choice of basis corresponds to.

A change of basis in this new kinematic space will mostly be the same as for Minkowski space, except for one exception : as the spacelike region is squashed into a plane, a boost will move the timelike element of a basis but not the spacelike ones (as a boost will simply move a spacelike vector into another part of the spacelike region without moving with respect to the spacelike basis e_i). This means that the boost of a vector will be something of the form

$$K_{\mu\nu} v^\mu = K_{t\nu} v_t + K_{i\nu} v^i \quad (140.4)$$

The timelike component of the boost needs to be contracted to 0. This can be done by considering a new basis of the Poincaré group by replacing only the two elements (K^i, P_0) by new ones (C^i, E)

$$K^i = cC^i \quad (140.5)$$

$$P_0 = \frac{1}{c}E \quad (140.6)$$

Given the usual relation of the Poincaré algebra, we obtain

$$\left[cC^i, \frac{1}{c}E \right] = -icE \quad (140.7)$$

which in the appropriate limit $c \rightarrow \infty$ gives us that those new generators obey the relation $[C^i, E] = 0$, or that boosts commute with time translation. Since trivially $[\partial_t, E] = 0$, we have that this new generator indeed drops the timelike part of the Lorentz boost.

We now need to show that this still forms an appropriate Lie algebra

[...]

The resulting Lie algebra is the *Galilean algebra*, $\mathfrak{gal}(n)$, which integrates to the *Galilean group* $\text{Gal}(n)$. Its stabilizer subgroup is the semidirect product of the boosts and spatial rotations, which is isomorphic to the Euclidian group $\text{ISO}(n-1)$.

[350]

Doing this change to the kinematic group has quite a large amount of consequences down the line for the physics, as almost every step of our derivation of physics relies on the kinematic group in some way.

The Galilean model space

Metric and inverse metric to the space metric and time metric

[Does it make sense to reduce the principal bundle by itself or do I need to do it around a section, up to some $SO(3)$ choice maybe]

First, the change in kinematic group will obviously change the principal bundle that directly relies on the kinematic group. While we started with the Cartan geometry $(\text{Poinc}(1, n-1), \text{O}(1, n-1))$, our new Cartan geometry will be $(\text{Gal}(1, n-1), \text{ISO}(n-1))$. This means that our original principal bundle $P_{\text{SO}(n-1,1)}$ is replaced by $P_{\text{ISO}(n-1,1)}$. The solder form and its inverse suffer a split due to this

Our connection, originally the reduction $F_{\text{O}(1,n-1)}^2 M \rightarrow F_{\text{O}(1,n-1)} M$, is now the reduction $F_{\text{ISO}(1,n-1)}^2 M \rightarrow F_{\text{ISO}(1,n-1)} M$

All the associated fiber bundles to the kinematic group will now have to change their structure group.

$$P_{\text{SO}(1,n-1)} \times S/\text{SO}(1, n-1) \rightarrow P_{\text{ISO}(1,n-1)} \times S/\text{ISO}(1, n-1) \quad (140.8)$$

In particular, the fake tangent bundle will now transform as

$$P_{G(1,n-1)} \times S/G(1, n-1) \quad (140.9)$$

Contraction of the EoM or Lagrangian

General relativity is based on the Cartan group of the Poincaré and Lorentz group, so that we need to consider the appropriate contraction for those

Poincaré algebra :

$$[J_m, P_n] = i\epsilon_{mnk} P_k \quad (140.10)$$

$$[P_m, P_n] = \quad (140.11)$$

Inönü-Wigner contraction of the connection

Galilean

"The system is small compared to the radii of the curvature or equivalently the maximum density of the mass energy and the linear dimensions over which the density of mass energy is large satisfies $\rho \ll \ell$ " or $U/c^2 \ll 1$

Contraction of the metric and inverse metric, resp. contraction of the solder form and its inverse

140.2 Galilean manifolds

The equivalent of the Lorentzian manifold for the Galilean group is the Galilean manifold.

Definition 140.1. A *Galilean manifold* (M, δ, β) is a manifold M equipped with a nowhere vanishing 1-form and a rank $(0, 2)$ symmetric tensor δ of

[349, 351]

140.3 Galilean dynamics

Fermat particles

140.4 Newton-Cartan theory

γ, t : Coriolis structure

140.5 Newtonian gravity

If we perform every possible simplifications of general relativity (short of making it a theory with no gravity, as in the Aristotelian case), we obtain *Newtonian gravity*.

Newtonian gravity is usually written with its own notation derived from classical mechanics. Vectors tangent to Σ are simply written as $\Sigma^*X = \vec{X}$, while the vectors tangent to the time flow are simply written as scalars t , since the cases are roughly similar in one dimension.

We will also generally confuse positions and vectors in Newtonian mechanics. A position vector $\vec{x}$ corresponds to the position given by the exponential starting at the origin point :

$$p = \exp_o(\vec{x}) \quad (140.12)$$

$$\vec{x} = \exp_o^{-1}(p) \quad (140.13)$$

In this context, the sum of two position by the sum of their two vectors correspond to parallel transport

Sums of position as parallel transport

Derivatives in Newtonian gravity simplify since we are considering

How to write the derivatives

Write the coordinate changes

Newtonian gravity's field equation is the gravitational Poisson equation :

$$\Delta\Phi = 4\pi G\rho \quad (140.14)$$

We also sometimes consider its *gravitational field* $\vec{g} = \vec{\nabla}\Phi$

$$\vec{\nabla} \cdot \vec{g} = 4\pi G\rho \quad (140.15)$$

As a linear partial differential equation, the equation can be solved by considering the Green's function of the Laplace operator Δ ,

$$\Delta G[\phi_1, \phi_2] = \delta() \quad (140.16)$$

$$\Delta G(x, y) = \delta(x - y) \quad (140.17)$$

Theorem 140.2. The Green's function of the Laplace operator is

$$G(x, y) = 4\pi \frac{1}{\|\vec{x} - \vec{y}\|} \quad (140.18)$$

Proof.

□

The solution for a mass distribution ρ is

$$\Phi(\vec{x}) = \int_{\Sigma} \frac{\rho(\vec{y})}{4\pi|\vec{x} - \vec{y}|} d^3y \quad (140.19)$$

In the limit of a point mass situated at $\vec{x}_0$, this leads to

$$\Phi(\vec{x}) = \int_{\Sigma} \frac{m\delta(\vec{x}_0 - \vec{y})}{4\pi|\vec{x} - \vec{y}|} d^3y \quad (140.20)$$

$$= \frac{m}{4\pi|\vec{x} - \vec{x}_0|} \quad (140.21)$$

Problems with renormalization (gravitational potential energy)

Solving via Green functions to get the appropriate result for point masses

140.5.1 Symmetries

Spatially homogeneous and isotropic system due to the flat structure : conservation laws for the momentum and angular momentum [Or does it? What if Φ depends on spatial coordinates]

Newtonian gravity as a geometric theory is formally not maximally symmetric except for the vacuum case, as we simply have a lack of Killing vectors etc

It is easy to see this by considering any two test particles in a classical gravitational field, as no change in coordinate will lead to a system where energy or momentum is conserved

the classical case is much easier to consider as a perturbation around the vacuum case, in which case we can simply consider the entire system of the perturbation and the rest acting against the flat background.

Conservation laws of the Galilean group : the moments $\mu = \{\vec{L}, \vec{g}, \vec{p}, E\}$

For a point particle : $\mu = \{m\vec{r} \times \vec{v}, \vec{r} - \vec{v}t, m\vec{v}, \frac{m}{2}\|\vec{v}\|^2\}$

Point particles in a gravitational field :

In the case where the gravitational field Φ has additional symmetries, we fall back on the usual case of Killing symmetries. The two most common cases that we will consider is the spherically symmetric case (which is the field generated by a spherically symmetric distribution of matter such as a point particle) and the static case

140.5.2 Coordinate forces

Coordinate forces : Born, Rindler

140.5.3 The two body problem

In the context of the classical limit of general relativity, there are two ways in which we can solve the two body problem : we can either consider the relativistic two body problem (in the test particle limit), such as a free particle in a Schwarzschild metric, and then consider its contraction to the classical limit, or we can use Newtonian gravity directly.

Unlike the case of general relativity, we can solve the two-body problem exactly (with some problems but still) without the use for any approximation such as the test particle limit, although we will still look at that approximation as well to compare it with the classical limit of Schwarzschild.

Consider the case of two point masses $\vec{X}_1, \vec{X}_2$, of masses m_1, m_2 . In some inertial frame, we have

$$m_1 \ddot{\vec{X}}_1(t) = \frac{Gm_1m_2}{\|\vec{X}_1 - \vec{X}_2\|^2} \frac{\vec{X}_2 - \vec{X}_1}{\|\vec{X}_1 - \vec{X}_2\|} \quad (140.22)$$

$$m_2 \ddot{\vec{X}}_2(t) = \frac{Gm_1m_2}{\|\vec{X}_1 - \vec{X}_2\|^2} \frac{\vec{X}_1 - \vec{X}_2}{\|\vec{X}_1 - \vec{X}_2\|} \quad (140.23)$$

We define the separation vector between the two objects as $\vec{r} = \vec{X}_2 - \vec{X}_1$, with $r = \|\vec{r}\|^2$ and the unit vector $\vec{e}_r = \vec{r}/r$:

$$m_1 \ddot{\vec{X}}_1(t) = \frac{Gm_1m_2}{r^2} \vec{e}_r \quad (140.24)$$

$$m_2 \ddot{\vec{X}}_2(t) = -\frac{Gm_1m_2}{r^2} \vec{e}_r \quad (140.25)$$

To simplify the problem, there are two motions that we can consider : the motion of the center of mass, and the relative motion of each object with respect to the center of mass. The center of mass motion is given by

$$\ddot{\vec{X}}_{\text{CM}}(t) = \quad (140.26)$$

Center of mass :

$$m_1 \ddot{\vec{X}}_1(t) + m_2 \ddot{\vec{X}}_2(t) = \quad (140.27)$$

[...]

If we now look at the limit in which $m_1 \gg m_2$, many simplifications occur. Let's rewrite our system as $X_1 \rightarrow 0$ and $X_2 \rightarrow X$, with $m_1 \rightarrow M \approx \infty$ and $m_2 \rightarrow m$.

If we disregard the mass m_2 from its contribution to the gravitational field, as far as the geometry is concerned the system is spherically symmetric around X_1 , giving us conservation of angular momentum around it.

As the reduced mass μ The center of mass will merge with X_1

Conservation of angular momentum :

$$\vec{L} = \vec{r} \times \vec{p} \quad (140.28)$$

of constant value

$$\frac{d}{dt} \vec{L} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = 0 \quad (140.29)$$

Since $\vec{r}$ and $\vec{p}$ are colinear,

$$\vec{r} \times \dot{\vec{p}} = 0 \quad (140.30)$$

As a flat space along the spacelike hypersurfaces, Newton-Cartan admits some coordinates of the default $SO(3)$ -structure on those slices, the Cartesian coordinates (t, x, y, z) , which give rise to the usual form of the Newton equations, but it is possible to use other coordinate systems which give us the usual coordinate forces of classical mechanics.

Rindler coordinates

If we consider the Born coordinates,

$$\theta = \theta - \omega(t)t \quad (140.31)$$

140.6 Limit process

Beyond simply considering the limit of the kinematics and structure of general relativity to the non-relativistic limit, a further thing that can be done is to consider the limit of given configurations as well.

This is a highly non-trivial process, as the exact meaning of what the "non-relativistic limit" of a given system is unclear. If we simply take all of our original objects and leave them "as is", so to speak, the limit would fundamentally change the behaviour of those objects. Consider for instance the following curves

[diagram]

If we simply go to the non-relativistic limit without changing those curves, the null curves and most of the spacelike curves will become timelike curves by the simple widening of the light cone.

On the flip side, if we try to contract the configurations themselves, we end up with contradictions of the following sort : consider two spacelike curves emanating from the same point p , such that each of those curves have a point p_i along them for which p_1 and p_2 are connected by a timelike curve. The contraction would mean that those two curves would get squashed to the same curve, making the distance p_1, p_2 null (this roughly stems from the fact that exchange processes go at infinite speed and therefore occur simultaneously). However, this is true for any arbitrarily distance point p , meaning that any timelike curve can be so contracted to zero.

Therefore we will have to be quite careful with what we exactly mean by such a limit.

Chapter 141

The optical approximation

In the case where we only consider the motion of light rays on a spacetime, it can be useful to approximate it so as to only contain the data relevant for this.

Chapter 142

Parameterized post-Newtonian formalism

As long as our theories of gravity remain some variation of a

For theories of a symmetric metric tensor, test bodies following the geodesics of that metric and locally like special relativity

The parameterized post-Newtonian formalism (also called PPN formalism) has two main goals : as a kinematic theory, it offers some tractable approximations of metric theories, and as a dynamical theory, it is flexible enough to accommodate a wide variety of different theories. It can thus be used both for practical predictions in a given theory, as well as offering a way of comparing different theories.

There are many conditions that need to be met for the PPN formalism to apply, most of which can be met under the condition that we consider a totally geodesic neighbourhood. We need the region of spacetime considered to be globally hyperbolic, having some space-time split given by a time function t and spacelike distribution ω , the metric should be asymptotically flat (if it is a finite region, in the sense that the metric decays) [is that necessary]

Using our split, we define some coordinates in which our metric decomposes as

$$g_{00} = -1 + \mathcal{O}(c^{-2}) \tag{142.1}$$

$$g_{0i} = \mathcal{O}(c^{-3}) \tag{142.2}$$

$$g_{ij} = \delta_{ij} + \mathcal{O}(c^{-2}) \tag{142.3}$$

We perform some Taylor expansion in which we cut off terms higher than third order.

$$g_{00} = -\exp\left(-\frac{2w}{c^2}\right) = -1 + \frac{2w}{c^2} - \frac{2w^2}{c^2} + \mathcal{O}(c^{-6}) \quad (142.4)$$

$$g_{0i} = -\frac{4w_i}{c^3} + \mathcal{O}(c^{-5}) \quad (142.5)$$

$$g_{ij} = \gamma_{ij} \exp\left(\frac{2w}{c^2}\right) \approx \gamma_{ij} \left(1 + \frac{2w}{c^2}\right) + \mathcal{O}(c^{-4}) \quad (142.6)$$

PPN bookkeeping (Will) : In the solar system, the gravitational potential Φ is nowhere larger than 10^{-5} in geometrized units

Planetary velocities are related to U by Virial relations which yield $v^2 < \Phi$

The matter of the sun and planets are under pressure p which is generally smaller than the matter's gravitational energy $\rho\Phi$, $p/\rho < \Phi$ (p/ρ is about 10^{-5} in the sun, 10^{-10} in Earth)

Other form of energy densities in the solar system (radiation, thermal, etc) are small : Π specific energy density (ratio of energy density to rest mass density) is $\Pi < \Phi$ (Π is $\approx 10^{-5}$ in the sun, $\approx 10^{-9}$ on Earth)

Scales involved : $\Phi \approx v^2 \approx p/\rho \approx \Pi \approx O(2)$, $v \approx O(1)$, $\Phi^2 \approx O(4)$, $\Phi v \approx O(3)$, $\Phi\Pi \approx O(4)$

$\partial_t \approx \vec{v} \cdot \vec{\nabla}$, ratio of time derivatives to space derivatives is $\approx O(1)$

For a point particle in Nambu-Goto action :

$$S[U; X] = -m \int \sqrt{-g(\dot{X}, \dot{X})} dt \quad (142.7)$$

$$= -m \int \sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j} dt \quad (142.8)$$

Formulate in terms of split? Observer? etc

Some restrictions on the fluid matter : The metric terms should be of post-Newtonian order, no higher order term. The terms should tend to 0 with the distance $|x - x'|$, x' a point inside the matter, becomes large (metric is asymptotically Minkowski). Coordinates chosen so that the metric is dimensionless. In our quasi-cartesian system, the spatial origin and initial moment of time are arbitrary, so the metric should contain no explicit reference to these quantities.

The metric corrections $h_{00}, h_{0,i}, h_{ij}$ should transform under spatial rotations as a scalar, vector and tensor.

In the rest frame of an isotropic universe, no vectors or anisotropic tensors can be constructed.

Metric functionals should be generated by rest mass, energy, pressure and velocity, not their gradients (subjective)

Functionals should be simple

Einstein–Infeld–Hoffmann equations

[353]

Hydrodynamical PPN

[354]

[355]

Chapter 143

Backreaction

We have so far mostly seen two cases when it comes to the interaction of gravity and other physical systems. Either we consider them to be in the test limit, where physical objects move on the geometry of spacetime but do not affect it, or we try to solve the entire system given by their interaction. Both methods have their pros and cons, the test field limit being much more tractable to solve but obviously not the actual solution we would expect, while the exact method is typically impossible to solve except for a handful of very symmetric cases.

There can however be cases where the exact solutions would be too complex to work out but the gravitational effects of the objects we consider are too large to ignore. One method we can use in this case is to consider the *backreaction* of the object on the geometry.

143.1 The MiSaTaQuWa equation

A typical case where we may use the backreaction is in the case of some material body that we would typically model as a geodesic. If we have consider some background spacetime (M, g) in which some massive object moves along,

That method from Visser by convergence

Chapter 144

Calculation tricks

Doing general relativity by hand can be extremely time consuming, as the average tensor of rank (r, s) can require the computation of up to n^{r+s} components, which even in realistic cases can very often include rank 4 tensors in 4 dimensions, for up to 64 components.

Most computations will however be much shorter than this, as we can exploit a variety of symmetries to remove them.

A fairly simple case of such tricks revolve around the case of *diagonal metrics*. A metric is diagonal in a coordinate system if its components in that coordinate chart are all diagonal :

$$g = \text{diag}(g_{00}, g_{11}, \dots) \quad (144.1)$$

dropping its number of components from $n(n-1)/2$ to n . In the case of a diagonal metric, two immediate results is that the inverse metric is the inverse of the diagonal :

$$g^{\mu\nu} = \frac{1}{g_{\mu\mu}} \quad (144.2)$$

and the determinant is the product of the diagonal,

$$\det g = \prod_{\mu=0}^{n-1} g_{\mu\mu} \quad (144.3)$$

If our metric is diagonal, the Christoffel symbols simplify

Theorem 144.1. For a diagonal metric, the only non-zero components of the Christoffel

symbols are of the form, for a given index x :

$$\Gamma^x_{bb} = -\frac{1}{2}g^{xx}g_{bb,x}(b \neq x) \quad (144.4)$$

$$\Gamma^x_{bx} = \frac{1}{2}g^{xx}g_{xx,b}(b \neq x) \quad (144.5)$$

$$\Gamma^x_{xx} = \frac{1}{2}g^{xx}g_{xx,x} \quad (144.6)$$

Proof. Given the form of the Christoffel symbols,

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\tau}(g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau}) \quad (144.7)$$

If we look at components all of the same index,

$$\Gamma^x_{xx} = \frac{1}{2}g^{\sigma x}(g_{x\tau,x} + g_{x\tau,x} - g_{xx,\tau}) \quad (144.8)$$

The only non-zero components are for $\tau = x$:

$$\Gamma^x_{xx} = \frac{1}{2}g^{xx}g_{xx,x} \quad (144.9)$$

□

This reduces the number of Christoffel symbols from $n^2(n+1)/2$ to $n(2n-1)$.

Another common scenario is that within the coordinates given, the metric is independent from some of them (meaning that the holonomic vector fields given by the coordinate lines are Killing vectors of the metric). For instance, for a spherically symmetric static metric,

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2d\Omega^2 \quad (144.10)$$

This leads the metric's derivative with respect to this coordinate to be 0. If we pick some coordinate x for instance, we would have $g_{\mu\nu,x} = 0$.

If a metric tensor stems from a product of pseudo-Riemannian manifolds

$$(M_1, g_1) \otimes (M_2, g_2) = (M_1 \times M_2, g_1 \oplus g_2) \quad (144.11)$$

with a given coordinate system in each, the metric tensor has the form

$$ds^2 = \sum_{\mu,\nu=0}^{k-1} g_{\mu\nu}^{(1)} + \sum_{\mu,\nu=k}^{n-1} g_{\mu\nu}^{(2)} \quad (144.12)$$

[in matrix form]

This gives us various simplifications stemming from the lack of some diagonal terms as well as the independence of each block to the other coordinates. By themselves, the original spaces (M_1, g_1) and (M_2, g_2) will have their own number of Christoffel symbols, respectively $k^2(k+1)/2$ and $(n-k)^2(n-k+1)/2$.

If the manifold is merely a warped product of two pseudo-Riemannian manifold,

$$(M_1, g_1) \times_f (M_2, g_2) = (M_1 \times M_2, g_1 \oplus f g_2) \quad (144.13)$$

Chapter 145

Misc

Schild's ladder?

Chapter 146

Numerical gravity

Numerical gravity is the field of application of numerical methods to general relativity

ADM formalism and numerical gravity

Choice of initial conditions [356]

[206, 357, 358]

BSSN

Part VIII

Alternative theories

While I tried so far to keep everything under the same broad category of geometric theories of gravity and looking at specific types of such theories, not all proposed theories of gravity admit such a nice interpretation under this umbrella. This chapter is then dedicated to all theories of gravity that cannot simply be considered as some geometric theory of gravity described by some Cartan connection.

If the theory of gravity is still a field theory in the broad sense, and still involves the Poincaré group as its kinematic group, there is a broad classification of such theories by the irreps of the Poincaré group those fields transform as. The most common types of such theories are the *scalar theories*, *vector theories*, *scalar-vector theories*, *tensor theories*, *scalar-tensor theories*, and *scalar-vector-tensor theories*. There also exists proposals for less common irreps for gravity, such as *spinor theories*, but those tend to be less common and mostly toy models.

Some other theories entirely bypass this classification by not being based on the Poincaré group or not being field theories.

Chapter 147

Scalar theories

Historically gravity was assumed to be a scalar theory, as the Newtonian theory of gravity is. As we saw previously, Newtonian gravity corresponds to general relativity in some limit, and its equation of motion is

$$\Delta\Phi = 4\pi G\rho \tag{147.1}$$

obeying the Poisson equation.

Historically, this idea naturally generalized into a relativistic theory in the traditional way by considering the natural translation into relativistic terms $\vec{\nabla} \rightarrow \nabla_\mu$:

$$\square\Phi = 4\pi G\rho \tag{147.2}$$

ρ here being some scalar function that should reduce in some classical limit to the mass density. The choice of what ρ is was usually what made the specific theory of gravity.

147.1 Einstein-Nordstrøm theory

The Einstein-Nordstrøm theory was one of the earliest attempt at a relativistic gravitational theory. In this case, the source of the Poisson equation was given as the trace of the stress-energy tensor,

$$\rho = \text{Tr}(T) \tag{147.3}$$

$$\square\phi = -4\pi G\text{Tr}(T) \tag{147.4}$$

The stress-energy tensor here is not defined quite in the same way that it would be in the metric case, since we cannot define it as the variation with respect to the metric. The Lagrangian of such a theory will have to be something of the form

$$L = \|d\phi\|^2 + T[\phi] + L_M \quad (147.5)$$

This is a fairly natural idea, since in the non-relativistic limit,

[show $T \approx \rho$]

$$\mathcal{L} = \frac{1}{8\pi} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \rho \phi \quad (147.6)$$

This theory can actually still be put into geometric form, [Volume structure?]

Equivalent to $g_{\mu\nu} = \omega^2 \eta_{\mu\nu}$, or $e^a{}_\mu = \omega \delta^a{}_\mu$, with curvature

$$R = -\frac{6\Box\phi}{\phi^3} \quad (147.7)$$

$$R = 24\pi T \quad (147.8)$$

147.2 Einstein's scalar theory

$$T_g^{\mu\nu} = \frac{1}{4\pi G} [\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\rho \phi \partial^\rho \phi] \quad (147.9)$$

$$T_m^{\mu\nu} = \rho \phi u^\mu u^\nu \quad (147.10)$$

Chapter 148

$f(R)$ gravity

Another rather straightforward way of generalizing the Einstein-Hilbert action is to simply consider a generic function of the Ricci scalar as the Lagrangian.

$$\mathcal{L} = \frac{1}{2\kappa} f(R) \tag{148.1}$$

From Lovelock's theorem, we can already see that this action will not lead to a theory with a locally conserved current.

Variation of the action :

$$\frac{\partial f(R)}{\partial R} R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + [g_{\mu\nu} \square - \nabla_\mu \nabla_\nu] \frac{\partial f(R)}{\partial R} = \kappa T_{\mu\nu} \tag{148.2}$$

Divergence

Chapter 149

Brans-Dicke gravity

[359]

Scalar-tensor theory

$$\mathcal{L} = \phi R - \frac{\omega}{\phi} \|d\phi\|^2 \tag{149.1}$$

Chapter 150

Massive gravity

The basic idea of massive gravity is to consider some theory of gravity interpreted as a regular matter field, such as the Pauli-Fierz theory, and introducing a mass term to it.

If we consider for instance the linear Pauli-Fierz theory

Pauli-Fierz with mass term [360]

Broken diffeomorphism invariance

Chapter 151

Non-commutative geometry

Elements $[uv]$ defined by the Lie bracket, defines a Lie algebra of vector fields (algebra of diffeomorphisms), called $\mathfrak{X}$.

Universal enveloping algebra $U\mathfrak{X}$ defined by the tensor algebra over $\mathbb{C} uv$ generated by $\mathfrak{X}$ and $uv - vu - [uv]$.

Natural Hopf algebra structure. For the unit element 1,

$$\begin{aligned}\Delta(u) &= u \otimes 1 + 1 \otimes u \\ \Delta(1) &= 1 \otimes 1 \\ \varepsilon(u) &= 0 \\ \varepsilon(1) &= 1 \\ S(u) &= -u \\ S(1) &= 1\end{aligned}$$

Twist of a Hopf algebra

Chapter 152

Vector theories

A *vector theory* of gravity, as the name implies, is simply a theory where the field responsible for gravity is simply a vector field, $\text{Conf} = \mathfrak{X}(M)$.

By their very nature, vector theories are fundamentally different from others in that they select a preferred direction and therefore cannot be a geometric theory

Chapter 153

Regge calculus

153.1 Simplicial manifolds

Define spacetime as a graph of spacetime points linked together by edges, metric is defined by the length of the edges or their angles

Definition : The lattice definition of a spacetime is a graph (v_i, e_i) with spacetime points for vertices and the edges defining the distances between two points. To every edge between points v_i and v_j is associated a real number which is the length between the two points.

It is a simplicial complex.

Curvature on the hinges : in an n -dimensional spacetime, the lattice is made of n -simplices, with the curvature concentrated on subsimplices of dimension $n-2$, aka hinges.

Deficit angle : for a point v_i , consider the angle made by every simplicial n -polytope around v_i . The deficit angle

$$S = \frac{1}{2\kappa} \int R d\mu[g] \rightarrow S_R = \sum_i |\sigma^i| \varepsilon_i \quad (153.1)$$

153.2 In two dimensions

$$S_R = \sum_i l^i \varepsilon_i \quad (153.2)$$

Chapter 154

Theories of a different signature

All theories of gravity have been so far under the assumption of a fundamental split between one dimension of time and other dimensions of space, such as in the usual case the Lorentz group $SO(1, n - 1)$. We can however easily generalize this to the more general case of an arbitrary signature (p, q) .

154.1 Euclidian gravity

The case of Riemannian gravity is not of particular interest for modelling the real world, but it is commonly used for the purpose of simplification of computations (via Wick rotation) and has also been used as a toy model [Greg Egan]

Basics

Most of the basics of the Lorentzian case are about the same. We have some Cartan connection

Riemannian metric and connection

lack of a fundamental preferred direction

154.1.1 Wick rotation

The main use of Euclidian gravity is to consider it as some transformation of Lorentzian gravity, usually written out as the passage to complex time :

$$t \rightarrow t' = it \tag{154.1}$$

which, in the Minkowski case, does indeed work :

$$ds^2 = d(it)^2 + \sum_{i=1}^{n-1} dx^i dx^i \quad (154.2)$$

$$= -dt^2 + \sum_{i=1}^{n-1} dx^i dx^i \quad (154.3)$$

wick rotation relates to timelike distribution?

Field propagation

$$\Delta\phi = 0 \quad (154.4)$$

Causality

Obviously causality cannot be enforced by geometry in the Euclidian case, as there are no preferred direction given.

Cauchy problem on the torus

154.2 Ultra-hyperbolic spacetimes

There have been some investigations in spacetimes with an arbitrary signature (p, q) , in particular with $q > 0$. These are called *ultrahyperbolic spacetimes*, as the resulting wave equation is the one in the ultrahyperbolic case.

[361, 362]

While this type of kinematics has a lot of strange effects, there is no fundamental issue with the functioning of time as we experience it, since our definition of time more properly relates to proper time, which remains the same in the ultrahyperbolic case. For a timelike curve γ ,

$$\tau = \int_{\lambda_0}^{\lambda} \sqrt{g_{\gamma(\lambda)}(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))} d\lambda \quad (154.5)$$

The obvious problem comes from the causal structure : the light cone $\mathcal{N}_p$ is the hyperquadric defined by

$$-t_1^2 - t_2^2 + \dots + t_p^2 + x_1^2 + x_2^2 + \dots + x_q^2 \quad (154.6)$$

An easy example of this is the $(2 + 1)$ dimensional case, which is the mirror image of the $(1 + 2)$ -dimensional case : the typing of timelike and spacelike vectors is interchanged

Since there exists a timelike subspace of dimension higher than 1, there exists smooth closed curves in that subspace, giving us an easy method of making closed timelike curves, of any size.

instability of photons

problems of cauchy

closed timelike curves even locally

Chapter 155

Einstein-Straus theory

[363–369]

One very early unified field theory was the *Einstein-Strauss* theory, based on the observation that the metric tensor is a symmetric tensor field while the electromagnetic tensor is antisymmetric, leading to some motivation to attempt a theory of a general tensor,

$$g = \gamma + \phi \tag{155.1}$$

This corresponds to the splitting of g into its symmetric and antisymmetric parts : γ is a symmetric tensor (of signature $(n - 1, 1)$), while ϕ is an antisymmetric tensor.

The inverse tensor g^{-1} does not correspond to the sum of inverses our two components

Irreducible decomposition of the ES metric?

Chapter 156

Polarizable vacuum

One of the early idea for general relativity was that rather than the geometry of space changing, it was simply the speed of light that did [370]. As the main evidence for relativity related to the behaviour of light, this was a fairly natural idea.

[371,372]

Coming from special relativity, the fundamental idea that seemed to be underpinning the theory was that the speed of light was invariant in all reference frames.

GR from space-dependent polarization

Puthoff

Chapter 157

Bimetric gravity

The motivation for *bimetric gravity* is to separate the terms of the metric tensor which correspond to gravity from those that correspond to geometry.

Two metric tensors g, γ [373, 374]

Chapter 158

MOND

The discovery of anomalies in the galactic rotation curve (cf. X) has prompted some to attempt alternative descriptions of gravity to explain those discrepancies in terms of a new theory rather than as being explained by dark matter.

The term itself refers to "Modified Newtonian Dynamics", as the basic idea behind the theory was simply a modified version of Newtonian mechanics. Rather than using Newton's law $\vec{F} = m\vec{a}$, the theory is based on the idea that this formula is only an approximation for low values of a of a more general formula,

$$\vec{F} = m\mu\left(\frac{a}{a_0}\right)\vec{a} \tag{158.1}$$

with μ the *interpolating function*

Tensor–vector–scalar gravity

[375]

Chapter 159

Hoyle's theory

After the discovery of cosmological inflation,

[376, 377] Hoyle attempted to formulate a variation of general relativity which would allow the steady state model to keep on working.

C field : $C \in \Omega^1(M)$, gives rise to the tensor ∇C

$$\text{Riem} + \frac{1}{2}Rg + \nabla C = T \quad (159.1)$$

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} + \nabla_\mu C_\nu = T_{\mu\nu} \quad (159.2)$$

[378, 379]

Chapter 160

Entropic gravity

Entropic gravity is one of the rare theory of gravity that is neither a geometric theory nor a field theory, and is instead proposed to be an emergent property of the entropy of matter [380]

160.1 Entropy

For a macroscopic configuration (or macrostate) X , the entropy associated is

$$S(X) = k_B \ln(\Omega(X)) \quad (160.1)$$

with $\Omega(X)$ the number of microstates associated with this macroscopic configuration.

160.2 Entropic forces

Entropic force : effective macroscopic force generated by the tendency to increase entropy.

No fundamental field associated with an entropic force.

In the canonical ensemble, entropic force $\vec{F}$ associated to a macrostate partition $\{X\}$ is

$$F(X_0) = T \nabla_X S(X)|_{X_0} \quad (160.2)$$

T the temperature, S the entropy.

Examples : colloid molecules suspended in a thermal environment of smaller particles, osmosis, polymer molecules

$$F \approx -\alpha k_B T \langle x \rangle \tag{160.3}$$

160.3 Entropic gravity

Timelike Killing vector $\approx$ temperature and entropy gradient

Geodesic motion $\approx$ entropic force

BTZ black hole in quantum gravity

Part IX

Quantum theory

Chapter 161

Quantum theories

So far we have only considered the case of classical mechanics as it relates to general relativity, but classical mechanics is known to not be a sufficient description of physical systems. From a variety of theorems, we know that there does not exist any modification of classical mechanics that can fully account for the phenomenon that we can observe.

What a quantum theory is can be a bit difficult to pin down as they may follow a variety of different formalisms which do not share obvious similarities, but for the most part, all quantum theories either follow the *von Neumann axioms* or are isomorphic (or close to) to a theory that obeys them :

1. Physical states are represented by rays in a Hilbert space, that is, for $\psi \in \mathcal{H} \setminus \{0\}$, a state is represented by $\psi/\|\psi\|$.
2. Observables are represented by self-adjoint operators on this Hilbert space.
3. For a system in a state ψ , an experiment to measure a quantity associated with the observable $\hat{A}$, with eigenvectors $\{\psi_n\}$ and eigenvalues A_n , the probability of measuring that system in a state ψ_k (with A_k as a result of this measurement) is

$$P(X = A_k) = \langle \psi, \psi_k \rangle$$

The von Neumann axioms are particularly broad as they do not even assume that the quantum theory occurs on a particular underlying space (it may in fact not even occur on one at all). As we are interested in the case of quantum theories as they relate to general relativity, we need to consider

161.1 Hilbert spaces

Most formulations of quantum mechanics involve the configuration space being some manner of (projective) Hilbert space

The basic definition of a Hilbert space is

Definition 161.1. A *Hilbert space* $\mathcal{H}$ over a field F is an F -vector space equipped with a positive-definite sesquilinear form $\langle -, - \rangle$ with respect to which it is complete.

So that a Hilbert space is some kind of structure $(\mathcal{H}, +, \cdot, \langle -, - \rangle)$. The form $\langle -, - \rangle$

$$\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \rightarrow F \quad (161.1)$$

which is sesquilinear, that is, for $X, Y, Z \in \mathcal{H}$ and $a, b \in F$,

$$\begin{aligned} \langle X, aY + bZ \rangle &= a\langle X, Y \rangle + b\langle X, Z \rangle \\ \langle aX + bY, Z \rangle &= \bar{a}\langle X, Z \rangle + \bar{b}\langle Y, Z \rangle \end{aligned}$$

$$\langle X, Y \rangle = \overline{\langle Y, X \rangle} \quad (161.2)$$

The field we will consider quantum mechanics over will typically be $\mathbb{C}$ [ref reason for it]

The inner product on the Hilbert space allows us to define the norm

$$\begin{aligned} \| - \| : \mathcal{H} &\rightarrow \mathbb{R}^+ \\ X &\rightarrow \|X\| = \sqrt{\langle X, X \rangle} \end{aligned} \quad (161.3)$$

$\|X\|$ is guaranteed to be positive real by $\langle X, X \rangle = \overline{\langle X, X \rangle}$

161.1.1 Properties

161.1.2 Linear operators on a Hilbert space

As the Hilbert space is our

Linear operator on a dense subset $D \subset \mathcal{H}$

$$A : D \rightarrow \mathcal{H} \quad (161.4)$$

Norm on operators :

$$|A| = \sup_{X \in D} [A(X)] \quad (161.5)$$

Adjoint of an operator A : An adjoint operator A^* is an operator such that

$$\langle AX, Y \rangle = \langle X, A^*Y \rangle \quad (161.6)$$

An operator is then defined as hermitian if it acts in the same way on vectors as its adjoints

$$\langle AX, Y \rangle = \langle X, AY \rangle \quad (161.7)$$

Self-adjoint if both hermitian and $D(A) = D(A^*)$

C^* algebra : Banach algebra over $\mathbb{C}$

Linear operators

Self-adjoint operators

Projection-valued measures

von Neumann algebra

161.2 Category of Hilbert spaces

The category of Hilbert spaces **Hilb** is the category where each object $H \in \text{Obj}(\mathbf{Hilb})$ is a Hilbert space and each morphism $\hat{O} \in \text{Mor}(\mathbf{Hilb})$ is a linear map between Hilbert spaces.

One important aspect of Hilbert spaces is that they form a *dagger category*.

Equipped with the tensor product $\otimes$, it is a symmetric

Categorical QM toy model : category of relations

[381]

Bohr topos?

161.3 C star algebras

One of the big issue of quantum mechanics if we hope to pass it to the relativistic case is that the Hilbert space $\mathcal{H}$ is usually defined as being related to some synchronization. The inner product $\langle \psi, \psi \rangle = 1$ is meant to be related to some spacelike hypersurface : at a given time, the total probability is 1.

This is a difficult aspect of quantum mechanics to remove, and quantizations will usually work best when this remains possible, such as in the globally hyperbolic case, but to at least delay the moment where we will have to deal with the issue, first let's define some aspects of quantum mechanics that will not rely so much on the Hilbert space.

Instead let's focus for now on the operators on that Hilbert space.

Definition 161.2. A C^* -algebra is a Banach algebra (ie an associative algebra with a norm $\|a\|$ such that $\|ab\| = \|a\|\|b\|$ and a complete space with respect to that norm) with an involution $*$ such that

$$a^{**} = (a^*)^* = a \quad (161.8)$$

$$(a + b)^* = a^* + b^* \quad (161.9)$$

$$(xy)^* = y^* x^* \quad (161.10)$$

$$(\lambda a)^* = \bar{\lambda} a^* \quad (161.11)$$

C^* algebra if $\|a^* a\| = \|a\| \|a^*\|$

161.4 Quantum logic

Solèr's theorem

Chapter 162

Symmetries and quantum theory

162.1 Wigner's theorem

Like in the classical theory, we need to define what it exactly means for a transformation to be a symmetry of the system.

Most of the operators of interest in quantum mechanics are the linear maps on the Hilbert space, so that the main focus will be on them. In this context, we define a *symmetry* on the quantum theory to be an endomorphism $f : \mathcal{H} \rightarrow \mathcal{H}$ on the Hilbert space preserving its inner product :

$$\forall \psi_1, \psi_2 \in \mathcal{H}, \langle \psi_1 | \psi_2 \rangle = \langle f(\psi_1), f(\psi_2) \rangle \quad (162.1)$$

As our measurable entities depend on the projective Hilbert space, we will broaden this definition to also include

Theorem 162.1. A transformation

In most quantum theories, we will generally only consider

Bohr category?

(Braided) dagger monoidal symmetric category

superselection rules

162.2 Quantum theory on a spacetime

By its raw definition, a quantum theory does not need to involve an underlying space, and it is quite common to consider systems which are entirely independent of such a notion, such as the finite dimensional Hilbert spaces $\mathbb{C}^n$ (which can be considered to be quantum theories over a point), commonly used in quantum computing for instance.

[382] If our quantum theory occurs on a spacetime, we need some way to connect the quantum theory to it in some way. One of the way to do it is with *local algebras*.

Given some manifold equipped with structures in some category **Loc**, and the category of $*$ -algebras we define a functor for locally covariant theories :

$$A : \mathbf{Loc} \rightarrow \mathbf{Alg} \quad (162.2)$$

As any open set of a manifold with some structure will itself be such a space, there exists an algebra for any neighbourhood U . We have a inclusion morphism

$$\iota_{M,U} : M|_U \rightarrow M \quad (162.3)$$

and our functor will map this morphism

$$A(M; U) = A(\iota_{M,U})(A(M|_U)) \quad (162.4)$$

If $U_1 \subset U_2$, $\iota_{M;U_1} = \iota_{M;U_2} \circ \iota_{M|_{U_2};U_1}$

$$A(\iota_{M;U_1}) = A(\iota_{M;U_2}) \circ A(\iota_{M|_{U_2};U_1}) \quad (162.5)$$

$$A(M; U_1) \subset A(M; U_2) \quad (162.6)$$

The kinematic net is isotonomous.

If $U_1, U_2 \subset M$,

$$A(M; U_1 \cap U_2) \subset A(M; U_1) \cap A(M; U_2) \quad (162.7)$$

$$A(M; U_1) \vee A(M; U_2) \subset A(M; U_1 \cup U_2) \quad (162.8)$$

For a morphism $f : M \rightarrow N$,

$$A(N; f(U)) = A(f)(A(M; O)) \quad (162.9)$$

Isomorphism of algebras

For an endomorphism $f : M \rightarrow M$, $A(f)$ is an endomorphism of the kinematic net. For isomorphisms, $f \in \text{Aut}(M)$, $A(f)$ is a net automorphism.

Causality : if U_1, U_2 are causally disjoint such that $U_1 \subset M \setminus \overline{J_M(U_2)}$,

$$[A(M; U_1), A(M; U_2)] = \{0\} \quad (162.10)$$

Timeslice : if $f : M \rightarrow N$ is Cauchy [$f(M)$ contains a Cauchy surface for N], $A(f)$ is an isomorphism.

This implies that the theory has a dynamical law

162.3 Kinematics of quantum mechanics

Representation of the kinematic group as 1-parameter endomorphisms whatever (see Simon)

Get a strongly continuous unitary representation of the restricted Poincaré group on $\mathcal{H}$.

$$\langle \psi, U(a)\psi \rangle = \langle \psi, \exp(i\langle a, P \rangle)\psi \rangle = \int_{\mathbb{R}^4} e^{i\langle a, \lambda \rangle} d(\psi, E_\lambda \psi) \quad (162.11)$$

P_0 is the Hamiltonian

Much like in the case of symplectic manifolds, the kinematic group's action on the quantum states has to be through a central extension of the group, but this time via the $U(1)$ group

Spectral condition : the projection-valued measure E_Ω on $\mathbb{R}^4$ corresponding to $U(a, I) = e^{i\langle a, P \rangle}$ has support in the closed forward light cone.

For $g \in \text{Poinc}(n)$,

$$U(g)\phi(f)U^{-1}(g)\psi = \phi(\rho_g f)\psi \quad (162.12)$$

Does it generalize to all kinetic groups

Polysymplectic quantization stuff mb

$$1 \rightarrow U(1) \rightarrow G \rightarrow PG \rightarrow 1 \quad (162.13)$$

This is called the *projective representation* of the group

Mass is measurable for free systems

Strocchi on why you need operator valued distributions for the QFT case

[383, 384]

Definition 162.2. A quantum algebra obeys the *split property* if for two causally independent

162.4 Wigner's classification

General case : classification by stabilizer

Little group

Poincaré case

classification by mass and angular momentum (Casimir invariants)

Conformal case?

162.5 The Stone-von Neumann theorem

A lot of quantum mechanics, at least in the non-relativistic regime, rests on the representations of the commutation relations of operators

$$[\hat{x}_i, \hat{p}_j] = -i\hbar \hat{I} \quad (162.14)$$

$$[\hat{x}_i, \hat{x}_j] = 0 \quad (162.15)$$

$$[\hat{p}_i, \hat{p}_j] = 0 \quad (162.16)$$

Theorem 162.3. If the Hilbert space $\mathcal{H}$ is finite dimensional, there exists no operators satisfying those relations.

Proof. Using the commutation relation of the trace operator, $\text{Tr}(\hat{A}\hat{B}) = \text{Tr}(\hat{B}\hat{A})$, we obtain

$$\text{Tr}([\hat{x}_i, \hat{p}_j]) = \text{Tr}(\hat{x}\hat{p}) - \text{Tr}(\hat{p}\hat{x}) \quad (162.17)$$

$$= \text{Tr}(\hat{x}\hat{p}) - \text{Tr}(\hat{x}\hat{p}) \quad (162.18)$$

$$= 0 \quad (162.19)$$

$$\text{Tr}(-i\hbar\hat{I}) = -i\hbar\text{Tr}(\hat{I}) \quad (162.20)$$

$$= -i\hbar n \quad (162.21)$$

Therefore operators obeying this relation would imply $0 = -i\hbar n$. $\square$

This will not have any influence on the infinite dimensional case, as the trace operator is simply not defined in this case.

Algebra is that of the Heisenberg group/algebra

Chapter 163

Composite systems

The basic way to perform some composition of systems in quantum mechanics is via the tensor product of the individual Hilbert spaces. For any two systems $\mathcal{H}_A$ and $\mathcal{H}_B$, with the algebra of operators $\{\mathcal{O}^A\}$, $\{\mathcal{O}^B\}$, the composition of those two systems is done by the tensor product

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \quad (163.1)$$

163.1 Fock spaces

tensor product of Hilbert spaces :

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \dots \otimes \mathcal{H} \quad (163.2)$$

Tensor product with braiding rules

$$\dots \quad (163.3)$$

Fock space as the direct sum of such n -particle systems

$$\mathcal{H}_F = \bigoplus_{i=0}^{\infty} \quad (163.4)$$

Fock space fails to be separable

Chapter 164

Quantum observables

Quantum observables are defined somewhat analogously to that of classical ones, except that in the quantum case, the physical data of a system is encoded in the wavefunction, so that a quantum observable is defined as a functional on the projective Hilbert space :

$$O : P(\mathcal{H}) \rightarrow \mathbb{R} \tag{164.1}$$

States of a system ψ_S to measure with an apparatus ψ_O , combined system is $\psi_S \otimes \psi_O$ (show)

The macro states of the machine are some map $\mathcal{H}_O \rightarrow \mathbb{R}$ or maybe $\mathbb{R}$. Splits the Hilbert space into subspaces?

Chapter 165

Superselection rules

The method of generating quantum systems for composite physical system does not work in the general case unfortunately.

If we consider a non-relativistic system of a single particle in some state ψ , it will gain an additional phase term

[385, 386]

Chapter 166

Quantum logic etc

The strangeness of quantum theory's treatment of statements regarding physical measurements has motivated some attempts to redefine the functioning of logic as it relates to such statements.

Hilbert lattice

Linear logic

Internal logic of the topos

Chapter 167

Quantization

While we will want to define quantum theories independently of their classical counterparts, it will be useful to generate some quantum theories from the current existing theories we have as a starting point. This is done by a process called *quantization*.

There are several ways to perform quantization, sometimes equivalent and sometimes not.

167.1 Canonical quantization

Canonical quantization is done by considering the Poisson algebra $(X, \{\cdot, \cdot\})$ of a theory and mapping it out to an algebra of linear operators on a Hilbert space. In other words, canonical quantization is a functor from Poisson manifolds to C^* -algebras.

The canonical quantization process will generally consider the following properties for the quantization procedure, called the *Dirac quantization conditions* :

1. The map $f \mapsto \hat{f}$ is linear
2. If the observable f is a constant function, then $\hat{f}$ is a multiplication operator
3. The Poisson bracket is mapped to the commutator,

$$\{f_1, f_2\} = f_3 \Rightarrow [\hat{f}_1, \hat{f}_2] = -i\hat{f}_3$$

The standard example of canonical quantization is given by the classical mechanics quantization to the Schrödinger equation. Consider the usual non-relativistic action for a point particle under a potential,

$$H = \frac{p \cdot p}{2m} + V(x) \tag{167.1}$$

With the Poisson algebra given by

$$\{x, p\} = 1 \quad (167.2)$$

$$\{x, x\} = 0 \quad (167.3)$$

$$\{p, p\} = 0 \quad (167.4)$$

We are therefore looking for an algebra of operators obeying

$$[\hat{x}, \hat{p}] = -i\hbar \text{Id} \quad (167.5)$$

$$[\hat{x}, \hat{x}] = 0 \quad (167.6)$$

$$[\hat{p}, \hat{p}] = 0 \quad (167.7)$$

From the Stone-von Neumann theorem, we know that any such algebra is going to be isomorphic to the algebra on the Hilbert space $L^2(\mathbb{R}^n, \mu_\ell^n)$

$$\hat{x}\psi = x\psi \quad (167.8)$$

$$\hat{p}\psi = -i\hbar \frac{\partial}{\partial x} \psi \quad (167.9)$$

prove x and p are self-adjoint

This algebra however fails to fulfill the Dirac conditions. Consider the element of the algebra given by $\hat{x}^4$

$$\langle \psi_1, \hat{x}^4 \psi_2 \rangle = \int_{\mathbb{R}^n} \bar{\psi}_1(x) x^4 \psi_2(x) d^n x \quad (167.10)$$

$$= \quad (167.11)$$

It is in fact impossible to perform canonical quantization

Groenewold–Van Hove theorem [387–389]

167.2 Deformation quantization

One of the way to get around the Groenewold–Van Hove theorem is to drop the Dirac requirement of mapping the Poisson bracket to the commutator.

Deformation of the Poisson algebra

*-product (star product)

Ordering of operators

Example : Schrödinger

Quantum master equation

167.3 Path integral quantization

The path integral of a function space X is a map from

$$\langle \psi, \hat{A}\psi \rangle = \int T[A\mathcal{D}\phi] \quad (167.12)$$

[390, 391]

Cecile De Witt's definition

Integrator theory

Gaussian measure in the Euclidian case

Cylinder sets

Case of the Wiener measure and Wiener process

Regularity of curves and their measure

ordering of operator via the what's his name stochastic integral

167.4 Geometric quantization

[392] The process of geometric quantization involves adding a variety of geometric structures to the usual classical geometric structures to obtain the appropriate quantum theory.

While quite an attractive procedure to use in the context of general relativity, geometric quantization still suffers from a number of problems that prevents it from being an effective quantization procedure in the general case.

The procedure of geometric quantization is to add the geometric structures necessary to replicate those of quantum mechanics.

Take a classical system described by a symplectic manifold (M, ω) with a Poisson structure defined by some Poisson bracket $\{-, -\}$

We add to it a *Hermitian line bundle* $\pi : H \rightarrow M$, which is simply a complex line bundle equipped with a Hermitian inner product $\langle -, - \rangle$

$U(1)$ connection on the Hermitian line bundle

$$\nabla = d - \frac{i}{\hbar} \theta \quad (167.13)$$

Quantization map for f a classical variable with Hamiltonian vector field X_f and $s \in \Gamma(H)$:

$$Q[f](s) = -i\hbar \nabla_{X_f} s + fs \quad (167.14)$$

Weil's integration condition : The cohomology class $(2\pi\hbar)^{-1}\omega \in \check{H}^2(M, \mathbb{R})$ lies in the image of $\check{H}(M, \mathbb{Z})$ if and only if there exists a hermitian line bundle and a connection with curvature $\hbar^{-1}\omega$

Lagrange submanifolds

Bohr-Sommerfeld condition

Quantomorphism

Definition 167.1. The *quantomorphism group* is the $U(1)$ -central extension of the Hamiltonian group

$$1 \rightarrow U(1) \rightarrow \text{Quant} \rightarrow \text{Ham} \rightarrow 1 \quad (167.15)$$

mechanics on the prequantum theory

Koopman von Neumann

Koopman-van Hove

Polysymplectic quantization

167.5 Polymer quantization

A much more specialized type of quantization is *polymer quantization*, which is usually used in the context of loop quantum gravity

[393–395]

From Stone-von Neumann : $[x_i, p_i] = 1$ has only one representation up to isomorphism

Dropping continuity

Stone-von Neumann for polymers

Chapter 168

Fock spaces

Chapter 169

Quantum theory of point particles

As usual, the most basic case of a physical theory is given by the case of a point particle

$$X : L \rightarrow M \quad (169.1)$$

with the appropriate action given by any of the process we saw for the given kinematic group that we have available.

169.1 The Schrödinger equation

The *Schrödinger equation* (not to be confused with the more general Schrödinger equation $\hat{H}\psi = i\partial_t\psi$ for the time evolution of a quantum state) is the quantization of the action for a Galilean point particle. If we consider the usual action

$$S[(t_1, t_2); \vec{x}(t)] = \int_{t_1}^{t_2} \left[\frac{m}{2} g(\vec{x}, \vec{x}) - V(\vec{x}) \right] dt \quad (169.2)$$

with its corresponding Hamiltonian :

$$H(\vec{x}, \vec{p}) = \frac{\delta(\vec{p}, \vec{p})}{2m} + V(\vec{x}) \quad (169.3)$$

and Poisson algebra :

$$\{\vec{x}, \vec{p}\} = \quad (169.4)$$

Application of the Galileo group to the Poisson algebra

The naive application of canonical quantization quantization

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad (169.5)$$

If we use the more general Dirac constrained action

$$S[(\tau_1, \tau_2); t,] = \int_{\tau_1}^{\tau_2} d\tau \frac{m}{2} \frac{\dot{\vec{x}}(\tau) \cdot \dot{\vec{x}}(\tau)}{\dot{t}(\tau)} \quad (169.6)$$

with its gauge group is $\text{Diff}((\tau_1, \tau_2))$

Constraint

$$\Phi = \frac{\vec{p} \cdot \vec{p}}{2m} + p_t = 0 \quad (169.7)$$

BRST

BRST quantization

solutions

dispersion

measurements

169.1.1 Free particles

The free particle case simply corresponds to the case $\hat{V} = 0$

$$\hat{H} = \frac{\hat{p}^2}{2m} \quad (169.8)$$

Solution from Brian Hall

Simon-Reed stuff and Fourier transform etc

Curved space case

Path integral

Deformation idk

169.1.2 The harmonic oscillator

One of the most famous case for the Schrödinger equation is the quantum harmonic oscillator, consisting of a particle subject to a quadratic potential

$$\hat{H} = \frac{\hat{p}^2}{2m} + \quad (169.9)$$

Ladder operators (Cartan-Weyl basis?)

169.2 The Pauli equation

quantization of superparticle

Hilbert space : $\mathcal{H} \otimes \mathcal{C}^2$

Transforms under the spin rep of the Galilean group

$$\hat{H}\psi = \frac{1}{2m} \quad (169.10)$$

169.3 Relativistic point particle

Consider the usual point particle Polyakov action :

$$S[U; X, e] = \int_{\Sigma} \llbracket d\mu[X^*g] \quad (169.11)$$

canonical quantization

Failure for the interacting case : no unitary reps or something

[396]

169.3.1 Malament's theorem

One important difference of relativistic quantum mechanics as opposed to the non-relativistic case is that wavefunctions cannot be localized.

Malament's theorem holds for a variety of cases, based on the following requirements :

-

[397]

BRST

Edge of the wedge theorem, Strocchi, unitarity of the quantization, etc etc

169.4 Carrollian particle

Chapter 170

Quantum field theory

Operator-valued distributions :

$$\hat{\phi} : \mathcal{D} \rightarrow \mathcal{O} \tag{170.1}$$

Proof that operator valued functions lead to a trivial QFT [398] 4.1

case of non-relativistic fields?

separable hilbert spaces

mass hyperboloid

time orientation

green function

170.1 Wightman's axioms

Operator valued distributions

- 1.
- 2.

[399]

Axioms : QM axioms of Dirac and von Neumann, rep of the Poincaré group, continuity etc, microcausality

170.2 Osterwalder–Schrader axioms

The *Osterwalder–Schrader axioms* is an axiomatization of free QFTs by mean of a path integral on the space of configurations.

1. Analyticity : For $k \in \mathbb{Z}$ and some set of test functions $(f_i)_{1 \leq i \leq k}$, and some set of complex numbers $z \in \mathbb{C}^k$, $z = (z_i)_{1 \leq i \leq k}$, the function

$$z \mapsto \sum_{i=1}^k z_i f_i \tag{170.2}$$

is entire analytic.

2. Regularity : For $p = 1$ or $p = 2$ and

[400]

Euclidian path integrals

170.3 Haag-Kastler

[401]

Chapter 171

The Spin-Statistics theorem

As our quantization will map objects to operators only up to a phase factor, we have some freedom for which representations we choose under $U(1)$

$$f \rightarrow e^{i\phi(f)} \hat{f} \tag{171.1}$$

$$U(g)U(h) = \omega(g, h)U(g, h) \tag{171.2}$$

[402]

Braiding

Chapter 172

Gauge quantization

If the classical theory that we consider has some gauge degrees of freedom, there will be issues at the quantization.

Failure of the Hilbert space to define a probability

There are two simple example we can pick for this, which are the relativistic point particle with its $\text{Diff}(L)$ gauge, and the electromagnetic theory with its $U(1)$ gauge.

Take the

EM example

Relativistic point particle example

BRST quantization

BRST operators have 0 eigenvalue, cohomology by the action of those

Chapter 173

Free quantum field theories

As a first application of QFT, let's first look at various systems corresponding to the quantization of free QFTs, as these are (for reasons we will see later on in X) the QFTs that have the most unambiguous definition.

173.1 Scalar QFT

The free scalar field QFT is generally the first example that is done of a QFT, as it has the benefit of being exactly solvable, not suffering from any issue stemming from gauge or complications from its Poincaré representation.

Scalar quantum field theories are generally written as the usual free Klein-Gordon theory plus some potential term depending on the field,

$$S[U; \phi] = \int_U [g(d\phi, d\phi) + V(\phi)] d\mu[g] \quad (173.1)$$

Free theory

$$\hat{\phi} = \int \quad (173.2)$$

173.1.1 Fock space

173.1.2 Hydrogen atom

The scalar hydrogen atom is considered by taking the potential of the scalar field to be some background potential of the form of the Coulomb field,

$$V(\phi) = \frac{e}{r^2} \quad (173.3)$$

173.1.3 Particle in a box

A few different physical configurations referred to as "particle in a box" are done in non-relativistic quantum mechanics, but the QFT version is less commonly discussed.

Klein-Gordon equation on an interval with infinite potential at the boundaries

Klein paradox

KG with a mirror?

173.1.4 Scalar field in de Sitter

cf. Birrel Davies

173.1.5 Scalar electrodynamics

Complex scalar field with U(1) gauge symmetry

$$P_{U(1)} \times C/U(1)$$

173.2 Dirac

Dirac equation

$$S[U; \psi] = \int_U \frac{1}{2}(\overline{\psi}) \quad (173.4)$$

From spin statistics, we consider the spinor fields to anticommute

173.2.1 The hydrogen-like atom

If we take a Dirac field in a fixed Coulomb potential

$$A = \left(-\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r}, 0 \right) \quad (173.5)$$

The corresponding Dirac Hamiltonian is

$$\hat{H} = \quad (173.6)$$

173.2.2 Dirac around a black hole

173.3 Electromagnetism

Electromagnetism is a gauge theory, BRST quantization, Gupta–Bleuler formalism

Poisson algebra

Chapter 174

Perturbative QFT

Quantum field theory is rarely solved exactly, as the number of exactly solvable QFT theory is both small and in general unrelated to real physical systems. Those systems tend to be either free theories, very low dimensional theories such as $(1+1)$ -dimensional systems and lower, or highly symmetric theories like conformal field theories.

The typical way to find approximate solutions to quantum field theories is perturbatively, by doing a perturbation of the theory around an exactly solvable system such as a free theory.

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_I \quad (174.1)$$

For the finite dimensional case : Stone-von Neumann theorem

For the infinite dimensional case : Haag theorem

[...]

The fundamental issue with perturbative theory in QFT is our mistreatment of some often ignored theorem of analysis, Fubini's theorem.

Theorem 174.1. For two complete metric spaces X, Y with measures μ_X, μ_Y and a function $f : X \times Y \rightarrow \mathbb{R}$ on the product space $X \times Y$ with the product measure $\mu_X \times \mu_Y$, the three following integrals are equivalent

$$\int_{X \times Y} f \mu_X \times \mu_Y = \int_X \left[\int_Y f \mu_Y \right] \mu_X = \int_Y \left[\int_X f \mu_X \right] \mu_Y \quad (174.2)$$

if the integral is absolutely convergent

$$\int_{X \times Y} |f| \mu_X \times \mu_Y < \infty \quad (174.3)$$

Our perturbation theory rests on the step

$$x \tag{174.4}$$

Radius of convergence etc

Resurgence, alien calculus, zero modes, instantons, asymptotic series

Chapter 175

Renormalization

[403]

So far we have only considered quantum field theories with linear equations of motion, and this for a very good reason. As we saw in 69.2, distributions cannot be multiplied together, and the same applies to operator-valued distributions.

Theorem 175.1. There is no differential algebra of operator-valued distributions

Proof.

□

Just as we had to renormalize our distributions in the classical case, we will need to perform some renormalization in the quantum case.

naive regularization and renormalization

175.1 Scaling degree

relation to Weyl scaling?

175.2 Epstein-Glaser renormalization

Chapter 176

Causality in QFT

Scharnhorst effect, tachyonic fields, Rarita-Schwinger fields, [404–407]

Chapter 177

Supersymmetric QFT

Chapter 178

Conformal field theory and the AdS/CFT correspondance

Definition 178.1. A field theory is *conformally invariant*

Part X

Quantum gravity

Chapter 179

Generalities

Due to various arguments (that will be exposed in details in the section on semiclassical gravity), it is likely that gravity itself needs to be formulated as a quantum theory. This presents some problems, as the notion of time is pretty deeply ingrained in quantum theory, even in curved spacetime, but the appropriate notion of time can be difficult to pin down properly in general relativity.

if we quantize the metric itself, time will itself become part of the equation. Problems of quantizing gravity : define time, diffeomorphism invariance

Quantization and background independence

There are a few heuristic arguments for getting the order of magnitude of the scale at which quantum gravity effects might become important. A few of them are :

- Consider a particle with a Compton wavelength equal to its own Schwarzschild radius,

$$\lambda = \frac{h}{mc} = r_s = \frac{2GM}{c^2}$$

The

All of these explanations use quantities of around the Planck scale, which let us define the Planck mass, Planck length, Planck time and Planck energy :

$$m_p = \sqrt{\frac{\hbar c}{G}}, \quad l_p = \sqrt{\frac{\hbar c}{G}}, \quad t_p = \sqrt{\frac{\hbar G}{c^5}}, \quad E_p = \sqrt{\frac{\hbar c^3}{G}}$$

with the simple relations between them of $l_p/t_p = c$, $E_p = m_p c^2$, $l_p = 2Gm_p/c^2$ and $E_p = \hbar 2\pi/t_p$

Chapter 180

Schrödinger-Newton theory

It will be of some interest to consider the non-relativistic limit of quantum gravity, as it is the only one that has been well verified experimentally.

The Schrödinger-Newton equation is quite simply the Schrödinger equation with a quantization of the Newtonian classical potential, that is

$$i\hbar \frac{\partial \Psi}{\partial t}(\vec{x}, t) = -\frac{\hbar^2}{2m} \Delta \Psi(\vec{x}, t) + V \Psi(\vec{x}, t) + m \Phi \Psi(\vec{x}, t) \quad (180.1)$$

with the Newtonian potential defined from the Poisson equation where the mass distribution is given by the wavefunction's density

$$\Delta \Phi(\vec{x}, t) = 4\pi G m |\Psi|^2 \quad (180.2)$$

Galilei invariance

extra symmetry :

$$m \rightarrow \mu m \quad (180.3)$$

$$t \rightarrow \mu^{-5} t \quad (180.4)$$

$$x \rightarrow \mu^{-3} x \quad (180.5)$$

$$\psi(x, t) \rightarrow \mu^{\frac{9}{2}} \psi(\mu^5 t, \mu^3 x) \quad (180.6)$$

$$(180.7)$$

Many body version

Center of mass motion

Chapter 181

Quantization of Newton-Cartan

A more proper theory of quantum gravity in the non-relativistic case is the quantization of theories of the type given by Galilean manifolds, and in particular Newton-Cartan theories. They have the benefit of being well-defined and illuminating what exactly will be the issues in terms of quantum gravity, as they otherwise share quite a lot of other traits with relativistic theories, such as general covariance, the equivalent principle, etc.

[408]

Chapter 182

Semi-classical gravity

The simplest form of unifying quantum field theory and general relativity is to leave them each as they are and simply connecting them by the stress-energy tensor (and the quantum field theory in curved spacetime that we have seen previously), with the form

$$T = \langle \psi, \hat{T} \psi \rangle \quad (182.1)$$

$$T_{\mu\nu} = \langle \psi, \hat{T}_{\mu\nu} \psi \rangle \quad (182.2)$$

where we define an operator for the stress-energy tensor. The Einstein field equation then just becomes

$$\text{Ric} - \frac{1}{2}Rg + \Lambda g = \langle \psi, \hat{T} \psi \rangle \quad (182.3)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \langle \psi, \hat{T}_{\mu\nu} \psi \rangle \quad (182.4)$$

This has the benefit of being the simplest possible theory of quantum gravity, and it is expected that any other theory of quantum gravity should be approximated by it within some limit.

182.1 Quantization of the stress-energy tensor

If we defined our quantum field theory with respect to a quantization procedure, the simplest way to obtain the stress energy tensor is via the quantization procedure of the classical stress-energy tensor, that is,

$$T^{\mu\nu}(\phi, x) \rightarrow \hat{T}^{\mu\nu}(\hat{\phi}, \hat{\pi}) \quad (182.5)$$

This has the problem of not being immediately well defined, as we saw previously, since the stress-energy tensor will usually be defined as the product of fields at the same point. We will instead start off by defining the stress-energy bi-tensor, which has the benefit of being well-defined at least for free fields.

$$T^{\mu\nu}(\phi, x, y) \rightarrow \hat{T}^{\mu\nu}(\hat{\phi}, \hat{\pi}, x, y) \quad (182.6)$$

As quantum fields are defined as operator-valued distributions, this will be well-defined in the case where the distribution defined by the expectation of the stress-energy tensor will itself be a bitensor (that is not too divergent). In this case, we'll define the stress-energy tensor as the coincidence limit

$$T^{\mu\nu}(x) = \lim_{y \rightarrow x} \langle \hat{T}^{\mu\nu}(x, y) \rangle_\psi \quad (182.7)$$

This will require some renormalization, even for the free field, similar to the renormalization of the Hamiltonian in QFT.

182.1.1 The scalar field

The simplest example of a semiclassical system is as usual the scalar field, with stress-energy tensor

$$T = d\phi \otimes d\phi - \frac{g}{2}(g^{-1}(d\phi, d\phi) + m^2\phi^2) \quad (182.8)$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + m^2\phi^2) \quad (182.9)$$

Its quantization will obviously lead to products of distributions. Instead, we will use the fact that for the propagator $D(f, g)$, we have the property

$$D(f, g) = \langle \phi[f]\phi[g] \rangle_0 \quad (182.10)$$

By linearity of the sesquilinear product, we also have that

$$\nabla_\mu D(f, g) = \langle (\nabla_\mu\phi[f])\phi[g] + \phi[f](\nabla_\mu\phi[g]) \rangle_0 \quad (182.11)$$

$$\nabla_\nu\nabla_\mu D(f, g) = \langle (\nabla_\nu\partial_\mu\phi[f])\phi[g] + \phi[f](\nabla_\nu\partial_\mu\phi[g]) + (\nabla_\mu\phi[f])(\nabla_\nu\phi[g]) + (\nabla_\nu\phi[f])(\nabla_\mu\phi[g]) \rangle_0 \quad (182.12)$$

$$\begin{aligned}
D_{\mu\nu}(x, y, \gamma) &= \frac{1}{6}(\nabla_\mu^x \gamma_\nu^\alpha(x, y, \gamma) \nabla_\alpha^y + \gamma_\mu^\alpha(x, y, \gamma) \nabla_\alpha^y \nabla_\nu^x) \\
&- \frac{1}{12} g_{\mu\nu}(x) \gamma^{\alpha\beta}(x, y, \gamma) \nabla_\alpha^x \nabla_\beta^y \\
&- \frac{1}{12} (\nabla_\mu^x \nabla_\nu^x + \gamma_\mu^\alpha(x, y, \gamma) \nabla_\alpha^y \gamma_\nu^\beta(x, y, \gamma) \nabla_\beta^y) \\
&+ \frac{1}{48} g_{\mu\nu}(x) (g^{\alpha\beta}(x) \nabla_\alpha^x \nabla_\beta^x + g^{\alpha\beta}(y) \nabla_\alpha^y \nabla_\beta^y) \\
&- R_{\mu\nu}(x) + \frac{1}{4} g_{\mu\nu}(x) R(x)
\end{aligned} \tag{182.13}$$

Classical limit :

$$T_c^{\mu\nu} = \lim_{\hbar \rightarrow 0} \langle \psi, \hat{T}_{\mu\nu} \psi \rangle \tag{182.14}$$

182.2 Domain of validity

It is generally assumed that semiclassical gravity will not be a correct description of quantum gravity. One of the common argument for this is the following :

Consider two quantum state $|\psi_1\rangle$ and $|\psi_2\rangle$, such that the support of $\langle T_{\mu\nu} \rangle_1$ and $\langle T_{\mu\nu} \rangle_2$ are disjoint. If we take the state to be

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) \tag{182.15}$$

then the initial stress-energy tensor will be

$$T_{\mu\nu} = \frac{1}{2} \langle T_{\mu\nu} \rangle_1 + \frac{1}{2} \langle T_{\mu\nu} \rangle_2 \tag{182.16}$$

After a measurement performed that would collapse the wavefunction to either ψ_1 or ψ_2 , the stress energy tensor will become (for let's say ψ_1)

$$T_{\mu\nu} = \langle T_{\mu\nu} \rangle_1 \tag{182.17}$$

The stress-energy tensor has doubled in the first region and totally disappeared in the second, and this regardless of their separation. Due to this non-local effect, it is assumed that the metric itself has to be described by a quantum field, and that semiclassical gravity can only be an appropriate description of when it nears classical behaviour.

Fluctuations of the stress energy tensor : cf "Gravitational radiation by quantum systems", should be

$$\langle T_{\mu\nu}(p)T_{\rho\sigma}(q) \rangle \approx \langle T_{\mu\nu}(p) \rangle \langle T_{\rho\sigma}(q) \rangle \quad (182.18)$$

182.3 Renormalization of the stress-energy tensor

As with most products of field operators, the stress-energy tensor operator requires some renormalization. Renormalization via the coincidence limit

Dimensional regularization

182.4 Effective action

It is also possible to express semiclassical gravity by defining an effective action W so that

$$T_{\mu\nu}^{\text{Eff}} = -2 \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle_\psi \quad (182.19)$$

We then need to find an effective action W such that its functional derivative will give us the appropriate stress-energy tensor.

182.5 Energy conditions

182.6 Quantum aspects of chronology protection

One of the application of semiclassical gravity is that we can see the effect of Cauchy horizons on the stress energy tensor.

As the equations for free fields will usually be second order hyperbolic partial differential equations, it helps to express the propagator in the Hadamard form :

$$G_R(x, y) = \hbar \sum_{\gamma} \frac{\sqrt{\Delta_{\gamma}(x, y)}}{4\pi^2} \left[\frac{1}{\sigma_{\gamma}(x, y)} + v_{\gamma}(x, y) \ln(|\sigma_{\gamma}(x, y)|) + \varpi_{\gamma}(x, y) \right] \quad (182.20)$$

The Hadamard form of the Green function has the benefit of outlining the singular structure, as it can be shown that $\varpi_\gamma(x, y)$ and $v_\gamma(x, y)$ both converge to proper functions in the limit $x \rightarrow y$, so that we may focus on the remaining terms for divergences.

Stress energy tensor :

$$T_{\mu\nu}(x) = \lim_{x \rightarrow y} T_{\mu\nu}(x, y) = \lim_{x \rightarrow y} D_{\mu\nu} G_R(x, y) \quad (182.21)$$

Stress energy tensor renormalization :

$$T_{\mu\nu}^{Ren}(x) = \quad (182.22)$$

Chapter 183

Stochastic gravity

Stochastic gravity is a compromise between semiclassical gravity and a full theory of quantum gravity. The metric remains classical, but a stochastic term that depends on the fluctuations of the stress-energy tensor is added.

$$\hat{t}_{\mu\nu} = \hat{T}_{\mu\nu} - \langle \hat{T}_{\mu\nu} \rangle \hat{I} \quad (183.1)$$

Noise kernel bitensor :

$$N_{\mu\nu\rho\sigma}(p, q) = \frac{1}{2} \langle \{ \hat{t}_{\mu\nu}, \hat{t}_{\rho\sigma} \} \rangle \quad (183.2)$$

$$N_{\mu\nu\rho\sigma}(p, q) = N_{\rho\sigma\mu\nu}(q, p) \quad (183.3)$$

$\hat{t}_{\mu\nu}$ is finite since we subtract the UV divergences.

T is self-adjoint $\Rightarrow N$ is real and positive semi-definite since it's the expectation value of an anticommutator. Classical gaussian stochastic tensor $\xi_{\mu\nu}$

$$\langle \xi_{\mu\nu}(p) \rangle_s = 0, \quad \langle \xi_{\mu\nu}(p) \xi_{\rho\sigma}(q) \rangle_s = N_{\mu\nu\rho\sigma}(p, q) \quad (183.4)$$

$\xi_{\mu\nu} = \xi_{\nu\mu}$ and $\nabla^m u \xi_{\mu\nu} = 0$. $\langle \cdot \rangle_s$ is a classical stochastic average. Correction to the metric tensor $g \rightarrow g + h$

Einstein-Langevin equation :

$$G_{\mu\nu}^{(1)}[g + h] = \kappa (\langle \hat{T}_{\mu\nu}^{(1)}[g + h] \rangle_{\text{ren}} + \xi_{\mu\nu}[g]) \quad (183.5)$$

g is a solution to the semiclassical equation.

Chapter 184

Canonical gravity

[409]

Canonical gravity is the theory of quantum gravity which can be obtained by some procedure similar to canonical quantization, in the sense that we start with the Hamiltonian version of relativity to obtain the appropriate quantization.

The usual process of canonical quantization for gravity is usually performed on some globally hyperbolic spacetime $M \cong \mathbb{R} \times \Sigma$, after having given the configuration space its appropriate conjugate variables and performed the space-time split, giving us the bundle $\pi : M \rightarrow \Sigma$ and the ADM variables and constraints $({}^3g, \pi)$ and (n, b) etc

Canonical quantization

Deformation quantization

A simple way to quantize gravity is to simply use canonical quantization on the Hamiltonian theory we already have, ADM gravity. As the GR Hamiltonian is identically zero, this will go through the usual process of BRST quantization.

First it is important to define the field variables we'll be using, by performing canonical quantization on the Lie brackets. For ADM, these were :

$$\{q\} = \tag{184.1}$$

Wheeler-DeWitt equation :

$$H\psi = 0 \tag{184.2}$$

Problem of interpretation of the equation

184.0.1 Thermal time hypothesis

Quantum theory generally uses the time

Chapter 185

Euclidian gravity

Euclidian path integral of the Riemannian gravitational action,

$$S[U; g] = \int_U \text{Ric}[g] d\mu[g] \tag{185.1}$$

Weyl transform $g \rightarrow \omega g$

Chapter 186

Causal sets

Causal set theory is an attempt to turn the structure of spacetime itself in a quantum theory. The basic idea stems from the abstract causal spaces defined in [x], in which the structure of spacetime itself is defined in the abstract. Such structures are a bit too general for causal set theory, as we want spacetime to be discrete as well. To guarantee discreteness, we will add some requirements of local finiteness.

Definition 186.1. A *causal set*, or *causet*, is a set C with a partial relation order $\prec$ with the following properties

1. Transitivity : $p \prec q \prec r \Rightarrow p \prec r$
2. Irreflexivity : $p \not\prec p$
3. Local finiteness : $\text{Card}(\{q \in C | p \prec q \prec r\}) \in \mathbb{N}$

The ordering roughly corresponds to the strict causal relation $p < q$ seen previously, but with the added condition that there are only finitely many points in the intersection of the future and the past of two points.

No spacetime is a causal set, since the axiom of local finiteness runs afoul of its manifold structure, but it is possible to embed of causal sets in them. For instance, if we pick any covering of $\mathcal{M}$ by non-overlapping compact subsets, and pick a finite number of points in each of these subsets, this will generate a causal set.

On the other hand, some causal sets cannot be embedded in any manifold.

To generate a causal set from a given spacetime, the method used is sprinkling : we select random points in it according to a Poisson process. That is, for any compact region of volume V , we select n elements with probability

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \quad (186.1)$$

where ρ is the fundamental density, a parameter of the theory, probably related to the Planck density.

A manifold $(\mathcal{M}, g)$ is said to approximate a causet if C has a relatively high probability of coming from a sprinkling process on it. C is then said to be faithfully embeddable in $\mathcal{M}$.

Causal set fundamental conjecture : two wildly different spacetimes on a large scale cannot be faithfully approximated by the same sprinkling.

Function $D(C)$: approximate dimension of the manifold

Estimators for timelike distances, volumes, etc

The volume is $\approx$ proportional to the cardinality of the causet, so

Chapter 187

Covariant quantum gravity

A simple quantization scheme for gravity is to treat gravity perturbatively around a given metric, treating our gravitational field as a spin 2 field around a background metric.

Covariant gravity is one of the first attempt at the quantization of gravity, based on the canonical quantization of the Pauli-Fierz theory of gravity. As a simple interacting QFT on flat space, it is expected to work by the exchange of gauge bosons, the graviton, represented by the perturbative metric field $h_{\mu\nu}$, or the perturbative frame field, h_μ^a , if fermions are to be added to the theory.

As we have seen earlier, Pauli-Fierz on a Minkowski background has the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}h_{\mu\nu,\sigma}h^{\mu\nu,\sigma} + \frac{1}{2}h_{\mu\nu,\sigma}h_{\sigma\nu,\mu} + \frac{1}{4}(\partial_\sigma h)(\partial^\sigma h) - \frac{1}{2}(\partial_\sigma h)(\partial_\nu h^{\nu\sigma}) + V(h)$$

With V a potential that will depend on h and its first derivatives. Its equation of motion being

$$\square h_{\mu\nu} - \dots \quad (187.1)$$

If we limit for now our attention to the linear part

Hamiltonian :

Let's define $N = -h_{00}$, $N_i = 2h_{0i}$

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}h_{\mu\nu,\sigma}h^{\mu\nu,\sigma} + \frac{1}{2}h_{\mu\nu,\sigma}h_{\sigma\nu,\mu} + \frac{1}{4}(\partial_\sigma h)(\partial^\sigma h) - \frac{1}{2}(\partial_\sigma h)(\partial_\nu h^{\nu\sigma}) \\ &= \end{aligned} \quad (187.2)$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \quad (187.3)$$

The operator $\hat{h}$ for the free theory obeys, in the de donder gauge :

$$\square h_{\mu\nu} = 0 \quad (187.4)$$

Free wave solutions for $k^2 = 0$:

$$h_{\mu\nu} = \varepsilon_{\mu\nu}(\vec{k}) e^{\pm i k^\alpha x_\alpha} \quad (187.5)$$

Polarization obeys [why]

$$k^\mu \varepsilon_{\mu\nu}(\vec{k}) = \varepsilon_{\mu\nu}(\vec{k}) k^\nu = 0, \quad \varepsilon_\mu{}^\mu = 0 \quad (187.6)$$

Choice :

$$\varepsilon_\nu = 0, \quad k^i \varepsilon_{ij}(\vec{k}) = 0, \quad \varepsilon_i{}^i = 0 \quad (187.7)$$

Propagator :

$$\tilde{h}_{\mu\nu} = k^{-2} \quad (187.8)$$

$$h_{\mu\nu} = \int k^{-2} \quad (187.9)$$

Graviton propagator :

We expect

$$\frac{i P_{\mu\nu,\rho\sigma}}{k^2 + i\varepsilon} \quad (187.10)$$

$$[e^{\mu\nu} p_\mu p'_\nu - \frac{1}{2} e^\rho (\eta^{\mu\nu} p_\mu p'_\nu - m^2)] \quad (187.11)$$

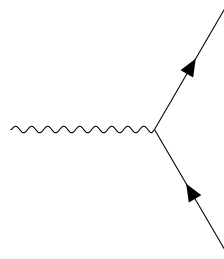


Figure 187.1: Tree-level gravitational process

187.1 Couplings of the graviton to matter

$$h_{\mu\nu} = \frac{\kappa}{k^2} (T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T) \quad (187.12)$$

Coupling to scalar fields

Coupling to EM

Coupling to fermions

187.2 Renormalization

Renormalization

1-loop

2-loop

187.3 Classical limits

Tree level approximation

Chapter 188

Causal Dynamical Triangulation

Variation on quantum Regge calculus

Chapter 189

String theory

String theory is a perturbative quantum theory that includes gravity as one of its limit.

189.1 Bosonic string theory

The simplest string theory model is *bosonic string theory*, where we just consider the bosonic action of a string (a 1-brane), using either the Nambu-Goto or Polyakov action for strings that we saw in 113.

In the Nambu-Goto case, the action we consider is

$$S[U; X] = \int_{X^{-1}(U)} d\mu[X^*g] \quad (189.1)$$

Or in a given coordinate patch for both Σ and M ,

$$S[U; X^\mu] = \int \sqrt{\det [g_{\mu\nu}(X(\sigma))(\partial_a X^\mu(\sigma))(\partial_b X^\nu(\sigma))]} d^2\sigma \quad (189.2)$$

Polyakov

$$S[X, \gamma, g] = \int_{\Sigma} (\|dX\| + R - \Lambda) d\mu[\gamma] \quad (189.3)$$

$$S[X, \gamma, g] = \int_{\Sigma} (h^{ab}(\sigma)g_{\mu\nu}(X)\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma) + R(\sigma) - \Lambda)\sqrt{-\gamma}d\sigma^p \quad (189.4)$$

In the $(1 + 1)$ dimensional case, the Ricci scalar curvature of the brane has a constant action, so that we can simply write this term as a constant parametrized by the Euler characteristic.

The canonical quantization of the bosonic string is done

$$\square \quad (189.5)$$

Open and closed strings

BRST

Virasoro algebra

Quantization

Light cone quantization

Hilbert space

Conformal symmetry

Dimension

Vertex algebra

D-branes

Superstrings

open and closed strings

5 types of string theories

Compactification of the target manifold

Calabi-Yau manifold

Chapter 190

String field theory

While quantum field theory is approximated as a theory of Feynman diagrams in the perturbative limit, string theory is usually constructed as a perturbative theory of string diagrams directly. One may then wonder if there is a non-perturbative theory which would give out such a theory. This type of theory is known under the name of *string field theory*.

Instead of taking the fundamental action to be a functional on fields, maps from points of the manifold to $\mathbb{R}$, the theory instead considers the notion of a *string field*, whose configuration space is the space of curves on the manifold, for instance $C^\infty(L, M)$, to $\mathbb{R}$.

$$\Phi : \tag{190.1}$$

Unlike the case of fields, which could be interpreted as being based on points, smooth maps from the point space to the manifold $C^\infty(\{\bullet\}, M) \cong M$, the dual space of the configuration space is not the manifold, but the space of all curves on it. The action functional is therefore not an integral on the manifold but a functional integral on that space.

Q operator

Chapter 191

Loop quantum gravity

Hamilton-Jacobi equation of general relativity :

$$F_{ab}^{ij}(\vec{x}) \frac{\delta S[A]}{\delta A_a^i(\vec{x})} \frac{\delta S[A]}{\delta A_b^j(\vec{x})} = 0 \quad (191.1)$$

Ashtekar variables

Spatial tetrads

$$E^a{}_\mu = e^a{}_\mu + e^b{}_\mu n^a n_b \quad (191.2)$$

Properties : $E^a{}_\mu n_a = 0$, $E^a{}_\mu n^\mu = 0$

Haar measure : measure on locally compact group G gives integral for $SU(2)$

Properties : for $S \subset G$, $g \in G$, $\mu(gS) = \mu S$

for compact S , $\mu(S) < \infty$

Normalization condition : $\mu(G) = 1$

191.1 Spin networks

Spin networks [410]

Part XI

Examples

Now that we know all that we need to know about the inner workings of spacetime, let's apply those tools to a variety of them.

Chapter 192

Minkowski space and variants

Minkowski space is the simplest example of a spacetime, as it is maximally symmetric, topologically trivial and almost every of its tensor quantities either vanish or are constant in Cartesian coordinates. It also offers the benefit of Cartesian coordinates being defined on a single coordinate patch for the whole manifold. As such, it presents many interesting properties.

From its topology as $\mathbb{R}^n$, we can tell that Minkowski space is orientable, parallelizable, contractible, simply connected.

It is a two-point homogeneous Lorentz manifold, with the full Poincaré group as its set of Killing vectors. Its exponential map is diffeomorphic on the entire manifold,

$$\exp : T_p M \rightarrow M \tag{192.1}$$

For some choice of a point p and some orthonormal frame e around that point, we have that the metric structure integrates uniquely with a coordinate patch for the whole manifold, the *cartesian coordinates* (t, x_i) , in which the metric is written in the canonical way,

$$ds^2 = -dt^2 + \sum_{i=1}^n dx_i^2 \tag{192.2}$$

The set of Cartesian coordinates given every possible points and frame at that point form the set of all Minkowski coordinates, related by the Lorentz pseudogroup of coordinate transformations between them. It is in this sense that the Minkowski spacetime is equivalent to the Minkowski affine space.

From the metric in Cartesian coordinates, we can compute the various other quantities associated with it. The inverse metric is simply

$$g^{-1} = -\partial_t^2 + \sum_{i=1}^n \partial_i^2 \quad (192.3)$$

As all the terms of the metric are constant, the Christoffel symbols will trivially be zero in this coordinate system

$$\Gamma^\sigma_{\mu\nu} = 0 \quad (192.4)$$

So that the covariant derivative reduces to the usual derivative

$$\nabla_X Y = \partial_\mu Y^\nu X^\mu \quad (192.5)$$

The curvature tensor will likewise reduce to zero, as well as all of its contractions :

$$\text{Riem} = 0 \quad (192.6)$$

$$\text{Ric} = 0 \quad (192.7)$$

$$\text{Gauss} = 0 \quad (192.8)$$

$$R = 0 \quad (192.9)$$

$$\text{Weyl} = 0 \quad (192.10)$$

The holonomy of Minkowski space is the trivial group :

$$\text{Hol}_\nabla(\gamma) = \{e\} \quad (192.11)$$

From the vanishing of all curvature components, we can tell that Minkowski space has a vanishing stress-energy tensor vanishing at every point, making it a vacuum spacetime, or at least one where the stress-energy of its various fields cancel each other out, although this would require [violation of EC]

Minkowski space allows

Geodesic congruences remain undisturbed

The causal structure of Minkowski space

The asymptotic structure is the one we saw in [X],

Existence of the Fourier transform

Complex structure in specific dimensions

192.1 Special properties of Minkowski space

192.2 Solutions of field equations

Many field equations are exactly solvable in Minkowski space, as the flatness will usually cast the linearized equation of motion into either some wave equation or the equation for free objects,

Klein-Gordon :

$$\square\phi = 0 \quad (192.12)$$

In two dimensions this problem is exactly solvable using *d'Alembert's formula*, where for some initial slice in Cartesian coordinates,

$$\square\phi = 0 \quad \phi(x, 0) = \phi_0(x) \quad (192.13)$$

$$\partial_t\phi(x, 0) = \pi_0(x) \quad (192.14)$$

$$\phi(t, x) = \frac{1}{2}(\phi_0(x+t)\phi_0(x-t)) + \frac{1}{2} \quad (192.15)$$

Fourier transform solution

Fields

Solution for Maxwell :

$$\partial_\mu F^{\mu\nu} \quad (192.16)$$

192.3 Other coordinates

All the usual coordinate systems of Euclidian $\mathbb{R}^{n-1}$ can be used for Minkowski space by considering the split (t, x_i) .

The most commonly used of such coordinate system is that of spherical coordinates, which while not properly manifold charts [X] are still mostly usable for that purpose.

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad (192.17)$$

Cylindrical

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2 \quad (192.18)$$

Bispherical

There are also some time-dependent coordinates that are of importance.

First is the case of an accelerated frame along a particular direction, which we can take to be along x without loss of generality by rotating our original Cartesian coordinates. The accelerated frame is given by an observer field of accelerated observers of constant acceleration α ,

$$\|D\gamma\| = \alpha \quad (192.19)$$

$$t' = x \sinh(\alpha t) \quad (192.20)$$

$$x' = x \cosh(\alpha t) \quad (192.21)$$

Born coordinates

Kottler–Møller coordinates

Radar coordinates (Lass coordinates)

Rindler coordinates

Null coordinates

192.4 The Minkowski cylinder

The Minkowski cylinder is one of a variety of quotients of Minkowski space up to some translation. For the equivalence relation

$$x^\mu \sim x^\mu + a^\mu \quad (192.22)$$

The Minkowski cylinder $\mathbb{R}^{(n-1),1} / \sim$

Translation is an isometry

Of particular importance is the case of the translation up to a timelike vector, called the *timelike cylinder*.

Very obviously this spacetime is not causal and has closed timelike curves

192.5 The Minkowski torus

The Minkowski torus is given by n quotients of Minkowski space by translations

$$T^n = \text{Mink}^n / \sim_a \dots \quad (192.23)$$

Langevin twins on the torus

192.6 The non-time orientable Minkowski space

$$M = L \times S^1 / (t \rightarrow -t, \theta \rightarrow \theta + \pi) \quad (192.24)$$

This is the simplest example of a non-time orientable manifold, giving us some example of the behaviour of its dynamic.

Consider the wave equation

$$\square \phi = 0 \quad (192.25)$$

As we've seen, the solutions on Minkowski space are of the form

$$\phi = \int d^3 \tilde{\phi} \quad (192.26)$$

The solutions on the non-time orientable version will therefore be solutions of the form

$$\phi(\theta, t) = \phi(\theta + \pi, -t) \quad (192.27)$$

Green functions

Chapter 193

de Sitter space

[411]

There's a few different definitions of de Sitter space that we can come up with. One of the most common way in textbooks is to define it as the induced spacetime on the hyperboloid quadric in $\mathbb{R}^{n,1}$:

We will consider mainly the $(3 + 1)$ -dimensional case, given by the implicit equation

$$-u^2 + w^2 + x^2 + y^2 + z^2 = R^2 \quad (193.1)$$

and the $(1 + 1)$ -dimensional case, which can be represented graphically, which is simply a hyperboloid of one sheet in $\mathbb{R}^3$:

$$-t^2 + x^2 + y^2 = R^2 \quad (193.2)$$

[Diagram of the 2D case]

This implicit equation can be parametrized using the induced coordinates $(t, \chi, \theta, \varphi)$:

$$u = R \sinh\left(\frac{t}{R}\right) \quad (193.3)$$

$$w = R \cosh\left(\frac{t}{R}\right) \cos(\chi) \quad (193.4)$$

$$x = R \cosh\left(\frac{t}{R}\right) \sin(\chi) \cos(\theta) \quad (193.5)$$

$$y = R \cosh\left(\frac{t}{R}\right) \sin(\chi) \sin(\theta) \cos(\varphi) \quad (193.6)$$

$$z = R \cosh\left(\frac{t}{R}\right) \sin(\chi) \sin(\theta) \sin(\varphi) \quad (193.7)$$

In those coordinates, the induced metric on the hypersurface is given by the first fundamental form $g = f_*\eta$:

$$g_{\mu\nu} = \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \eta_{\alpha\beta} \quad (193.8)$$

As the original metric is diagonal, computations simplify somewhat, as for any pair of indexes μ, ν of g , we will only have to consider the five terms corresponding to $\mu\nu\nu\mu$, $\mu w\nu w$, etc.

$$g_{tt} = \frac{\partial f^\alpha}{\partial t} \frac{\partial f^\beta}{\partial t} \eta_{\alpha\beta} \quad (193.9)$$

$$= -\cosh^2\left(\frac{t}{R}\right) + \sinh^2\left(\frac{t}{R}\right) [\cos^2(\chi) + \sin^2(\chi) \cos^2(\theta)] \quad (193.10)$$

$$+ \sin^2(\chi) \sin^2(\theta) \cos^2(\varphi) + \sin^2(\chi) \sin^2(\theta) \sin^2(\varphi)] \quad (193.11)$$

$$= -\cosh^2\left(\frac{t}{R}\right) + \sinh^2\left(\frac{t}{R}\right) [\cos^2(\chi) + \sin^2(\chi) [\cos^2(\theta) + \sin^2(\theta) [\cos^2(\varphi) + \sin^2(\varphi)]]] \quad (193.12)$$

$$= -\cosh^2\left(\frac{t}{R}\right) + \sinh^2\left(\frac{t}{R}\right) [\cos^2(\chi) + \sin^2(\chi) [\cos^2(\theta) + \sin^2(\theta)]] \quad (193.13)$$

$$= -\cosh^2\left(\frac{t}{R}\right) + \sinh^2\left(\frac{t}{R}\right) [\cos^2(\chi) + \sin^2(\chi)] \quad (193.14)$$

$$= -\cosh^2\left(\frac{t}{R}\right) + \sinh^2\left(\frac{t}{R}\right) \quad (193.15)$$

$$= -1 \quad (193.16)$$

Therefore using this parametrization of the surface, de Sitter space

$$ds^2 = -dt^2 + R^2 \cosh^2\left(\frac{t}{R}\right) [d\chi^2 + d\Omega^2] \quad (193.17)$$

Those are called the *closed slicing coordinates*, as the spacelike hypersurfaces generated by slices of constant t are simply spheres

Generalize in n dimension with $d\Omega_{n-2}^2$

[412, 413]

De Sitter space can also be defined as the model space of the Klein geometry $(O(n, 1), O(n-1, 1))$.

Equivalence as the hyperboloid being the coset space of such a thing? Different values of R corresponding to different "boosts"?

De Sitter space is also (at least as far as its local geometry goes) a homogeneous space defined by its curvature

$$\text{Riem} = Rg \oslash g \quad (193.18)$$

In the case of positive curvature, the only possible homogeneous geometries are that of De Sitter space or one of its quotients [139].

De Sitter space

Different coordinates and foliations

$$ds^2 = \quad (193.19)$$

de Sitter space is the

193.1 Universal cover

Since de Sitter space is not simply connected and contains closed timelike curves, it is common to consider its universal cover $\tilde{\text{dS}}$

193.2 Causal structure

193.3 Elliptic de Sitter space

There exists some involutions of de Sitter space

Theorem 193.1. Elliptic de Sitter space is non-time orientable.

Chapter 194

Anti-de Sitter space

Anti-de Sitter space is the equivalent of de Sitter space with negative curvature

Like the case of de Sitter space, we can consider it as a quadric of a higher dimensional space, but here we are using $\mathbb{R}^{2,(n-1)}$:

$$ds^2 = -dt^2 - du^2 + \sum_{i=1}^{(n-1)} (dx_i)^2 \quad (194.1)$$

Quadrics

Klein geometry definition

194.1 Global structure

Chapter 195

The Schwarzschild metric

The *Schwarzschild metric* describes various cases of spherically symmetric vacuum solutions. Its fundamental features are that :

- The spacetime is static (with some nuances to the exact meaning of this) and admits a Killing vector k
- It is spherically symmetric and admits some isometry group $SO(3)$ whose orbits are orthogonal to k
- It is a vacuum solution
- It is asymptotically flat

Straumann 8.2

All of those together are well enough to prove a unique class of spacetimes parametrized by their mass M (up to some topological considerations that we will see later), but as we'll see those are in fact somewhat superfluous as the Schwarzschild metric can get by on a smaller amount of assumptions.

By its static nature, we know that the spacetime can be decomposed as $M = \mathbb{R} \times \Sigma$, with the warped product metric

$$ds^2 = -\langle k, k \rangle dt^2 + g_\Sigma \quad (195.1)$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \quad (195.2)$$

Those are the *Schwarzschild coordinates*, which are only well-defined in the range $r \in (r_s, +\infty)$ or $r \in (0, r_s)$, corresponding to what are referred to as the exterior and interior

Schwarzschild solutions. It is also quite common to consider a further restriction of the range of the exterior solution to $r \in (R, +\infty)$, $R > r_s$, when we consider the behaviour of spherical objects which are not vacuum for $r < R$, such as the spacetime around stars, where the full spacetime will be some gluing of this solution with another at its boundary.

For now we will focus on the exterior solution in Schwarzschild coordinates. The determinant of the metric is given by

$$\det(g) = \quad (195.3)$$

Birkhoff theorem

Israel theorem

Pasting inner and outer Schwarzschild?

Maximally extended Schwarzschild

Different coordinate systems

Horizon

Singularity

195.1 Geodesics of the exterior solution

Using the Killing vectors of the Schwarzschild metric, the geodesic equation in Schwarzschild coordinates ends up being

$$\left(1 - \frac{r_s}{r}\right)E^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - \frac{L^2}{r^2} = 1 \quad (195.4)$$

$$\ddot{r} \dots \quad (195.5)$$

Observers, infalling, orbiting, etc

195.2 Extended solutions

The exterior Schwarzschild solution is well-known to be singular at $r = r_s$, as a simple infalling radial geodesic starting at some point $r = r_0$ will have a total proper time of

$$\tau = \tag{195.6}$$

However, this singularity is simply a regular boundary point, as the exterior solution admits extensions. It can already be seen that the curvature tensors seem to be well-behaved at the horizon

There are several methods to construct extensions to the exterior metric, the most commonly used one being to consider a coordinate transformation from the Schwarzschild coordinates to

195.3 Singular structure

The maximally extended Schwarzschild solution (and a fair bit of its important subsets) is famously singular, as can be seen by simply looking at a radial geodesic

195.3.1 Singularities on the horizon

Another type of singularity that can be found [?] on a Schwarzschild spacetime is [...]

Eddington coordinates :

$$t^\pm = t \pm 2M \ln(|r - 2M|) \tag{195.7}$$

$$ds^\pm = (-dt_\pm^2 + dr^2 + r^2 d\Omega^2) + \frac{2M}{r} (k_\pm)_\mu dx^\mu \tag{195.8}$$

$$k_\pm = -dt \pm dr \tag{195.9}$$

195.4 Boundaries

195.5 Fields

195.5.1 Klein-Gordon

The Klein-Gordon action in a fixed Schwarzschild background is given by

$$S[U; \phi] = \int_0^{2\pi} \int \quad (195.10)$$

In terms of Schwarzschild coordinates,

$$\square_{LB}\phi + m^2\phi = 0 \quad (195.11)$$

$$= \quad (195.12)$$

195.6 Hawking radiation

[414,415] From the black hole thermodynamics, we saw that a black hole's area can only increase under the assumption of the theorem, but those assumptions can be shown to be broken under the use of quantum fields

Hawking radiation

Solution of Klein-Gordon in the Schwarzschild metric

$$\square\phi = \quad (195.13)$$

Asymptotic flow

Inward flow violating the NEC

Gravitational squeezing

Evaporation : Approximation as a Vaidia metric

Chapter 196

The Reissner–Nordström metric

The *Reissner–Nordström metric* is a class of spherically symmetric static electrovacuum solutions. Given the same geometric constraints as we had on the Schwarzschild metric (static, spherically symmetric with spacelike orbits, spherical orbits orthogonal to the timelike ones, etc), but replacing the vacuum condition by an electrovacuum of some electromagnetic field, the Reissner–Nordström metric will then be given by some metric of the form

$$ds^2 = -f(r)dt^1 + g(r)dr^2 + r^{n-2}d\Omega^2 \quad (196.1)$$

Spherically symmetric EM field?

and obeying the Einstein-Maxwell equations

$$G = T \quad (196.2)$$

$$d \star F = \quad (196.3)$$

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2d\Omega^2 \quad (196.4)$$

$$F = -\frac{e}{r^2}dt \wedge dr \quad (196.5)$$

Magnetic charge case

Naked singularity

Chapter 197

The Kerr metric

The Kerr metric is a vacuum solution

197.1 The Tomimatsu metric

Despite the impression one may get, the Kerr metric is not the unique vacuum solution of an axisymmetric spacetime in the same way that a Schwarzschild metric is, such spacetimes are classified under the *Tomimatsu-Sato metrics*. If we consider the general case of a stationary axisymmetric vacuum metric,

[416]

Chapter 198

Colored black holes

Black holes for Yang-Mills fields

Chapter 199

The FRWL universe

The Friedmann-Robertson-Walker-Lemaître universe (or FRWL) is a cosmological model in which the universe is considered as homogeneous and isotropic.

Let's first consider that our universe M has some foliation into spacelike hypersurfaces so that it makes sense to talk of homogeneity [IS THAT TRUE]. We can consider it to be some bundle structure over a one dimensional manifold [IS THAT THE MOST GENERAL POSSIBILITY FOR SUCH A FOLIATION],

$$\pi : M \rightarrow L \text{ or } S^1 \tag{199.1}$$

As L is contractible we can assume the bundle over L to be trivial, $M = \Sigma \times \mathbb{R}$, while a bundle over S may not be.

A further case we can consider is a general bundle, where the fibers over time are not necessarily homeomorphic, where the spacelike hypersurface may change topology.

If we consider a spacetime that can be foliated by spacelike hypersurfaces with diffeomorphic leaves, and where the induced metric on those leaves leads them to be homogeneous manifolds, this is a Friedmann–Lemaître–Robertson–Walker spacetime, or *FLRW spacetime*, also called FRW.

The FLRW spacetimes are going classified by a few possible things. First if we consider the bundle of spacelike slices, the base space will be either $\mathbb{R}$ or S^1 .

The possible geometries for the spacelike hypersurfaces are classified by the homogeneous Riemannian manifolds in $(n - 1)$ -dimensions. These are all based on homogeneous spaces from Klein geometries. If we require those manifolds to be symmetric in addition to being homogeneous, their classifications is given by [417] :

1. Euclidian space $\mathbb{R}^n = \text{Euc}(n)/\text{O}(n)$

2. The n -sphere $S^n = \text{SO}(n+1)/\text{SO}(n)$
3. The real projective space $\mathbb{R}P^n = \text{SO}(n+1)/\text{O}(n)$
4. The complex projective space $\mathbb{C}P^n = \text{SU}(n+1)/\text{U}(n)$
5. The quaternionic projective space $\mathbb{H}P^n = \text{Sp}(n+1)/\text{Sp}(n) \times \text{Sp}(1)$
6. The Cayley projective plane $\text{Cay}P^2 = \text{F}_4/\text{Spin}(9)$
7. The real hyperbolic space $\mathbb{H}^n(\mathbb{R}) = \text{SO}(1, n-1)/\text{SO}(n)$
8. The complex hyperbolic space $\mathbb{H}^n(\mathbb{C}) = \text{SU}(1, n-1)/\text{U}(n)$
9. The quaternionic hyperbolic space $\mathbb{H}^n(\mathbb{H}) = \text{Sp}(1, n-1)/\text{Sp}(n) \times \text{Sp}(1)$
10. The Cayley hyperbolic plane $\mathbb{H}^n(\text{Cay}) = \text{F}_4^*/\text{Spin}(9)$

All those spaces are further classified by their Ricci scalar curvature, which is always 0 for Euclidian space, a positive constant for the spherical and projective spaces, and a negative constant for the hyperbolic spaces.

If we drop the assumption of symmetry for our homogeneous manifolds, there exists further homogeneous manifolds, such as the flag manifolds G/P .

If we restrict our attention to globally hyperbolic spacetimes, with a topology $\mathbb{R} \times \Sigma$, Σ a homogeneous slice,

In the $(3+1)$ -dimensional case, we are reduced to four possible topologies, removing all the homogeneous spaces which are impossible in odd dimensions (such as spaces based on complex Klein geometries) or 3 dimensions specifically. These are the 3-sphere $S^3 = \text{SO}(4)/\text{SO}(3)$, the Euclidian space $\text{Euc}(3)/\text{SO}(3)$, the real hyperbolic space $H^3 = \text{SO}(1, 2)/\text{SO}(3)$, and the real projective space $\mathbb{R}P^n = \text{SO}(4)/\text{O}(3)$.

Of these, the sphere is the cover of the real projective space, $\mathbb{R}P^n = S^3/\mathbb{Z}_2$, which fails to be spatially orientable, generally making it an unlikely candidate as a cosmological model, so that cosmological models are usually limited to the three other spatial topologies. We speak of the *open universe* $\mathcal{H}^3$, the *closed universe* S^3 and the *flat universe* $\mathbb{R}^3$, open and closed referring to the topology of those spaces.

The FRW metric is the metric on one of those three possible universes given the assumption of homogeneity, given by the warped product of the (unique) Riemannian manifold on the interval (I, d) and the homogeneous manifold with bi-invariant metric (Σ, g_Σ) ,

$$I \times_a \Sigma = (I \times \Sigma, g = \text{pr}_1^*d + (a \circ \text{pr}_1)^2 \text{pr}_2^*g_\Sigma) \quad (199.2)$$

Given what we know of these manifolds, the metric is, in the appropriate canonical coordinates for both the interval and the homogeneous spaces,

$$ds^2 = -dt^2 + a^2(t)g^\sigma \quad (199.3)$$

For our three different topologies in the $(3 + 1)$ -dimensional case, this gives rise to the following three metrics :

The flat universe simply has the flat space geometry on its spatial section,

$$ds^2 = -dt^2 + a^2(t) \left[\sum_{i=1}^{n-1} (dx^i)^2 \right] \quad (199.4)$$

For easy comparison with the other models, it is common to put it in hyperspherical coordinates,

$$ds^2 = -dt^2 + a^2(t) [dr^2 + r^{n-2}d\Omega^2] \quad (199.5)$$

If we write the open universe of radius parameter R using standard hyperspherical coordinates, we get

$$ds^2 = -dt^2 + a^2(t) [R^2 d\Omega^2] \quad (199.6)$$

with $d\Omega^2$ the standard hyperspherical volume element^{16.1}. In the $(1 + 3)$ -dimensional case, this is simply

$$ds^2 = -dt^2 + a^2(t) [R^2 d\theta + R^2 \sin^2(\theta) d\varphi_1 + R^2 \sin^2(\theta) \sin(\varphi_1) d\varphi_2] \quad (199.7)$$

If we define a point at the pole $\theta = 0$, we can define the coordinates given by the radial distance from that point.

Closed universe

General formula

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad (199.8)$$

Conformal time

Slicing

Connection and geodesics

Riemann curvature

3+1 decomposition of the Riemann curvature

As the FRW spacetime is a warped product of two homogeneous manifolds, the decomposition of its curvature into the various parts of this product is quite interesting

$$R = \frac{6}{a^2}(\ddot{a}a + \dot{a}^2 + k) \quad (199.9)$$

To make calculations easier, we will also define the *Hubble parameter* H , which can roughly be considered as the relative rate of expansion of the spatial distances.

$$H = \frac{\dot{a}}{a} \quad (199.10)$$

199.1 Geodesics and light

The geodesics of the FRWL metric are given by

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \quad (199.11)$$

199.2 Matter

As a homogeneous spacetime, we expect the matter field to be homogeneous as well [True?]

Restrictions of the type in Hawking classification?

Equation of states

In coordinates that diagonalize the SET

Friedmann equations :

$$\dot{\rho} \quad (199.12)$$

199.3 Global structure

That one article that has all the Penrose diagrams

199.4 Classical limit

Contraction of the FRW solution

[418]

Chapter 200

The Einstein universe

The Einstein static universe [419] was an early cosmological model based on the assumption of a static universe, both in the general relativistic sense and in the earlier cosmological sense of an eternal universe.

Chapter 201

The Milne universe

$$ds^2 = -dt^2 + t^2(d\xi^2 + \sinh^2(\xi)d\Omega^2) \tag{201.1}$$

Chapter 202

The Bianchi universes

We can drop the requirement that our universe is spatially two-points homogeneous, and instead only require one-point homogeneity. The classification of such manifolds is known as the Bianchi classification

202.1 Bianchi classification

Chapter 203

Lemaître-Tolman metric

The *Lemaître-Tolman metric* [420] is an example of a non-homogeneous cosmological model, in this case one that is spherically symmetric

[421]

Chapter 204

Kerr-Schild metrics

[422–426]

Kerr-Schild metrics are a class of metrics that are the linear combination of a flat metric and a bilinear form composed of null vectors in that flat space. In other words, if we have some coordinate patch on which we have a flat metric η , and we have some null vectors k such that $\eta(k, k) = 0$, a Kerr-Schild metric has the form

$$g = \eta + Hk^{\flat} \otimes k^{\flat} \quad (204.1)$$

$$= (\eta_{\mu\nu} + Hk_{\mu}k_{\nu})dx^{\mu}dx^{\nu} \quad (204.2)$$

Here $\flat$ is the musical isomorphism with respect to the Kerr-Schild metric g , $k^{\flat} = g(k, -)$. From this, we have that the null vectors k of the flat space are also null vectors in the Kerr-Schild metric :

$$g(k, k) = \eta + Hk^{\flat} \otimes k^{\flat} \quad (204.3)$$

$$= \eta(k, k) + Hk^{\flat}(k) \otimes k^{\flat}(k) \quad (204.4)$$

$$= 0 + Hg(k, k) \otimes g(k, k) \quad (204.5)$$

$$= 0 \quad (204.6)$$

And furthermore, the musical isomorphisms also function with the flat metric :

If we look at the inverse metric g^{-1} , we can see that it itself admits such a decomposition. Using the musical isomorphisms, we have that

$$g^{-1}(k^{\flat}, k^{\flat}) = g(k, k) = 0 \quad (204.7)$$

[CHECK IF THIS WORKS PROPERLY BECAUSE THE ISOMORPHISMS CAN BE g or η]

Chapter 205

pp-wave solutions

Exact wave solutions

$$g = g_R(x) + 2dudv + H(x, u)du^2 \quad (205.1)$$

Gravitational wave solutions

EM wave solutions

205.1 Monochromatic wave solution

One example of a pp-wave solution [200] is the case in which the metric is generated by a "monochromatic" (we will see exactly what is meant by this) wave, such as an electromagnetic wave.

$$ds^2 = -2dudv + \text{ce}_{\omega u}\left(\frac{q^2}{\omega^2}, 2\frac{q^2}{\omega^2}\right) \sum_{i=1}^{n-1} (dx_i)^2 \quad (205.2)$$

where $\text{ce}_m(x)$ is the elliptic cosine [427], one of the elliptic Jacobi function, which is the solution of the differential equation

$$\frac{d^2}{dx^2}y + (1 - 2m)y + 2my^3 = 0 \quad (205.3)$$

pp-wave spacetimes

wave collision

Chapter 206

The Gödel spacetime

The Gödel spacetime was one of the first spacetime admitting closed timelike curves that was studied (the actual first was the timelike cylinder by Reichenbach [182], and another was the van Stockum dust solution [428] which happened before Gödel but whose closed timelike curves were not noticed until Tipler).

It is a cosmological dust solution, homogeneous and axisymmetric, and topologically trivial. We simply have the manifold $\mathbb{R}^4$.

$$ds^2 = \frac{1}{2\omega^2}[-(dt + e^x dz)^2 + dx^2 + dy^2 + \frac{1}{2}e^{2x} dz^2] \quad (206.1)$$

This corresponds to the frame field

$$\begin{aligned} e_0 &= \sqrt{2}\omega\partial_t \\ e_1 &= \sqrt{2}\omega\partial_x \\ e_2 &= \sqrt{2}\omega\partial_y \\ e_3 &= 2\omega(\exp(-x)\partial_z - \partial_t) \end{aligned}$$

Einstein tensor :

$$G^{ab} = \omega^2\eta^{ab} + 2\omega^2\partial_t \otimes \partial_t \quad (206.2)$$

206.1 Symmetries

Killing vectors : $\partial_t, \partial_y, \partial_z, \partial_x - z\partial_z, -2\exp(-x)\partial_t + z\partial_x + (\exp(-2x) - z^2/2)\partial_z$

206.2 Causality

The Gödel spacetime is most famous for its causal structure, as it has closed timelike curves going through every point, although all of them are accelerated curves.

Lack of spacelike hypersurfaces

Chapter 207

Wormholes

In this chapter we will look more specifically at *traversable wormholes*, as opposed to the more general meaning of the word that also includes some black hole spacetimes.

There are a few definitions one can give of wormholes, although none that encompass all possible variants without being too broad. For instance here are a few rough examples of spacetimes that are usually called wormholes :

First is the Morris-Thorne wormhole, which is made by the connected sum of two planes :

[diagram]

Then there is the cut and paste intrauniverse wormhole, given by the cut and paste procedure of two timelike tubes, or can be given by the connected sum of the Cauchy surfaces with a torus :

[diagram]

and we also have the bag of gold spacetime, given by the connected sum of $\mathbb{R}^n$ and S^n :

[diagram]

We will see in more details what those spacetimes are, but note that for instance, the cylinder is also the connected sum of two planes, the toroidal universe is also the connected sum of $\mathbb{R}^n$ and T^n , and Minkowski space is also the connected sum of $\mathbb{R}^n$ and S^n . This tells us that a wormhole is not just a topological notion, but also a geometric one. There needs to be a throat

Wormholes are sometimes defined as multiply connected spacetimes : the fundamental group of the manifold $\pi_1(M)$ fails to be trivial. While this does meet the definition of many wormholes, this does not fit the case of the bag of gold spacetime, but it does fit a number of cosmological models we probably wouldn't consider wormholes, such as the

flat toroidal spacetime.

207.1 Morris-Thorne wormholes

One of the simplest case of wormholes is the *Morris-Thorne wormhole*. It is a spherically symmetric spacetime with two asymptotic Minkowski regions such that an observer can go from one region to the other.

Topologically it is of the shape $S^{n-2}\mathbb{R}^2$. Due to its rather simple shape, it is typically studied under the obvious coordinates (t, r) along with the $n - 2$ hyperspherical coordinates, where (r, t) spans the entire range of $\mathbb{R}^2$. Asymptotically, those coordinates appear to simply be spherical coordinates, as the exterior part of the usual spherical coordinates of it outside of a sphere, but its non-trivial topology allows it to continue those all the way around to negative values, ending up on the "other" universe.

As a spherically symmetric spacetime, it can be expressed as

$$ds^2 = -f(t, r)dt^2 + g(t, r)dr^2 + Y(r, t)d\Omega^{n-2} \quad (207.1)$$

It is furthermore defined to be nowhere singular, lacking any horizon, so that we can if we wish stay in these coordinates all we want. The constraints are that [429]

- The metric is asymptotically flat
- There are no event horizons
-

207.2 Intra-universe wormholes

The Morris-Thorne wormholes are in some sense wormholes between two different universes, which can be seen by considering that it has two disconnected asymptotic regions, or that removing the wormhole's throat leaves us with two disconnected manifolds. If we consider the manifold obtained by removing the hypersurface spanned by $r = 0$,

[excision theorem, relative homology]

Making wormholes that connect two different point of the same universe is usually done by the cut and paste procedure, as the topology of such wormholes would usually make them too difficult to work with otherwise.

The simple case to work with is to consider some original spacetime foliated into spacelike hypersurfaces $M = L \times \Sigma$, take two diffeomorphic tubes $L \times S_i$, $i = 1, 2$, such that those tubes are embedded submanifolds of M obeying [foliation]

A particularly useful example of this is Wheeler's *bispherical wormhole*, obtained by considering the bispherical coordinates on a $\mathbb{R}^n$ manifold, given by the coordinates (τ, σ, φ)

$$x = a \frac{\sin(\sigma)}{\cosh(\tau) - \cos(\sigma)} \cos(\varphi) \quad (207.2)$$

$$y = \quad (207.3)$$

$$z = \quad (207.4)$$

[430] Cut and paste

Chapter 208

Closed timelike curves

We have already seen a few non-causal spacetimes, such as the Kerr spacetime, the de Sitter spacetime, the Gödel spacetime or a few of the wormhole spacetimes.

208.1 Tipler cylinder

The Tipler cylinder is a specific solution of the more general van Stockum dust metric discovered by van Stockum and Lanczos.

$$ds^2 \tag{208.1}$$

208.2 Gott solutions

Two rotating cosmic strings

208.3 van Stockum dust

Cylindrically symmetric solution of rotating dust

$$ds^2 \tag{208.2}$$

Chapter 209

Faster-than-light travel

209.1 What is an FTL spacetime

It can be difficult to define what a FTL spacetime is, as defining a speed in any given spacetime is already a complex task, let alone what it means for an object to move "faster than light" despite travelling in a lightcone. The notion is usually based on more practical grounds, although it also does not go too far the other way usually : if an observer travels slightly faster than expected due to the gravitational length contraction of a nearby star, it will not generally be counted as being FTL since this is the "default" geometry.

[220]

209.2 The Alcubierre drive

The most well known example of a FTL spacetime is the original Alcubierre drive, given by a piecewise metric consisting of Minkowski space on the outside of a timelike tube $D^{n-1} \times \mathbb{R}$, and inside of that timelike tube, a transition to another flat metric with a wider lightcone.

Tophat function

Alcubierre drive

Hiscock coordinates

Energy conditions

Careful that the contraction of space is an artefact of the specific metric

209.3 Lobo drive

Part XII

Measurements and philosophy

So far, all the previous chapters have mostly concerned purely mathematical objects despite these notions sometimes sounding like physical ones or hinting at their physical interpretations. While habit may make us think of fields and particles as physical objects, we have only defined a variety of mathematical objects and their properties.

A physical theory is more than simply its mathematical structure, however. It also requires some mapping of those elements to the real world. and this will be (among other things) what we'll attempt to do in the following chapters.

One may ask whether such a section has a place in a book on general relativity, but in my experience, such topics eventually come up sooner or later no matter what. Short of treating physics as essentially a mathematical theory, it can be otherwise confusing to see how everything relates to the real world, especially in such abstract theories. It is a common question for people studying it to wonder what is "real", etc. Even the common interpretation of physical theories as simply tools to arrive at results without any inherent meaning is in general not enough to avoid such issues, as we still need to explain *how* to arrive at the results, which is not a commonly treated topic in most physics books.

In particular it is quite common for "breakthrough" experiments to come out regularly, calling into question existing theories or seeming to confirm widely accepted hypothesis or implementing new technologies out of them.

Something about the type of assumptions we must make for a theory to apply : the actual formalism, the correspondence rules, some approximations for the boundary conditions, some operational rules, etc etc

Hypothesis on statistics

Discussion on prior geometry [355]

[431,432]

Chapter 210

What is a physical theory

We will not enquire too much here about how a physical theory comes to life, the psychological processes involved or the metaphysical principles that may be underpin it, topics on which there are no lack of sources [433, 434]. We will simply consider a given theory, see how it is actually constructed, and how we may judge it.

210.1 Analytic and synthetic statements

In philosophy there is a difference between on one hand *analytic* statements and on the other *synthetic* statements (despite having their roots in similar ideas, this is somewhat different from the notion of analytic and synthetic we saw for geometry). An analytic statement is a statement that can be determined to be true or false by its own meaning (usually within the context of some logical or mathematical theory), while the truth of a synthetic statement can only be determined from observations of the real world. As with any notion in philosophy, there are of course counterarguments for this distinction [435], but we will not nitpick quite that hard.

All that we saw previously have been analytic statements : the theorems we saw are true because of our assumptions of various mathematical axioms and rules of inference, and will remain so even if they do not describe the actual functioning of gravity. We even saw a variety of alternatives to general relativity which are known to not actually describe gravity, but the theorems we saw about them are still true in this sense.

The fundamental rules of logic and mathematics themselves can be debated, but for this chapter, let's stick to some rather basic classical logic : if any theory, analytical or synthetic, contains a contradiction, ie. if the theory contains both $\vdash p$ and $\vdash \neg p$, it is considered invalid. This is the main basic criteria of validity for such statements, called consistency, disregarding more esoteric notions like ω -consistency, provability or constructibility (those are typically less of a problem in the context of physical statements). Short of some errors introduced by some rather unfortunate physicist-based mathematics, or more dire scenarios like set theory itself being inconsistent, the theories that we have

seen previously should all be consistent, being mostly based on standard mathematics. This is true even of the theories that we have seen that are considered to be "false".

A physical theory therefore also needs synthetic statements. If we consider all possible statements relating to the real world, a physical theory should have some statements relating to those, either affirming their truth or falsehood. As an example, if we have a statement regarding the result of an experiment, for any outcome, we should have the proposition of that outcome and its negation, and only one of them can be true within the physical theory.

Example : Detector lights up or does not light up.

Logical v. empirical content

210.2 Theory

[...]

One specific aspect of the theory is the *phenomenology* of the theory, which are roughly speaking theoretical statements that are meant to be as close to the "real" systems meant to be studied as possible. In the context of gravitational physics for instance, this would be phenomenological models of the Earth for instance, which would often be the case of a spherically symmetric gravitational source. This phenomenon is quite close to what we would expect the physical system to be, and the correspondence rules between the two will be rather minimal, simply requiring us to plug a few different values from experiment into the model, such as the total mass, the radius, etc.

Phenomenon at differing levels of abstraction

Example of GR phenomenon : Langevin twins, geodesic in spherically symmetric space-time, etc

210.3 Observations

The basic observation statements are for instance atomic statements about what can be observed during an experiment.

Assuming that we have a list of all such possible statements, the domain of discourse of our physical theory should be that entire set of statements, and the role of the physical theory is to give us rules of inference from some set of such statements to others. In terms of a deduction, this would be given for instance by this simple system :

$$\frac{\vdash \text{Obs}_1 \quad \vdash \text{Obs}_1 \rightarrow \text{Obs}_2}{\vdash \text{Obs}_2}$$

The original observation Obs_1 is some physical observation we have done, the implication $\text{Obs}_1 \rightarrow \text{Obs}_2$ is given to us by our physical theory, and Obs_2 is the observation we should be expecting if the theory is true. The physical theory is of course larger than a single implication, so that our deduction should be something more akin to

$$\frac{\vdash \text{Obs}_1 \quad \frac{\text{Theory}}{\vdash \text{Obs}_1 \rightarrow \text{Obs}_2}}{\vdash \text{Obs}_2}$$

Universality (class of observations it applies to), precision (what are the constraints on the observations)

Universality and precision correlates with empirical (or logical) content, and therefore falsifiability

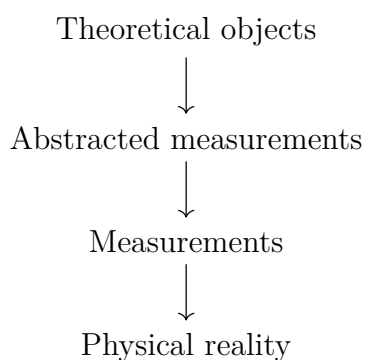
p is more precise than q if $\forall x, f_p x \rightarrow f_q x$ is tautological

contradiction

causality

singular and universal statements

domain of applicability



Statements, basic statements, existential, universal, contradiction

synthetic v. analytic

simplicity, example of curve fitting, schlick naturwissenschaften weyl

Induction

Uniformity of nature

Degrees of confidence [Keynes]

Axiomatization from just the list of observations

Naturalness?

causality

"We can now distinguish between sensations and those mental operations which follow so closely upon the senses [Francis Bacon, *Novum Organum*] and which are so firmly connected with their reactions that a separation is difficult to achieve." Natural interpretation, Kant is on the side of them being a priori presuppositions, Bacon is on the side of them being prejudices to be removed

Difference between laws and initial conditions

"Il est difficile de ne pas supposer que l'influence des corps très éloignés est tout à fait négligeable, que les petits mouvements obéissent à une loi linéaire, que l'effet est une fonction continue de sa cause. J'en dirai autant des conditions imposées par la symétrie."
[436]

[437]

Confirmation holism, duhem-quine thesis

Methodological/observational/semantic incommensurability

210.4 Falsifiability

One commonly used criteria used in physics is the notion of falsifiability.

falsifiability

Definition 210.1. A statement p is *falsifiable* if there exists another observational statement q such that the falsification of q (ie q is false, $\vdash \neg q$) implies the falsification of p , which can be expressed via

$$\vdash p \rightarrow q \quad (210.1)$$

Falsifiability was proposed as an alternative to a previous notion for the validity of a physical theory, called verificationism, where statements (in general, but in our context relating to a physical theory) were only considered meaningful if they were either true analytic or synthetic statements.

[...]

However, the contradiction of experiment and theory do not necessarily invalidate the theory at once, and in practice it is common to view them as anomalies rather than refutations [431, 438]. This is due to the fact that there are typically a very large amount of hypothesis beyond the physical theory itself that go in the experiment, which we will see in more details in the following chapters, such as the type of observables our apparatus measure, the initial condition given by our experimental setup, the boundary conditions given by the rest of the universe, etc etc. The physical theory in the narrow sense that we saw could therefore not be the culprit in a negative result.

Example : the actual set of propositions we are using will be something of the form

$$\vdash \quad (\text{Content of the theory} \quad (210.2)$$

$$\wedge \text{ Assumptions relating to measurements} \quad (210.3)$$

$$\wedge \text{ Assumptions on the initial conditions} \quad (210.4)$$

$$\wedge \dots) \rightarrow \text{ Experimental results} \quad (210.5)$$

The actual falsifiable statement is the entire conjunction of the theory with our various other assumptions, and a single false assumption there could render the entire proposition false.

210.5 Pragmatics

The few defining attributes we've seen for a physical theory are not generally that helpful to narrow things down out of a collection of different physical theories, as all that we are asking is that they output all of the appropriate observations for a given observation input. If we have a collection of experiments,

$$(\text{Obs}_1 \rightarrow \text{Obs}_2, \text{Obs}_1 \rightarrow \text{Obs}_2, \dots) \quad (210.6)$$

all verified by experiment, their continuation by some arbitrary collection of other observations will be just as valid as any other. As it is not typically too hard to find even an axiomatic basis for such a theory [Craig reaxiomatization], we need some additional criterias to pick a theory out of the space of all possible theories.

Unlike the previous sections, this will unfortunately have some possibly arbitrariness to it, notions that may turn out to be wrong or subjective, but these are the type of notions that tend to come up

210.5.1 Prediction

One type of pragmatic criteria that is sometimes asked of a given theory is that it can predict novel phenomenons and not merely accurately reproduce existing results.

Accomodationism v. predictionism

Bayesianism

How this applies to string theory

A good example of the tension between the two extremes of accomodationism and predictionism is th case of string theory. While string theory will reproduce so far at least general relativity in its low energy limits,

210.5.2 Research Programs

Lakatos research program, normal science by Kuhn, etc

What is the core v. auxilliary for GR

If we consider general relativity for instance, the core principles might be that the theory rests on the existence of spacetime as a single unified object, general covariance, the geometric nature of gravity, the equivalence principle, etc etc. Auxilliary ideas might be the specific form of gravitational dynamics, the presence of other possible fields to influence this, the exact relation of gravity to other forces and so forth. A theory that would violate some of those core principles for instance might be entropic gravity [shape dynamics? Ether theories?]

210.5.3 Aesthetics

The space of all theories that will still fit experimental data is still rather large (infinite in fact), so that it is quite common for further constraints to be added. These are aesthetics criteria, which are not a logically necessary constraint on the theory but criterions that are believed to form "good" theories in some sense.

Some basic examples of such criterions are that theories should derive from an action principle, that their equations of motion should be local, that the time evolution should be deterministic, the naturalness of constants, etc

There is no fundamental logical reasons why theories should be, at the very least, isomorphic to a theory obeying all those criterions, and in fact, those types of criterions vary quite a lot in history, as the belief in what constitutes a good physical theory changes as theories develop and die, no matter the certainty people had in those criterions from

what was assumed to be impossible to disprove assumptions. Modern theories would not try to assume such ideas as the principle of sufficient reason, such as the atomists arguing that the number of atoms in the universe has to be infinite because any finite number of atoms would be arbitrary or Leibniz arguing for the relativity of space because God would not pick an arbitrary location for every object.

[List examples]

The exact status of those types of criterias is unclear, as they may not be safe from being proven wrong, but they seem to at least be productive in producing theories. It is however important to try not to

The set of all commonly held aesthetic criterions in physics is the *aesthetic canon*, which usually corresponds to broad criterions that form the basis of most of the main candidates for physical theories.

aesthetic induction

What are the modern aesthetic criterions

Many of the aesthetic criterions for relativity specifically get their own sections, such as general covariance, the equivalence principle, etc.

Some other types of aesthetic criterions that occur in modern physics

Simplicity, symmetry, principle of least action

210.5.3.1 Simplicity

A very common if vague aesthetic requirement in physics is *simplicity*, typically meaning that given theories of equivalent predictive power, or at least equivalent for all known experiments so far, we will usually assume that the "simplest" one will be more likely to be correct.

210.5.3.2 Naturalness

One of the most obvious aesthetic criterion in modern times is that of *naturalness*. Physical theories are typically characterized by a variety of dimensionless parameters. The typical example of this is the fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \quad (210.7)$$

$$\text{Theory}(\alpha, \beta, \dots) \quad (210.8)$$

Naturalness is the requirement that all such constants should be of "order unity"

210.6 Explanatory models

What is an explanatory model

While in modern times, the notion that a physical theory should only be used for prediction of observations, in practice it is quite common for people to believe at least informally, use as a model to generate more theory or for pedagogical purpose an explanatory model.

Causality

Explanatory models can sometimes be suggested by the mathematical model a theory is built on, although in many cases, the opposite is true. A lot of the mathematical involved in physics derive fundamentally from ancient explanatory models, such as the ancient notion that there are natural motions and forced motions, corresponding in our modern models to geodesics versus accelerated curves.

Types of explanatory models :

Deductive-Nomological

Chapter 211

Correspondence rules

Most modern physical theories are not made of synthetic statements in their entirety, or even any random mix of synthetic and analytic statements. There is a rather sharp split between the set of all synthetic and all analytic statements, which are related by the *correspondence rules*.

On the synthetic side, we have what are called *observation sentences*, which are about phenomenon that can be directly relayed by the senses : we can observe objects at some location, the coincidence of two objects, etc (the typical observation sentences tend to be more complex than that but we will see the nuance in 211.1). 213

On the analytic side, we have the *theoretical terms*, which are usually mathematical objects with no fundamental interpretation as real objects.

The correspondence rules are sets of mixed sentences containing both observational and theoretical terms used to translate one into the other. For instance, if we were observing a ball within the laboratory, and measured it to be within a given distance d_z from the floor and two other distances d_x, d_y from two walls (we'll see in later chapters what this means in an observational sense), this would be translated into some reference frame $\{e_1, e_2, e_3\}$ defined by the laboratory, and we associate some theoretical object of the position of the ball within the laboratory's frame as (d_x, d_y, d_z) .

There is some general sense, although not necessarily in any formal way, that some theoretical objects of a theory are more "fundamental" than others. In the physical theories that we've seen, these can be for instance the configurations that we've established. However, as we'll see in the following chapters, those objects tend to not be directly translatable into observable terms, or vice-versa. There will be some tension between the use of easy to manipulate theoretical terms but hard to interpret as physical phenomenon, or objects with obvious interpretation but no simple rules relating them. This will bring up quite a lot of difficulty in our treatment of physics, as we will see that even sticking to the assumption of theories, there are no objects that will model their fundamental objects such as geodesics, null curves, etc, in which case the sense in which this describes the real world becomes less clear.

Physical reality to interaction with measuring apparatus to sensation to the abstraction of that sensation into numeric form to more "useful" quantities

Logical v. observable v. theoretical ("constructible") terms

Logical sentences, observation sentences (contain O terms but not T terms), mixed terms (O and T), purely theoretical terms (T but no O)

Theoretical postulates (pure T), and correspondence rules (mixed)

Reichenbach : correlative definition

A good example of correspondence rules in action are the *operational rules* of Bridgman [439], for which he gives series of physical manipulations to perform a given measurement.

Initial conditions as singular statements versus as statements from the theory if they are not measurable

"Theory-laden observations"

Gunther's model and pre-theories

This corresponds to the notion that any model of measurement in the real world we try to propose to fit within a theory will rely on some assumptions about the objects and processes involved that cannot be checked without some further experiments which would themselves require a measurement process. For instance the basic operation of measuring a length with a rod will rely on the assumption that the rod itself is an approximation of a straight line and of constant length, something which we cannot verify without a further process to measure lengths. We could perhaps check whether the operation is at least coherent within the framework of the theory, for instance check that the congruence of objects is transitive, ie if an object A is congruent to an object B and B with C , A should be congruent with C), which is something that could easily be false if the rod was of varying length in time, but this

This means that physical theories in practice are usually somewhat tied to various assumptions about experimental measurements. There are ways to try to get rid of this problem, such as simply considering the operation itself as the fundamental object of the theory rather than some theoretical quantity to derive from it (the length *is* the coincidence with the rod and not some quantity we try to extract from it), but this method has quite a lot of issues as far as tractability goes, since many completely unrelated processes can be used to measure the same quantity, which would require rules for every possible combinations of those methods.

[examples in the operation section]

211.1 Low-level correspondence

The correspondence rules can be somewhat multi-tiered in practice, as some observables have very simple types of correspondence rules that will put them into theoretical terms immediately. This is quite often done immediately during experiments, as experimental results (at least in physics) are usually put into numerical forms.

One such obvious translation is the idea of *counting*, where we simply put objects within a given class in correspondence with natural numbers. If we have no such object of that class ($\nexists x \in A, f(x)$), this is put in correspondence with having 0 such objects, and if there exists an object belonging to that class which has not yet been added to the count ($\exists x. \forall y \in C, x \neq y$), we have one more object than before.

Another easy concept with a clear mathematical equivalent is that of *incidence*. From our mereology section, we have a clear mathematical model of what it means for an object to be part of another object, which is something that can often be easily deduced from observation.

As our vision is for the most part a two-dimensional projection, a related notion is the vision of one object's 2D projection overlapping with that of another, for instance the case of a dial overlapping with a graduation

[diagram of a dial]

Possible issues [parallax etc]

Notion of order, congruence, etc for coincidence

Mapping to various categories

211.2 Apparatus

Most measurements of modern physics are not done by the raw observations of whatever phenomenon we wish to measure. Instead we complement the system we wish to study with additional objects (called an *apparatus*) that will interact with this system in a way that will change the observables we can glean from the apparatus, either with the methods we previously saw, or in many cases today, with symbols directly.

[Diagram of the readout of an apparatus, multimeter maybe]

The goal of those apparatus is often to measure some theoretical quantity that does not have any direct correspondence rule from the raw observables of the system. A mercury thermometer's goal is to measure temperature under the assumption that the dilation of the mercury in the tube is a function of only the temperature (and a roughly linear one at that), so that we can graduate it to use the incidence method to find congruent

temperatures : if the mercury rises up to a level previously observed, it is at the same temperature as that time.

One issue with apparatus is that what we measure is therefore never what we set out to measure, but the system interacting with the apparatus. In both theoretical and experimental grounds, we can see that these are not the same thing.

Another issue is that the usefulness of an apparatus is only as good as its capacity to measure what it purports to measure. While it is a common saying in physics that the definition of a given physical quantity is what its apparatus measures (ie. "Temperature is what is measured by a thermometer" or "Time is what you read on a clock"), these sentiments never quite express practices. First this only pushes back the question to what is and isn't a clock, and while any method that gives us an increasing value will technically count as a clock in some relativistic sense, we still have some notion that a given apparatus may be wrong. Apparatus can be broken, miscalibrated, etc.

There is also the problem that each apparatus is unique, but we still require some unified notions. If we simply assumed that each apparatus was giving its very own type of measurement (and even worse, that those measurements could not be compared at different points in time), it would be nearly impossible to say much about most experiments, but we generally assume that most apparatus belong to a certain class giving us similar types of measurements.

Bacon : "making manifest, things not directly perceptible"

211.2.1 Experimenter's regress

The difficulty of ascertaining whether an apparatus measures what it purports to measure can lead to the problem of the experimenter's regress, where it can become impossible to show that it indeed does. [440,441] [...]

Weber's bar

211.3 Measurable quantities

All the previous methods will give us the quantities we have decided are directly measurable from the system [cf. Carnap on the arbitrariness of that split]

Our correspondence rules proper should give us mapping in both directions from those observables to theoretical objects and back

Definitely not bijective

Philosophical observables : Phenomenon directly accessible to observation : "blue",

"hard", "hot"

For physicists : quantity measurable "directly" : temperature, length, etc (debatable how direct this is)

Carnap for the utility of numeric representations [442]

Ludwig : Pre-theory

Empirical v. theoretical law

Physical objects that are concrete representations of theoretical objects

211.4 Ramsay sentences

One aspect of physics that can be questioned is the presence of theoretical terms. Certain philosophical schools will perhaps be uncomfortable with the presence of fictitious objects such as theoretical terms and would prefer to only speak of observational terms, and it can be asked whether or not such terms are actually required.

Ramsay sentence, Craig reaxiomatization [442–444]

Theorem 211.1. Any recursively enumerable theory is recursively axiomatizable.

Proof. For S some deductively closed set of sentences,

□

Chapter 212

Modelling space and time

All throughout this book, we have modelled spacetime as mostly the same very specific kind of object, with a few varying models sprinkled throughout, and it is now time to consider how well-founded this assumption is.

It is generally assumed that spacetime obeys some kind of mereological rules, ie that there is some concept of parthood to regions of spacetime. These are usually assumptions that are in common with every model of space

[...]

Modelled mathematically, this would give us a specific type of locale, depending on the exact mereological rules that we consider.

One point of contention about this locale, going back all the way to antiquity, is whether or not this is a spatial locale, ie whether or not we may define it by a set of points.

understanding space(time) as an abstract space (topos) v. understanding it as the result of observation (scheme), define points in both cases

Discrete v. continuous

Is there a notion of congruence at all

Chapter 213

Observation terms

Structuralism?

The observation terms of our theory are the ones that we will be able to get directly from experiment (up to some identification with more complex ideas that we will see later on)

The formalization of such observations can be done in a few different ways. At its core, we want to be able to talk about various properties that our studied system can have, and at minimum, they should obey the two following principles : a system can have or not have a given property, and it can have multiple properties.

Ludwig Gunther [445] does this using a synthetic theory of such properties

Set M of objects, structure $\mathcal{E}$ such that $\mathcal{E} \subset \mathcal{P}(M)$. For $a, b \in \mathcal{E}$,

Axiom 213.1. 1. If $a \in \mathcal{E}$, then $M \setminus a \in \mathcal{E}$

2. If $a, b \in \mathcal{E}$, then $a \cap b \in \mathcal{E}$

All objects which do not have the property a have the property of $\neq a$, objects that have the property a and b have the property $a \cap b$.

(boolean ring of sets)

A categorical description of this is done using monads

Urs Schreiber or Lawvere : monads etc

What the hell is a monad here

Are the two equivalent

Chapter 214

Observables

Chapter 215

Modelling objects and observers

[446, 447]

As we saw previously in chapter 72, observers and objects are typically modelled by causal curves, but it is very much obvious that this is quite a big abstraction that deserves to be looked into.

An actual measuring apparatus is going to be made of some continuum of matter, without any clear distinction from its environment. If our matter is modelled by some set of fields on a bundles E with configuration space Conf for instance, the differing states of the apparatus will correspond to different configurations of those fields, $\{\phi_i\}$. Not all different configurations will correspond to a different measured state, as differences in the configuration of our apparatus will not always be due to the measuring it performs : whether some given atoms are slightly ajar in a ruler will not change the observed result. If we consider that our apparatus measures a set of observables $\{O_i\}$, assuming those are faithfully measured by the apparatus (every observable O_i corresponds to a state of the apparatus itself O'_i), then our measurements will correspond to a subspace of the configuration space that are the preimage of those observables :

$$O'_i : \text{Conf}_{\text{Apparatus}} \rightarrow \mathbb{R} \quad (215.1)$$

The corresponding subspace for a given value of the measurement m_i will be $O_i^{-1}(m_i) \subset \text{Conf}$.

A first approximation we can do then is to consider our apparatus as sitting in some compact open set $U \subset M$. We can model this domain in a variety of ways. The actual act of measurement will be done at some given time

Due to the typical length and time scale of an experiment, this region is likely to be much more stretched in the timelike direction than in space, as the scale of time compared to that of space is given by c , so that any given second of an experiment will correspond to

[Existence of a timelike curve such that the region is in between two Fermi or whatever neighbourhoods of small radius?]

[448]

Modelling matter as curves [cf. Geroch, Synge]

Tracking

[449, 450]

Modelling light as null geodesics

observers as timelike curves

215.1 Quantum objects

If we want to furthermore model our observers using quantum theory, we have to model them this time using a Hilbert ray corresponding to the appropriate measurement.

Apparatus etc modelled by some subspace of the projective Hilbert space

Everett's thesis, split between system and apparatus, measurement as the projection on some eigenvalue

Chapter 216

Spacetime as measurements

Isbell duality, spacetime as a scheme, Poincaré's idea, etc

Chapter 217

Operational measurements

While we may have difficulties defining how real life measurements and theoretical objects relate properly, we still have to attempt to do it if we wish to even try anything resembling physics. To do this, we will outline a variety of methods to perform measurements of basic observables in general relativity, their strengths and weaknesses as well as the assumptions that go in their use.

Operational rules are a set of procedures that one can perform for the measurement of some property in physics. Those rules tend to not often be quite explicit and often have many subtleties [438], but we will try to give an exposition of those in some detail.

[439]

217.1 Clocks

One fundamental tool in the study of general relativity is that of a *clock*. As an abstraction, a clock corresponds to some parametrization of a (usually future-directed, if orientable) timelike curve. An actual clock will more properly assign integers for open sets of the spacetime

A good first approximation of a clock is a timelike curve γ with a sequence of events $(e_i)_{i \in I}$ along that curve

If we define one such event as the origin, the clock simply counts the number of such events in a sequence. The time defined by that clock is that if we look at the clock at a given point $p \in \text{Im}(\gamma)$, its time is simply the cardinality of all events between e_0 and the

$$t(p) = \text{Card}(\{e_i | e_i > e_0 \wedge e_i < p\}) \quad (217.1)$$

This type of clock can only ascertain that the events occur one after the other, at least

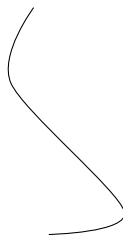


Figure 217.1: Three subsets of the plane of identical cardinality

within the context of the clock, and any globally hyperbolic neighbourhood we can define around it. In such a neighbourhood we can define for instance that the time defined by the clock corresponds to whether

A first nuance we can apply here is that we do not

approximation of a clock via interpolation and assuming those open sets to reduce to a curve

Atomic clocks

Light clock

Markze-Wheeler clock

EPS, Perlick, standard clock

Kundt-Hoffmann clock

A clock in general relativity will be any process which, approximated as happening along a single timelike curve $\gamma(\tau)$ of unit speed, has recurring events such that if this event occurs at some τ_p , then it will only occur at $\tau_p + kT$, for some $T \in \mathbb{R}_{>0}$, $k \in \mathbb{Z}$. As the proper time along a geodesic of unit speed is just $\tau_2 - \tau_1$, the proper time between any two events will be kT , and the proper time between two consecutive events will be T , the *period* of the clock.

We can then define a parametrization of the curve

$$\tau \mapsto (\tau - \tau_p)/T \tag{217.2}$$

In this parametrization, every event will occur at $\tau = k$. This is the time as measured by the clock.

An important clock is the one defined by the maximal amplitude of the radiation emitted by the transition between the two hyperfine levels of the ground state of caesium 133, at rest and at a temperature of $0K$. For this, we define additionally the parametrization

$$\tau \mapsto t = (\tau - \tau_p) / (9\,192\,631\,770 \text{ } T) \quad (217.3)$$

for which we go from t to $t + 1$ after 9 192 631 770 consecutive events. We then say that in this parametrization, the time elapsed between two events $\gamma(t_1)$ and $\gamma(t_2)$ is $t_2 - t_1$ seconds.

If we have to use two different clocks, we can calibrate them to measure approximately the same time. If we have two clocks of period T_1 and T_2 starting at λ_p , and we find that the last tick N_2 of the second clock occurs between the ticks N_1 and $N_1 + 1$ of the first clock, we then know that the relation between the two periods will be between $N_1 T_1 = N_2 T_2$ and $(N_1 + 1) T_1 = N_2 T_2$, or

$$T_2 \in [T_1 \frac{N_1}{N_2}, T_1 \frac{N_1 + 1}{N_2}] \quad (217.4)$$

which we can write as the uncertainty

$$T_2 = T_1 \left(\frac{N_1}{N_2} + \frac{1}{2N_2} \right) \pm T_1 \frac{1}{2N_2} \quad (217.5)$$

We can then parametrize, up to uncertainty, any timelike curve with the parametrization of that new clock. This will allow us to measure any time interval using SI units.

If the spacetime is globally hyperbolic, we can define the temporal function from a foliation such that $\nabla t = n$ has timelike curves as a flow with this parametrization for hypothetical clocks going through each points. This will be the global SI parametrization of time for this spacetime. If the spacetime isn't globally hyperbolic, we can simply pick a globally hyperbolic neighbourhood of the region of interest.

Light clocks

217.1.1 Measuring rod

Measuring rod : two timelike curves such that their intersection in an achronal spacelike hypersurface is of constant separation.

Meter : a measuring rod is of 1 m if a light rays leave point A at λ_A and reaches point B at λ_B such that, in the SI parametrization,

$$(\lambda_B - \lambda_A) = \frac{1}{299\,792\,458} \text{ s} \quad (217.6)$$

or for a round trip AB, BA , the time measured along A is twice that.

Observer carries a clock and measuring rods

217.1.2 Atomic clocks

A very common type of clocks in operational measurements is the *atomic clock*, based on the supposed periodicity of the emission of atomic electronic transitions

See the QM part for electronic transition

An atomic clock obviously defines a clock in the broad sense, but do they in fact measure proper time, at least in theory? If we consider some entirely unperturbed atomic system in a fixed background metric, using the same approximations as in flat space, we obtain the equation

$$\gamma^\mu \nabla_\mu + \dots \tag{217.7}$$

217.1.3 Conformal clocks versus metric clocks

There are two possible types of clocks that one can define, either by leaving the definition of the scale up to initial conditions, or by using some objective scale to set it up.

Every clock built using classical mechanics will be of the first kind. The period spanned by a clock built from a simple harmonic oscillator (a pendulum) will have its period depending from its length, the period spanned by an orbit such as Earth will depend on its distance from the sun, and so on. Without already having some scale to work from, we cannot guarantee the length of time spanned by such a clock, but we can observe the count of such a clock with any event, and compare the clock measurements of any two intervals to determine their ratio.

In particular, this kind of conformal clock does not allow us to communicate its scale to a distant observer. Without physically carrying the device to synchronize some other clock with it, we cannot have a similar scale.

On the other hand, most modern scales used as standards for modern clocks depend on objective scales, and they are all quantum mechanical in nature.

Some example of such scales are

- Bohr radius
- Compton wavelength
- Classical radius of the electron

- Planck mass

The Bohr radius is the most common natural length scale commonly used experimentally, as it relates to both the processes involved in atomic clocks as well as defining the length of a given prototype length in solids. As we have seen, scales can be defined from a mass scale, an interaction

217.1.4 Matter wave clock

As mass relates to scale, there is an extra way of building clocks when we consider the quantum theory of a massive particle.

De Broglie wavelength

Related to the central extension of the kinematic group?

217.1.5 Synchronization

One operational aspect of clocks is the synchronization of multiple clocks. In the abstract, two observers equipped with a clock intersecting at some point p of spacetime are *synchronized* if their clocks are measuring the same value at that point :

$$h_1(p) = h_2(p) \tag{217.8}$$

[do the proper delineation of what it means for that function to be evaluated at p]

The actual operational synchronization of two observers is however far from this abstraction. As the two observers are not actual points, nor can they actually intersect, the

217.2 Rods

As there is no canonical split of a spacetime into spacelike hypersurfaces, the measurement of length is a more complicated task than that of time

The operational method that we used so far for the measurement of distances has been the light bounce, using the *Einstein convention* that for an observer γ emitting a ray of light ℓ_1 at time τ_1 towards some object and receiving it back at time τ_2 as ℓ_2 , then the distance between the mid-point $p = \gamma((\tau_1 + \tau_2)/2)$ and the point q where that ray bounced will be

$$d(p, q) = \frac{\tau_2 - \tau_1}{2} \quad (217.9)$$

In a radar neighbourhood, this does indeed define an appropriate foliation into spacelike hypersurfaces.

Reichenbach convention

However, in many practical cases, this is not the method we use for measuring distances, but what is generally described as a *rod* : the distance between two points is described by congruence with that of an object.

In terms of Euclidian geometry, the rod can be modelled by a segment. The measurement of two points are then separated by a length equal to that of the segment if there exists an isometry (a rigid transformation) taking the segment such that its endpoints coincide with the two points.

However, this notion fail as soon as we try to apply it to general relativity, even special relativity. A segment in that space can only represent a physical object if that object is a point on some spacelike hypersurface, as that segment can only be timelike, which is what we used for our clock construction. An actual rod in the Minkowski case will have to be represented by a worldsheet.

List of various length measurements used in geodesy : [451]

217.2.1 Rigid objects

The fundamental problems of rods in general relativity is that of *rigidity*.

See the previous chapter on the topic114.1

Even assuming flat spacetime,

Impossibility of rigid objects

A very extreme case of this problem can be seen Rigid objects and horizons in the extreme case

Approximation of rigid objects

Rigid motion defined in a given foliation? [See Lightman, maybe? also]

Born rigidity [452–454] :

Definition 217.1. For an object X moving with a velocity described by a velocity field u , and the projection of the metric tensor onto the orthogonal complement of that velocity

$h = u \otimes u - g$, the motion is said to be *rigid* if

$$\mathcal{L}_u h = 0 \quad (217.10)$$

The strain rate tensor vanishes, Killing motions are rigid

Theorem 217.2. The only rigid motions are

217.3 Angle measurements

Protractor construction

The basic idea to measure plane angles is to consider a central point around which we distribute a number N of points which are both equidistant from the center o and to their two nearest neighbours : $d(p_i, p_{i+1}) = d(p_{i+1}, p_{i+2})$. The angle defined by $\widehat{p_i o p_{i+1}}$ will therefore be the same for any value of i , which we will call α , and the angle $\widehat{p_i o p_{i+k}}$ will be equal to $k\alpha$ [show it]. In particular, if our system of point is such that the first and last point are either identical or colinear on different sides of o , we have that $\alpha = 2\pi/N$ or $\alpha = \pi/N$.

If we define one of the point p_i , let's say p_0 , as a reference point, we can define angles relative to that point, such that the angle of any line op_k will be equal to $k\alpha$ relative to p_0 . The absolute angle $\widehat{p_i o p_{i+k}}$ can also be defined by $\widehat{p_0 o p_{i+k}} - \widehat{p_0 o p_i} = (i+k)\alpha - i\alpha = k\alpha$ [check order]

Two radius op_k and op_{k+1} define a sector on this circle, and by prolongation, extending out into infinity, for which any incident line to that sector has an angle of relative angle $\theta \in [k, k+1]\alpha$. Relative angles between two incident lines of different sectors will therefore be in $\theta_2 - \theta_1 \in \square$ (widest one of $k_2 - k_1, k_2 + 1 - k_1, k_2 - k_1 - 1, k_2 + 1 - k_1 - 1$)

Consider how to form a basis in n dimensions : usage of $n + 1$ points

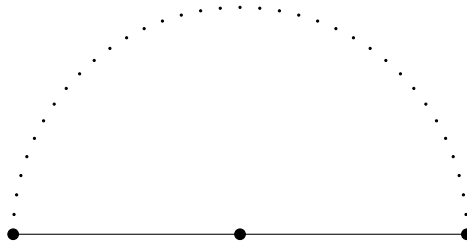


Figure 217.2: Protractor

A very similar method can be used in general relativity, using a set of curves

To form a (spatial) basis in GR : $(n + 1)$ objects

Full spacetime basis :

System of n objects [cf Synge]

[455]

217.4 Gravimetry

Direct measures of the gravitational field are often done in geodesy using gravimeters, which are essentially just accelerometers, exploiting the principle of equivalence for this effect.

Most gravimeters function on the principle of a mass m resting upon a spring, with some function relating its elastic force to its length, $\vec{F} = \vec{f}(l)$. Many commonly used springs in gravimeters do not obey Hook's law, with the force proportional to the length of the spring

How gravimeter works

Does it relate to the gravitational compass and Synge's tetrahedron thing

"The relativistic geoid is the surface where precise clocks run with the same speed and the surface is nearest to mean sea level."

217.5 Other stuff

Gravitational compass?

217.5.1 The EPS process

[456] [457–459]

EPS

Chapter 218

Dimensional analysis and units

Thus far we have mostly not included at all the topic of dimensions.

Dimensions for physical quantities roughly represent some map from the various quantities we have considered so far to a tuple of real numbers (although more likely rational numbers),

$$[-] : ? \rightarrow \mathbb{R}^d \tag{218.1}$$

The common notion used for dimensionality in physics is that of a *graded algebra*, where all of our terms are

There are already a few such gradings that we have naturally imposed on our physics through a variety of requirements. Every

[460, 461]

Some quantities that we have seen so far that have a natural interpretation as a dimensioned quantity have been the lengths, times and masses. We would like our system of dimension to have some space-time dimension for curves,

The action S is given the dimension of an action, $[A]$. As its appearance in $\exp(iS/\hbar)$ should be dimensionless, we will also define $\hbar$ to have the dimension $[A]$.

As an integral, it is probably natural to define the volume form as being of the space-time length dimension $[S^n]$, and if we perform the appropriate space-time split, $[L^{(n-1)}T]$, so that our Lagrangian should be of dimension $[AS^{-n}]$.

Every field may have its own type of dimension, but we will ask that the derivatives with respect to the coordinates have the dimension $[S^{-1}]$. If we require that the kinetic term be without any dimensional factor to it, this means that the average bosonic field component will be of dimension $[A^{1/2}S^{-1}]$

$$[\partial_\mu \phi \partial_\nu \phi] = [S^{-2}][\phi]^2 = [AS^{-n}] \quad (218.2)$$

The Weyl weight of our fields should also give us some information on the

Dimension $[L]$ for curves

One dimension per bundle?

Dimension for the action $[A]$

Graded algebra

Dimension related to the algebra of observables

Chapter 219

The inverse problem

A lot of physics is dedicated to the problem of obtaining measurable data from theoretical entities, such as solving the equations of motion as an initial value problem and then getting the observables from it, but the inverse problem, of getting those theoretical entities from measurements, does not necessarily get quite as much attention.

While already not a trivial matter even for simple theories like physical geometry or classical mechanics, things get much more complicated in the case of field theories like general relativity, where the configuration space is infinite dimensional, and a finite number of measurements will not give us any consequential reduction of the possible configurations that could fit such measurements.

Consider for instance any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. There are $\mathfrak{c}$ such functions. If we want to consider a finite number of measurable quantities from it, we consider a sampling :

Definition 219.1. A *sampling* of a function f for a set of points $(x_i)_{1 \leq i \leq n}$ is the corresponding values of the function at those points :

$$(f(x_i))_{1 \leq i \leq n} \tag{219.1}$$

An example of a sampling in a physical experiment would be for instance to measure the temperature in a room at various points. If we assume that the thermometer measures the temperature at a single point, we have a sampling of the temperature field $T : U \rightarrow \mathbb{R}^+$.

If we stick to the one dimensional case for now, given a sampling of our function, we would like to try to reconstruct our original function given a sampling. That is, find a function f fitting the sampling (f_i) .

For any finite sampling, the cardinality of functions obeying those relations still remain of cardinality $\mathfrak{c}$. We could simply pick the space of all functions satisfying our sampling as union of the spaces of all functions between any two points such that their boundaries match the sampling point, which up to rescaling is simply going to be isomorphic to $C(\mathbb{R}, [0, 1])$, itself with the cardinality of the continuum.

We could consider already for the sampling $(i)_{i \in \mathbb{N}}$ equal to some constant value $(0)_{i \in \mathbb{N}}$ the function 0 as well as any periodic function of period $1/k$ for $k \in \mathbb{N}$

[...]

Therefore we have no hope of finding any exact match for our configuration space without further assumptions.

reproducing kernel Hilbert space

Define regions of the configuration space [dense?]

The *inverse problem*

[462]

219.1 The inverse problem for the Coulomb field

As a first simple case, let's consider the case of the Coulomb field. We have a non-relativistic electric potential ϕ (let's assume $\phi(\infty) = 0$ to discard any remaining gauge freedom).

What we can measure of the Coulomb field is for instance the motion of a charge in it. If we have a selection of spots in which we can measure the passage of this charge at given times separated by a given distance, we could measure its velocity over that interval (for instance by filming its passage in a cloud chamber, the time interval being given by the frame rate).

What we have is its average speed over this interval

approximation as speed at a point

We are therefore left with a selection of the particle's speed at various point of the

Chapter 220

Conventionalism

One fundamental problem of measurements in geometric theories is the decision of what qualifies as the objects of this geometry. If we ignore for a moment the problem of idealizing an object as a point, in the case of spatial geometry, it is the choice of what a straight line is, and in the case of spacetime geometry, it is the choice of a free object.

If we consider what a straight line might be in physical Euclidian geometry, this may be for example the curve defined by a taut piece of string, the boundary of a rod that happens to be a similar figure to such a taut string, or the path traced by a ray of light. If it turns out that the curves defined by those objects do not conform to the axioms of Euclidian geometry, we can react in two different ways.

First we can reject that they indeed define rigid straight lines. A piece of string or a rod is not perfectly rigid, its profile is not a perfectly straight line, lengths can dilate under the influence of temperature and so forth. There are many factors that we can imagine why a given operational procedure would not conform to the definition of the geometric object we wish to use.

In the case of kinematics, this problem can be that of finding something to approximate a free object. Actual objects are never free, as they will always interact even if very faintly with electromagnetic forces for instance, even technically neutral objects like neutrinos will still interact with electromagnetic fields that fill the entire universe via higher order QFT processes. Objects are also never test objects, and they will influence the geometry that we try to measure. They also in general fail to be well-localized objects [...]

This can pose a problem as we may end up with a geometry whose fundamental objects we can never look at directly, which is the case for general relativity. Does it make sense to talk of the behaviour of lines if there exist nothing that we could use to check their behaviour?

But if we can choose freely what belongs to the geometry and what belongs to other physical processes, there is always the possibility of assigning the consequences that deviate from our current theories of geometry and kinematics to other processes, and this

is what was done historically. The deviations from Galilean kinematics were attributed in the Lorentz ether theory to actual physical contractions of the body or speeds to account for the change of time. Or we can attribute the changes of geometry in general relativity to a spin 2 field as done in the Pauli-Fierz theory.

Perturbative gravity around a given metric

The other solution is to change the geometry that we consider

Helmholtz, Reichenbach

The difficulty of these two solutions is where to draw the line, as modern theories would typically not claim that temperature will change the geometry of space even if it changes length measurements, but gravitation does.

While this choice is somewhat arbitrary on a formal level, the criterion for modern theories of geometry is that of general covariance. The action of a "force" upon an object is considered geometric as opposed to physical if there exist some coordinate system in which the motion is considered free [DEFINE FREE MOTION FROM INERTIA ETC KINMATIC GROUP]

This can still be a difficult position, as we never actually have pointwise measurements of such things, as any geometric measurement is actually

Universal forces

What a single observer sees

What a collection of observers in contact see

What the scale of the probing may see

Conventionality of dimensions? [cf. Poincaré]

Chapter 221

Mach's principle

The theory of general relativity did not appear in a theoretical vacuum, but was in part largely inspired by the ongoing debate in the 19th century on the topic of what can be considered absolute and relative when it comes to space and time, primarily from Ernst Mach [463], but also from quite a few other authors such as Poincaré [437], Hertz [464],

[465]

221.1 Newton and absolute space

Absolute space up to Galilean group

Accelerations are real as departure of this absolute space of motions

Newton's bucket problem

If we consider a rotating bucket in an otherwise empty universe, what is it rotating against?

221.2 Mach's principle

Mach's book on mechanics [463] had quite a lot to say about various aspects of mechanics, including talking at length about the issue of absolute and relative space and time, although it is generally considered that, despite what the name of "Mach's principle" implies, he did not seem to have any specific thesis

The ideas of Mach overall do not seem to form a particularly cogent whole [465]. The various ideas to be found in Mach tend to be more criticisms of Newton's theory than a

particularly well thought out alternative to it, so that it is difficult to obtain a specific principle out of it.

[Citations o Mach's various ideas]

This has however motivated plenty of its readers to attempt to make such theories that they thought would align closer with Mach's idea of a completely relative space [Schrödinger etc].

Einstein in particular attempted to model his own theory of general relativity on some of those ideas, trying to formulate the ideas of Mach in a more explicit way (if perhaps idiosyntratic to Einstein), of what is called today *Mach's principle* [466].

221.3 Machian theories

It can be difficult from the various statements of what would constitute a specific Machian mechanics, although fortunately there does exist a few models for it which, if not necessarily physical, at least claim to embody Mach's general idea of what mechanics should be.

221.4 General relativity and Mach's principle

Is general relativity Machian? What does it mean for a theory to be Machian

221.5 Shape dynamics

One attempt to formulate a more Machian alternative to general relativity has been *shape dynamics*.

Chapter 222

General covariance and the hole argument

[467]

We have already seen [X] that there are many deep connections between the diffeomorphism group $\text{Diff}(M)$ and the geometric theories of gravity that are not found in other theories.

If we take a spacetime (M, g) equipped with some coordinate system (x^μ) , There are two kinds of transformations on it that can be considered. We can first consider a diffeomorphism as just a change in the labelling of spacetime points, a purely conventional thing. This is a *passive transformation*. Our passive transformation leaves the underlying spacetime the same, but the chart is moved a certain way by f .

We can also consider a case where we physically transform the given spacetime, for instance by considering a spacetime with the same topology but where all the matter and fields have been moved a given distance. This is an *active transformation*. We keep the same coordinate system on the manifold, but the physical content of the universe is moved.

Q group?

nlab stuff?

$$[M//\text{Diff}(M), \text{Fields}] = [M, \text{Fields}] // \text{Diff}(M) \quad (222.1)$$

[76, 468]

222.1 The hole argument

The distinction between a passive and active transformation rests on the notion that there is some sense in talking about specific spacetime points as having an independent existence. If we have a spacetime with a single object in it, we perform an active transformation by removing that object from a given region and putting it in another region, making it a spacetime different in some sense from the original one.

[the hole argument] [469, 469]

The *hole argument* looks into

Take some spacetime (M, g) with a hole H in it, where H is some open set $H \subseteq M$ in which all other fields vanish. In other words, in H , we have (if we disregard the cosmological constant)

$$G = 0 \tag{222.2}$$

222.2 The hole argument in general

The hole argument does not rest on the specificities of general relativity, or even on geometric theories. It is a rather general statement on the theory of G -equivariant spaces [76]

If we have some G -equivariant

222.3 Absolute objects

[470, 471]

Chapter 223

Principle of equivalence

Chapter 224

The Ether

One fairly esoteric issue that must be discussed is the notion of an *ether*. For almost as long as there has been inquiry of a scientific nature, there has been some notion of an ether, which was somewhat connected to a variety of physical phenomenon, including quite a lot relating more or less directly to topics in general relativity, such as gravity, light propagation, electricity and magnetism. The ether doesn't have a lot of constant characteristics throughout history, but the two main attributes it seems to always possess is that it fills the entire space and has somewhat mechanical properties.

Ether, Αἰθήρ, was originally a divinity of the Greek pantheon, son of Erebus (Ἔρεβος), darkness, and Nyx (Νύξ), the night, themselves born of Chaos, one of the primordial divinities. Ether

From Chaos came forth Erebus and black Night; but of Night were born Aether and Day, whom she conceived and bore from union in love with Erebus.

In natural philosophy, the Ether is originally simply described as some kind of air, cf. Empedocles and Plato :

What is an ether (ex. Aristotle etc, versus the fluid, versus other conceptions)

Aristotle's On the Heavens holds that there are two types of natural motions, rectilinear and circular, and that the ether is the type of substance whose natural

Ether in Cartesian theories

Lorentz ether

Can we call anything in modern times an ether? CMB, fields, etc

Mechanism as explanations [436]

Chapter 225

Ontology

While not a topic commonly discussed in most physics book, there is still quite a large literature regarding the ontological status of various objects in spacetime theories that should at least get looked into.

While the modern point of view with respect to physical theories is that a physical theory is purely a tool to predict observations, it is pretty common for them to have an associated *ontology*, where we assign some of the theoretical objects in the theory a reality

The basic element of spacetime theories to consider is the spacetime itself, as one of the structure that we have defined it by, ie. the set, topological space, manifold, locale, etc. of spacetime. The basic ontological question regarding spacetime is whether spacetime is a *substance*, that is, some object that exists independently of the matter that is occupying it. On a formal level, it is perhaps possible to entirely get rid of spacetime as a basic entity, for instance by instead defining it as the

[472,473]

Chapter 226

Theory of time

Another philosophical aspect of general relativity is that it relates to the various *theories of time* that can be found there. Those theories can be about a variety of different properties of time

226.1 A theory and B theory

A theory and B theory [474]

226.2 Ontology of time

block universe growing block presentism

226.3 Time structure

The notion of a *time structure* relates to the "topology" of time as a whole. The basic notion, going back to antiquity, is what can we say about time globally. Does it have a beginning, an end, does it loop back upon itself, etc

In terms of general relativity, this question is complicated by the lack of a clear split between time and space.

An attempt at a definition of a *time structure* of a spacetime (M, g) is [475]

Definition 226.1. A time-oriented spacetime (M, g) has an *open time structure* if there is a continuous time function $t : M \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}$, $t^{-1}(x)$ is a time slice,

and $p \ll q$ implies $t(p) < t(q)$

[Is that already in the chapter on causality]

Equivalently, a spacetime has an open time structure if it has a collection of time slices F that partition M such that $M/F \cong \mathbb{R}$. [proof equivalence]

226.4 The arrows of time

As a geometric structure, spacetimes can be given a time orientation τ , but this is a purely arbitrary convention. There is however a seemingly preferred direction of time, at least within our own perspective, so that it can be of interest to consider what could be causing this.

There are a few different ways in which certain processes can pick a preferred time direction.

226.4.1 Time arrow of fields

If our spacetime is time orientable, as we've seen [X], it is possible to split the Green function into two parts, the advanced and retarded potential :

$$G = G^+ + G^- \tag{226.1}$$

In various aspects of our theories, there is a preference to disregard the advanced potential to only keep the retarded one.

226.4.2 The thermodynamical time arrow

One of the most commonly observe preference for time orientation is the *thermodynamical time arrow*, which corresponds to the notion that for any observed isolated system in the universe, the averaged entropy measured will always increase along a specific time direction.

Initial conditions

226.4.3 Psychological time arrow

As the final choice of the time orientation

How memory works wrt entropy [476]

Chapter 227

Practical measurement issues

Beyond the many epistemological problems associated with the measurement of observables in general relativity, there are much more prosaic issues associated with the measurement process. All the various abstractions that we presented (as well as some more implicit ones) for the measurement processes can fail in different ways, leading to an experiment's possibility to fail in a variety of different ways, either giving false positives or false negatives.

The types of those kinds of issues can be split into various ways, depending on how we consider them. A common way is to split it by the "source" of the error, between instrument error, human error and natural error, or we could differentiate between the type of error that stems from the fundamental limitations of the measurement and the ones stemming from flawed assumptions, or errors due to systematic error v. random error

The LEP and train

Influence of the experimenter on the measurements (that example from the Manhattan project)

EM Drive

FTL neutrinos

227.1 Instrument error

The way we defined our various measurement apparatus in the previous chapters is rife for error sources, simply from the ways they can fail to obey the assumptions that we set out for them. Every apparatus comes with its very own theory of measurement associated to it, which could be as simple as saying that it is an object of rigid shape and as complicated as requiring an entire theory of electronics, optics, thermodynamics etc to guarantee that

its function does indeed measure what it sets out to measure.

First among these is simply the assumption that we can define a class of identical measurement apparatus.

A very common source of error is from the breaking of our assumptions on congruence. If we assume that we have some actual way of making graduations with congruent intervals, we can simply fail to make it properly. This can be simply due to the limit on precision that we can achieve in their construction, it can be a manufacturing defect, or it can be other circumstances such as thermal dilation etc.

Manufacturing error

Thermal dilation

Bad calibration

One basic observable we saw [ref] was the overlap of two objects when projected onto the vision plane, but this process is dependent on the position of the observer with respect to these two objects. If we have some 2D surface and two 3D objects that are to be projected upon it,

[proof]

This is called a *parallax error*.

227.1.1 Manufacturing error

FTL neutrinos

227.2 Modelling error

A large amount of work on modelling measurements is to simplify many of the actual realities of it to make it both tractable and predictable, but those assumptions are not necessarily justified, and those can later lead to errors that are

A lot of our measurement processes have rested on the assumption of geodesic motion on the part of our objects. Beyond the issue of the approximation of test objects and approximation as curves, real objects are never in geodesic motion. Due to a variety of effects, as we are never in a total vacuum (especially in the modelling by quantum field theory), every object deviates from geodesic motion to some degree.

Let's take the example of light, which is one of the big tool used in general relativity. Light's common modelling is that it is a geodesic null curve carrying a vector representing

its polarization and wavevector, which is parallel-transported along its path.

List of potential issues for geodesic motion for light :

[451] [477] Light refraction from the medium (particularly air)

The atmosphere's refractive index will vary with temperature, pressure and composition, such as its humidity and carbon dioxide. If we have a continuum of differing refractive indices, we have to use gradient index optics.

$$L = \int_p^q n(x, y, z) ds \quad (227.1)$$

In addition to light failing to be perfectly modelled by null geodesics that parallel transports its polarization, light will interact with the medium it goes through.

Modelling the atmosphere as a medium

When light propagates through a medium such as the atmosphere,

Continuous media

In terms of the geometric optic limit, this corresponds

$$\int n(x) \quad (227.2)$$

Diffraction Diffusion Anisotropic media Chromatic aberration Hall effect? Interaction with the CMB Nonlinear EM

Deviation of physical objects from geodesic motion

Problems with measurement apparatus

Far-off influences

Another typical problem that may occur is that most experiments are modelled in some way as being local : we do not consider what is happening outside of a rather small region in which we consider that we have a good idea of what is happening there. There are a few heuristic reasons for this, such

Inverse square law

Mean fields

Cluster decomposition

However, if outside perturbation are large enough or measurements precise enough, there is still the possibility that those influences may be detected.

A famous example of this was the Large Electron-Positron collider (or LEP). [478]

LIGO vibrations

Modelling of the universe

It is very common for experiments to simply assume some laws that are not directly based on direct measurements and the basic model of physics that we consider, as it is typically impossible to get such informations directly. A lot of informations about stars are not directly available but are however extremely important, which is why we generally assume that all stars fall into just a few categories that can be guessed at with the experimental data we have.

Finally, there may simply be deviations from our assumptions that are simply unknown. New physical effects that have yet to be discovered

227.3 Human error

Science is mostly done as a human endeavour. Any experiment is fundamentally designed, assembled and interpreted by humans, and even in the case where we could assume such a process perfect, science is done by a large network of them, as a single person could not do the necessary work to perform all possible experiments for it on their own. Part of the scientific method will have to rely on how they will perform at every level of the process.

There are some obvious basic problems that could occur of simply failing to note properly what the measuring apparatus are reporting. Typically not as dire as hallucinations and delusions, these could simply be inattention errors, clerical errors,

Confirmation bias, Kuhn and normal science

A famous example of this is the *Millikan oil drop experiment* to measure the charge of the electron.

N-rays from Prosper-René Blondlot [479, 480]

in some cases, the problem of reporting could even be outright fraud [Cold fusion, torsion field]

replication crisis

beyond those psychological aspects, experimenters are also physical objects themselves,

no different from the apparatus and other sources of perturbations in any experiment. It is not uncommon for them to influence experiments simply by their own presence.

A famous example of this was Otto Frisch's incident during the Manhattan project [481, p. 575]

"Bending over the Uranium blocks, Frisch involuntarily performed as a neutron reflector and thus provoked a CI [Criticality Incident]. Frisch felt a heat stroke, immediately jumping aside thus stopping the reaction, and, by his estimations, just another couple of seconds, and he could have got a lethal dose."

[482] "Frisch had been stacking blocks of enriched uranium without a reflective assembly - they were "naked", and so he called this the Lady Godiva experiment - when he leaned over the bricks to holler an order to a nearby graduate student. The red light on the neutron counters started glowing continuously, and Frisch realized what was happening : the white cloth of his lab coat and the water inside his chest were reflecting neutrons back into the uranium blocks. Frisch immediately knocked several of the blocks onto the floor with his forearm. After checking the meters, he saw he had already given himself a full day's allotment of radiation. Two more seconds and he would have been dead."

Chapter 228

Knowledge of boundary conditions

As one of the fundamental ingredient in the actual practice of physics is the knowledge of boundary conditions, and that in the real world, we cannot actually know any reasonable amount of them, it can be useful to investigate what the consequences of this can be.

If we stick to fairly well-accepted models, at most what an observer can hope to get as far as knowledge of initial conditions is whatever is contained in the observer's past lightcone at that point, $J^-(p)$, which is very much not the usual assumption of the knowledge of the initial data on something like an initial Cauchy surface if our spacetime is well-behaved enough, or at least a small spacelike region. Even if we try to use our knowledge of the past lightcone to try to reconstruct the initial data, if our spacetime is past-distinguishing, any other point q that is causally independent from p will depend on some unknown data (the data will "seep in" from outside the past light cone), as the past light cone of a causally independent point q will contain points not present in $J^-(p)$. We also can't consider a small spacelike hypersurface S contained in $J^-(p)$, as its domain of dependence will necessarily be inside $J^-(p)$ itself [prove].

Therefore, for any attempt at knowing the initial conditions of a physical system, we will have to make some simplifying assumptions regarding what kind of informations come in from outside our past light cone.

The simplest assumptions we can make is that the content of the past light cone is the only thing that we will consider. While this is an assumption that can work somehow well in the case of extended objects, where we can assume that all our objects are the ones present in the past light cone, this logic does not extend simply to fields, as we cannot really consider the absence of fields outside the light cone. At best we can try to assign it a value, but the simplest solution to simply assign a zero value to the field outside the light cone is also not reasonable, as it would likely lead to a discontinuity of the field.

Mean field, other such hypothesis?

[...]

228.1 Approximating objects

The structure of actual objects would be quite difficult to model perfectly even if we had perfect knowledge of them. Even assuming that our modern theories are entirely accurate as to the general structure of matter, this would mean that the stress energy tensor would resemble something like a smattering of localised peaks of

228.2 Faraway influences

An extremely difficult aspect of physics is that of considering the influence of faraway objects. If we were to consider the actual totality of the universe in terms of atoms modelled as some kind of point particles, the typical estimate is that there are about 10^{80} of them [Eddington?], which would give us about as many coupled nonlinear PDEs to solve. This number would not go down by a significant enough number if we tried one of our previous trick of simplifying large number of objects as single objects such as stars ($\approx 10^{22}$)

For tractability issues, as well as our general lack of knowledge on the exact characteristics of all those objects, we need some method to condense the influence of all those objects on the behaviour of the system we study.

A common way to do this is to simply ignore every object beyond a certain range. In the Newtonian approximation [should do PPN], the influence of those objects (at least gravitationally) is of the form

$$\ddot{x} = \sum_i \frac{GM_i}{r_i^2} \quad (228.1)$$

If we assume a maximal mass

[...]

Inverse square law etc

Despite our current experiences with various phenomenon, it is not completely ruled out that objects of arbitrarily far distances would provoke arbitrarily high disturbances [Wave of death example]. It should therefore only be used as a heuristic that we can model the configurations beyond a certain distance as either a vacuum or a mean field of a given value, which could turn out to be false.

It is unfortunately pretty rare that we have analytical estimates of the error produced by objects at a given distance, as we usually only have its asymptotic behaviour, so that our heuristic will usually have to rely on the size of a given order rather than having a hard boundary.

Chapter 229

Probabilities and errors

All the difficulties that we have seen with regard to measurement have only given us at best the possibilities of what the true measure could be. If we can estimate the maximals of a parallax error, the most that graduations could dilate under thermal stress, etc, we could get a range of the true values of the measurement. However, this is assuming that such bounds are known to us, and if we only obtain boundaries from the very worst case scenarios every time, we may have extremely large intervals that could be near meaningless in a falsifiability sense.

If we cannot gain any certainty on the various bounds that a source of error may have, we are left in a much worse position where performing the calculations could become impossible. If our theory would allow some arbitrarily large perturbation to occur for some given boundary condition that we cannot ascertain, our predictions for some observable could conceivably be any value in $\mathbb{R}$, giving it a null use for falsifiability.

In the case of our field theories for instance, we could consider the case [X] of the thunderbolt metric, where a perturbation coming from outside our light cone could reach us with such a force as to send the curvature diverging. We could consider the case of electromagnetic waves of arbitrary intensities from infinity. Or we could imagine some thermodynamical configurations of such unlikely probability that

[example of highly unlikely configuration?]

Example of the temperature in an experiment

Hypothesis on some random parameters of an experiment

229.1 What is probability

What are probabilities, esp. in regard to physics and measurement

Keynes

Carnap

Von Mises

No interpretation for probabilities

Popper : probabilities are not falsifiable

Probability as insufficient information

Can we know when we have the same initial conditions up to random error

Probability as applying only to sequences of experiments

229.2 Classical probability

Kolmogorov probability

229.3 Quantum probability

quantum probability

quantum logic

commutativity of measurements

229.4 The theory of error

[483–487]

Chapter 230

All together

While not all physical theories are required to follow the same model, as the content of a physical theory are still a fairly contentious issue [431, 488], here is an outline of the steps we can take to apply a physical theory.

If we have some set of theories $\{T_i\}$

The various assumptions we have collected together

Along with these theories, we consider various other sets of assumptions. We consider the various purely mathematical assumptions that we have made without proof, the mathematical assumptions that we have made with the belief that any deviation from the correct result will be below the reach of our measurements, the theory of our measuring apparatuses, the pre-theories with which we

Part XIII

Experiments and Applications

Now that we have just about every element about both general relativity as a mathematical formalism and as it relates to the physical world, it is time to actually apply this to physical experiments.

Unlike most other chapters, where I have strived to keep general relativity rigorously, practical applications may not always have an easy mapping to the language of general relativity, and in some cases this may not have much impact on actual results. It is difficult to know exactly what size this may have, as typical derivations for this only gives the asymptotic behaviour of the error and not its size, but a system as complex as the observation of such diverse and complex objects with various instruments and unknown conditions makes it impractical to treat everything in the same formalism as the theoretical part.

For this reason I may switch between general relativity, linearized gravity, parameterized post-Newtonian gravity and Newtonian gravity depending on the circumstances, and a given experiment may even include a variety of those theories together. This is therefore another assumption that we must add to the context of the experiment that could be wrong. The various extra assumptions will usually be something of the form "The error generated by this approximation is smaller than the expected error from various other considerations". As we will typically not actually compute the size of the error, those will be some potential sources of errors in our experiments,

$$\frac{\begin{array}{c} \text{Theory} \\ \hline \vdash \text{Obs}_1 \quad \vdash \text{Obs}_1 \rightarrow \text{Obs}_2 \end{array} \quad \vdash \text{Valid approximations}}{\vdash \text{Obs}_2}$$

This type of behaviour is quite common practice indeed, as quite often experiments will attempt to get by on a reasonably simple model and only try more elaborate ones if this analysis fails to replicate the experiment, although this could have the problem of an experiment that succeeds with the simple model but might fail with the "real" model.

Since this section will involve physical measurements, we will put back in every SI constant in the equations, with the measured values of

$$\begin{aligned} c &= 299\,792\,458 \text{ m s}^{-1} \\ \kappa &= 2.076\,579 \times 10^{-43} \text{ s}^2 \text{ m}^{-1} \text{ kg}^{-1} \\ G &= 6.674\,08(31) \times 10^{-11} \text{ s}^{-2} \text{ m}^3 \text{ kg}^{-1} \\ \hbar &= 1.054\,571\,800(13) \times 10^{-34} \text{ J s} \\ M_{\oplus} &= 5.9724(3) \times 10^{-24} \text{ kg} \\ R_{\oplus} &= 6.3781 \times 10^{-6} \text{ m} \\ M_{\odot} &= 1.9885 \times 10^{-30} \text{ kg} \\ R_{\odot} &= 6.957 \times 10^{-8} \text{ m} \end{aligned}$$

Our Einstein field equation is defined in this context by

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad (230.1)$$

with

$$\kappa = \frac{8\pi G}{c^4} \quad (230.2)$$

Every spacetime vector in a basis composed of a timelike vector and $(n - 1)$ spacelike vectors gets a factor of conversion of c :

$$(v^0, \vec{v}) \rightarrow (cv^0, \vec{v}) \quad (230.3)$$

where the timelike component v^0 has the dimension of time $[T]$ and the spacelike components v^i have the dimension of length $[L]$. This leads to the actual measurements of

Does the metric tensor get c or the vectors

There may also occasionally be other systems of units used, either due to historical reasons, such as the cgs (centimetre–gram–second) system which was widely in use until relatively recently, or domain-specific units such as astronomy, particle physics, etc.

Dicke framework [355, 489]

Chapter 231

Gravitational theories in terms of PPN

Most of those experiments as they relate to verifying the validity of our gravitational theories can give us bounds on the values of the PPN approximation of each theory.

list of theories :

GR, Einstein-Cartan, Whitehead, Weyl, Dicke, Einstein-Nordström, vector theories, Lovelock, Hoyle

|

Table 231.1: PPN values for various theories

Chapter 232

The Earth as an experimental setting

As almost every non-astronomical gravitational experiment is done using either the Earth's gravitational field, or is greatly influenced by it, we are first going to look at what the geometry around the Earth is expected to be. [490]

The pre-theory that we have of Earth, from various fields like classical mechanics, geology, geodesy, astronomy, aeronautics, etc, is that Earth is a roughly spherical object. This is known from evidence going all the way back to Erathostene [491] and Ptolemy [492], and in modern times by geodesic data such as [X], etc

[some experimental proof of Earth's shape]

Its density is about isotropic (ignoring the ellipsoidal shape) [geological data]

The Earth is also rotating. As far as the kind of frame an observer on the Earth's surface can tell, [...]

[Foucault's pendulum etc]

A natural model for such an object is to pick a solution which externally resembles some form of the Tomimatsu-Sato metric outside of its mass.

$$ds^2 = \tag{232.1}$$

As the Earth is

If we neglect the Earth's flattening and multipolar moments, we can simply consider Earth as being modelled by a Schwarzschild metric above its surface

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \quad (232.2)$$

With the Earth's total mass given by its Komar mass

$$M = \int \quad (232.3)$$

[Does this correspond to the mass as computed via other means]

232.1 The mass of the Earth

The mass of the Earth can be difficult to estimate outside of the context of a gravitational theory, making it somewhat fraught to estimate to do our experiments without some prior assumptions.

Seismology estimations

Geological structure

232.2 The PPN approximation of Earth

If we deviate from this very simplified model, we will remain unable to say much about the Earth. It will therefore be necessary to switch to the PPN model to simplify calculations.

The basic model of the Earth is the *reference ellipsoid*, the one currently in use being the WGS 84 [493, 494], an ellipsoid of semi-major axis of 6 378 137 m and inverse flattening of 298.257223563,

Coriolis, centrifugal, Eotvos effect

Hartle–Thorne metric

232.3 The Moon and Sun

The Earth does have a rather large perturbation to its gravitational field nearby in the form of the Moon, with a smaller one given by the sun (anomalies from other celestial bodies are generally ignored as far as the gravity on Earth is concerned).

We will see later on the exact trajectories of the sun and moon around the Earth, but for now we will only require their instantaneous angular position, distance, and momentum.

Both the Sun and the Moon have negligible velocities in the Earth's frame compared to the speed of light (about 1 km s^{-1} and 30 km s^{-1} , less than 0.01% the speed of light), so that we can probably ignore them. If we consider their velocities as roughly constant for the duration of the experiment, we can see that the potentials involving them are

$$A = \quad (232.4)$$

$$\Phi_3 = \quad (232.5)$$

$$= \quad (232.6)$$

The Moon is very nearly spherical (with a flattening of 0.0012), with a radius of about 1737.4 km, and a total mass of about $7.342 \times 10^{22} \text{ kg}$. Its distance from the Earth varies from about 362 600 km to about 405 400 km.

The Sun is also very nearly spherical, with an almost zero flattening, and a radius of about 695.700 km, a total mass of about $1.9885 \times 10^{30} \text{ kg}$.

Compute the various potentials and currents

232.4 Gravitational anomalies

[495] Any term of the gravitational potential which deviates from the International Reference Ellipsoid is termed a *gravitational anomaly*. In addition to the gravitational interference of the sun and moon, this contains every deviation from the approximations that we previously discussed.

Normal gravity :

$$g_n = g_e(1 + \beta_1 \sin^2(\lambda) + \beta_2 \sin^2(2\lambda)) \quad (232.7)$$

g_e is the standard gravitational acceleration $9.780\,327 \text{ m s}^{-2}$ (gravity at the equator),

- Tidal correction
- Terrain correction
- Free-air correction
- Bouguer plate correction
- Isostatic correction

232.4.1 The structure of the Earth

Some of the large anomalies are due to Earth's geological structure, as these are a place where we can find some of the largest differences in mass distribution.

Some stuff about Earth's crust and its various layers

232.4.2 Terrain correction

Terrain correction

Density contrast

The same density contrast at different depths has variations of the potential as Δh^2

Free air correction

Bouguer plate correction

Isostatic correction

Show that the closer we are to the experiment site, the more important the distribution of mass is

Classical case :

$$\Delta\Phi = 4\pi G\rho \quad (232.8)$$

Mass anomalies

Harmonic representation

While it is a good thing to keep in mind all these elements, one important thing to remember is that a lot of the data concerning the gravitational profile of the Earth were measured partly by the gravitational data itself in a classical model [via gravimetry?], so that it may be ill advised to try to use these data for another model, although this will likely not be of much importance in the Earth's case.

[What does a gravimeter measure and can it be translated into a PPN model?]

Compare this with the classical orbit.

$$m\ddot{x}_i = G \frac{Mm}{\|\vec{x}_1 - \vec{x}_2\|^2} \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|} \quad (232.9)$$

$$-M\ddot{x}_i = -G \frac{Mm}{\|\vec{x}_1 - \vec{x}_2\|^2} \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|} \quad (232.10)$$

Reduced mass and center of mass :

$$m_{12} = \frac{mM}{M + m} \quad (232.11)$$

Displacement :

$$\vec{r} = \vec{x}_2 - \vec{x}_1 \quad (232.12)$$

Center of mass :

$$\vec{R} = \frac{\vec{x}_1 + \vec{x}_2}{2} \quad (232.13)$$

Chapter 233

Physical Euclidian geometry

The origins of geometry go back to prehistoric times, so that it can be difficult to know exactly what kind of problems motivated its invention. People before the iron age were not particularly interested into treating geometry as much more than a set of practical methods, at least in writing. The few written traces on the topic that we have tend to be only various formulas and methods to be applied, without much in the way of justification.

Even works meant to motivate the rules of geometry, like Plato's *Meno* [496], tend to rely more on their intuitiveness than on experimental verification.

All the following experiments rely entirely on the propagation of light for their measurements, meaning that up to non-gravitational effects, this will only serve to measure the conformal structure of spacetime. Furthermore, these were all conceived in pre-relativistic times, so that the assumption for all of these was roughly equivalent to the Galilean approximation case.

233.1 Gauss' mountains

There exists a common myth regarding non-Euclidian geometry that Gauss, one of the early mathematician to work on non-Euclidian spaces, attempted to physically measure the curvature of space by measuring the angle formed by three mountains in Hanover. This is very likely to be a myth [497], as nothing suggests it in Gauss' original papers, but he did measure a various number of triangles during his geodesic mission, and he did use the example of his biggest measured triangle as an illustration of the triangle postulate [498].

However, while the survey itself was not for the purpose of testing the accuracy of the triangle postulate, we can still use it to test out this hypothesis.

Gauss was tasked by the Royal Ministry of the Interior to survey the land of the Kingdom

of Hanover from 1821 to 1825. [...]

The exact instruments used are not necessarily well documented, but one specific instrument we do know of during Gauss' survey is the *heliotrope*

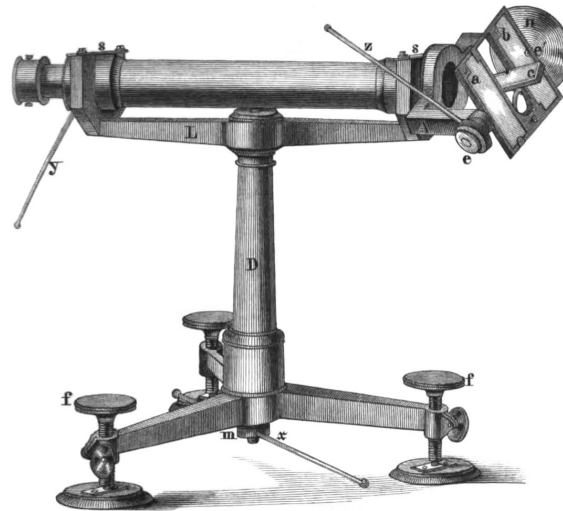


Figure 233.1: Gauss' heliotrope

The famed triangle defined in this survey is the one defined by three elevated points, Hoher Hagen (a volcanic hill near Göttingen), Brocken (a mountain peak) and Großer Inselsberg (another mountain). Each of these elevated point is in view of the other despite their great distance, making them ideal spots for such a large triangle.

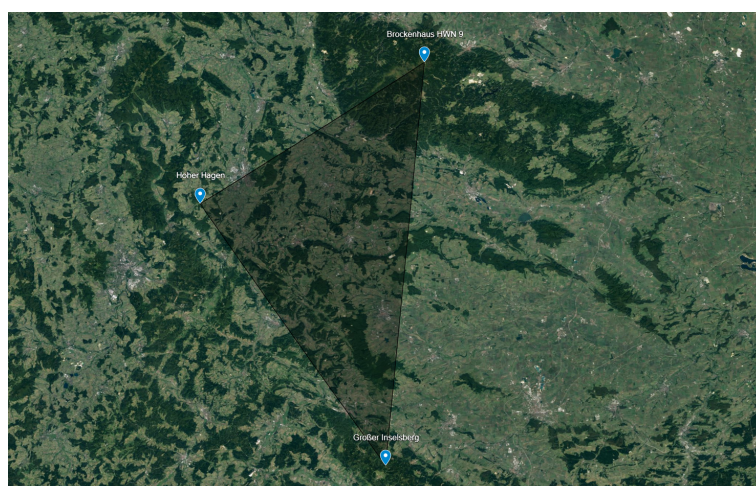


Figure 233.2: Area of Gauss' triangle

The angles measured by Gauss are given as follow :

Location	Angle
Hoher Hagen	$86^{\circ}13'38.366''$
Brocken	$53^{\circ}6'45.642''$
Inselberg	$40^{\circ}39'44.473''$

Theodolite measurements : Index error, Horizontal axis error, Collimation error [451]

Gauss' method of repeating measurements, mean error, etc

mean error only indicates the error by the user, not so much the distortion due to distances

The famed Gauss triangle is a bit hard to put into context with the rest of his triangulations, as it is much larger than all other triangles, and therefore has likely a larger error : if we assumed for now that the space over Hanover is Euclidian, and that the excess angle is entirely due to error, the second largest triangle of Falkenberg-Deister-Lichtenberg has the second largest error at around $8''$, while the smaller ones tend to be more under $4''$.

233.2 Lobachevski stellar experiment

As one of the first mathematician to hold non-Euclidian geometry somewhat seriously, Lobachevski attempted to come up with some experimental way of checking the geometry of space, whether Euclidian or his own Lobachevskian geometry (what we refer to nowadays as hyperbolic geometry).

Lobachevski's geometry is the geometry of hyperbolic space $\mathbb{H}^3$. As a spacetime, it would therefore be the ultrastatic spacetime $(L \times \mathbb{H}^3, -dt^2 + g)$

[499, 500]

Chapter 234

Free fall

The most basic of all gravitational experiment is that of *free fall*, where we simply consider the motion of an object under only gravitational forces.

234.1 Classical case

The basic classical treatment of this in the Earth's gravitational field is to look at the Newtonian gravitational acceleration of an object

$$\vec{a}_g = -G \frac{M_\oplus}{r^2} \vec{e}_r \quad (234.1)$$

The position of an object in free fall will then be the integration of this acceleration. If an object is at a height h from the ground, it is at $R_\oplus + h$ from the center of the Earth

$$\ddot{\vec{x}}(t) = -G \frac{M_\oplus}{(R + h)^2} \quad (234.2)$$

[...]

We typically only look at the case of free fall on Earth in the approximation of a small variation of the height h , as for any experiment done in the atmosphere, $\Delta h \ll R_\oplus$, since objects are not expected to leave the atmosphere or go below ground level. This will not be true for some types of free fall such as the case of a comet landing on Earth for instance, but if we can assume that the entire experiment will be constrained between two very close together heights,

[...]

Another way of considering the case of free fall is to consider

234.2 Friction

234.3 Schwarzschild case

Free fall as it was done historically is done by considering the gravitational field of Earth, which is at its simplest modelled by the Schwarzschild metric

If we consider a free-falling object with no initial velocity, we can simply take the radial geodesics as the paths of our objects, as we have $L = 0$:

$$c^{-2}\dot{r}^2 + \left(1 - \frac{r_S}{r}\right) = E^2 \quad (234.3)$$

The initial conditions of this differential equation will be that our object is at a distance equal to the radius of the Earth plus the height of this object from that level, $r_0 = R_{\oplus} + h_0$. If we recast our r variable into $h = r - R_{\oplus}$, with initial condition $h_0 = r_0 - R_{\oplus}$, we can more easily solve this by a Taylor expansion

$$\frac{1}{R_{\oplus} + h} = \frac{1}{R_{\oplus}} - \frac{h}{R_{\oplus}^2} + \mathcal{O}(h^2) \quad (234.4)$$

$$c^{-2}\dot{h}^2 + \left(1 - \frac{r_S}{R_{\oplus} + h}\right) - E^2 = 0 \quad (234.5)$$

$$\approx c^{-2}\dot{h}^2 + \left(1 - \frac{r_S}{R_{\oplus}} - h \frac{r_S}{R_{\oplus}^2}\right) - E^2 \quad (234.6)$$

If we take the proper time derivative of this expression,

$$\frac{d}{dt} \left[c^{-2}\dot{h}^2 + \left(1 - \frac{r_S}{R_{\oplus}} - h \frac{r_S}{R_{\oplus}^2}\right) - E^2 \right] = c^{-2}\ddot{h}\dot{h} - \dot{h} \frac{r_S}{R_{\oplus}^2} \quad (234.7)$$

$$= \dot{h} \left[c^{-2}\ddot{h} - \frac{r_S}{R_{\oplus}^2} \right] \quad (234.8)$$

Therefore either the velocity of the object is 0, or it has to obey the approximate equation

$$\ddot{h} = \frac{r_S}{c^2 R_{\oplus}^2} \quad (234.9)$$

This is indeed the Newtonian equation of free fall, which for vertical motion with the approximate constant value of the gravitational field $\vec{F} \approx -mg\vec{e}_z$,

$$\ddot{z} = g \quad (234.10)$$

$$\frac{r_S}{c^2 R_\oplus^2} = \frac{2GM_\oplus}{c^4 R_\oplus^2} \quad (234.11)$$

Check the c 's

Case of a thrown object

234.4 Rotating case

The actual experiment of free fall as it is done on Earth however is not quite perfectly modelled by the Schwarzschild metric. Beyond the deviations from spherical symmetry, Earth is a rotating system, and our laboratory setting is typically a rotating frame on its surface.

Coriolis, Eotvos, centrifugal, etc

Difference in latitude

Example : Galileo's experiments on inclined planes

Chapter 235

Pendulum experiments

One of the main way to perform gravitational experiments in pre-modern time was via the use of a *pendulum*. The pendulum is a mass m at the end of a string affixed to a central point p , standing in the Earth's gravitational field. We usually consider the following simplifications for the pendulum :

- The string itself has negligible mass
- The string is rigid
- The Earth's gravitational field is a constant downward force
- There are no friction forces involved.

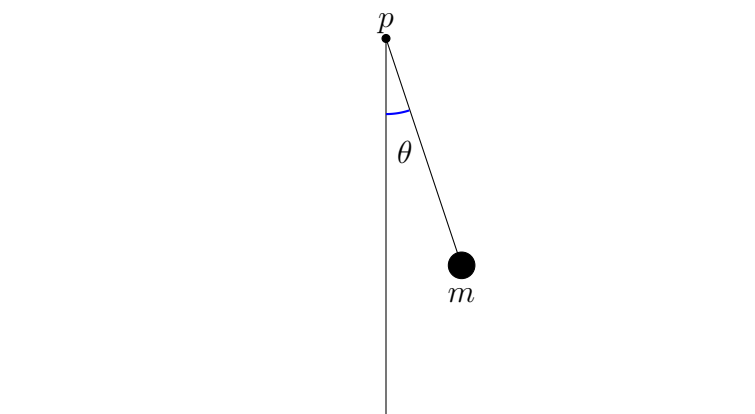


Figure 235.1: The simple pendulum

If we consider all our approximations, we end up classically with the *simple harmonic oscillator* (or SHO), which is under the force of the string's tension and the gravitational field.

$$m\ddot{x} = -mg \dots \quad (235.1)$$

This system is invariant under rotations along the z axis, so that by conservation of angular momentum, the pendulum will stay in the same plane. By the rigidity of the string, we can also assume a constant distance between the fixed point p and the point mass. Using these constraints, we can define coordinates so that the pendulum is only dynamic in one of them, the angle between the string and the vertical θ . In these coordinates,

$$\ddot{\theta} + \frac{g}{l} \sin(\theta) = 0 \quad (235.2)$$

This differential equation has an exact solution as a special function, if we rewrite it first as

$$\frac{d}{dt} \left[\frac{1}{2} \dot{\theta}^2 - \frac{g}{l} \cos(\theta) \right] = 0 \quad (235.3)$$

Therefore giving us the integral of motion

$$\frac{1}{2} \dot{\theta}^2 - \frac{g}{l} \cos(\theta) = C = -\frac{g}{l} \cos(\theta_0) \quad (235.4)$$

If we assume $\dot{\theta}(0) = 0$

$$\dot{\theta}^2 = \sqrt{\frac{2g}{l} (\cos(\theta) - \cos(\theta_0))} \quad (235.5)$$

$$\dot{\theta}^2 = \sqrt{\frac{2g}{l} (\cos(\theta) - \cos(\theta_0))} \quad (235.6)$$

[...]

Jacobi elliptic function

$$\theta(t) = 2 \arcsin(\sin(\frac{\phi_0}{2}) \text{cd}(\omega_0 t, \sin(\frac{\phi_0}{2}))) \quad (235.7)$$

Small angle approximation

Method by Lagrangian constraints :

$$L = \frac{m}{2} \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{mgl}{2} \quad (235.8)$$

The approximations we used have a variety of error sizes. A massive string would have its own force acting on it, except that this force would be applied halfway up the string instead of at its end, giving an equivalent system except with a shifted length depending on the weighed average : $l' = (\frac{m_s+2m}{2m})l$, so that the error will be roughly $\approx m_s/m$. An elastic string stretching linearly with stress [...]

If we consider the gravitational field to vary with distance to the Earth's barycenter, the order of the error for the frequency is about

$$\Delta T \approx \frac{l}{2R_{\oplus}} \sqrt{\frac{l}{g}} \quad (235.9)$$

Friction forces

Now if we consider some additional gravitational source $\Delta\Phi$ in addition to the field of the Earth, at a typical distance $\approx d$ and of total mass M_2 , what will be the change to the motion of the pendulum, and how does it compare to the approximations that we made to the pendulum?

Does it have any influence if we only consider the angle

The simple pendulum also has a general relativistic version, although with some complications due to the notion of rigidity of the the string [501,501].

Pendulum defined by rigid motion

Chapter 236

Schiehallion experiment

If we consider some region on Earth that deviates in some major way from the assumption of spherical symmetry (or the oblate spheroidal approximation), we should see some deviations from the usual gravitational behaviour of matter.

This can be done for instance by looking at the gravitational behaviour near a mountain. If we know the mass distribution ρ of a mountain, in the Newtonian approximation, the gravitational field should be given by

$$\Phi = \int_{\oplus} \rho_{\oplus}(r) + \int_{\text{mountain}} \rho \quad (236.1)$$

We therefore have some deviation $\Delta\Phi$ from the usual gravitational field, which should attract objects towards the mountain.

This experiment was performed with the mountain of Schiehallion in Perthshire, Scotland, in the 1770's. From 1775 to 1776, a survey team was sent to perform a variety of measurements to attempt to decompose the mountain into a set of vertical prisms [502] [...]

Determination of the gravitational attraction of each prism until 1778

Finding : the attraction of a plum-bob by the Earth would be 9933 times larger than that of the mountain at the north and south stations, if they have the same density.

"Since the actual deflection of 11.6 implied a ratio of 17,804:1 after accounting for the effect of latitude on gravity, he was able to state that the Earth had a mean density of $\frac{17,804}{9,933}$, or about $\frac{9}{5}$ that of the mountain

Further experiments by Cavendish

Survey by John Playfair in 1811 for the rock strata, suggesting a density of $4.560 \text{ kg} \cdot \text{m}^{-3}$

to $4.870 \text{ kg} \cdot \text{m}^{-3}$

Chapter 237

Tidal forces

The earliest evidences for gravitational tidal forces were from the tides themselves. It can be observed in various regions that water levels from large bodies of water will rise and fall twice a day. Furthermore, the highest and lowest levels of the tide can be observed to vary over the span of a month, reaching their widest and narrowest separations about twice a month each. The widest is called the *spring tide*, while the narrowest is called the *neap tide*.

From Pliny's *Naturalis Historia* book 2

XCIX. About the nature of bodies of water a great deal has been said. But the rise and fall of the tides of the sea is extremely mysterious, at all events in its irregularity; however the cause lies in the sun and moon. Between two risings of the moon there are two high and two low tides every 24 hours, the tide first swelling as the world moves upward with the moon, then falling as it slopes from the midday summit of the sky towards sunset, and again coming in as after sunset the world goes below the earth to the lowest parts of the heaven and approaches the regions opposite to the meridian, and from that point sucking back until it rises again; and never flowing back at the same time as the day before, just as if gasping for breath as the greedy star draws the seas with it at a draught and constantly rises from another point than the day before; yet returning at equal intervals and in every six hours, not of each day or night or place but equinoctial hours, so that the tidal periods are not equal by the space of ordinary hours whenever the tides occupy larger measures of either diurnal or nocturnal hours, and only equal everywhere at the equinox. It is a vast and illuminating proof, and one of even divine utterance, that those are dull of wit who deny that the same stars pass below the earth and rise up again, and that they present a similar appearance to the lands and indeed to the whole of nature in the same processes of rising and setting, the course or other operation of a star being manifest beneath the earth in just the same way as when it is travelling past our eyes.

Moreover, the lunar difference is manifold, and to begin with, its period is seven days: inasmuch as the tides, which are moderate from new moon to half-moon, therefrom rise higher and at full moon are at their maximum; after that they relax, at the seventh day being equal to what they were at first; and they increase again when the moon divides

on the other side, at the union of the moon with the sun being equal to what they were at full moon. When the moon is northward and retiring further from the earth the tides are gentler than when she has swerved towards the south and exerts her force at a nearer angle. At every eighth year the tides are brought back at the hundredth circuit of the moon to the beginnings of their motion and to corresponding stages of increase. They make all these increases owing to the yearly influences of the sun, swelling most at the two equinoxes and more at the autumn than the spring one, but empty at midwinter and more so at midsummer. Nevertheless this does not occur at the exact points of time I have specified, but a few days after, just as it is not at full or new moon but afterwards, and not immediately when the world shows or hides the moon or slopes it in the middle quarter, but about two equinoctial hours later, the effect of all the occurrences in the sky reaching the earth more slowly than the sight of them, as is the case with lightning, thunder and thunderbolts.

Origin of tidal theory

In a classical theory, the tide can be understood by considering the dynamics of the masses of fluids on Earth. Let's consider a spherical mass of water with the moon rotating around it. The motion of the moon is extremely complicated [503], but as a first approximation we can consider it as a classic two body problem. Our total system is a ball of water in hydrostatic equilibrium with the moon rotating around their center of mass :

$$\frac{D\vec{u}}{Dt} = -\vec{\nabla}w + \vec{g} \quad (237.1)$$

$$\text{div}(\vec{u}) = 0 \quad (237.2)$$

Laplace's tidal equation [504]

$$\frac{\partial}{\partial t}\vec{u} + 2\vec{\Omega} \times \vec{u} = -\vec{\nabla} \left(\frac{p}{\rho} + \Phi + \chi \right) \quad (237.3)$$

$$\text{div}(\vec{u}) = 0 \quad (237.4)$$

u the fluid velocity, p the fluid pressure, Φ the grav potential, χ the centrifugal potential

Moon and Earth system

GR system for a fluid

Relativistic Euler equations

Tidal tensor

Chapter 238

The two body problem

In the relativistic two body problem, we consider the motion of two bodies such that one body is particularly heavier than the other, $M \gg m$, so that the contribution of the second body to the stress-energy tensor can be neglected and we can consider it as a test body (it is in fact exactly solvable in classical gravity but will not be so in the relativistic case).

The two body problem here will also be a case of solving Schwarzschild geodesics, but in this case we consider the bound geodesics with $L \neq 0$.

Geodesics

Kepler laws

Theorem 238.1. The Kepler laws for a gravitational two-body problem are

1. The orbit of a planet is an ellipse with one of the foci being the center of mass of the system.
- 2.
- 3.

Version with Einstein-Infeld-Hoffmann

Chapter 239

The Cavendish experiment

The Cavendish experiment, performed between 1797 and 1798 by Henry Cavendish,

The purpose of the Cavendish experiment was to measure the strength of the gravitational constant G independently of the mass of the Earth, which was not known at the time.

Torsion balance

[505]

[506]

Bricks of lead part of the experiment

Chapter 240

The equivalence principle

[507]

One of the fundamental principle behind general relativity, as well as most metric theories of gravity, is the equivalence principle, which comes in two variety :

Weak equivalence principle : The trajectory of a test particle only subject to gravity will only depend on its initial position and velocity.

Strong equivalence principle : The outcome of any local experiment in a freely falling frame is independent of its position and velocity.

Problem with the QM case due to the dependence on the smearing of the wavefunction depending on its mass

From [354]: WEP : If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition

Uncharged test body : electrically neutral body that has a negligible self-gravitational energy (estimated using Newtonian theory) and small enough in size so that its coupling to inhomogeneities in external fields can be ignored

Local nongravitational test experiment : any experiment performed in a freely falling laboratory that is shielded and is sufficiently small that inhomogeneities in the external fields can be ignored throughout its volume, in which self-gravitational effects are negligible

Einstein equivalence principle : WEP is valid, the outcome of any local nongravitational test experiment is independent of the velocity of the freely falling apparatus, the outcome of any local nongravitational test experiment is independent of where and when in the universe it is performed

Equivalent(?) to the postulates of metric theories of gravity : Spacetime is endowed with a metric g , the worldline of test bodies are geodesics of that metric, in local freely falling frames called local Lorentz frames, the nongravitational laws of physics are those of special relativity.

Proof : 'The validity of WEP endows spacetime with a family of preferred trajectories, the world lines of freely falling test bodies. In a local frame that follows one of these trajectories, test bodies have unaccelerated motions. Furthermore, the results of local nongravitational test experiments are independent of the velocity of the frame. In two such frames located at the same event, q , in spacetime but moving relative to each other, all the nongravitational laws of physics must make the same predictions for identical experiments, that is, they must be Lorentz invariant. We call this aspect of EEP Local Lorentz Invariance (LLI). Therefore, there must exist in the universe one or more frame. In two such frames located at the same event p , in spacetime but moving relative to each other, all the nongravitational laws of physics must make the same predictions for identical experiments, that is, they must be Lorentz invariant. We call this aspect of EEP Local Lorentz Invariance (LLI). Therefore, there must exist in the universe one or more second-rank tensor fields that reduce in a local freely falling frame to fields that are proportional to the Minkowski metric, $\phi^i(p)\eta$, where ϕ^i are scalar fields that can vary from event to event. ...'

(cf. Will p. 23)

Schiff's conjecture : Any complete, self-consistent theory of gravity that embodies WEP necessarily embodies EEP. Arguments for it : Dicke (1964), Nordtvedt (1975), Haugan (1979)

240.1 The Eötvös-Dicke experiment

The Eötvös experiment was one of the largest test of the equivalence principle

objects of various materials dropped in a vacuum

Eötvös experiment :

From Dicke :

"Before discussing this, it is important to note that gold and aluminium differ from each other rather greatly in several important ways. First, the neutron to proton ratio is quite different in the two elements, varying from 1.08 in aluminium to 1.5 in gold. Second, the electrons in aluminium move with non-relativistic velocities, but in gold the k -shell electrons have a 12 per cent increase in their mass as a result of their relativistic velocities. Third, the electromagnetic negative contribution to the binding energy of the nucleus varies as Z^2 and represents 1/2 per cent of the total mass of a gold atom, whereas it is negligible in aluminium. In similar fashion, the virtual pair field around the gold

nucleus would be expected to represent a far bigger contribution to the total energy than in the case of aluminium. Also the virtual pion field, and other virtual fields, would be expected to be different in the two atoms. We would conclude that in most physical aspects gold and aluminium differ substantially from each other and that the equality of their accelerations represents a very important condition to be satisfied by any other theory of gravitation.”

240.1.1 The environment

For an extremely precise experiment like the Eötvös experiment, it is interesting to consider the possible gravity anomalies that may be involved.

The Eötvös experiment took place in the Faculty of Science of the University of Budapest, between 1906 and 1909.

Geological study of the region, laboratory

240.1.2 The experiment

Chapter 241

Astrophysics

For a book about general relativity, a discipline quite steeped in astronomy, we have so far not considered what a star is. As the behaviour of stars will be extremely important later on, let's spend a little time to find out how stars actually work.

Weinberg's book?

Collapse of clouds of hydrogen etc

241.1 Stellar structure

Let's consider the case of some spherically symmetric matter distribution in hydrostatic equilibrium as a first model for our stars.

If we consider a globally hyperbolic static spacetime (M, g) (or with a slowly varying metric with respect to some timelike vector field), we can define its Komar mass within a given region

$$M = \int_U (2T(u, \xi) - \text{Tr}(T)g(u, \xi)) d\mu[g] \quad (241.1)$$

$$\stackrel{\text{coord}}{=} \int_U (2T_{\mu\nu} - T^\sigma{}_\sigma g(u, \xi)) u^\mu \xi^\nu d\mu[g] \quad (241.2)$$

If we consider a spherically symmetric distribution of gas,

[508, 509]

Star of radius r and mass M_r

$$M_r = \int_{B_r} \rho d\mu[g] \quad (241.3)$$

$$dM_r = M_{r+dr} - M_r$$

$$dM_r = 4\pi r^2 \rho dr$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad (241.4)$$

Stellar structure equations :

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad (241.5)$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \quad (241.6)$$

Vogt–Russell theorem

Tolman–Oppenheimer–Volkov equation of relativistic hydrostatic equilibrium

$$p' = -\frac{1}{2}(p + \mu) \frac{2M(r) + r^3(p - \mu)}{r[r - 2M(r)]} \quad (241.7)$$

241.2 Fusion

The main source of energy for stars, beyond the brief window of time where the initial collapse gives it its initial temperature, is that of *fusion*.

Very brief outline of what happens in fusion

A full presentation of the underlying theory for nuclear processes is well outside the scope of this book. As nuclear processes involve quantum field theory with the strong interaction (the SU(3) Yang-Mills theory), a full presentation of this could take an entire book.

Fortunately for stellar physics, we can use much simpler models. In our case we will use the *liquid drop model*. In this very rough model of the details between the strong interactions and the electromagnetic interactions in a nucleus, the binding energy of the nucleus is contributed by the strong interaction in terms of volume. As the strong interaction is a very short range one, we consider its contribution to be proportional to

the number of nucleons in contact with other nucleons. This makes this contribution roughly equal to the volume, and also roughly equal to the total number of nucleons, except for the nucleons on the boundary. Their missing nuclei contribution is removed with a term proportional to the surface. If the volume is proportional to the number of nucleons, $V \propto A$, and nuclei are roughly spherical, this means its radius will behave as

$$V \propto A \rightarrow R \propto A^{\frac{1}{3}} \quad (241.8)$$

And the area will therefore be $S \propto A^{\frac{2}{3}}$.

Every proton is also getting a repulsion coming from its electromagnetic interaction with other protons. This means that every proton is feeling a charge of $(Z-1)$ proton charges, at an average distance [...] $A^{\frac{1}{3}}$, for a total contribution of Z such

There are in addition a few contributions coming from quantum and QCD effects

$$E_B = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(N, Z) \quad (241.9)$$

”von Weizsäcker mass formula (also known as the semi-empirical mass formula – SEMF)
”

Binding energy plot

This means that if a nuclei manages to capture another

Collision of a proton and nucleus :

$$\hat{H} \quad (241.10)$$

Range of capture for protons

Energy required for this

Different fusion

Proton–proton chain reaction, CNO cycle, and Deuterium fusion



pp1, pp2, pp3

241.3 Starlight

At first approximation, the light given off by a star can be approximated classically by the emission of a blackbody.

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT)} - 1} \quad (241.13)$$

The emission of light in a given frequency range $\nu \in [\nu_{\min}, \nu_{\max}]$ is given by

$$\frac{P}{A} = \int_{\nu_{\min}}^{\nu_{\max}} I(\nu, T) d\nu \int_{A \subset S^2} \cos(\theta) d\Omega \quad (241.14)$$

Stefan-Boltzmann law :

Luminosity at radius r : L_r

Third stellar structure equation

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon \quad (241.15)$$

ε rate of energy generation (power per volume)

Fourth stellar structure equation : Heat transport inside the star (via radiative transfer)

$$\frac{dT}{dt} = -\frac{3}{4a_B c} \frac{\chi \rho}{T^3} \frac{L_r}{4\pi r^2} \quad (241.16)$$

Chemical composition : X_i , decomposed into X_H (hydrogen), X_{He} (helium), X_m (heavier elements)

Equation of state $P(\rho, T, X_i)$, opacity $\chi(\rho, T, X_i)$, nuclear energy generation rate $\varepsilon(\rho, T, X_i)$

Stellar structure equations :

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad (241.17)$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \quad (241.18)$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon \quad (241.19)$$

$$\frac{dT}{dr} = -\frac{3}{4a_B c} \frac{\chi \rho}{T^3} \frac{L_r}{4\pi r^2} \quad (241.20)$$

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr} \quad (241.21)$$

Homologous stellar models

$$L \propto M^3 \quad (241.22)$$

Eddington luminosity limit

241.3.1 Light spectrum

On their surface, we can consider stars to radiate roughly as a blackbody, following the *Stefan-Boltzmann law*

Roughly blackbody on the surface :

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \quad (241.23)$$

$$L \propto R^2 T^4 \quad (241.24)$$

$$L \propto T_{\text{eff}}^6 \quad (241.25)$$

Spectra from chemical composition on the surface

Depends on temperature

Sequences

Main sequence

Compact stars

It can be difficult to fully classify the stars from observable quantities, even in the best of cases, as deviations from the assumptions we laid out can easily happen, and even pretty wild deviations in some cases [examples of stars with very different behaviours]. But as we will see from the observational data, those classifications are fine enough to predict the behaviours of most stars.

Chemically peculiar star

collision processes

mass transfer

241.4 Other emissions

In addition to light, we are expecting (in modern astronomy) three other types of emissions to be detected from stars : neutrinos, stellar rays, and gravitational waves.

241.4.1 Neutrinos

As we've seen from X, stars will generate neutrinos during the fusion process, with about two electronic neutrinos for every four protons in the *pp1* chain, and similar reactions for the other fusion cycles. As those neutrinos stem from fairly well studied nuclear reactions, we know the distribution of their average momenta upon emission.

What happens to them in the stars

After exiting the star, neutrinos are likely to be mostly unaffected during their trip, with a mostly geodesic motion.

241.4.2 Cosmic rays

Cosmic rays are the emission of stars from every other type of particles. As for astronomical purpose, we are mostly interested in particles that are likely to survive such distances, this means that we will mostly look at nucleons and electrons.

By proportion [510], most of the cosmic rays we receive on Earth are bare nuclei (about 99%), of which most of them are hydrogen nuclei (protons), at about 90% of all nuclei, followed by alpha rays (helium) at about 9%, and then in much smaller quantities carbon, oxygen, magnesium, silicium, iron, and so forth (HZE ions). The rest of the cosmic rays are mostly electrons (about 1%). There is also a very small fraction of antimatter (antiprotons and positrons).

Cosmic rays are much harder to interpret than other types of star emissions, as they are charged particles and their path is much more likely to change during their trip.

[Is there any direct thing we can infer for GR from cosmic rays?]

Chapter 242

Astronomy

[511, 512]

Quite a lot of experiments relating to gravity involve astronomy, as it is the field most likely to involve both objects massive enough as well as distances long enough for the non-Euclidian nature of spacetime to manifest. We therefore need before we consider them to look at how the practice of astronomy looks like, both in the classical and general relativistic regime.

242.1 The solar system

All observations we can currently make are based in the solar system. Currently the furthest space probe that we have launched is Voyager 1, at roughly 299 792 458 au from the sun, making it still fairly close to the solar system. Therefore, our frames of reference will typically involve the solar system itself.

The simplest coordinate system to study the planets of the solar system is the *ecliptic coordinate system*, defined in reference to the *ecliptic plane*, the plane defined by the orbit of the Earth around the sun. If we consider that the Earth's orbit is roughly in the same plane (as it is when approximated as a two body problem), the ecliptic plane is defined by every point colinear with one of the lines defined by the Earth-Sun barycenter and the center of the Earth at some time of the year.

[Image of stars on the ecliptic]

The heliocentric ecliptic coordinate system defines its origin as the barycenter of the solar system (close to the center of the sun itself), with the z axis normal and northward to the ecliptic plane (northward in the sense of being in the same direction as the Earth's northern direction). The x axis is defined by the intersection of the ecliptic plane and the solar equatorial plane, and the y axis is the natural third axis in a right-handed coordinate system.

242.2 Astronomical instruments

From the importance of measurement apparatus we've seen in [X], a good description of the instruments used in astronomy seems necessary.

A lot of the basic instruments in astronomy are some form of the basic notion of optical (something something) [455] [See also the operational chapter], where we try to ascertain the direction in which some light source is coming from. Historically these mostly took the form of

Gnomon

[513]

242.2.1 The telescope

An optical *telescope* is some system of optical components such as lenses, mirrors and prisms with the primary goal of magnifying images, hopefully helping with the appropriate measurement of angles by centering the image of the star, as well as allowing the better observation of other characteristics such as color, magnitude, etc.

There are two basic designs for telescopes, the refracting telescope and reflecting telescope

A *refracting telescope* simply uses a system of lenses for its magnification of images. For instance, the Galilean telescope is composed of two lenses in the following configuration :



Figure 242.1: The Galilean telescope

A *reflecting telescope*

The two basic characteristics of a telescope as an abstract optical system are its *aperture* and its *field of view*

Limitations : aperture diffraction,

Geometric aberrations : The five Seidel aberrations : Spherical aberration, Coma, Astigmatism, Petzval field curvature, Distortion

Parasitic aberration

Chromatic aberrations : Longitudinal chromatic aberration, axial chromatic aberration, magnification chromatic aberration

Telescope mounts : Altitude-Azimuth Mount, Equatorial Mount

242.2.2 Radiotelescope

242.2.3 Photometry

Color filters for photometry [514]

Photography for telescope

242.3 Ground-based astronomy

Until fairly recently, most of the astronomy done for experiments was ground-based astronomy, where the observer is somewhere at the surface of the Earth.

242.3.1 Horizontal coordinate systems

A natural coordinate system for Earth-based observations is to consider the observatory as the origin point of our system

Definition 242.1. A *horizontal coordinate system* is a coordinate system for an observer on Earth, using the plane of the horizon as the basis for two angles (θ, φ) , the *altitude* and *azimuth*, which are the angle between the observed object and the horizon, and the angle projected on the horizon's plane between the object and a predefined direction, either true north or south, with the angle oriented as increasing by moving either east or west.

The most common convention is that true north is the direction of 0 azimuthal angle, and that the azimuthal angle increases eastward.

This is the kind of coordinate that we will get from any primitive observation of the stars, as it can be obtained directly after calibrating the instruments so as to be aligned with the

To compare it properly with other coordinate system based on the surface of the Earth, we need to know the altitude, longitude and latitude of the origin.

242.3.2 Equatorial coordinate system

A coordinate system that is less dependent on the

equatorial coordinate system

242.3.3 Time standards

To supplement these purely spatial coordinates, we also need a time coordinate.

Time standards in astronomy :

UT1 UTC GPS TT TAI

"For applications at the Earth's surface, ET's official replacement was Terrestrial Dynamical Time (TDT), which maintained continuity with it. TDT is a uniform atomic time scale, whose unit is the SI second. TDT is tied in its rate to the SI second, as is International Atomic Time (TAI), but because TAI was somewhat arbitrarily defined at its inception in 1958 to be initially equal to a refined version of UT, TDT was offset from TAI, by a constant 32.184 seconds. The offset provided a continuity from Ephemeris Time to TDT. TDT has since been redefined as Terrestrial Time (TT).

For the calculation of ephemerides, Barycentric Dynamical Time (TDB) was officially recommended to replace ET. TDB is similar to TDT but includes relativistic corrections that move the origin to the barycenter, hence it is a dynamical time at the barycenter.[24] TDB differs from TT only in periodic terms. The difference is at most 2 milliseconds. Deficiencies were found in the definition of TDB (though not affecting Teph), and TDB has been replaced by Barycentric Coordinate Time (TCB) and Geocentric Coordinate Time (TCG), and redefined to be JPL ephemeris time argument Teph, a specific fixed linear transformation of TCB. As defined, TCB (as observed from the Earth's surface) is of divergent rate relative to all of ET, Teph and TDT/TT;[25] and the same is true, to a lesser extent, of TCG. The ephemerides of Sun, Moon and planets in current widespread and official use continue to be those calculated at the Jet Propulsion Laboratory (updated as from 2003 to DE405) using as argument Teph. "

242.4 Stellar coordinates

242.5 Stars

The informations we have of a star are rather limited (except for the sun which is much more readily available). We have access to the light spectrum it emits, the position it

occupies in the sky, as well as possibly other particles they might emit such as cosmic rays, neutrinos and gravitational waves.

From what we saw previously, we can classify stars by their spectrum with reasonable accuracy if we are given their original spectrum. The light we receive however suffers from a variety of distortions of this spectrum as it travels through space, due to redshifting from the star's own motion, the interstellar medium, cosmological redshift, the gravitational influence of other nearby stars, etc.

242.5.1 The trip of light in space

Let's consider some stellar spectrum originally close to the blackbody spectrum with some absorption lines, emitted by a roughly spherical star. Its original blackbody spectrum will be some distribution

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad (242.1)$$

power per unit solid angle and per unit of area normal to the propagation

[Graph representing such a spectrum]

From this, we can determine what the original intensity and spectra of those stars is with some degree of accuracy, allowing us to reconstruct what circumstances the light emitted by those stars encountered in between its emission and its detection.

As we've seen, the evolution of the light amplitude with respect to the distance is given by the van Vleck determinant etc

A star's light will therefore be redshifted crawling out of its own gravitational potential, as well as any other it may cross through (although this is typically much rarer to see)

In addition to the deformation of the spectrum from gravity, the starlight interacts with the interstellar medium.

Most of the interstellar medium is composed of a mixture of hydrogen and helium (metallic composition?)

What happens to the light when it traverses a medium : general absorption, absorption of specific wavelengths, diffraction,

What happen in the atmosphere : extinction correction

What happens to the different wavelengths

242.5.2 Detection of light

Once the light has reached our telescopes and other apparatus, we can finally attempt to reconstruct it.

The amplitudes of our star are given by a *photometric system*

Types of photometric system : UBV, UBVRI, etc

Monochromatic : AB magnitude

Measurements on starlight : Apparent magnitude in U, B and V bands

The various color amplitudes can be compared to give us the *color index* of our star (in some sense they are what define its color) $B - V$: star color

[sample of stars by their $B - V$ color index]

light temperature : pyrometry : thermocouples or optical pyrometry

Peak of the spectrum determines T_{eff}

Beam geometry

Composition from spectral lines (but not as easy)

"If the stellar surface temperature is too high, then hydrogen is completely ionized and such atomic transitions do not take place."

"On the other hand, a low surface temperature would imply that all hydrogen atoms are mostly in the ground state $n = 1$, with very few atoms occupying the states $n = 3, 4, \dots$."

Stellar spectra classification : OBAFGKM

Mnemonics

"The star's velocity component along the line of sight would cause a Doppler shift of spectral lines, and by measuring this Doppler shift, the line of sight velocity of a star can be measured. Again, if the star is strongly magnetic, then one can hope to detect the Zeeman effect in the stellar spectra which would give information about the magnetic field."

Blue and redshifting

eclipsing binaries

Hertzsprung-Russell diagram

Star spectra

Inverse square law and the van Vleck determinant for light propagation

242.5.3 Stellar rays

The cosmic rays produced by the star will not come to us in a direct manner

242.5.4 Neutrino astronomy

Neutrino astronomy is a fairly new field, starting out in 1976 with the DUMAND project (Deep Underwater Muon And Neutrino Detector Project)

ICE Cube etc

Neutrino detectors [515,516]

Neutrino detectors all work by exploiting one of the several possible weak interactions between neutrinos and normal matter, usually either reactions with nucleons

$$\nu_e + p \rightarrow n + e^+ \quad (242.2)$$

$$\nu_e + n \rightarrow p + e^- \quad (242.3)$$

$$\bar{\nu}_e + p \rightarrow n + e^+ \quad (242.4)$$

$$\bar{\nu}_e + n \rightarrow p + e^- \quad (242.5)$$

Water Cerenkov Technique

Determination of neutrino type and energy

242.5.5 Gravitational wave astronomy

For a few decades by now, we have also had access to the data given to us by the gravitational waves

Probably the most famous gravitational wave observatory is LIGO, the Laser Interferometer Gravitational-Wave Observatory

Principle : laser interferometry

Show the influence of the G-wave on the laser beams

propagation and detection issues

242.6 Galaxies

242.6.1 Galactic rotation curve

If we assume that the gravitational potential of a galaxy is roughly axisymmetric,

The *rotation curve* of a galaxy is the function assigning the circular velocity of a star as a function of its distance to the center

Dark matter

242.7 Relativistic coordinate system

[517]

242.8 Orbit determination

Orbit determination is a set of methods one can use to determine the characteristics of an object's orbit given some observations of it

"Observations made by a ground-based observer typically consist of time-tagged azimuth, elevation, range, and/or range rate values. "

242.8.1 Gauss' method

[518]

Data to input : Observer's position vector, direction cosine of the star, 3 times of such measurements

$$\tau_1 = t_1 - t_2 \quad (242.6)$$

$$\tau_2 = t_3 - t_2 \quad (242.7)$$

$$\tau_3 = t_3 - t_1 \quad (242.8)$$

For the three direction cosine vectors :

$$p_1 = \rho_2 \times \rho_3 \quad (242.9)$$

$$p_2 = \rho_1 \times \rho_3 \quad (242.10)$$

$$p_3 = \rho_1 \times \rho_2 \quad (242.11)$$

$$D_0 = \rho_1 \cdot (\rho_2 \times \rho_3) \quad (242.12)$$

Given the p quantities and the position vectors, compute the 9

$$D_{ij} = \vec{R}_i \cdot p_j \quad (242.13)$$

242.8.2 Relativistic methods

Chapter 243

The cosmic distance ladder

As we have seen throughout this book, defining a notion of distances in general relativity can be complex and will often rely on specific operational procedures that we use. Therefore an important step for astronomical experiments is the setting of a distance scale by defining some measure of distances for astronomical bodies.

There are unfortunately no methods that will work at all scales, even assuming a classical universe. This is why we talk of a *cosmic distance ladder*, where we consider different processes for different scales, that we need to match on the overlap of their ranges.

Beyond some of the processes we already saw specifically for general relativity disconnected from all other physics, a lot of the cosmic distance ladder rests on the assumptions from other parts of physics, that we can recognize specific types of objects, that those objects have more or less constant properties, and that those properties give different measurements depending on the distance.

- Rods
- Parallax
- Standard candles
- Cepheid variable stars

243.0.0.1 Parallax measure

Parallax measure is the oldest method available for the measuring of distances that cannot be reached physically, going back to 189 BCE with Hipparchus' measurement of the distance of the moon.

Perí megethōn kaí apostēmátōn ("On Sizes and Distances") by Pappus and in Pappus's commentary on the Almagest V.11

Aristarchus measurement of the distance of the sun

First successful measurement : Friedrich Bessel in 1838 for the star 61 Cygni.

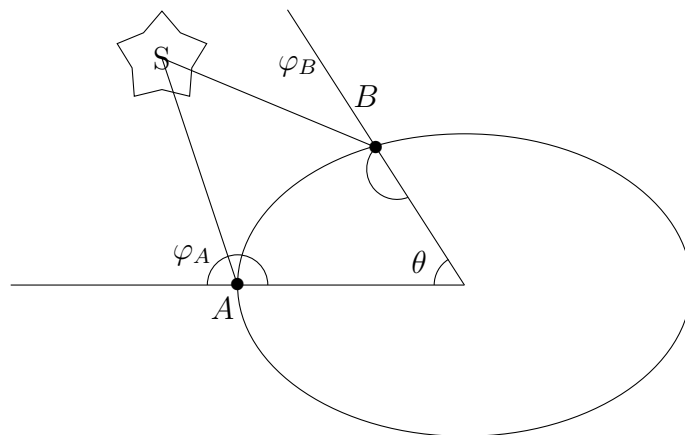


Figure 243.1: Measurement of distances using the parallax

If we assume the space sufficiently flat in between the points A , B and the star S , we can use to a good approximation Euclidian geometry. If we consider the spherical coordinates formed by the center of the point of origin that we consider, with the coordinates

$$\begin{aligned} A &= (\varphi_A, \theta_A, r_A) \\ B &= (\varphi_B, \theta_B, r_B) \\ S &= (\varphi_S, \theta_S, r_S) \end{aligned} \tag{243.1}$$

Distance A to S , B to S , distance A to B known

Simplest case : $r_A = r_B = R$, $\theta = \pi/2$, $\varphi_A = 0$, $\varphi_B = \pi$

$$x \tag{243.2}$$

parsecs

Parallax in GR

Assumption of static or nigh-staticness?

Distance derived from a foliation

243.0.0.2 Standard candles

As stars follow for the most part the same model, we usually have a good enough idea of their mass and luminosity simply from their composition (spectral rays).

If we can consider the curvature small enough, from the van Vleck determinant, we know that the light intensity of a source decays roughly as

$$I(r) \propto r^{-2} \quad (243.3)$$

Square decay

Distance modulus

$$m - M = 5 \log_{10}(d) - 5 \quad (243.4)$$

M absolute magnitude, m apparent magnitude

[Some Hertzsprung russell diagram]

Cepheid variable stars

Stars that pulsate at a regular interval

Period of pulsation proportional to the luminosity

243.0.1 Cosmological redshift

If all other methods fail, the last remaining method available to us is the measurement of the cosmological redshift.

Redshift is a somewhat

Redshift from original spectrum to redshifted spectrum

Chapter 244

The perihelion of Mercury

For the classical two body gravitational problem, if we consider the bound case, the resulting orbits are elliptical in an inertial frame. If we consider some inertial Cartesian coordinate system where two of the axis pass through the vertex and covertex of the resulting ellipse, those vertices and covertices will not move through time.

For orbits around the sun, the two vertices of an elliptical orbit are called the aphelion for the vertex nearest to the focus point (roughly the sun), while the vertex furthest from the sun is the perihelion.

244.1 Measure of the perihelion

Gauss algorithm

What the perihelion looks like experimentally

If we consider the position of this perihelion against the background stars, supposed fixed on the time considered, we can verify whether or not this perihelion is indeed fixed by the coordinate system made from the lines connecting Earth

The solar system is very much not a two body problems, even only considering objects big enough to influence measurements within experimental uncertainty. The actual system to solve for the motion of a planet will be some n -body problem, so that this initial method for orbit determination may be off

244.2 Classical precession

There is no reason Mercury should have precession of its perihelion as a two body problem, but the solar system is far from one.

If we consider the contribution of

$$\Delta\Phi = \sum \tag{244.1}$$

Influences of other planets

244.3 Relativistic precession

The relativistic precession is given by the geodesic equation

Extra inverse cubic term etc

Computation using the influence of other planets etc

Chapter 245

Clocks

As the clock is one of the main property of observers in general relativity, clock-based experiments are an important part of experimental relativity.

245.1 Clock effects

The classic clock experiment of relativity is that of time dilation, which technically only depends on the relativistic kinematics of spacetime, although due to the high precision involved and the lack of relativistic speeds experimentally available for actual clocks will typically also involve gravitational effects.

Given two observers γ_1, γ_2 initially at the same point p , synchronized so that $\gamma_1(0) = \gamma_2(0) = p$, with clocks initially ticking at the same rate [show how], the *first clock effect* is the change in their measured proper time after the two curves diverging and then reuniting, while the *second clock effect* is the change in their ticking rates even after reuniting.

245.1.1 Test of the first clock effect

The first clock effect comes in many different variants, as the causes affecting it can be numerous.

245.1.1.1 Muon decay

An example of the first clock effect can be done by considering the decay of muons in Earth's atmosphere. A muon can be used as a rough clock for such experiments as they will decay with a half-life of $2.196\,981\,1(22)\,\mu\text{s}$ via the process

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \quad (245.1)$$

Muons are generated in Earth's atmosphere via a variety of process,

$$C + p \rightarrow \dots + \pi^- \quad (245.2)$$

Altitudes at which muons are produced, their momentum when ejected and the distance they should travel if non-relativistic v. where we can detect them

245.1.1.2 Hafele–Keating experiment

First clock effect experiment : atomic clocks on planes etc



Figure 245.1: The Hafele experiment [519]

see the relativistic geodesy book?

Tests of accelerations on clocks : [354], Newman et al 1978

245.1.2 Test of the second clock effect

The other clock effect relates to the conformal structure and the Weyl vector of the geometry. While the Weyl vector is typically assumed to be zero, if the connection is supposed to be metric, this assumption can be tested using the second clock effect.

[is it real [520]]

If we consider two observers γ_1, γ_2 , each equipped with a clock h_1, h_2

Depends on the curvature of the Weyl potential

”An immediate physical consequence is that the existence of sharp spectral lines would not be possible in the presence of a non-zero field strength H , since the rate of atomic clocks, as measured by some periodic physical process, would depend on their past history.”

Chapter 246

Lense-Thirring effect

The Lense-Thirring effect (or Lense–Thirring precession)

Rotating body around the Kerr metric :

Modelled by some curve carrying a vector/rotating frame?

”The Juno spacecraft’s suite of science instruments will primarily characterize and explore the three-dimensional structure of Jupiter’s polar magnetosphere, auroras and mass composition. As Juno is a polar-orbit mission, it will be possible to measure the orbital frame-dragging, known also as Lense–Thirring precession, caused by the angular momentum of Jupiter.”

Chapter 247

Gravitational redshift

EM field under a gravitational field

Astronomical experiments

Chapter 248

Gravity Probe B

[521]

Chapter 249

Gravitational waves

For any spacetime with dimension $> (1 + 2)$, there are propagating degrees of freedom for the gravitational field in a vacuum, which allows us

249.1 Binary compact stars

As the strength of gravitational waves is roughly proportional to the quadrupole moment, a good source of them is to consider massive rotating objects at small distances. This can be obtained by considering the case of binary compact stars, such as binary systems of neutron stars.

Binary star test

Orbit decay

Hulse–Taylor binary pulsar

Gravitational waves detectors

LIGO

249.2 Laboratory gravitational waves

It has been proposed to generate gravitational waves in the laboratory [522]

Quadrupolar momentum of electrons under EM waves

Chapter 250

Light deflection and gravitational lensing

Light deflection in Newtonian mechanics

In a Newtonian context, we first consider the case of a typical two body problem where the two bodies are not gravitationally bound, in the limit of the smaller of the two bodies being of a much smaller mass, $m \ll M$. As far as the two body problem goes, a good example of deflection is given by hyperbolic orbits, corresponding to the case of the smaller body having enough kinetic energy to escape the gravity well.

The unbound orbits in such conditions are the hyperbolic orbits, with eccentricity $e > 1$.
238

[523]

Does it make sense as an EM field?

Newton-Cartan version

Deviation of starlight during eclipses [524]

250.0.1 The Eddington expedition

The earliest experiment of light deflection (and the one that gave general relativity its fame) is the Eddington experiment in 1919.

Taking advantage of the total solar eclipse of May 29, 1919, Eddington, meaning to test the predictions of light deflection from relativity, set up two expeditions to observe the eclipse, one in Sobral, Brazil, and the other on the island of Principe.

Coordinates : Sobral : 2h47m25s W 3°41'33" S, altitude 230 feet

"In Plate 1 is given a half-tone reproduction of one of the negatives taken with the 4-inch lens at Sobral. This shows the position of the stars, and, as far as possible in a reproduction of this kind, the character of the images, as there has been no retouching. A number of photographic prints have been made and applications for these from astronomers, who wish to assure themselves of the quality of the photographs, will be considered as far as possible acceded to."

Results :

No.	Names	Magnitude	Coordinates	Displacement
1	B.D., 21°, 641	7.0	(+0.026, -0.200)	0

[525]

"In principle then to determine the deflection one should photograph a field of stars at a season when the sun is not between them and the Earth, and photograph the same field during a total solar eclipse when the sun is so located that light going from the stars to the Earth must pass near the sun's limb. Superposing the two photographs, one should be able to measure the displacement of each star image in fractions of millimeters and, knowing the scale of the photograph, calculate the displacement in seconds of arc.

In practice, the slightest mechanical change in the telescope between the taking of the two sets of photographs will alter the scale: one millimeter on the eclipse photographs will correspond to a different number of seconds of arc than will a millimeter on the comparison photographs. A displacement will occur because of the change of scale. Since the eclipse and comparison photographs are ordinarily taken months apart, one set during the day and the other at night, significant scale differences are always to be expected. In addition, small rotational and translational shifts of star images occur in the course of superposing the eclipse photograph on the comparison plate. Further, besides the displacement due to scale differences (traceable chiefly to a change in focal length of the telescope), there are displacements due to changes in the orientation of the photographic plates to the optical axis. Altogether, modern treatments of the deflection involve at least a dozen parameters (six for displacement in the direction of each of two orthogonal axes) that require the images of at least six stars for their determination. [526] The number of parameters required is sometimes reduced by the use of check plates, photographs of an arbitrary field of stars taken usually the night before or after the eclipse and again when the comparison photographs are taken. Assuming that the focus has not changed between the day of the eclipse and the night before or the night after, or between the comparison exposures and those of their check plates, the scale factor between eclipse and comparison plates can be determined from the check plates. The assumption is not always a good one. The eclipse photographs are taken during the day and the accompanying check plates at night, when different temperatures prevail; and the telescope must usually be moved to obtain check plates with the comparison plates. Since changes in the effective focal length as small as a hundredth of a millimeter in a standard astrographic telescope can have significant effects on the scale value determined from check plates, temperature changes and mechanical deformations can be serious sources of error."

Reanalysis of the results [527]

250.0.2 1939

250.0.3 1947

250.0.4 1952

250.0.5 The 1973 eclipse

Due to the rather ambiguous results of the Eddington experiment, a similar one was conducted in 1973.

250.0.6 Radioastronomy

While the history of eclipse experiments for the testing of light deflection in general relativity with visible light has been fairly difficult due to the very short window of time during which this can be done, those constraints are not as strict for radioastronomy.

250.0.7 Einstein cross

Einstein cross

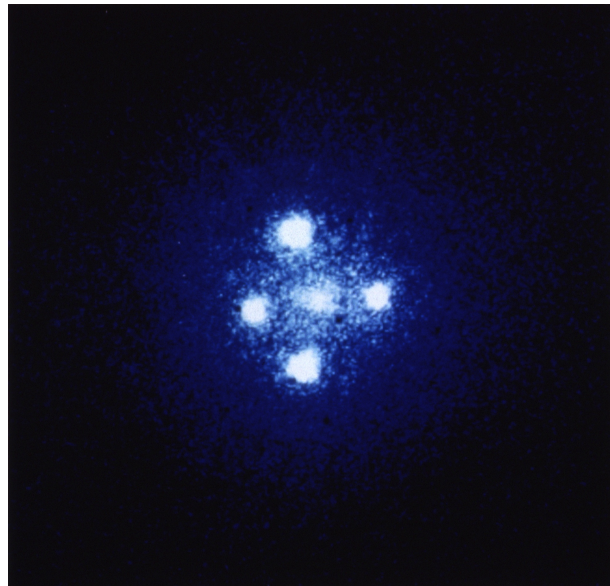


Figure 250.1: The Einstein cross [NASA credit etc]

Chapter 251

The GPS timing delay

Perhaps one of the most often quoted application of general relativity is to GPS satellites. As GPS satellites

[528]

Chapter 252

The Pound-Rebka experiment

The Pound-Rebka experiment was an experiment carried out in 1959 by Robert Pound and Glen Rebka to measure the gravitational redshift of Earth's gravitational field on gamma rays [529].

The setup was to consider a source of gamma rays at a certain altitude h , with a wavelength known (up to some effects we will get into) and then measure their wavelengths once they have travelled all the way to another given altitude h' . Under the approximation of geometric optics with light rays travelling radially in a Schwarzschild metric, the redshifting (or blueshifting) given by this setup should be

$$f' = \tag{252.1}$$

252.1 The environment

The Pound-Rebka experiment took place at Harvard's Jefferson laboratory [...]

252.2 Radiation source

The gamma ray source for the experiment was an iron disk of ^{57}Fe containing some ^{57}Co , a radioactive isotope of cobalt with a half-life of 271,74(6) days, which undergoes the decay to iron via electronic capture exclusively



decaying to an excited state of ^{57}Fe , either to its energy level of 706.42 keV (0.183%) or

136.47 keV (99.80%) before settling to a lower energy state of 14.4 keV and then to its ground state via the emission of gamma rays.

The emissions concerned by the experiment are the 14.4 keV gamma rays.

Gamma ray emissions have their own uncertainty stemming from Heisenberg uncertainty

Thermal noise

Internal conversion

Mossbauer effect

Experiment to check the redshifting of some photons travelling up or down a gravitational potential.

Consider an EM wave/null particle travelling radially on a Schwarzschild metric

Blueshifting of the photon :

$$f_r = \left(\frac{1 - \frac{2GM}{(R+h)c^2}}{1 - \frac{2GM}{Rc^2}} \right)^{\frac{1}{2}} f_e \quad (252.3)$$

Emitter placed at the top of the tower, receiver at the bottom.

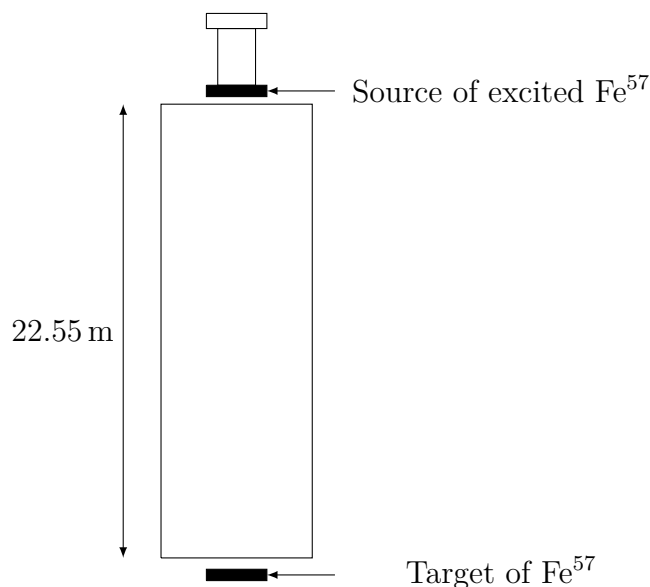


Figure 252.1: The Pound-Rebka experimental setup

Mossbauer effect with iron 57 : Nucleus in a lattice emits or absorbs gamma rays without significant loss of energy to nuclear recoil. The emitted gamma ray can be reabsorbed by another nucleus at rest with respect to the first one due to the very small difference.

Emitter : a source of $0.4 \text{ } ^{57}\text{Co}$, decaying to ^{57}Fe in an excited state

Emission : 14.4 keV by iron 57 atoms

Distance between source and receiver : 22.55 m

Chapter 253

Galactic rotations and dark matter

[530]

Chapter 254

Satellite geodesy

Modern gravitational geodesy

[531]

LAGEOS

Chapter 255

Cosmology

One of the main area of use for general relativity is that of cosmology,

255.1 The Copernician principle

Attempting to guess the overall distribution of matter in the universe is a fairly ambitious project given the scales and numbers involved. Current rough estimates of the total stellar population in the observable universe is at around 10^{22} - 10^{24} (ref)

Furthermore, without knowing the global geometry of the universe, estimating the distance of those stars can be difficult. As we've seen, beyond a certain scale, we can only guess at the distance of given celestial bodies by their redshift, which relies on some initial assumption about the cosmological geometry.

To simplify this, most cosmology is done by assuming the *Copernician principle*, stating that there is no preferred point for the distribution of matter in cosmology, and no specific orientation either : the matter is distributed, at a large enough scale, in a homogeneous and isotropic manner.

There are some obvious problems with this. First of which is that this is obviously false at scales that are smaller than the largest structures in the universe (in fact, if there are still structures at that scale, it would fail the homogeneity assumption). This is wrong at our own scale, at the scale of stars, galaxies, clusters of galaxies and so on.

If the very largest distances are only measured by their redshift, which is obtained by the assumption of the FRW metric, then we may be falling to conventionalism, where our results only agree due to the initial assumption of our geometry.

For now, we will stick with the assumption of homogeneity and isotropy, and we will investigate later what the failure of this may imply.

Assuming this, our universe would have some FRW metric (cf. 199).

Theorem 255.1. If the matter distribution is homogeneous and isotropic, then so is the metric tensor.

Experimental data that seems to confirm homogeneity

255.2 Non-homogeneous models

While the universe seems to be roughly homogeneous and isotropic, it remains possible that even on average our universe fails to be so.

The dropping of any number of degrees of symmetries should have the obvious consequences of being spacetimes with metrics befitting such isometry groups. Every degree of symmetry dropped will however leave more and more free parameters for the cosmological model to have to fit.

On the extreme side of things, we can also consider spacetimes with no isometries. While probably closer to what actual spacetimes are like, this makes them fundamentally intractable and only really open to numerical calculations (example : Mixmaster universe)

[421]

255.2.1 Axisymmetric cosmologies

Axis of evil

255.3 Matter fields

In cosmological terms, we consider our stress-energy tensor in terms of various forms of matter rather than fields, as those would not typically be possible to measure directly.

For the moment, we will assume that all those sources of matter are perfect fluids, obeying some equation of state

$$s(p, \rho) = 0 \tag{255.1}$$

255.3.1 Baryonic matter

Baryonic matter is composed of all the ordinary matter, in cosmological terms, mostly by hydrogen and helium. As far as the bulk of its composition goes, it is mostly constituted specifically by the nuclei of these atoms, which is simply single protons for the vast majority of that matter.

Described as a perfect fluid, baryonic matter is described by the equation of state $\rho = 3p$

255.3.2 Radiation

The radiation part of the cosmological stress-energy tensor is composed of all the electromagnetic fields present in the universe.

Equation of state :

The CMB

255.3.3 Dark matter

The modelling of dark matter in cosmology is complicated by the fact that we so far do not have much certainty as to what dark matter is, but there are a few model at our disposal

The most common model for dark matter is that of *cold dark matter*

255.3.4 Neutrinos

While neutrinos are an omnipresent matter source in the universe, their very small mass typically gives them a rather small contribution to the total stress-energy tensor in cosmology.

255.3.5 Cosmological constant

It is common in cosmology to rewrite the Einstein field equation to include the cosmological constant on the side of the matter fields :

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = T'_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu} \quad (255.2)$$

In this sense, the cosmological constant can be interpreted as a matter source. If we diagonalize the metric, we have that the cosmological constant is some fluid of energy $\rho = \Lambda$ and of isotropic momentum $p = -\Lambda$ [check signs], making it a fluid with equation of state $w = -1$.

255.3.6 Other sources

255.3.7 The assumption of perfect fluids

A rather natural extension of the models we have considered so far is to simply look at the case where the stress-energy tensor is not abstracted to a perfect fluid, since we would not even expect this to be case.

255.4 Consequences of homogeneity

To check the assumption of the homogeneity of the universe, one can look at the consequences of

EGS theorem

[532–534]

255.5 Approximate homogeneity

That the approximate large scale homogeneity and isotropy of the universe implies its approximate resemblance to the FRW cosmology is a linchpin of the standard model of cosmology.

Counter example

cf. Ringström

255.6 The standard cosmological model

The currently favored standard model for cosmology is the Λ -CDM *model*, named after the cosmological constant Λ and the cold dark matter that were the main additions to previous standard models.

Assuming the homogeneity of the universe, the standard model of cosmology is considered as a mixture of a variety of fluids

Cold matter Hot matter Photons Neutrinos Dark energy

[535, 536]

Quantum effects at some point (expansion)

Λ -CDM model

Minimal lambda CDM model : $(A_s, n_s, \omega_b, \omega_m, \Omega_\Lambda, \tau)$

A_s global amplitude of the primordial spectrum n_s : spectral index of primordial scalar perturbations of density ω_b : baryon density, $\omega_b = \Omega_b h^2$ ω_m : non-relativistic matter density $\omega_m = \Omega_m h^2$ Ω_Λ ! fractional density of the cosmological constant τ optical depth of reionisation

Photon density fixed by the CMB temperature T_0

Neutrinos of mass 0

flat universe

Extended models

255.6.1 Timeline of the universe

The exact timeline of our universe in the Λ CDM model is subject to some variations,

We can consider the timeline of our universe in two ways, either by trying to rollback the configuration of the present days, or by trying to evolve some initial assumptions.

Since we are assuming something close to homogeneity, it is somewhat possible to try to guess at a reasonable initial condition at a given time, which would mostly be constrained by the proportions of our various forms of matter in the model.

Recombination, photon last scattering

255.7 Cosmic redshifting

Redshifting in some FRW metric

Ignoring other sources of redshift for now, the redshift given by the FRW metric that

we've seen is such that

$$1 - z = \dots \tag{255.3}$$

255.8 The Cosmic Microwave Background

From our assumptions, whatever fluids we consider in our model will have to be roughly homogeneous

FRWL cosmology

Motivate the homogeneity and isotropy

Polarization of the CMB

Origin of the CMB

Chapter 256

Compact stars

The area in which general relativity will intersect heavily with quite a lot of the rest of physics is that of *compact stars*, stars of particularly high densities such as white dwarves, neutron stars and some more exotic types of compact stars (we will look at the case of black holes in the following chapter).

Neutron star physics

Chapter 257

Ultracold neutron experiment

One of the few effects of quantum gravity that we can test for directly is the case of the non-relativistic limit, using the Newton-Schrödinger equation.

schrödinger eq for cold neutrons in a gravitational field

$$\hat{H} = \frac{\hat{p}}{2m} + m_N g \hat{z} \tag{257.1}$$

Approximation of a wall too

[537]

Chapter 258

Analogue gravity

Parts of the behaviours of gravity can be replicated with other physical systems sharing similar equations of motions. This allows for the simulations of some general relativistic effects without numerical simulation

This is called *analogue gravity*

258.1 Acoustic gravity

The propagation of sound waves in a medium

Acoustic gravity

Bose Einstein gravity [538]

Chapter 259

Computation and relativity

259.1 Hypercomputation

One niche application of general relativity that has been occasionally investigated is that of *hypercomputation*.

One of the fundamental limit of physical computation is expressed by the Church-Turing thesis, which roughly state that any feasible computation must be equivalent to a Turing machine

One workaround of the Church-Turing

259.1.1 Malament-Hogarth spacetimes

Malament Hogarth spacetime

Definition 259.1. A *Malament-Hogarth spacetime* is a spacetime (M, g) for which there exists a timelike curve γ and a point p such that $\text{Im}(\gamma) \subset I^-(p)$, and the curve γ has infinite proper time : $l(\gamma) = -\infty$.

In other words, an observer located at p would be able to receive a signal from γ emitted after an arbitrary length of proper time.

runaway blueshift

max memory

259.1.2 Computation with closed timelike curves

Another application of general relativity for computation is the use of closed timelike curves for computation.

Malament

259.1.3 Quantum computing along closed timelike curves

Chapter 260

Gravitational shielding and antigravity

As nothing stops us from considering any possible metric, it is not a particularly harduous tasks to find some some solutions of the Einstein field equations that would correspond to either gravitational shielding or antigravity

260.1 Gravitational shielding

The notion of gravitational shielding goes all the way back to Newtonian gravity [X]

Frontiers in propulsion physics

260.2 Antigravity

[...]

Unfortunately, the notion of antigravity in general relativity can only refer to the production of antigravity from a matter source, and not the reaction of a test particle to any existing one, as by the principle of equivalence, particles of any type have to follow the same path.

Chapter 261

Odd and ends

There's a variety of applications of general relativity that are not of particular use either for experiment or engineering, but for which we still have elements.

261.1 Brachistochrone curves

[539]

261.2 Ballistic suicide

Synge

Chapter 262

Conclusion

From the various experiments that we have reviewed, we can get a better feeling of the various concepts that we broached in XII.

For the idea of confirmation holism, we have seen how the various assumptions forming the base of those experiments have relied, in addition to relativity, on optics, geology, seismology, astrophysics, thermodynamics, nuclear physics

Part XIV

History

As this history goes back rather far and encompasses quite a wide variety of topics, it should be noted that the focus on most of the premodern geometry, mechanics, etc is on the *conceptions* of those topics that people of those era had more than specific theorems and applications. In the case of geometry, we will be looking more into

Due to the very large amount of sources one can find in a very large variety of languages, there will inevitably be a lot of holes in the coverage of this history, being mostly restricted to being able to read French and English sources, especially for topics which are less likely to be well-covered by reviews of sources in other languages such as the popular reception of some of those ideas. There will therefore be quite likely some particular focus on European and American history due to this.

[488, 540]

Chapter 263

Prehistory

It can be difficult to say anything about the conception of space and time before writing, and even for most of the early literate period, as we do not have any explicit statements of the people involved. Anything about the prehistoric conception of space, time and dynamics will therefore have to rely on pretty questionable informations on what we can hope to either generalize to the past in some way or try to extract from actual archeological evidence.

such as neuroscience, animal models, material culture, premodern groups, linguistic evidence, etc

[541]

263.1 Neurological basis for space

At its core, the human conception of space and time stems from how humans (and animals in general) have to relate to their environment.

As far as human senses go, the main ones where the conception of space plays a major role are vision, proprioception (the awareness of mechanical informations on the muscles and joints), equilibrioception and in a lesser role, hearing.

There are on the other hand no senses which are directly tied to the perception of time.
[437, 542]

263.1.1 Vision

Vision functions by a variety of photoreceptor cells in the eyes

[light sensitivity distribution]

Retinal mosaic

The brain receiving the raw informations from the photoreceptors does not have any particular way to reconstruct the original pattern of the signal from these, as the inputs do not carry any information on their disposition.

Visual cortex

Reconstruction of the topology (from the commonality of sequences of informations? closely identical informations? Does it relate to convolutional NN)

Estimation of distances

Optical illusions

263.1.2 Proprioception

Proprioception : muscle spindles for muscle tension, golgi tendon organs, joint receptors

”The proprioceptive sense is believed to be composed of information from sensory neurons located in the inner ear (motion and orientation) and in the stretch receptors located in the muscles and the joint-supporting ligaments (stance). ”

The various informations of the tensions of our muscles allows us to reconstruct a map of our body. All the various informations of those tensions in a given articulation of the body are interpreted under the previous experience of the body that those receptors are typically activated together and will coincide with some given visual information, or coincide with some other informations such as this body part touching another, giving us roughly the experience of the limb being in one given configuration that the muscles allow, the angles [roll, pitch, yaw, idk]

We will have such informations about every such articulations, which, simplifying the mechanics of the brain, should give us about three degrees of freedom per articulation, assuming that all such articulations can move in all directions, for a total of $3N$ degrees of freedom in our body. However, we also have the constraint that some of those configurations we know are the same : given some configurations of our body, we know that our left hand could be made to be in the same place as our right hand in another configuration, making those configurations equivalent in some sense. The kinesthetic conception of space is the quotient of the total configuration space of our body modulo those equivalences.

constraints from our limbs being of roughly constant lengths

263.1.3 Equilibrioception

Humans also have a sense of their general orientation with respect to the gravitational field given by equilibrioception, measured by the vestibular system in the inner ear.

263.1.4 Hearing

Hearing is overall not the most commonly used sense for our conception of space (unlike other species such as bats), but there are still some informations we can glean from it.

Delay between sounds

Case of bats

263.2 Geometry of primitive cultures

It can be difficult to try to model what prehistory could be like based on primitive cultures [explain primitive] [find source for the well foundedness of that]. Very few cultures described into such details are entirely isolated from other cultures.

These are however the rare few detailed accounts we have of what geometric constructions look like in societies without formal geometry.

Hortense Powdermaker in "Life in Lesu: The Study of a Melanesian Society in New Ireland" :

"Putting the posts in the ground in a straight line is a skilled job and not every man can do it. There were ten men in a village of two hundred and thirty people who were skilled in this. If a man is notable to do it he gets one of these skilled men to mark off the ground in a straight line with a vine."

Haddon, "Canoes of Oceania", parallel line construction

Rectangular houses, usually by visual inspection

Methods for right angles : Best ("The Maori as he was"), Te Rangi Hiroa, Levar

Lebar :

"In order that the building may assume the correct shape and proportions, the *souim* [construction chief] makes a series of measurements and readjustements both before the building is started and during the construction. First the wall and end plates are placed on the ground in the relative position they will assume in the completed building. To

ensure that all corners of this frame will be right angle corners the *souim* stretches a line of coconut fiber cord diagonally the length of the frame, from one corner to the other, and the timbers are readjusted until the same distance obtains between the two remaining corners.”

For a given parallelogram, the diagonals between two opposing corners define two triangles with angles opposed to this diagonal α_1, α_2 , with $\alpha_1 + \alpha_2 = \pi$. The length of those diagonals will be

$$l_i = \sqrt{a^2 + b^2 + 2ab \cos(\alpha_i)} \quad (263.1)$$

and will indeed be a rectangle in the case of those two diagonals being equal.

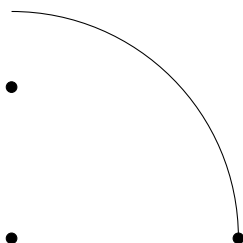


Figure 263.1: The rectangle construction

Hose, Charles and McDougall "The pagan tribes of Borneo" [543], measurement of the elevation of the midday sun by the Kenyah tribe

Kayan tribe determining that by making a hole in the roof of a house, distances on a special plank on the floor from the point below the hole to the spot of sunlight on the plank to determine etc

Sea Dayaks determine the season using the elevation of the Pleiades at dusk

Long bamboo tube closed at one end is filled with water, when tipped back the level of water is noted, when agreed with a predetermined mark left from previous time of observation this is the season

The bamboo is originally filled with a volume of water $\approx b \times h$, assuming a cylindrical shape, filled to the top. When tipped at an angle θ with respect to the vertical, the surface of the liquid still flat with respect to the vertical will form an angle of $\pi/2 - \theta$ with respect to the perpendicular to the tube, forming an obliquely cut cylinder, of volume $b \times (h' + h)/2$, where h is the height of the tube and h' is the point it reaches.

We have some right triangle of lengths equal to the diameter of the tube, $h - h'$, with the hypotenuse unknown, allowing us to compute $h - h'$ from the tangent

$$\tan(\theta) = \frac{h - h'}{d} \quad (263.2)$$

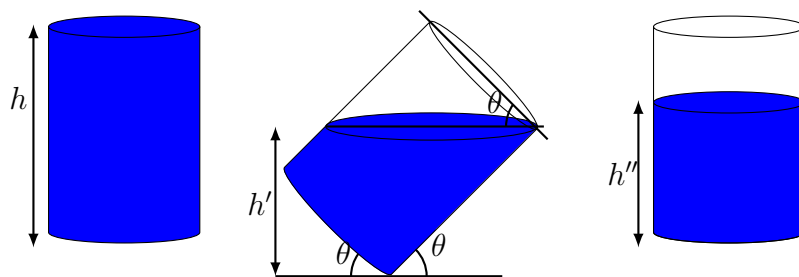


Figure 263.2: Bamboo

$$h' = h - d \tan(\theta) \quad (263.3)$$

The new volume once the tube is righted will be the same as when inclined, but it will now reach a different height from before : as they have the same base, the new height h'' will be equal to the average of the other two,

$$h'' = h - \frac{d}{2} \tan(\theta) \quad (263.4)$$

We therefore have a relation between the new volume and the tipped angle, allowing the marking of specific angles inside the container to determine the measured angle.

Navigation map in the Bernice P Bishop museum, still in the McDougall book

[544]

263.3 Material culture

A lot of the evidence we are left with in prehistoric cultures is the material culture, the artefacts constructed by that culture. The main use of geometry in ancient times, and presumably in prehistoric times if such usage had its root there, was for construction and geodesy. As we would probably not found much evidence for the use of geodesy in prehistoric times if it was used (the example of maps of pre-agrarian cultures that we have tend to be made from wood), the main artefacts we will have to consider are the ones related to constructions.

263.3.1 Archeoastronomy

One method of astronomical observation we are aware of in historical time was the practice of observing the coincidence of various heavenly bodies with well-known landmarks to

give the time of the year. As it can be difficult to have interpretations of prehistoric buildings, we will first look at those historic cases where we have written sources.

We can for instance find it in some stele in the city of Itanos in Crete, where an inscription reads [545]

Patron set this up for Zeus Epopsios. Winter solstice. Should anyone wish to know: off
 'the little pig' and the stele the sun turns.

The "little pig" was a rock formation

From such practices, the field of archaeoastronomy attempts to find similar prehistoric landmarks specifically placed and oriented to potential times of the year of interest with the stars and sun. In particular, turning points of the year for the seasons such as the winter and summer solstice, and spring and fall equinox.

Perhaps the most famous such prehistoric construction is Stonehenge

ther megalithic structures

Thirteen Towers solar observatory

263.4 Linguistics

Analogies in languages between time and space [546]

Chapter 264

Antiquity

As general relativity covers such domains of ancient thought as geometry, time, kinematics, dynamics, cosmology and gravity, there are quite a lot of material one could relate to it. The main sources of the ancient world that we have concerning those topics are mainly those of Ancient Egypt, Sumer, Mesopotamia, Persia, Greece, Rome, India and China.

The few sources we have on Ancient Egyptian geometry are from the Rhind mathematical papyrus [547], the Berlin Papyrus 6619, the Moscow Mathematical Papyrus and the Lahun Mathematical Papyri. These mostly seem to be oriented towards education or practical applications, being mostly made of mathematical problems to train scribes. They mostly concern the computation of areas, volumes and lengths, and do not contain any derivations of the formulas involved (a few of the formulas included seem to only be approximations) or anything concerning their ideas about geometry.

Sesotris

Sources in Ancient Greece do seem to indicate that there was quite a lot of geometric knowledge to be found in Egypt [548], as Thales was reputed to have learned it there

”And Pamphile relates that he, having learnt geometry from the Egyptians, was the first person to describe a right-angled triangle in a circle, and that he sacrificed an ox in honour of his discovery.”

Proclus :

Since, then, we have to consider the beginnings of the arts and sciences with reference to the particular cycle [of the series postulated by Aristotle] through which the universe is at present passing, we say that, according to most accounts, geometry was first discovered in Egypt, having had its origin in the measurement of areas. For this was a necessity for the Egyptians owing to the rising of the Nile which effaced the proper boundaries of everybody’s lands.’

Democritus in Clement of Alexandria' Stromata :

in the putting together of lines, with the necessary proof, no one has yet surpassed me, not even the so-called harpedonaptae (rope-stretchers) of Egypt

but those may not have been preserved to modern days.

Sumerian [549, 550]

264.1 Ancient geometry

One area where the people of antiquity had quite a lot to say about was that of geometry. It should be noted here is that while we previously had some distinction between the notion of abstract geometry and physical geometry, the abstract notion did not exist (or at least was considered to only be valid if it reflected physical geometry), so that all the geometry here will be that of physical space.

While the earliest Greek philosophers all apparently had many opinions on geometry, starting with Thales himself, no writings on the topic remain, so that we mostly have to rely on later accounts of those philosophers, and even then some of those later accounts (such as Eudemus' History of Geometry) being lost.

[551]

Thales from Proclus :

"Thales, who had travelled to Egypt, was the first to introduce this science into Greece. He made many discoveries himself and taught the principles for many others to his successors, attacking some problems in a general way and others more empirically."

Pliny : "Thales discovered how to obtain the height of pyramids and all other similar objects, namely, by measuring the shadow of the object at the time when a body and its shadow are equal in length."

264.1.1 The Pythagoreans

Pythagoras is generally regarded as one of the main figure in the foundation of geometry,

The exact beliefs of Pythagoras can be hard to pin down as he left no writings behind, and the later sources tend to be a bit ambiguous as to exactly who held those beliefs between Pythagoras and later Pythagoreans,

The Pythagorean conception of space ties with their more general conception of the universe as being fundamentally made of numbers, and space is in some sense discrete.

Space and objects in that space such as lines are composed of discrete elements, which are sometimes referred to as points (or "partless signs" [1]). Those partless signs aren't necessarily points in the sense of being of infinitesimal size

"For they construct the whole universe out of numbers-only not numbers consisting of abstract units; they suppose the units to have spatial magnitude. "

Aristotle :

"of those who have maintained the existence of indivisibles, some, as for example Leucippus and Democritus, believe in indivisible bodies, others, like Xenocrates, in indivisible lines"

But in the sense of them being atomic, ie having nothing smaller than them.

Finite segments are in this context made of a finite number of points, and their lengths stem from being composed of a certain number N of points. If we have two segments A, B , those segments are congruent if they have the same number of points : $N_A = N_B$.

This belief has some important consequence as any two segments in this model are meant to be *commensurable*, that is to say, there exists some segment C such that their two lengths are multiples of C . In modern terms we would say that the ratio of their length is rational, ie

$$|A|/|B| \in \mathbb{Q} \quad (264.1)$$

In which case $|A|/|B| = p/q$ for two integers p, q , so that $q|A| = p|B|$. A length to which those two segments are multiples of will then be a common divisor of p and q , c , so that $p = mc$ and $q = nc$, in which case we simply have that a line C made by splitting A into m components will have both A and B being integer multiples of these.

cf. Euclid : "Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure."

This led to the big geometric crisis of the incommensurable segments, what we would call today irrational numbers. It is not known exactly who made the discovery [551]. The discovery is sometimes attributed to Pythagoras, or Hippasus of Metapontum, an early Pythagorean, who was drowned by other Pythagoreans for his sin of discovering it, according to legend. What we do know is that Theodorus of Cyrene, a 5th century mathematician, is said to have proven the incommensurability of $\sqrt{3}$ (and every non-square number up to 17) to a unit length, suggesting that the irrationality of $\sqrt{2}$ came somewhat before him.

This proof can for instance be found in Euclid

"To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line."

Theorem 264.1. Given a square, the segment composed by one of its diagonal is incommensurable to the sides.

Proof.

□

"We know that the early Pythagoreans thought that every segment was made up of finitely many points"

"This can reasonably be deduced from several elements: the fact, reported by many sources, that the Pythagoreans based geometry on the integers; the Pythagorean theories of "figurative numbers" (for which see [Heath: HGM], vol. I, pp. 76–84, and [Knorr: EEE], Chapter 5); and above all Aristotle's assertion that the Pythagoreans attributed a magnitude to the units that made up material bodies (Aristotle, *Metaphysica*, XIII, vi, 1080b:16–21 + 1083b:8–18 = [FV], I, 453:39 – 454:9, Pythagoreans B9, B10). Sextus Empiricus seems to still be thinking in Pythagorean mode when he says that it is impossible to bisect a segment formed by an odd number of points: *Adversus physicos* I (= *Adv. dogmaticos* III = *Adv. mathematicos* IX), §283; *Adversus geometras* (= *Adv. math.* III), §§110–111"

Metaphysics book XIII :

And the Pythagoreans, also, believe in one kind of number-the mathematical; only they say it is not separate but sensible substances are formed out of it. For they construct the whole universe out of numbers-only not numbers consisting of abstract units; they suppose the units to have spatial magnitude.

Geometric algebra (rel. between numbers and lengths and areas) from Euclid and the pythagoreans

Plato's *Timaeus*?

264.1.2 The Point and commensurability

There were two common debates regarding the nature of space during Greek antiquity, those being the status of the point (and other abstractions like lines and surfaces), and the commensurability of lengths.

As we already exposed in [X], the notion of a *point*, and any further abstracted lower dimensional shape like a line or a plane, can be a difficult one to justify. This

The notion of *commensurability* in ancient geometry is that two geometric objects are such that there exists an integer ratio between them, the basic example being that if two lines A, B are commensurable, their lengths are such that

(or in a synthetic sense, that there exists a finite number of transformations)

proof of arbitrarily small lengths from incommensurability

the Pythagoreans were the first to address themselves to the investigation of commensurability, having discovered it as the result of their observation of numbers ; for, while the unit is a common measure of all numbers, they were unable to find a common measure of all magnitudes, . . . because all magnitudes are divisible ad infinitum, and never leave a magnitude which is too small to admit of further division, but that remainder is equally divisible ad infinitum,'

the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck '

264.1.3 Infinitesimals

The notion of approximating objects in space by an infinite number of infinitely small objects goes back fairly far in the Greek tradition, although the exact start is a bit blurry, as this started out as rather vague statements regarding the part and parthood relationship

Antiphon, Euclid, Eudoxus

Zeno :

"To prove the latter of these contentions, Zeno relied on the infinite divisibility of bodies as evident ; assuming this, he easily proved that division will continually give smaller and smaller parts, there will be no limit to the diminution, and, if there is a final element, it must be absolutely nothing. Consequently to add any number of these nil elements to anything will not increase its size, nor will the subtraction of them diminish it ; and of course to add them to one another, even in infinite number, will give nothing as the total. "

264.1.4 Euclid

It is believed by modern historians that Euclid's Elements is simply a compilation of various other authors, mostly by various Pythagoreans, Eudoxus, etc

The main foundational importance of Euclid's Elements for our purpose is its organization around a few definitions (), postulates (A) and common notions (K) (what we would call today axioms)

Euclid's book provoked quite a lot of reactions from various other philosophers

Epicurians, Proclus, Sextus Empiricus, Aristotle, Euclid [496]

A non-Western source in which we can find some similar discussions on the topic of geometry is Ancient China, where we can find some axiomatic definitions

264.1.5 The Mohist canon

One part of the ancient world where we find such thinking on the topic of space is in the Mozi (墨子) in ancient China during the Warring States period.

The Mozi is not a book particularly centered on a given topic, but in

Mohists and the Mozi

264.2 Ancient mechanics

The fundamental idea that was at the center of ancient mechanics was the notion of *motion* and *change* : what does it mean for an object to be in motion.

264.2.1 The Eleatics

The very oldest extent text that we have on that topic [552]

Parmenides, Zeno, impossibility of motion,

According to the Eleatics, the notion of motion was expressed by that of Being and Non-Being. An object in motion as evolving in time would exist in a location and then cease to exist in this location, passing from being into non-being. However, the position of the Eleatics was that transition from being to non-being was impossible, making motion (as well as any type of change) impossible in their view. This basic argument, while considered enough for the eleatics themselves, was supplemented by various other demonstrations on the impossibility of changes of various types.

The most famous of these are *Zeno's paradoxes*, a set of paradoxes meant to show by contradiction the impossibility of motion

Achilles and the tortoise

Arrow paradox

A lot of the early conceptions of motion and mechanics are essentially reactions to the Eleatics

264.2.2 The Atomists

One of the earliest reaction to the Eleatics was the Atomists, such as Leucippus and Democritus, who tried to find a concept of motion that did not contradict itself in this way.

The atomist conception of motion involves much like the Eleatics that of being and non-being, in the form of the atoms and the void.

Plato's *Timaeus*?

264.2.3 Aristotelian mechanics

The notion of mechanics that was perhaps the most influential from Antiquity until the late medieval era was that of Aristotle, detailed mostly in his books *Physics* (Φ), *On the Heavens* (Π) and *On Generation and Corruption*

Aristotle's mechanics, Impossibility of vacuum [553]

From [3] : "A given weight moves a given distance in a given time; a weight which is as great and more moves the same distance in a less time, the times being in inverse proportion to the weights. For instance, if one weight is twice another, it will take half as long over a given movement. Further, a finite weight traverses any finite distance in a finite time."

"A thing moves naturally to a place in which it rests without constraint, and rests naturally in a place to which it moves without constraint."

264.2.4 The Mechanicians

A less metaphysical attempt at describing mechanics was that of the mechanicians, a tradition that was more rooted in the practical application of those principles.

Statics with Archimedes, mechanics of Pappus, Hero of Alexandria

264.3 Ancient cosmology

264.3.1 Earliest models

A lot of ancient cosmologies are based on the difficulties to explain what one can observe from the night sky : there are what appears to be objects in the sky remaining there (or at least not appearing to fall in a way that most objects would), and those objects have motions of various types, either regular and periodic (the stars), or more complex motions (the planets), or simply rapidly disappearing (such as comets).

This led to a variety of ways of considering how stars could remain up without falling from the sky.

A lot of times, those solutions involved the firmament as some kind of physical object. [example of China, Greece, Sumer?, India?] The stars are simply fixed to some kind of shell, the firmament, above Earth. In the earliest models, this is often a god of some kind, like Nut in Ancient Egypt

Some of the Chinese models for the firmament were that it was some kind of plate resting above the Earth, supported by mountains [Needham]

"Such a constrained movement would necessarily involve effort the more so, the more eternal it were-and would be inconsistent with perfection. Hence we must not believe the old tale which says that the world needs some Atlas to keep it safe-a tale composed, it would seem, by men who, like later thinkers, conceived of all the upper bodies as earthy and endowed with weight, and therefore supported it in their fabulous way upon animate necessity. "

264.3.2 Presocratic models

Get all the models listed in Aristotle etc

Thales with the floaty Earth

"Hermesianax, one of the older generation of Alexandrine poets (about 300 B.C.), is quoted as saying ' What inspiration laid forceful hold on Pythagoras when he discovered the subtle geometry of (the heavenly) spirals and compressed in a small sphere the whole of the circle which the aether embraces."

Anaximander : "According to him the earth is a short cylinder with two bases (on one of which we live) and of depth equal to one-third of the diameter of either base. It is suspended freely in the middle of the universe without support, being kept there in equilibrium by virtue of its equidistance from the extremities and from the other heavenly bodies all round. The sun, moon, and stars are enclosed in opaque rings of compressed air concentric with the earth and filled with fire : what we see is the fire shining through

vents (like gas-jets, as it were). The sun's ring is 27 or 28 times, the moon's ring 19 times, as large as the earth, i.e. the sun's and moon's distances are estimated in terms (as we may suppose) of the radius of the circular face of the earth ; the fixed stars and the planets are nearer to the earth than the sun and moon."

"He is also credited, like Thales before him, with having constructed a sphere to represent the heavens."

264.3.3 The Ptolemaic model

The model that would be the basis for most astronomy in the West until around the Renaissance is that of Claude Ptolemy as described in the *Almagest* (originally *Μαθηματικὴ Σύνταξις*, the astronomical composition).

Motion of the stars (Do they land on Earth or not) [492]

Rotation of the firmament

system of concentric spheres holding the various celestial objects together,

Very early cosmology tends to include the intervention of various gods for the celestial motions

Ptolemy, Idea of Ptolemy on why the firmament does not fall, Relative notion of down

Socrates and the Republic

Plato

Structure of time, Cyclical time

From Aristotle's perspective, the universe was a finite space, ending at around the sphere of fixed stars [3]

"It is therefore evident that there is also no place or void or time outside the heaven."

Chapter 265

Medieval

The medieval era is commonly treated as

265.1 The theory of impetus

The notion of impetus (what we would call today momentum) did not technically originate during the medieval era, [John Philoponus]

Ibn Sīnā's Book of Healing and impetus

Buridan on impetus

Aquinas and cyclical time

The Oxford calculators (Merton School)

Buridan and boat

A ship drawn swiftly in the river even against the flow of the river, after the drawing has ceased, cannot be stopped quickly, but continues to move for a long time. And yet a sailor on deck does not feel any air from behind pushing him. He feels only the air from the front resisting him. [554]

Can God move the universe (Locke, Barrow, Gassend)

Coordinates from Oresme

Vacuum (John of Jandun)

Chapter 266

Galilean relativity and Newtonian mechanics

The first theories that are considered modern on the topic of space, time and gravity arrive during the late 16th century and onward, most famously with the type of theories given by Descartes, Galileo, Newton and Leibniz.

266.1 Galilean relativity and motion

Galileo : Boat example

[555]

Corpuscular theories

Descartes

[556]

266.2 Absolute and relative space

Absolute time, absolute space, Newton v. Leibniz

Bucket argument

Kant on space

Chapter 267

Newtonian gravity

Gravitational shielding

LeSage theory

267.1 The nature of space

Leibniz and Newton [540]

267.2 LeSage theory of gravity

Due to the popularity of corpuscular and fluid theories in the 17th century, Newton's theory which seemed to imply action at a distance seemed like a return to the notion of occult qualities and natural tendencies that had fallen out of favor.

One theory to try to circumvent this idea was to turn Newtonian gravity into a corpuscular/fluid-based theory of gravity, originally proposed by Nicolas Fatio de Duillier in 1690 [557] and expanded upon in X [558],

[559]

Homogeneous fluid applying isotropic pressure

Mass shields the pressure in between two massive bodies

267.3 Gravitational shielding

Experiments on gravitational shielding

Chapter 268

Classical cosmologies

For a rather long period after antiquity, the standard western cosmology was some variant of the Ptolemaic cosmology.

Giordano Bruno : *La Cena de le Ceneri* and *De l'infinito universo et mundi*

Copernicus : *De revolutionibus orbium coelestium*

Kepler

Galileo

primum mobile

268.1 Newtonian cosmology

One of the earliest formal cosmology post-Ptolemaic cosmology was the Newtonian cosmology.

Infinite Euclidian space, steady state, roughly homogeneous mass distribution

[560, 561]

[562] Newtonian cosmology

"As to your first query, it seems to me that if the matter of our sun and planets and all the matter of the universe were evenly scattered throughout all the heavens, and every particle had an innate gravity toward all the rest, and the whole space throughout which this matter was scattered was but finite, the matter on the outside of this space would, by its gravity, tend toward all the matter on the inside and, by consequence, fall down into the middle of the whole space and there compose one great spherical mass. But if

the matter was evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered at great distances from one to another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matter were of a lucid nature.”

268.2 Olbers' paradox

The Olber's paradox is rather simple to understand. If there are an infinite number of stars uniformly distributed, and the universe is infinitely old, statistically speaking, wherever you look, there should be a star. On the other hand, the night sky is mostly black.

Olber's paradox can most easily be seen by considering the case of stars distributed evenly on a grid with coordinates $(i, j, k) \in \mathbb{Z}^3$ and sides of length ℓ , with Earth placed at the vertex $(0, 0, 0)$. In a perfect vacuum, the starlight arrives to us with an intensity proportional to r^{-2} of their original luminosity. Therefore, the total intensity as seen from Earth should be something along the lines of, assuming some standard luminosity L_0

$$L = \sum_{i,j,k \in \mathbb{Z}^3} \frac{L_0}{4\pi\ell^2(i^2 + j^2 + k^2)} \quad (268.1)$$

A better proof is to consider some uniform distribution of stars. Let's consider that on a large enough scale, our universe is filled with a uniform distribution of stars. Let's say that we split our universe into cubic regions of the same size, with sides ℓ and volume ℓ^3 , each cube on a lattice so that Earth, at $(0, 0, 0)$, is at the center of cube $(0, 0, 0)$. Any cube (i, j, k) is centered around the point of the lattice (i, j, k) , at a distance $d = i^2 + j^2 + k^2$ of Earth, and the borders of the cube extend as $[i - \ell/2, i + \ell/2] \times [j - \ell/2, j + \ell/2] \times [k - \ell/2, k + \ell/2]$ so that the nearest point of the cube is

There were several attempts at a resolution of Olber's paradox

268.2.1 Effect of the interstellar medium

One resolution of the paradox was assuming that for far enough stars, the intensity of each star's light would eventually fall off until the total intensity on Earth would be finite.

Simple model of absorption :

$$\frac{dI_\nu}{dS} = \rho\kappa_\nu I_\nu \quad (268.2)$$

κ_ν mass absorption coefficient or opacity

$$I(r) = I_0 \frac{e^{-\rho\kappa r}}{r^2} \quad (268.3)$$

Heating of the interstellar medium until homogeneous temperature and glow

268.2.2 Non-homogeneous distribution

Another resolution of the paradox is to simply reject the hypothesis that the star distribution is homogeneous. However, we still cannot simply have them in a compact region of space or even tapering off in density, as otherwise they would be at risk of collapse.

[Not stable under Newton]

One model to solve this issue was to assume a fractal distribution

268.3 Divergence of the Newtonian cosmology

If we consider the Newtonian cosmology on its own, it is quite easy to show that something is wrong even there. If we consider the Laplace equation with constant source term, corresponding to the roughly homogeneous mass density of the universe

$$\Delta\Phi = 4\pi G\rho \quad (268.4)$$

using the Green function of the Laplace operator,

$$G(\vec{x}, \vec{y}) = \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|} \quad (268.5)$$

The solution will be, in spherical coordinates,

$$\Phi(r, \theta, \varphi) = \frac{4\pi G}{4\pi} \rho \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{r} \dots \quad (268.6)$$

Boundary conditions Poisson equation

[560, 563]

Many attempts were made to try to solve this issue analytically, to simply find some convergence process that would converge for all relevant quantities [Seeler]

Modifications to Newton's gravitation

Hierarchical distribution

curved space cosmologies

Schwarzschild

268.4 Thermodynamics

One rather large issue for the steady state model that appeared in the 19th century was the issue of thermodynamics. Using the conception of entropy as it was understood [Clausius?], the most logical outcome would be that the universe would reach thermodynamic equilibrium (or at least approach it arbitrarily closely) in finite time, which is obviously not the case of our universe.

Boltzmann and recurrence time

Chapter 269

Absolute and relative space

One area of Newtonian mechanics that has been in dispute since its inception is the concept of absolute space and absolute time.

Descartes, Newton, Hertz, Mach, Poincaré, Andrade, etc

One such attempt at a more relative formulation of

A further issue concerning the absolute v. relative distinction is the concept of inertia, where acceleration is considered to be absolute, even in the models of mechanics ditching the notion of absolute space and time.

Mach and inertia

Chapter 270

Non-Euclidian geometry

[564]

The axioms of Euclid have been contested as far back as they have been created, by such people as Proclus

[Parallel postulate from Proclus]

On a more constructive side, analogies between spherical geometry and plane geometry were made as far back as Menelaus of Alexandria, comparing the great circles of a sphere to the lines of Euclidian geometry

For quite a while however, the axioms of Euclid went by unchallenged as far as their truth, but one issue that kept coming up was the status of the parallel postulate. Many mathematicians have considered the parallel postulate to be much more complex than all other of Euclid's postulates, and have suspected that it could be derived, either from the existing ones or from a new, simpler postulate.

Attempts at proving the parallel postulates through history

Proof via similar triangles

270.1 Kant

270.2 Early non-Euclidian geometry

Gauss

Lobachevsky, Lobachevsky geometry

Riemann, theorema egregorium

Proof that theorems from the non-Euclidian geometry can be translated into Euclidian statements, making them as strong as Euclid's geometry

Helmoltz on the conventionality of geometry

Poincaré, analysis situs, dimensions and the geometry of space

270.3 The dimension of space

The question of the dimensionality of space did not come up much as a question up until Kant (except for a very marginal appearance in the writings of [17th century guy]), as most sources on the topic of the dimensionality of space before that would simply take it as granted that space was three-dimensional, the reason for which was usually some numerological idea, ie Aristotle's "On the heavens" :

A magnitude if divisible one way is a line, if two ways a surface, and if three a body. Beyond these there is no other magnitude, because the three dimensions are all that there are, and that which is divisible in three directions is divisible in all. For, as the Pythagoreans say, the world and all that is in it is determined by the number three, since beginning and middle and end give the number of an 'all', and the number they give is the triad.

Along with the rise of non-Euclidian geometries in the sense of breaking of the parallel postulate came the idea of geometries with a number of dimensions higher than the three dimensions of Euclidian geometry, either in the sense of more spatial dimensions, or in considering time itself as another dimension.

This had a few different sources. First was the use of analytic geometry in mechanics, where the number of dimension of the geometry relates to the n -tuple of numbers considered. In the case of mechanics, this has led to some people starting to talk about time in a way [cf. d'Alembert]

The speculations on non-Euclidian geometry also naturally led to such ideas as, despite Gauss' Theorema Egregium pointing the lack of necessity of curved spaces to be embedded in a larger space, this was still how the notion was often explained and conceived

Klein

Another source was complex geometry. quaternions [565]

Non-Euclidian geometries as embedded surfaces in a higher geometry

Higher dimensions [566],

Unodim, Flatland

[221, 436, 567]

270.4 Spacetime

The notions of time and space have always been closely associated with each other, even going back to Ancient Greece, but the attempt to unify them is a fairly more recent endeavour.

The oldest trace of such an idea [D'Alembert] [568]

Hamilton

Chapter 271

Light, electromagnetism and the ether

Three important fields that eventually culminated in the development of special relativity were that of optics, electricity and magnetism, originally three entirely distinct phenomenon

271.1 Electricity and magnetism

Electricity and magnetism have been known for quite a while, attempts at describing their behaviour going as far back as Thales of Miletus

De Magnete, Magneticisque Corporibus, et de Magno Magnete Tellure in 1600 by William Gilbert

Coulomb's law

"It follows therefore from these three tests, that the repulsive force that the two balls — that were electrified with the same kind of electricity — exert on each other, follows the inverse proportion of the square of the distance."

$$\vec{F} = k \frac{q_1 q_2}{r^2} \frac{\vec{r}_1 - \vec{r}_2}{\|\vec{r}_1 - \vec{r}_2\|} \quad (271.1)$$

Theory of the electric fluid

Benjamin Franklin

Gauss

Ampère

271.2 Optics and light

Newton : [569]

Ether for light propagation in optics, Fresnel, Huygens, etc

Olaus Roemer's speed of light measurements

This measurement was done by considering the observation of Io, Jupiter's satellite. Io's orbit around Jupiter is very nearly in the same plane as Jupiter's orbit around the sun (an inclination of 0.05°), orbiting Jupiter every $42.45930686h$. Due to this, eclipses of Io are a fairly common phenomenon

Under Fresnel's interpretation of light, one way to measure the speed of light is

Interferometer :

271.3 Electromagnetism

Maxwell "On physical lines of force" [570]

$$\vec{\nabla} \cdot \vec{E}(t, \vec{x}) = \frac{\rho(t, \vec{x})}{\varepsilon_0} \quad (271.2)$$

$$\vec{\nabla} \cdot \vec{B}(t, \vec{x}) = 0 \quad (271.3)$$

$$\vec{\nabla} \times \vec{E}(t, \vec{x}) = -\frac{\partial \vec{B}(t, \vec{x})}{\partial t} \quad (271.4)$$

$$\vec{\nabla} \times \vec{B}(t, \vec{x}) = \mu_0(\vec{J}(t, \vec{x}) + \varepsilon_0 \frac{\partial \vec{B}(t, \vec{x})}{\partial t}) \quad (271.5)$$

The equation of Maxwells give rise to, after some rewriting of terms, into a system of wave equations

$$(\vec{\nabla}^2 - \varepsilon\mu\partial_t^2)\vec{E}(t, x) = 0 \quad (271.6)$$

$$(\vec{\nabla}^2 - \varepsilon\mu\partial_t^2)\vec{B}(t, x) = 0 \quad (271.7)$$

The natural interpretation of this, in analogy with wave equations for sound or elastic

medium, is that the Maxwell equation describes motion in some kind of medium, which when disturbed, produces waves.

Breaking of Galilean invariance

Ether

While the Maxwell equations do seem to break Galilean invariance, this did not raise much alarm under the assumption of the ether theory, as this was simply a manifestation of the ether fluid breaking the boost invariance, the same way that the distribution of a fluid would break it for acoustic laws.

If we expressed the laws of electromagnetism in terms of the ether itself, and added whatever equation of state the ether itself obeyed, this

[571]

The Ether jelly (Kelvin) [572]

Chapter 272

Special relativity

The Maxwell equations for electromagnetism

Total current (p, q, r) , conduction current (p', q', r') , magnetic potential (F, G, H) , magnetizing field (α, β, γ) , electric field (P, Q, R) , displacement field (f, g, h) , electric potential Ψ

$$\begin{aligned}p' &= p + \frac{df}{dt} \\q' &= q + \frac{dg}{dt} \\r' &= r + \frac{dh}{dt}\end{aligned}$$

$$\begin{aligned}\mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz} \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx} \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy}\end{aligned}\tag{272.1}$$

Galilean invariance : measurable quantities should be invariant under the transformation $x \rightarrow x + vt$, $v \in \mathbb{R}^3$

Break of Galilean relativity in electromagnetism due to the magnetic term

This meant that either the principle of Galilean invariance was wrong or that the laws of electromagnetism were only valid in a specific frame, the later being the theory that was adopted at the time, as it was supposed that electromagnetism was the consequence of the mechanical deformation of a fluid, the aether, in a similar manner to sound waves.

If true, this would mean that the speed of light will vary in frames in motion with respect to the aether.

272.0.0.1 The Michelson-Morley experiment

To try to measure the relative motion of earth to the aether, Albert Abraham Michelson proposed an experiment in 1881 [573]. The goal was to measure the speed of light on Earth at two points

Consider the speed of light c , speed of earth with respect to the aether v , take two points A and B separated by a distance D . During the time t_1 it takes for light to travel from A to B , Earth moves a distance d_1 , while during the time t_2 it takes to travel from B to A , the earth will move a distance d_2 . Were the Earth at rest in the aether, we would simply have $t_1 = t_2 = t = D/c$, but as the earth will move by d_1 during the first trip, the point B will move the same distance, hence the true distance travelled by light will be $D + d_1$, and similarly on the other way, $D - d_2$.

$$\begin{aligned} t_1 &= \frac{D + d_1}{c} = \frac{d_1}{v} \\ t_2 &= \frac{D - d_2}{c} = \frac{d_2}{v} \end{aligned} \tag{272.2}$$

$$\begin{aligned} d_1 &= D \frac{v}{c - v} \\ d_2 &= D \frac{v}{c + v} \end{aligned} \tag{272.3}$$

$$\begin{aligned} t_1 &= \frac{D}{c - v} \\ t_2 &= \frac{D}{c + v} \end{aligned} \tag{272.4}$$

From this, we get

$$t_1 - t_2 = 2D \frac{c}{c^2 - v^2} = 2t \frac{1}{1 - \frac{v^2}{c^2}} \tag{272.5}$$

Michelson and Edward Morley performed this experiment (later called the Michelson-Morley experiment) [cf "On the Relative Motion of the Earth and the Luminiferous Ether", 1887] between april and july 1887. [574]

Interferometer to measure the speed of light in two directions

Experimental setup :

Aether dragging

272.1 The Lorentz ether theory

To account for the constancy of the speed of light in apparently all inertial frames, the effect of Lorentz contraction was formulated by Hendryke Lorentz, in the general framework called the Lorentz ether theory.

Poincaré and simultaneity [575]

Einstein's theory of special relativity (≈ 1905)

Minkowski

Chapter 273

The birth of general relativity

A variety of anomalous effects existed by the early 20th century that would be later on explained by special and then general relativity, although few of them were considered anomalous enough to require such a radical change to physics at the time.

One of the earliest of those effects was the anomalous precession of the perihelion of Mercury. The best computations of Urbain Le Verrier, by the analysis of the transit of Mercury from 1697 to 1848, showed that the value was of about $5600''$ per century, while the computed value

Newtonian cosmology

Olber's paradox points out that if we assume a roughly homogeneous distribution of stars in the universe, as well as an eternal cosmology, the night sky cannot be

Mach's principle

273.0.1 Gravity in special relativity

The translation of classical theories to special relativistic theories is usually fairly straightforward.

Problem with the equivalence principle

attempts to have theories of gravity with it

Scalar gravity : Einstein, Nordstrom

Vector gravity

Tensor gravity : Pauli-Fierz

273.0.2 General relativity

1908 : "On the relativity principle and the conclusions drawn from it" (acceleration in SR)
1909 : Ehrenfest : "Uniform rotation of rigid bodies and the theory of relativity"
1911 : "On the influence of gravity on the propagation of light"
1913 : "Outline of a generalized theory of relativity and of a theory of gravitation" General relativity

Weyl's theory of length gauge

Teleparallelism

Eddington and grav. waves

Reichenbach, Carnap

Chapter 274

Big bang cosmology

A general relativistic cosmological model was one of the earliest application of general relativity, going back to Einstein's static model of the universe [576]

Lemaître model, de Sitter,

Chapter 275

Nazi Germany

The rise of the Nazis in the 1920's and 1930's happened during an era where Einstein was already quite well-known for his theories.

Reactionary opposition to Einstein's physics

100 authors against Einstein [577]

[578]

[579]

Deutsche Physik movement

As Einstein was the figurehead of the relativity theory, a lot of the attacks on it got mixed in with clear antisemitism in the 1930's [580].

German versus Jewish physics (intuitionism v. theoretical physics)

Philipp Lenard

275.1 World ice theory

Under Nazi germany, the reaction to Einstein overall was to simply stick to the previously accepted models for kinematics, gravity and cosmology. One of the few exception to this trend was a cosmological theory by Hanns Hörbiger under the name of *world ice theory* (Welteislehre).

Chapter 276

General relativity in the Soviet Union

Hegel, Marx, Engel, Lenin on epistemology and science [581]

Opinions on the validity of cosmology

[582] Dialectic materialism

[581, 583, 584]

Steady state universes

Boycott of the Vatican congress

Torsion scam

Chapter 277

Relativity in China

[585]

Chapter 278

The relativity renaissance

It is commonly said that there was somewhat of a lull in the development of relativity around the 1940's and 1950's.

One big event that triggered the relativity renaissance was the publication of Kruskal's maximally extended solution to the Schwarzschild metric, [586]

Hawking, Penrose, Singularity theorem

Chapter 279

Quantum gravity

[587] Original by Russian guy

Pauli-Fierz?

De Witt

Feynman's covariant QG

That history paper from Rovelli

Chapter 280

String theory

The origins of string theory are entirely removed from that of a gravitational theory

280.1 Regge poles

To understand the origins of string theory, it is necessary to first look at the theory of Regge poles [588].

Regge poles of the Coulomb potential :

$$V(x) = -\frac{e^2}{4\pi\epsilon_0 r^2} \quad (280.1)$$

Binding energy between two particles of mass m_1, m_2 , reduced mass

$$M = \frac{m_1 m_2}{m_1 + m_2} \quad (280.2)$$

$$E_N = -\frac{2M\pi^2 e^4}{h^2 N^2 (4\pi\epsilon)^2} \quad (280.3)$$

$$N = n + l + 1$$

Angular momentum l :

$$l(E) = -n + g(E) \quad (280.4)$$

$$g(E) = -1 + i \frac{\pi e^2}{4\pi\epsilon_0 h} \sqrt{\frac{2M}{E}} \quad (280.5)$$

The path in the complex plane for the complex angular momentum as parametrized by values of E is the *Regge trajectory*

280.2 Pomerons

280.3 The string model

Nambu-Goto

Barbashov

Polyakov

Witten etc

Chapter 281

Wormholes and warp drives

Wormholes have been somewhat considered in general relativity as far back as 1917 [Flamm],

Einstein-Rosen bridge [589]

Wheeler, Ellis drainhole

The modern era of theoretical work on wormhole started with Arthur C. Clarke asking

Morris-Thorne, Eric Davies [590] Blue Sky projects

Krasnikov, Alcubierre

Harold White, Warp Drive company

Chapter 282

Relativity in popular culture

The history of relativity that I have presented so far has been of a nature pretty limited to academic circles, occasionally intersecting with politics, but beyond that, relativity has also been an idea that gained a lot of popularity with the wider public during the early 20th century.

When the various notions of what would later become relativity first came around in the late 19th century, this did not seem to have had much of an impact on people outside of academia

First appearances of relativity in newspapers and pop science?

Popular Science was, before its purchase by another company in 1915, a journal fairly directed at other academics, but at least not specifically physicists, and we can find references to relativity in it in 1910, and later on in a much more extensive article in 1914. [Other earlier articles about non-Euclidian geometry]

Nothing post-1915, when the journal was aimed at the wider public

Times article of 1919

[591]

[592–594]

282.1 Popular science

Much of the early reception of relativity accented its abstract character and the elite character of understanding it, commonly claiming that only a few people on Earth were capable of doing so.

The Boy Scientist

282.2 Magic

Strangely enough one of the oldest aspect of the popularization of ideas as they relate to the various elements that would eventually form relativity has been in relation to magic, going all the way back to 1875 with Zöllner's "Transcendental Physics"

The Secret Doctrine madame blavatsky

[595–597]

Tao of physics

282.3 Science fiction

One of the bigger impact of relativity in popular culture was in how it affected science fiction. The type of settings and of consequences of relativity theory, with its extremely high speed, focus on stars, manipulation of time and the great energy held by inert matter, along with its popularity, made it a fairly easy fit for science fiction scenarios, even in the very thought experiments that spawned it, such as the case of the Langevin twins.

The premises of the ideas of relativity that we saw previously (such as early ideas of spacetime and higher dimensions) can already be found in 19th century science fiction. The idea of higher dimensions can be found most famously in Edwin Abbott's *Flatland*, which, among other things, provides the analogy of the functioning of higher dimensions by considering the lives of two dimensional beings, as already found in Hinton [566], and Hinton himself later made more such stories. H. G. Well's *The Time Machine* has its titular time machine working on the principle of space and time being part of a same whole, with the time machine allowing motion through time rather than space.

The pre-relativistic era (and the era before it became standard to worry about relativity in science fiction) can be found in some books, such as A. E. van Vogt's *Voyage of the Space Beagle* (1950)

Heinlein, *Time for the stars* (1956), *La planète des Singe* (1963), Poul Anderson, *Tau Zero* (1970)

Chapter 283

Fringe ideas

General relativity managed to be a well-publicized theory, wildly unintuitive to most and a strong departure from many of the ideas that were commonly accepted before it. Due to this, it attracted quite a large literature of fringe theories, either to continue along those lines or as reactions to try to salvage classical physics.

Deciding what counts as a fringe idea is not entirely uncontroversial, as an idea simply being wrong is not enough. Most people would not consider the proposal of a new theory that is later found to not work to qualify, nor conservatism denying a new theory that is later found to be correct (at least until it is part of the paradigm [598]), or simply vague proposals that are not pushed with much rigor or conviction like the one electron universe theory [599]. Ideas may also widely differ in their acceptability depending on the context of the current paradigm, in that we would probably not consider ancient ideas mixing science and religion [463] to be fringe, but we would probably do so today. Many fringe ideas in fact start out as simply unsuccessful proposals that are later held onto beyond their reasonable point.

Demarcation problem cf. [438] [600]

Finding such theories can be hard, as many of them are typically not published in academic publications which are well archived, and tend not to be cited very often by others, so that it is common for only a few of the most famous such theories to remain easily available. [601]

283.1 Heim theory

Burkhard Heim

Heim's theory can be somewhat difficult to find, as most of his work on his theories remains untranslated

283.2 Torsion field

283.3 Scalar weapons

During the cold war, various technologies have been attributed to either sides of the conflict based on fairly thin reasoning.

Soviet Scalar Electromagnetic Weapons, Fer de Lance, Bearden

Chapter 284

The current era

FERMI

Gravity probe B

PLANCK

LIGO

Nanograv [602]

Chapter 285

The Future

WHAT DOES THE FUTURE HOLD

Part XV

Conclusion

Appendix A

Appendix chapter

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