The big book of spacetime

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Contents

Notation and conventions

Conventions on general relativity

As with most general relativity textbooks, we will use the Einstein notation, where summation is implied for identical upper and lower indices. That is, for two quantities X and Y, with components $X^{a_1 b_1 c_2 d_1 \ldots p \ldots}$ and $Y_{a_2 b_2 c_2 c_2 \ldots p \ldots}$, with a single common index p, we have the equality

$$X^{a_1 b_1 c_2 d_1 \ldots p \ldots} Y_{a_2 b_2 c_2 c_2 \ldots p \ldots} = \sum_{p=0}^{n} X^{a_1 b_1 c_2 d_1 \ldots p \ldots} Y_{a_2 b_2 c_2 c_2 \ldots p \ldots} \tag{0.1}$$

The sign conventions for general relativity are the following :

- The metric signature will be $(+ - - -)$.
- The Riemann tensor is defined as $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$.
- The Einstein equation is defined as $G_{\mu\nu} = \kappa T_{\mu\nu}$.
- The Levi-Civitta tensor's sign is defined by $\varepsilon_{123\ldots} = 1$.

In terms of the Misner-Thorne-Wheeler notation, this corresponds to the $(+ + +)$ convention.
Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.2}$$

0.1 Units and dimensions

The mathematical calculations will all be in natural units, $G = c = \hbar = 1$, while any practical calculations will be in SI units.
Dimensions :
g : dimensionless Γ : L^{-1} R : L^{-2} $\mathscr{L}$: J/m^3

1 Introduction

This book aims to provide a rather comprehensive set of theorems regarding spacetime, general relativity and related gravitational theories.
Structure of the book :
Part I : definition and properties of spacetimes without any reference to general relativity
Part II : General relativity and its action on spacetime
Part III : Alternative classical theories of gravitation
Part IV : How matter behaves on spacetime, both as test fields and with backreaction.
Part V : Cauchy problem
Part VI : Quantum theory in GR
Part VII : Attempts at quantum theories of gravitation.
Part VIII : Specific examples of spacetimes
Part IX : History and experiments
As the title implies, this is a fairly large book. I would not recommend reading it in its entirety, especially if you already have some familiarity with the topics involved, and would recommend instead to use it more as a reference book.
While every structure is explained, some familiarity with general relativity and quantum field theory is recommended.

Part I

Spacetime

This first part will be to define the structures of spacetime independently of general relativity. A spacetime will be defined in full as the 4-tuple $(\mathcal{M}, \mathcal{A}, g, \nabla)$, with $\mathcal{M}$ a topological manifold, $\mathcal{A}$ a smooth structure on that manifold, g a metric tensor and ∇ a connection.
For most spacetimes which are not too pathological, we will also define a time orientation τ and a volume form ε, bringing it to a 6-tuple $(\mathcal{M}, \mathcal{A}, g, \nabla, \tau, \varepsilon)$. In almost all cases, this notation will be shortened to $(\mathcal{M}, g)$, as the other structures usually either stem naturally from them or are obvious enough.

Constructing examples

Throughout this book, a lot of theorems will be proven using examples and counterexamples of various spacetimes. A useful checklist in general for the construction of such examples is Geroch's [13] list of methods for constructing spacetimes

- Check the known solutions
- Tip the light-cones in some way
- Take a covering space or product
- Isolate what makes an example fail in a local region and push it off to infinity
- Introduce a conformal factor
- Patch spacetimes together across boundary surfaces
- Cut holes of various types

Those methods will work to construct most examples used in this book. While we will investigate specific spacetimes and field theories in greater details later on, a few example of the well-known solutions that will be used commonly are :

1.0.0.1 Minkowski space

The most basic example of flat spacetime, with its metric in Cartesian coordinates as

$$ds^2 = -dt^2 + \sum_i dx_i^2 \tag{1.1}$$

Common variations on Minkowski space will be Minkowski space with some identified coordinates, usually either spacelike (where the spacelike hypersurface will be some cylinder or torus) or timelike (also called the timelike cylinder, the standard example of a spacetime with causality violations).
The full details on Minkowski space can be found in chapter 71.

1.0.0.2 Misner space

Misner space is a standard example of a spacetime with closed timelike curves, as it is simply Minkowski space identified by a boost.

$$ds^2 = \tag{1.2}$$

1.0.0.3 Schwarzschild spacetime

The Schwarzschild spacetime is the solution for a spherically symmetric matter distribution.

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\varphi^2) \tag{1.3}$$

1.0.0.4 (Anti) de Sitter space

1.0.0.5 The Einstein universe

Common matter fields

Another important list of example will be the common sources of matter used in the Einstein field equations. These are either realistic fields or useful toy models.

1.0.0.6 Scalar field

The generic form of the scalar field is composed of a kinetic term, a mass term (of either sign), a possible potential term and a coupling to curvature,

$$\mathscr{L} = \nabla_\mu \phi \nabla^\mu \phi + m^2 \phi^2 + \xi R \phi + V(\phi) \tag{1.4}$$

This gives rise to the Klein-Gordon equation

$$\Box \phi - m^2 \phi - \xi R - \frac{\partial V(\phi)}{\partial \phi} = 0 \tag{1.5}$$

1.0.0.7 Electromagnetism

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu \tag{1.6}$$

This gives us the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = j^\nu \tag{1.7}$$

1.0.0.8 Dirac field

$$(e^\mu_a \gamma^a \partial_\mu + m)\psi = 0 \tag{1.8}$$

2 The manifold

2.1 Charts and atlases

At its most basic level spacetime is defined, within the framework of general relativity, by a manifold, based on the assumption that spacetime, while shown to be non-Euclidian, still remains approximately Euclidian at a small enough scale, as can be determined by local experiments. This is described mathematically by the concept of a chart.

Definition 2.1. A *chart* on a topological space S is a homeomorphism $\phi : U \to O$, from an open set U of S to an open set O of $\mathbb{R}^n$. The chart is C^k if the function ϕ is C^k.

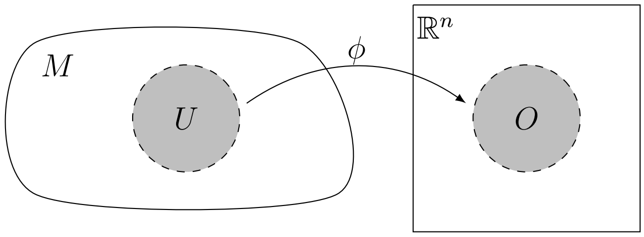


Figure 1: Chart on a manifold

Throughout this book, we will see that this fact, along with some Taylor expansions, will allow us to show that measurable quantities will be locally Euclidian. That is, for a small enough neighbourhood of a point, the deviation from the Euclidian quantity can be made arbitrarily small.

A single chart will usually not be enough to cover an entire manifold, unless this manifold is itself homeomorphic to a subset of $\mathbb{R}^n$. We will require a collection of charts.

Definition 2.2. A C^k atlas is an indexed collection of C^k charts $\{U_\alpha, \phi_\alpha\}$, $\alpha \in A$, such that $\bigcup_{\alpha \in A} U_\alpha = S$, and if $U_\alpha \cap U_\beta \neq \varnothing$, the transition map $\tau_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ is C^k as well.

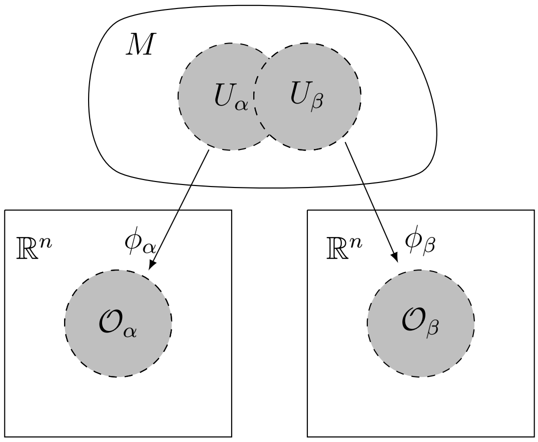


Figure 2: Transition map between two charts

2.1.1 Equivalence of manifolds

The atlas poses the problem that the same manifold could be covered by two different atlases, even $\mathbb{R}^n$. To remedy this, we will need some way of comparing manifolds with different atlases. There are two ways to do this. The first one is to define an equivalence relation

Definition 2.3. Two C^k atlases on the same set are C^k-equivalent if their union is also a C^k-atlas.

For instance, if we consider the two atlases $(\mathbb{R}^n, \phi_1)$ and $(\mathbb{R}^n, \phi^2)$, those are equivalent if $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ are both C^k on $\mathbb{R}^n$.
The other method is to equip every manifold with a complete atlas.

Definition 2.4. A C^k atlas is complete if every possible C^k chart that intersects it with a C^k transition function is included in it.

The equivalence between the two is simply that if we consider the equivalence class of every C^k-equivalent atlas on a manifold, the complete atlas is the union of all those atlases. A non-complete C^k atlas will then just be a representative of a complete atlas. We will usually just refer to a representative of the atlas for simplicity.

Definition 2.5. A C^k *manifold* is a topological space associated with a C^k complete atlas.

A C^0 manifold is also called a topological manifold (so called because it only has a topological structure and no differential structure). A C^∞ manifold is also called a smooth manifold.

2.1.2 Topology

One of the benefit of the charts on the manifold are that they define a topology on the manifold.

Lemma 2.1. A subset U of M is open if for all charts, $\phi_\alpha(U_\alpha \cap U)$ is open in $\mathbb{R}^n$.

We may have more than one topology defined on our manifold, either because we defined it from another structure or because we wish to add more to it. In this case, we can compare this topology to the manifold topology.

Theorem 2.6. A topology τ on M coincides with the manifold topology if and only if for all charts, U_α is open in τ and all maps ϕ_α are homeomorphisms with respect to τ.

Proof. If the two topologies coincide, any open set V of τ □

Proposition 2.7. An atlas can be equivalently defined as homeomorphisms between open subsets of M and open balls of $\mathbb{R}^n$, or $\mathbb{R}^n$ itself. [cf Lee 2.13]

Proof. An open subset O_α of $\mathbb{R}^n$ can be described as the union of a set of open balls $\bigcup B_{x_i,r_i}$. We can then consider the image

$$U_{\alpha,p_i,r_i} = \phi_\alpha^{-1}(B_{x_i,r_i}) \tag{2.1}$$

with $\phi(p_i) = x_i$. The restriction of the chart map to this subset ϕ_{α,p_i,r_i} will then form along with this open subset a chart. □

This will allow us to have fairly well behaved atlases on a manifold. A particularly useful type of atlas will be atlases of countably many simply connected sets.

Corrolary 2.1. A second-countable manifold admits a countable atlas mapping to simply connected subsets of $\mathbb{R}^n$.

If A is compact and C is closed $A \cap C$ is compact
proof : $\{U_\alpha\}$ an open cover of $A \cap C$. $U_\alpha + M \setminus C$ covers A. Since A is compact, $A \subset (M \setminus C) \cup U_{\alpha_1} \cup ... \cup U_{\alpha_N}$ so $A \cap C \subset U_{\alpha_1} \cup ... \cup U_{\alpha_N}$

Proposition 2.8. If M is a Hausdorff manifold, then every compact subset A is closed.

Proof. $p \in M \setminus A$, for $q \in A$, there are disjoint open neighbourhoods $p \in U_{p,q}, q \in U_q$. The set of all U_q for $q \in A$ are an open cover of A, hence there is a finite subcover $U_{q_1}, ..., U_{q_N}$. The intersection $U_{p,q_1} \cap ... \cap U_{p,q_N}$ is an open subset of M disjoint from $U_{q_1} \cup ... \cup U_{q_N}$. So every $p \in M \setminus A$ has an open neighbourhood $U \subset M \setminus A$. The union of all those neighbourhoods is open, hence the complement A is closed. □

2.1.3 Coordinates

The map from a spacetime point to individual coordinates of a chart will just be the composition of the map from the manifold to the coordinate patch and the projector π_μ from $\mathbb{R}^n$ to $\mathbb{R}$, with $\mu \in \mathbb{N}, 1 \leq \mu \leq N$, such that $\pi_\mu(\langle x_1, ..., x_\mu, ..., x_n \rangle) = x_\mu$.

$$\phi^\mu = \pi_\mu \circ \phi \tag{2.2}$$

The coordinate components x^μ of a point p will then be

$$x^\mu(p) = \phi^\mu(p) \tag{2.3}$$

usually just written x^μ if no confusion arises. Be aware that in general relativity, indexes generally go from 0 to $N-1$, so that we will need to actually use $\phi^\mu = \pi_{\mu+1} \circ \phi$.
The transition map defines a way to pass from one coordinate system to another. That is, if we have $p \in U_\alpha, U_\beta$, then its coordinates in both charts will be $x^\mu = \phi_\alpha(p) \in \mathcal{O}_\alpha$ and $y^\mu = \phi_\beta(p) \in \mathcal{O}_\beta$, in which case we have

$$y^\mu = (\phi_\beta \circ \phi_\alpha^{-1})(x^\mu) = \tau_{\alpha\beta}(x^\mu) \tag{2.4}$$

2.1.4 Dimensions

With this link between manifolds and $\mathbb{R}^n$, we can define the dimensionality of a manifold by referring to the dimensionality of the domain of its charts.

Definition 2.9. A manifold is said to have *dimension* n, noted $\dim M = n$, if any of its chart is to a subset of $\mathbb{R}^n$.

Since homeomorphisms between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ can only occur if $n = m$ (outside of the empty set), this means that a manifold of dimension n has all of its charts to subsets of $\mathbb{R}^n$. In other words, any manifold that isn't the empty set will only have a single dimension n. The empty set, which has the complete atlas $\{\varnothing, \varnothing\}$ will be a manifold of every dimension n.
In particular, this implies

Proposition 2.10. A non-empty manifold M cannot be homeomorphic to a manifold N unless $\dim M = \dim N$.

2.1.4.1 Manifolds of dimension 0

The set $\mathbb{R}^0$ is simply the singleton $\{b\}$. There is then obviously only one connected zero-dimensional manifold (outside of the empty set), up to homeomorphism, defined by the chart

$$(U, \phi) = (\{a\}, \langle a, b\rangle) \tag{2.5}$$

where the full atlas is the set of every possible b to map a to, and the homeomorphism between any two such manifolds is $f(a) = a'$.
Any disconnected 0-manifold will then just be a set of such charts, classified only by their cardinality.

2.1.4.2 Manifolds of dimension 1

Two simple examples of manifolds of dimension 1 are the line, which has the canonical chart $(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})$, and the circle, with the two charts $((-1,1), \phi_1)$ and $((-1,1), \phi_2)$, where the intervals $(-1,0)$ overlap and so do the $(0,1)$ interval, with transition charts

$$\tau_{12} = \begin{cases} 0 \end{cases} \tag{2.6}$$

Proposition 2.11. There are only two Hausdorff, second-countable, connected manifolds of dimension one.

Proof. □

Dimension 0 and 1 will be the only trivial cases for the classification of manifolds (although if the manifold isn't assumed paracompact, there are 4 1-dimensional manifolds, and if it is not assumed Hausdorff, there is an uncountable number of them). More details on the classification of manifolds of higher dimensions will be present later in the book.

2.1.5 Manifolds with boundaries

A generalization of manifolds that will be of some use is manifolds with boundaries

Definition 2.12. A *manifold with boundaries* is a manifold where the atlas maps subsets of the manifold to $\mathbb{H}^n$, the Euclidian half-space $\{(x_1, \ldots) | x_1 \geq 0\}$, rather than $\mathbb{R}^n$.

Also related are manifolds with corners

Definition 2.13. A *manifold with corners* is a manifold where the atlas maps subsets of the manifold to the positive section of Euclidian space $\{x \in \mathbb{R}^n | x^\mu >= 0\}$, rather than $\mathbb{R}^n$.

Proposition 2.14. Any manifold is a manifold with boundaries, any manifold with boundaries is a manifold with corners.

Proof. A simple map from $\mathbb{R}^n$ to the half space $\mathbb{R}_{\geq 0} \times \mathbb{R}^{(n-1)}$ is

$$(x_1, x_2, \ldots) \mapsto (\exp(x_1), x_2, \ldots) \tag{2.7}$$

Then we can compose this homeomorphism with the atlas maps to obtain maps from the manifold to subsets of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{(n-1)}$. The same argument applies to manifold with corners and a similar map to $\mathbb{R}^n_{\geq 0}$.

$$M \xrightarrow{\phi} \mathbb{R}^n \xrightarrow{\exp} \mathbb{R}_{\geq 0} \times \mathbb{R}^{(n-1)} \xrightarrow{\exp} \mathbb{R}^n_{\geq 0}$$

□

Proposition 2.15. The boundary ∂M of a manifold with boundaries is an $(n-1)$-dimensional manifold without boundaries.

Proof. If $p \in \partial M$ with $U \ni p$ a neighbourhood homeomorphic to $\mathbb{H}^n$, as $\partial(f(A)) = f(\partial A)$ under any homeomorphism f, we have $\phi(\partial M) \subset \partial H^n = \mathbb{R}^{n-1}$. If we restrict ϕ to ∂M, this defines a subspace homeomorphism to $\mathbb{R}^{n-1}$ and so an $(n-1)$-dimensional manifold. □

Collar neighbourhood theorem :

Theorem 2.16. For a smooth manifold M with compact boundary ∂M, there is a neighbourhood of ∂M diffeomorphic to $\partial M \times [0, 1)$

2.1.6 Homeomorphisms between manifolds

Manifolds are often referred to by the equivalence class that two manifolds are the same if they are homeomorphic, that is,

Definition 2.17. Two manifolds M and N are homeomorphic if there exists a homeomorphism $f : M \to N$ between them. It is noted $M \approx N$.

Since both the identity, the inverse of a homeomorphism and the composition of two homeomorphism are homeomorphisms, it is easy to show that $\approx$ is an equivalence relation.

2.1.7 Smooth manifolds and smooth structures

A manifold is called smooth if it is a C^k manifold for every $k \in \mathbb{N}$, or in other words if it is a C^∞ manifold. In this case, an atlas is also referred to as a smooth structure.
Whitney atlas smoothing theorem :

Theorem 2.18. For a manifold $(M, \mathcal{A}_k)$ with a C^k atlas, $k > 0$, the maximal atlas contains a C^∞ atlas on the same underlying set.

Because of this, and due to the fact that C^0 manifolds lack a lot of the important structures for general relativity (such as tangent vectors, as we will see), from now on all manifolds will be considered smooth unless otherwise specified.
Smooth structures on manifolds are not necessarily unique. Consider the two atlases on $\mathbb{R}$ $(\mathbb{R}, \phi_1)$ and $(\mathbb{R}, \phi_2)$, with chart maps

$$\phi_1 : x \mapsto x$$
$$\phi_2 : x \mapsto x^3$$

The union of those two charts do not form a smooth chart, as $\phi_1 \circ \phi_2^{-1} = \sqrt[3]{x}$ is not a smooth function at 0. Hence the maximal charts formed by those two atlases will not be equivalent.
However, it will be shown that as long as there exists a diffeomorphism between two manifolds $f : (M_1, \mathcal{A}_1) \to (M_2, \mathcal{A}_2)$, the two manifolds will be physically equivalent. From now on, unless otherwise specified, the equivalence of two manifolds $M_1 \approx M_2$ will be taken to mean that there is a diffeomorphism relating the two.

Theorem 2.19. There are homeomorphic manifolds with smooth structures that are not diffeomorphic.

The simplest example is that $\mathbb{R}^4$ is the only $\mathbb{R}^n$ that admits smooth structures that are not equivalent up to diffeomorphism (it in fact admits infinitely many). Beyond the obvious $(\mathbb{R}^4, \mathrm{Id})$ smooth structure, there exists manifolds noted $\mathbb{R}^4_\Theta$ such that there is no diffeomorphism $f : \mathbb{R}^4 \to \mathbb{R}^4_\Theta$.
The proof of this is rather long and involved
The smooth structures of manifolds will generally not be explicitly stated, but if a manifold admits more than one smooth structure (such as $\mathbb{R}^4$ or some S^n), it is assumed that the standard smooth structure is used, unless otherwise stated. For $\mathbb{R}^4$ this will be of course the smooth structure that includes $(\mathbb{R}^4, \mathrm{Id})$, while the standard smooth structure of the sphere includes the stereographic charts (U_S, ϕ_S) and (U_N, ϕ_N), with U_N and U_S the sphere with the north and south pole removed, respectively, with the charts defined by

$$\tau_{SN}(x^\mu) = (\frac{x^\mu}{\sum_\mu (x^\mu)^2}) \tag{2.8}$$

2.2 Maps between manifolds

Definition 2.20. A mapping between two manifolds M and N is a function $f : M \to N$. It is a continuous (resp. C^k, smooth) mapping at p if for every pair of charts of M and N, $\phi_{M,\alpha} : U_\alpha \to \mathcal{O}_\alpha$ and $\phi_{N,\beta} : U_\beta \to \mathcal{O}_\beta$, such that $p \in U_\alpha$ and $f(p) \in U_\beta$, the function $\phi_{N,\beta} \circ f \circ \phi_{M,\alpha}^{-1}$ is continuous (resp. C^k, smooth) at p. That property holds for the whole manifold if it is true for all $p \in M$.

Map $f : M \to N$ is smooth if for every charts in M and N, $\phi_N \circ f \circ \phi_M$ is smooth in the usual Euclidian sense
The set of all C^k mappings will be noted as $C^k(M, N)$, and the set of smooth mappings as $C^\infty(M, N)$.
We'll call the set of maps from M to the manifold $\mathbb{R}$ functions on M. If they are smooth, they'll be noted as $C^\infty(M)$ rather than $C^\infty(M, \mathbb{R})$.

Proposition 2.21. If a map f is smooth in a given atlas (U_α, ϕ_α), it is smooth.

Proof. For f to be smooth it must be smooth for every chart (U, ϕ). For any $p \in U$, there is a chart (U_α, ϕ_α) such that $p \in U_\alpha$. As it is part of the smooth structure, $\phi_\alpha \circ \phi^{-1}$ is smooth, and then so is $f \circ \phi^{-1} = (f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1})$. □

2.2.1 Diffeomorphisms

An important class of mappings between manifolds will be diffeomorphisms, which are defined in the usual way.

Definition 2.22. A smooth mapping f with an inverse f^{-1} that is also smooth is called a diffeomorphism.

The set of all diffeomorphisms from a manifold M to N will be noted $\mathrm{Diff}(M, N)$. As we saw earlier, two manifolds are considered equivalent if there exists a diffeomorphism between them. That is, for $f \in \mathrm{Diff}(M, N)$, $f(M) \approx N$. Which means that we can consider the set of diffeomorphisms on a manifold to be set of automorphisms on it.

$$\mathrm{Diff}(M, N) = \mathrm{Aut}(M) = \mathrm{Aut}(N) \tag{2.9}$$

Since $M \approx N$, we can just simplify the notation as $\mathrm{Diff}(M)$. Given that the identity is a diffeomorphism, as well as the inverse and composition of diffeomorphisms, the set of all diffeomorphism has a group structure. We will see later on the details of this group.
In the same way as for the transition between two atlases, the transition between the coordinates of two manifolds related by a diffeomorphism will be

$$y^\mu = (\phi_\beta \circ f \circ \phi_\alpha^{-1})(x^\mu) \tag{2.10}$$

This will be the most general form of coordinate transformations that we will consider. For brevity, we will usually denote it as

$$(\phi_\beta \circ f \circ \phi_\alpha^{-1})(x^\mu) = y^\mu(x^\mu) \tag{2.11}$$

A quantity we will commonly use throughout this book is the Jacobian of a coordinate change, defined by

$$J^\mu_\nu = \frac{\partial}{\partial x^\mu}(\phi_\beta \circ f \circ \phi_\alpha^{-1})(x^\nu) \tag{2.12}$$

or, in a more evocative form,

$$J^\mu_\nu = \frac{\partial y^\mu}{\partial x^\nu} \tag{2.13}$$

2.2.2 The pullback

The pullback is the composition of two maps between manifolds to form a third map, that is, if we consider the maps

$$f : M \to N,\ g : N \to P \tag{2.14}$$

the pullback of f by g, f^*g is the function

$$f^*g : M \quad \to \quad P \tag{2.15}$$

$$p \quad \mapsto \quad f \circ g(p) \tag{2.16}$$

in other words, the function f "pulls back" g from a function over N to a function over M.

2.2.3 Inclusion maps

Inclusion maps are maps between a manifold and a subset of itself.

Definition 2.23. A map $\iota : S \hookrightarrow M$ from a subset $S \subset M$ to M is an inclusion map if it is injective and its restriction on $\iota(S)$ is the identity on M

In other words, the inclusion map associates to points of a subset their equivalent point in the orignal manifold.
Subspace topology

2.3 Constructing manifolds

While it is not possible to find all possible manifolds for a dimension $n > 3$, there are many methods to construct quite a wide variety of manifolds for the study of spacetimes. We will show here a few methods to construct manifolds that will be useful for our study of spacetimes.
The first obvious example of a manifold is $\mathbb{R}^n$, using the chart $(\mathbb{R}^n, \mathrm{Id})$. One can also check that any covering of $\mathbb{R}^n$ by open sets will form a proper smooth chart, as any intersection will also be the identity on that open set.
By the same logic, any open subset O of $\mathbb{R}^n$ will also form a manifold, with the atlas (O, Id). As the empty set is itself always a subset of $\mathbb{R}^n$, it is a manifold, for every possible dimension. The n-ball (such as the disk for $\mathbb{R}^2$) and the n-cube $(0,1)^n$ are the simplest example of such manifolds.
Similarly, any open set U of a manifold is itself a manifold, as if we consider the pre-images $\phi_\alpha^{-1}(U)$ on each coordinate chart, each will form an open set that we can use as a new chart by considering the restriction of the transition maps to that set, $(\phi|_{\phi_\alpha^{-1}(U)}, \phi_\alpha^{-1}(U))$. As the complement of any closed set is an open set, removing a closed set from a manifold will also produce a manifold, and in particular, the removal of a point.

2.3.1 Abstract manifold

It is possible to define manifolds completely independently from any interpretation as another mathematical structure, simply by defining the transition maps $\tau_{\alpha\beta}$. If we have an index set I, with a subset $O_\alpha \subset \mathbb{R}^n$ for every $\alpha \in I$

2.3.2 Products of manifolds

Theorem 2.24. If M and N are manifolds of dimension m and n, there is a manifold $M \times N$ of dimension $n + m$.

Proof. The charts

$$\phi_M \times \phi_N : U_M \times U_N \to O_M \times O_N \tag{2.17}$$

maps points of the resulting manifold to open sets (with the product topology) of $\mathbb{R}^{n+m}$. □

A manifold $M = N^k$ will correspond to the k-fold product manifold $N \times N \times \ldots \times N$, of dimension $k \dim N$.

Example 2.25. A few important product manifolds are the cylinder $\mathbb{R} \times S$ and the torus $T^n = S^n$.

2.3.3 Lie groups

All Lie groups are manifolds by their very definition (cf. Appendix)
Lie's third theorem :

Theorem 2.26. Every finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ can be associated with some simply connected Lie group G.

2.3.4 Submanifolds

Definition 2.27. A subset

2.3.5 Quotient manifolds

Quotient manifolds can be defined by the folding of manifolds.

Definition 2.28. For an equivalence relation $\sim$ between points of a manifold M, the quotient space $M/\sim$ is defined to be the space of equivalence classes $[p] = \{q \in M | p \sim q\}$, with the topology

Unfortunately, this definition does not necessarily give us a manifold, for instance if we consider the equivalence relation $\sim$ on $\mathbb{R}^2$ where $(x, y) \sim (x', y')$ if both points are in the same quadrant and $x = y'$ or $x' = y$. This corresponds to an X shape that is not homeomorphic to $\mathbb{R}$ around 0.

Theorem 2.29. For a connected manifold M and a discrete group Γ that acts smoothly, freely and properly on M, then the quotient space M/Γ is itself a manifold.

Free group action : if $g(p) = g(q)$ then $p = q$ proper group action : inverse images of compact subsets are compact.
Example : $\mathbb{R}^n/\mathbb{Z}^n$,

2.3.6 Cut and pasting

It is possible to "glue" different portions of a manifold together by identifying them. There are two methods to do this : gluing along boundaries, or gluing along open sets. The generic definition of gluing a single set is the quotient again : if we have two subsets U_1, U_2 of a set A, with some homeomorphism $f : U_1 \to U_2$,

Definition 2.30. The gluing of two sets A and B is defined by $A \cup_f B = A \sqcup B/\sim$, where $x \sim f(x)$.

Proposition 2.31. The gluing of two manifolds M_1, M_2 along their boundaries is a manifold.

Proof. Consider the disjoint union $M_1 \sqcup M_2$ and the quotient $M = M_1 \cup_f M_2$ with the identification map $\pi : M_1 \sqcup M_2 \to M$. If we consider the collar of each boundary $e_i : \partial M_i \times [0, 1) \to M_i$, we can define a map

$$\begin{aligned} e : \partial M_1 \times (-1, 1) &\to M \\ (p, t) &\mapsto e(q, t) = \begin{cases} \pi(e_1(p, t)) & t \geq 0 \\ \pi(e_2(h(p), t)) & t \leq 0 \end{cases} \end{aligned} \tag{2.18}$$

As M identifies the boundary points $p \sim f(p)$, the function does indeed give one unique answer for $t = 0$. $\partial M_1 \times (-1, 1)$ defines a manifold, and from this we can define an atlas over M, by combining the coordinate charts $\phi_{M_1} \circ \pi$, $\phi_{M_2} \circ \pi$ and $\phi_{\partial M_1 \times D^1} \circ e$ [DO THE DETAILS ON THE HOMEOMORPHISMS] □

Proposition 2.32. The gluing $M_1 \cup_f M_2$ is unique up to diffeomorphism if ∂M_1 is compact.

Proof. The only non-unique part of the construction is the collar e_i of the boundary. As every collar neighbourhood around a boundary is diffeomorphic, it follows that every identical gluing will be diffeomorphic. □

Lemma 2.2. Gluing lemma : For two smooth manifolds M_1, M_2 with boundaries and a diffeomorphism $f : \partial M_1 \to \partial M_2$, consider the gluing of those two manifolds $M = M_1 \cup_f M_2$ and two smooth collars $h_i : \partial M_i \times \mathbb{R}_+ \hookrightarrow M_i$. Then :

1. There is a unique smooth structure $\mathfrak{A}(h_1, h_2)$ on M such that

2.3.7 Connected sums

The connected sum of two manifolds corresponds to the gluing of two manifold along the boundary of a hole cut out of both.

Definition 2.33. Given two manifolds of the same dimension M and N, their connected sum $M\#N$ is constructed by removing n-balls B_M and B_N from each of them and identifying the boundary ∂B_M with ∂B_N via some homeomorphism.

As the n-sphere minus an n-ball is itself an n-ball, the connected sum of a manifold with a sphere is homeomorphic to the manifold itself.
The connected sum of a manifold and a torus is referred to as the addition of a handle.
The connected sum with a non-orientable equivalent of the torus such as the Klein bottle or projective plane us referred to as the addition of an Alice handle.

Proposition 2.34. The connected sum of a manifold M and $\mathbb{R}^n$ is homeomorphic to $M \setminus \{p\}$.

Proof. □

A few examples of connected sums :

- plane with a handle $\mathbb{R}^2 \# T^2$ (homeomorphic to the punctured torus)
- The surface of genus 2 $T^2 \# T^2$

2.3.8 Exotic manifolds

Some manifolds can be fairly pathological, at least within the context of spacetimes, and are not easily constructed from basic manifold examples.
The long ray $\mathbb{L}^+$: first uncountable ordinal ω_1, $\omega_1 \times [0, 1)$, lexicographic topology, remove the smallest point
Long line $\mathbb{L}$: Long ray $\mathbb{L}^+$, define $\mathbb{L}^-$ as the long ray with the order relation reversed, glue at 0
Manifolds may also fail to obey the Hausdorff property. More details on this and many examples in section 25.4.

2.4 Topological properties

One important way to classify manifolds is by looking at their general topological properties, as many of them are preserved by diffeomorphisms, meaning that they characterize the manifold itself.

2.4.1 General topological properties

Some generic properties of topological spaces that can help the classification of manifolds

2.4.1.1 Second countability

All manifolds are first-countable (every $p \in M$ has a countable number of neighbourhoods $\{N_i\}$ of p such that for any neighbourhood $N \ni p$, there exists an $i \in \mathbb{N}$ such that $N_i \subset N$), as this property is inherited directly from $\mathbb{R}^n$. In addition, we say that a manifold is *second-countable* if it admits a countable basis, that is

Definition 2.35. A manifold is second-countable if there exists a countable set of open sets $\{\mathcal{B}_i\}$ (the basis) such that every open set U is the union of a subset of the basis.

$\mathbb{R}^n$ for instance has the countable basis $\{B_{q,r}\}$ of open balls situated at the points $q = (q_i)$, $q_i \in \mathbb{Q}$ and with rational radiuses. The long line is an example of a manifold that isn't second-countable.
Compactness
Connectedness
Hausdorff property
Paracompactness
Metrizability

2.4.2 Homotopy groups

Homotopy is a continuous deformation

Definition 2.36. A *homotopy* is a continuous map from the manifold to itself with respect to some parameter t

$$\begin{aligned} h : M \times I &\to M \\ (p, t) &\mapsto h_t(p) \end{aligned} \tag{2.19}$$

We will be interested in the behaviour of k-spheres submanifolds of M under homotopy.

Definition 2.37. The *homotopy group* of a manifold

For the manifold M, $\pi_k(M)$ is the set of homotopy classes of maps $f : S^k \to M$
$\pi_0(M)$: Number of connected components $\pi_1(M)$: Fundamental group, homotopy classes of loops (ie 1-spheres). $\pi_2(M)$:

Example 2.38. Euclidian space $\mathbb{R}^n$ has for all defined homotopy groups $\pi_k(\mathbb{R}^n) = I$

Proof. Since it is a vector space, the map $h_t(p) = (1 - t) \cdot p$ is continuous and maps every point of the manifold to 0, hence all spheres are homotopically equivalent. □

2.4.3 Betti numbers

2.4.4 The Euler characteristic

The Euler characteristic χ is a quantity defined for compact manifolds by the alternating sum of their Betti numbers, if they form a converging sum.

$$\chi = \sum_{i=0}^{n} (-1)^n b_i \tag{2.20}$$

Properties :

$$\chi(M \sqcup N) = \chi(M) + \chi(N) \tag{2.21}$$

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N) \tag{2.22}$$

$$\chi(M \# N) = \chi(M) + \chi(N) - \chi(S^n) \tag{2.23}$$

$$\chi(M \times N) = \chi(M)\chi(N) \tag{2.24}$$

For a k-sheeted covering space $\bar{M} \to M$

$$\chi(\bar{M}) = k\chi(M) \tag{2.25}$$

Fiber bundle $\pi E \to M$:

$$\chi(E) = \chi(M)\chi(F) \tag{2.26}$$

2.5 Orientation

Definition 2.39. A manifold M is *orientable* if for each pair of charts ϕ_α, ϕ_β in an atlas, the Jacobian determinant $J(\phi_\alpha, \phi_\beta) = \det(\frac{\partial x^\mu_\alpha}{\partial x^\nu_\beta})$ is positive.

Any manifold covered by a single chart (such as $\mathbb{R}^n$) is orientable, since the Jacobian of a single chart will just be $J(\phi, \phi) = \det(\delta^\mu_\nu) = 1$.
Example of a non-orientable manifold : Moebius strip, defined by the charts (I, ϕ_1), (I, ϕ_2), with $I = (-1, 1) \times (0, 1)$. The overlap of the two charts are over $(0, 1) \times (0, 1)$ and $(-1, 0) \times (0, 1)$, and vice versa. The transition functions are

$$\tau_{12}(x_1, x_2) = \begin{cases} (x_1 - 1, x_2) & x_1 \in (0, 1) \\ (x_1 - 1, 1 - x_2) & x_1 \in (-1, 0) \end{cases} \tag{2.27}$$

This has the Jacobian

$$\frac{\partial \tau_{12}}{\partial x_1} = (1, 0) \tag{2.28}$$

$$\frac{\partial \tau_{12}}{\partial x_2} = (0, \mathrm{sgn}(x_1)) \tag{2.29}$$

With determinant $\mathrm{sgn}(x_1)$, which is not positive on the whole overlap, and hence isn't orientable.

Definition 2.40. A map $f : M \to N$ is said to be orientation preserving if

2.6 Spacetime manifold

For our study of general relativity, we will have to restrict somewhat the class of manifolds to consider if we want them to admit all the necessary structures. We will say that M is a spacetime manifold, noted $\mathcal{M}$, if it is at least of dimension 2 (and to simplify matters, of finite dimension), connected, second countable, paracompact, and Hausdorff. It also specifically excludes the empty set, which vacuously fulfills those conditions.
The various requirements of the manifold are explained as follow :

Connected

While the spacetime manifold may not be connected, any disconnected component will have no influence on the physics of the component we inhabit. A rough argument for this is that if a manifold is path-connected, it is also connected. Since we can approximate a physical object as a continuous (timelike) curve, we will only be able to probe the path-connected part of this manifold. Due to this, it is generally not of great interest to consider any disconnected manifolds for a spacetime, at least classically.

Theorem 2.41. Connected components of manifolds are path-connected.

Proof. Assume M is a connected manifold. It is enough to show that a single $p \in M$ can be connected to all of M by paths. To do this, we will show that $\mathcal{C} = \{q \in M :$ there is a path from p to $q\}$ is open. It will then be closed, because its complement is a union of sets of the same form. Let $q \in \mathcal{C}$, and γ a path between p and q. Around q we may find a coordinate ball, a chart domain that is homeomorphic to the unit ball in $\mathbb{R}^n$. This is path connected since it is convex. By concatenating with a path in this ball, we can reach any point in it from p. Thus $\mathcal{C}$ is a union of open sets, hence open. □

Hausdorff

The Hausdorff condition is there both to allow analysis to be performed properly (it guarantees the uniqueness of limits). It will also along with paracompactness allow for the definition of a partition of unity, as will be seen later, allowing for the definition of a metric.
While spacetime is usually considered to be Hausdorff, there has been a few attempts at working with spacetimes that do not obey the Hausdorff property, called Y-manifolds or branching spacetimes. The analysis of such spacetimes is more complex since they do not admit bump functions for every open set, a partition of unity and their vector fields are not equivalent to derivations. Some details on this, as well as theorems on the matter, can be found in section 25.4.

Paracompact

As manifolds that are not paracompact do not admit a metric, a partition of unity, and generally have odd behaviours regarding their real-valued functions and vector fields, they are not considered viable manifolds as models of spacetime.
In particular,

Theorem 2.42. A second-countable Hausdorff manifold is paracompact.

Proof. Construction of a sequence $\{G_i\}_{i\in\mathbb{N}}$ such that it forms a cover of the manifold, $\bar{G}_i$ is compact and $\bar{G}_i \subset G_{i+1}$
As the manifold is second-countable, it topology has a countable basis $\{U_i\}$ □

2.7 Real-valued functions

Real-valued functions (generally just called functions) on a manifold are just mappings from the original manifold to the manifold $\mathbb{R}$, with the same definitions we have used previously. In other words, it is a map

$$f : \mathcal{M} \rightarrow \mathbb{R} \tag{2.30}$$

$$p \mapsto f(p) \tag{2.31}$$

Rather than denote it by $C^\infty(M, \mathbb{R})$, we will just use $C^\infty(M)$. The smoothness of a real-valued function is then defined similarly as a map between manifold

Definition 2.43. A real-valued function is smooth if for every chart (U, ϕ_U), we have that $f \circ \phi_U^{-1}$ is smooth in the usual sense of the term for functions between $\mathbb{R}^m$ and $\mathbb{R}^n$.

It will be quite often useful to also define functions on manifolds as being functions on their coordinates, in which case, for a chart neighbourhood U with map ϕ_U, the function becomes

$$f \circ \phi_U^{-1} : U \rightarrow \mathbb{R} \tag{2.32}$$

$$x^\mu \mapsto f(\phi_U^{-1}(x^\mu)) \tag{2.33}$$

We will usually simply note it $f(x^\mu)$ if no confusion arise. This form has the benefit of offering a simple definition of the derivative of a function, by simply using the usual definition of the derivative on $\mathbb{R}^n$. Hence, the derivative of a function f with respect to the coordinate x^μ on the coordinate patch U will be

$$\partial_\mu(f \circ \phi_U^{-1}) = \lim_{h\to 0} \frac{(f \circ \phi_U^{-1})(x^\mu + h) - (f \circ \phi_U^{-1})(x^\mu)}{h} \tag{2.34}$$

Real-valued functions form a ring on M, and in particular, every class of C^k functions (including C^∞ functions) form a subring of this ring.

2.7.1 Bump functions

A bump function is a function on a manifold that vanishes outside of a compact region, which is useful to generalizes local theorems to global ones. Definitions of bump functions may vary a bit, but we will use the most common one for working with manifolds :

Definition 2.44. A bump function at p is a smooth function of compact support, that is, for a given compact set $\mathrm{supp}(f)$, all points $q \notin \mathrm{supp}(f)$ are such that $f(q) = 0$, such that there exists a neighbourhood U of p where $f(U) = 1$.

We can construct a bump function on $\mathbb{R}^n$ [38], starting with the function on $\mathbb{R}$

$$f(x) = \begin{cases} \exp(-\frac{1}{x}) & x < 1 \\ 0 & x \geq 1 \end{cases} \tag{2.35}$$

This function is non-negative, smooth and positive for $x > 0$. Then we define

$$g(x) = \frac{f(x)}{f(x) + f(1-x)} \tag{2.36}$$

which is non-negative, smooth and has value 1 for $x \geq 1$, and 0 for $t \leq 0$. We then obtain a bump function by mirroring it

$$h(x) = g(2+x)g(2-x) \tag{2.37}$$

The bump function $\Pi(x^\mu)$ on $\mathbb{R}^n$ will then simply be a product of those functions.

$$\Pi(x^\mu) = h(x_1)h(x_2)\dots h(x_n) \tag{2.38}$$

More generally, we can define a bump function at any point y^μ with compact support on the ball $B_{y,R}$ by translating and rescaling this function.

$$\Pi_{y,R}(x^\mu) = \Pi(\frac{2\|x-y\|}{R}) \tag{2.39}$$

This can be generalized to arbitrary (Hausdorff) manifolds by the following theorem

Theorem 2.45. For a Hausdorff differential manifold, for every point p and every open set $U \ni p$, there's a bump function f with support $\operatorname{supp}(f) \subset U$ that is equal to 1 in an open neighbourhood V of p, $V \subset U$.

Proof. For an atlas $\{(U_\alpha, \phi_\alpha)\}$ on the manifold, with an index $\beta \in I$ such that $q \in U_\beta$, consider the neighbourhood $O' = \phi_\alpha(U \cap U_\beta)$, with the coordinates $y = \phi_\beta(p)$. For $R, \epsilon > 0$, consider the open ball $B_{y,R+\varepsilon}$ of center y and radius $R + \varepsilon$ in O'. This is always possible for a small enough radius since O' is open. We will also need the open balls $B_{y,R/2}$, $B_{y,R}$ and the closed ball $\bar{B}_{y,R}$. By Heine-Borel, $\bar{B}_{y,R}$ is compact.
We then have in M the sets

$$\begin{aligned} V &= \phi_\beta^{-1}(B_{y,R/2}) \\ W &= \phi_\beta^{-1}(B_{y,R}) \\ K &= \phi_\beta^{-1}(\bar{B}_{y,R}) = \bar{W} \end{aligned}$$

The bump function will then be defined by

$$f(p) = \begin{cases} \Pi_{y,R}(\phi_\beta(p)) & p \in U_\beta \\ 0 & p \notin U_\beta \end{cases} \tag{2.40}$$

Then we have that $f = 1$ in V, which is an open set of M. By the Hausdorff condition, if K is closed in U_β, it is also closed in M, so that $\operatorname{supp} f = K \subset U_\beta$. W is open in U_β, hence $W \cap U_\alpha$ is open for any α.
[blablabla]
Hence $f(p)$ is smooth on M.

□

2.8 Partition of unity

One important tool for the study of manifolds, to help make local structures into global ones, is the partition of unity.

Definition 2.46. A partition of unity on a manifold $\mathcal{M}$ is a set of smooth functions $\{f_\alpha | \alpha \in A\}$, $f_\alpha : \mathcal{M} \to [0,1]$ such that

- for every point p, there is a neighbourhood of p that intersects only a finite number of $\mathrm{supp}(f_\alpha)$
- $\sum_\alpha f_\alpha(p) = 1$

A particular example of a partition of unity is if we have a cover $\{U_\alpha\}$. If the partition of unity $\{f_\alpha\}$ is such that $\mathrm{supp}\, f_\alpha \subset U_\alpha$, we say that the partition of unity is subordinate to this cover.

Theorem 2.47. For a Hausdorff, second-countable manifold, every open cover has a countable partition of unity subordinate to it.

Proof. □

2.9 Curves

Definition 2.48. A curve on a manifold is a function $\gamma(\lambda)$ mapping a connected subset S of $\mathbb{R}$ to the manifold.

$$\gamma : S \to \mathcal{M} \tag{2.41}$$

The coordinates of a curve on a chart will sometimes be written as $x^\mu(\lambda) = \phi^\mu \circ \gamma(\lambda)$

Definition 2.49. A point $p \in M$ is a terminal accumulation point of λ if for every open neighbourhood U of p, and for every $\lambda_0 \in S$, there exists a $\lambda \in S$, $\lambda > 0$ such that

Definition 2.50. p is an endpoint of a curve γ if for every neighbourhood U of p there exists a $\lambda_0 \in S$ such that for all $\lambda > \lambda_0$, $\gamma(\lambda) \in U$.

While every endpoint is an accumulation point, the converse isn't true. For instance, take in $\mathbb{R}^2$ the curve

$$\begin{aligned} y(\lambda) &= \sin(\lambda^{-1}) \\ x(\lambda) &= \lambda \end{aligned}$$

0 is an accumulation point since for the open set B_r, for $\lambda_0 < r$ and $\lambda_0 = [2n\pi]^{-1}$, $\gamma(\lambda_0) \in B_r$, but if $r < 1$, this is not true for $\lambda_0 < r$ and $\lambda_0 = [(2n+1)\pi]^{-1}$. The curve leaves and re-enters any neighbourhoods of 0 no matter how close.
In general, we will only consider curves for which the terminal accumulation points are endpoints.
A curve γ is called endless if it has no endpoints.
A curve γ is inextendible if there's no curve that contains it as a proper subset.

Proposition 2.51. Inextendible curves have no endpoints.

Proof. If a curve has an endpoint $p \in M$, we can find a neighbourhood U of p. In a small enough neighbourhood, we can extend the □

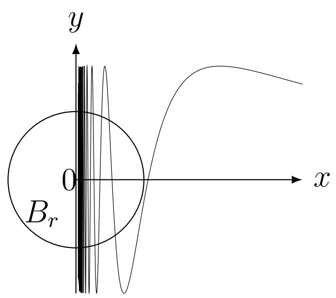


Figure 3: $\sin(x^{-1})$ with no endpoint

2.9.1 Reparametrization

For a curve γ and a smooth function $h : S' \to S$ between two connected subsets of $\mathbb{R}$, we say that $\gamma' = \gamma \circ h$ is a reparametrization of the curve γ.

Proposition 2.52. A continuous curve that contains two endpoints is compact and can be covered by a finite number of neighbourhoods.

Proof. By the definition of an endpoint, if a curve contains two endpoints, its range has an upper and lower bound, making it a closed subset of $\mathbb{R}$, which is compact. Continuous map of a compact set gives a compact set every point p of that curve admits some open neighbourhood U_p, with $U_{\gamma,p} = U_p \cap \gamma$ being a cover of the curve itself. Since it is compact, it admits a finite subcover $\{U_n\}$ such that $U_n = U_{\gamma,p}$ for some p, hence we have a finite set of neighbourhoods U_p covering the curve. □

Since a loop is an image of S^1, which is also compact, the same applies.

3 Vectors and vector fields

As manifolds will not generally be vector spaces, unless they are diffeomorphic to $\mathbb{R}^n$, we need another definition for the notion of a vector field. This can be done in a few different ways by analogy with vectors on $\mathbb{R}^n$.

3.1 Vectors as tangents of curves

The basic notion of vectors in manifold is via the notion of tangent of a curve. If we consider a curve γ, with some coordinate expression $x^\mu(\lambda)$ in some coordinate basis (U, ϕ), then this curve is a function

$$x^\mu : \mathbb{R} \to \mathbb{R}^n \tag{3.1}$$

We can then define a tangent vector in the usual way for curves in $\mathbb{R}^n$. The tangent vector to the curve γ in the chart U at the point $p = \gamma(u)$ is defined by

$$v_{\gamma,U,p} = (\frac{d}{d\lambda}\phi_U \circ \gamma(\lambda))|_{\lambda=u} \tag{3.2}$$

with $v \in \mathbb{R}^n$. Because of this, any manifold is required to be at least C^1 to admit vectors at every point.
For the expression in another chart V such that $V \cap U \neq \varnothing$,

$$v_{\gamma,V,p} = (\frac{d}{d\lambda}\phi_V \circ \gamma(\lambda))|_{\lambda=u} = (\frac{d}{d\lambda}\tau_{UV} \circ \phi_U \circ \gamma(\lambda))|_{\lambda=u} \tag{3.3}$$

So that, by the chain rule, we have

$$v_{\gamma,V,p} = \frac{d\tau_{UV}}{d\phi_U(p)} v_{\gamma,U,p} \tag{3.4}$$

If the chart U has the coordinate system $x^\mu(p) = \phi^\mu_U(p)$ and the chart V has the coordinate system $y^\mu(p) = \phi^\mu_V(p)$, the transition map will be the expression of y^μ in terms of x^μ, $\tau_{UV}(p) = y^\mu(x^\nu(p))$. This corresponds to

$$\frac{d\tau_{UV}}{d\phi_U(p)} = \frac{\partial y^\mu(x)}{\partial x^\nu} \tag{3.5}$$

This is the Jacobian matrix of the transformation of coordinates, as one could expect from a change of coordinates.
If we take a slightly more general transformation, with the use of a diffeomorphism $f \in \mathrm{Diff}(M)$, the coordinate expression of the curve in $f(M)$ will be $f \circ \phi_U \circ \gamma(\lambda)$
A tangent vector at $p \in M$ is an equivalent class of vectors v where two curves at p give rise to the same vector in the same chart if $v_{\gamma,V,p} = v_{\gamma',V,p}$, and vectors in two different charts are equivalent if they obey the transformation law.
The set of all such tangent vectors at p is the tangent space, denoted T_pM.

Proposition 3.1. The tangent space has the structure of a vector space.

Proof. Check that tangent vectors obey tangent space rules :

- The product of a tangent vector X by a scalar $a \in \mathbb{R}$ is a tangent vector : Consider the reparametrization of a curve with tangent vector X by $\lambda \to a\lambda$. By the chain rule, aX is the tangent vector of this curve.

- The sum of two tangent vectors is a tangent vector : Consider the curve in a small enough neighbourhood of p formed by the sum of the components of $\gamma_1(\lambda)$ with tangent X_1 and $\gamma_2(\lambda)$ with tangent X_2. By the linearity of the derivative the resulting curve has tangent vector $X_1 + X_2$.

□

Proposition 3.2. The vector space structure is preserved by diffeomorphism.

Proof. bla

- Multiplication by a scalar : If we have

- Sum of two vectors :

□

3.1.1 Basis

Since the tangent space at p is a (finite-dimensional) vector space, it can be equipped with a basis. Arbitrary basis are noted by a set of n tangent vectors (e_μ). We can show that given coordinates in a patch around p, there exists a basis formed from them, called the coordinate basis.
For a point p in a chart (U, ϕ), consider the family of n curves γ_ν defined by the coordinates $\phi^\mu \circ \gamma_\nu(\lambda) = (\phi^0(p), \phi^1(p), ..., \phi^\nu(p) + \lambda, ..., \phi^{n-1}(p))$, where only the ν-th coordinate varies with respect to λ (this correspond to the axis of that coordinate). This is called a coordinate curve. With this we can define

Definition 3.3. A coordinate basis vector ∂_ν at p for the coordinate chart (U, ϕ) is the tangent vector defined by the tangent of the ν-th coordinate curve going through p.

Theorem 3.4. The set of all coordinate basis vectors at p forms a basis for the tangent at p.

Proof. Since the curve is C^1 at p, by Taylor's theorem, we can expand around λ_p, the value for which $\gamma(\lambda_p) = p$:

$$\phi \circ \gamma(\lambda) = \phi(p) + (\lambda - \lambda_p)X(\lambda_p) + (\lambda - \lambda_p)^2 h(\lambda_p) \tag{3.6}$$

Since $X(\lambda_p)$ can be decomposed in some basis of $\mathbb{R}^n$, pick the canonical basis $\{e_\mu\}$ and then

$$\phi(p) + \lambda X(\lambda_p) = \phi(p) + \sum_{\mu=0}^{n} \lambda X^\mu e_\mu = \sum_{\mu=0}^{n} X^\mu \phi \circ \gamma_\mu(\lambda) \tag{3.7}$$

So that the derivative becomes

$$\begin{aligned} X &= [\sum_{\mu=0}^{n} X^\mu \frac{d}{d\lambda}(\phi \circ \gamma_\mu(\lambda)) + 2(\lambda - \lambda_p)\frac{d}{d\lambda}h(\lambda_p)]_{\lambda=\lambda_p} \\ &= \sum_{\mu=0}^{n} X^\mu \partial_\mu \end{aligned}$$

□

Hence any tangent vector can be decomposed, in Einstein notation, as $X = X^\mu \partial_\mu$, with X^μ called the components of the tangent vector. By some abuse of notation, X^μ will often be referred to as the vector itself.
The coordinate basis vectors transform in the same way as any vector
Transformation of components :

$$X^\mu \partial_\mu \to X^\mu J^{\mu'}{}_\mu \partial_{\mu'} \tag{3.8}$$

We can then define vector fields as the associations of a vector to every point of the manifold. While it is possible to define C^k vector fields with this definition of vectors (for instance by defining them as the vectors associated to a C^k foliation of the manifold by curves), it will be much simpler to do so with the definition of vectors as derivatives in the next section.

3.1.2 Left and right derivatives

As we will also consider curves that are only piecewise C^1, it will be useful to also require the usual notion of left and right derivative.
The left tangent vector $X^- = \gamma'^-$ is the vector defined by, in the coordinate patch U,

$$X^-(\lambda) = \lim_{h\to 0^-} \frac{1}{h}[\phi_U \circ \gamma(\lambda + h) - \phi_U \circ \gamma(\lambda)] \tag{3.9}$$

while the right tangent vector $X^+ = \gamma'^+$ is the vector defined by, in the coordinate patch U,

$$X^+(\lambda) = \lim_{h\to 0^+} \frac{1}{h}[\phi_U \circ \gamma(\lambda + h) - \phi_U \circ \gamma(\lambda)] \tag{3.10}$$

In the case of a piecewise C^1 curve, we may have some point $p = \gamma(\lambda)$ that $X^+(\lambda) \neq X^-(\lambda)$. Such a point is called a corner of the curve. Further properties on corners will be defined once the metric tensor is defined.

3.2 Vector fields as derivatives on smooth functions

Another way to define vector fields on manifolds is as directional derivatives on the set of smooth functions $C^\infty(M)$. While less intuitive than the definition via curves, it will be a much easier definition to deal with to prove theorems regarding them.
A derivation $\mathscr{D}$ can be defined as a function

$$\mathscr{D} : C^\infty(M) \to C^\infty(M) \tag{3.11}$$

Its properties are that, for any $a, b \in \mathbb{R}$, and $f, g \in C^\infty(M)$, we have

$$\begin{aligned} \mathscr{D}(af+bg) &= a\mathscr{D}(f)+b\mathscr{D}(g) && (3.12)\\ \mathscr{D}(fg) &= g\mathscr{D}(f)+f\mathscr{D}(g) && (3.13) \end{aligned}$$

which correspond to the linear property and the Leibniz property of derivatives. We will note the set of all derivations on M by $\mathrm{Der}(C^\infty(M))$, or $\mathrm{Der}(M)$ for short.

3.2.1 Germ of a function

First, if we wish to define individual vectors, we will need the notion of a germ of a function, which is an equivalence relation

Lemma 3.1. For $v \in T_pM$, if $f, g \in C^\infty(M)$ such that $f = g$ in a neighbourhood U of p, then $v(f) = v(g)$.

Proof. By linearity, $v(f) = v(g)$ is equivalent to $v(f - g) = 0$, so that we need only to show it true for a function locally equal to 0. For a bump function g with support in u, we have $fg = 0$ on all of M. By the properties of the derivation

$$[v(fg)](p) = g(p)[v(f)](p) + f(p)[v(g)](p) = 0 \tag{3.14}$$

Since $f(p) = 0$ and $g(p) = 1$, we have $[v(f)](p) = 0$. □

Lemma 3.2. For $v \in T_pM$ and a constant function h on a neighbourhood of p, $v(h) = 0$.

Proof. By the previous lemma, $v(h)$ will be the same as $v(c)$, for a constant function of value c, or the function $c \cdot \mathrm{Id}$. Then we have

$$v(c\,\mathrm{Id}) = cv(\mathrm{Id}) = cv(\mathrm{Id}\cdot\mathrm{Id}) = 2cv(\mathrm{Id})\,\mathrm{Id} = 2cv(\mathrm{Id}) \tag{3.15}$$

Which means that $v(\mathrm{Id}) = 0$ and so is $v(h)$. □

3.2.2 Vector fields

A vector field is defined as a derivation of smooth functions $C^\infty(M)$. The easiest way to picture this derivative as defining vectors fields is to consider an expression of one :

$$[V(f)](p) = \sum_\mu V^\mu(p)\frac{\partial}{\partial x^\mu}(f \circ \phi^{-1}(p)) \tag{3.16}$$

It can easily be checked that this is a derivation. We will see that in a chart, all vector fields can be expressed in such a way, with n functions $V^\mu(p)$ corresponding to the components and a basis ∂_μ.
Vector space structure on $\mathrm{Der}(M)$: we define the addition and product of derivatives, for $\mathscr{D}_1, \mathscr{D}_2 \in \mathrm{Der}(M)$

$$\begin{aligned} (\mathscr{D}_1+\mathscr{D}_2)(f) &= \mathscr{D}_1(f)+\mathscr{D}_2(f)\\ (a\mathscr{D}_1)(f) &= a\mathscr{D}_1(f) \end{aligned}$$

Vector field is smooth on an open set $U \in M$ if for every $f \in C^\infty(V)$, $V(f)$ is $C^\infty(U \cap V)$. Likewise, a vector field is $C^k(U)$ if for every $f \in C^\infty(V)$, $V(f)$ is a function $C^k(U \cap V)$. Tangent bundle :

$$TM = \bigsqcup_{p \in \mathcal{M}} T_p M \tag{3.17}$$

We will delve into the tangent bundle in greater details, such as its topology, in the chapter on fiber bundles. For now we will only consider it as the set of all tangent spaces. As it is the association of a tangent space to every point of the manifold, we can identify the space of vector fields $\mathfrak{X}(M)$ with the tangent bundle. Hence a vector field will just be a map $X : M \to TM$.

3.2.3 The coordinate basis

The basis of a vector field considered as a derivative is much simpler to consider. Take the basis vectors ∂_μ in a coordinate patch U with coordinates x^μ defined by

$$\partial_\mu(f) = \frac{\partial}{\partial x^\mu}(f \circ \phi_U^{-1})|_{\phi(p)} \tag{3.18}$$

As a derivative, it trivially fulfils the properties of a derivation.

Proposition 3.5. The coordinate basis forms a basis of the tangent space T_pM, such that any vector can be written as

$$X = X^\mu(p)\partial_\mu|_p \tag{3.19}$$

Proof. Since $X(c) = 0$, we can assume that $\phi(p) = 0$. If g is a smooth function on U, we define for $i \in \mathbb{N}$

$$g_\nu(x^\mu) = \int \frac{\partial g}{\partial x^\nu}(tx^\mu)dt \tag{3.20}$$

$$g = g(0) + g_\nu x^\nu \tag{3.21}$$

If we set $g = f \circ \phi$

$$f = f(p) + f_\nu x^\nu \tag{3.22}$$

□

Proposition 3.6. If the chart of a manifold is C^∞, the components of a C^k vector field are $C^k(M)$ functions.

Proof. A C^k vector field maps smooth functions to C^k functions, hence for a C^k vector field V,

$$V[f] = V^\mu \partial_\mu[f] = V^\mu \frac{\partial}{\partial x^\mu}(f \circ \phi_U^{-1})|_{\phi(p)} \tag{3.23}$$

As f and ϕ are smooth, $V[f]$ will only be C^k if V^μ is □

Conversely,

Proposition 3.7. For an atlas (U_α, ϕ_α) and a family of C^k functions V^μ_α, each defined on U_α, the vector fields $V^\mu_\alpha \partial_\mu$ on each U_α form a single C^k vector field on M if and only if, on $U_\alpha \cap U_\beta$, we have

$$V^\mu_\beta = J^\mu_\nu(\alpha, \beta) V^\nu_\alpha \tag{3.24}$$

Corrolary 3.1. Any family of n C^k functions on a manifold covered by a single chart defines a vector field.

Proposition 3.8. A C^k manifold is only guaranteed C^k vector fields at most.

Proof. Since the application of a vector field to a function $V(f)$ is represented by the $R \to \mathbb{R}^n$ function $V(f) \circ \phi^{-1}$ for differentiability, it will be at most C^k. □

3.3 Link between the two

Theorem 3.9. On a Hausdorff manifold, there is a canonical isomorphism

$$\tau : \mathfrak{X} \to \mathrm{Der}(M) \tag{3.25}$$

between the $C^\infty(M)$-module of vector fields on M and the derivatives of $C^\infty(M)$.

3.4 The pushforward

If we have a map ϕ between two manifolds M and N, we can map scalar fields to N via the pullback. In addition, it is possible to uniquely map vector fields via the pushforward, or differential map.

Definition 3.10. For a map $\phi : M \to N$ a C^k map between two C^k manifolds, and a C^{k-1} vector field X on M, the pushforward of X by ϕ is done by the *differential map* $d\phi$, defined by

$$\begin{aligned} d\phi : TM &\to TN \\ [d\phi(X)](f) &\mapsto X(\phi^* f) \end{aligned}$$

Described by the commuting diagram

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow{\scriptstyle \pi_{TM}} & & \downarrow{\scriptstyle \pi_{TN}} \\ M & \xrightarrow{f} & N \end{array}$$

Figure 4: Diagram of a differential map

The differential map is also sometimes noted by ϕ_* or Tf, but we will stick with the df notation here.
If we get a basis ∂_μ in a coordinate patch of TM and a basis ∂_a in a coordinate patch of TN, the components of the differential maps are

$$[df(X^\mu \partial_\mu)] = [df(\partial_\mu)]^a X^\mu \partial_a \tag{3.26}$$

Written as

$$[df(\partial_\mu)]^a = df^a_\mu \tag{3.27}$$

$\dim(M) \times \dim(N)$ dimensional matrix

Definition 3.11. The rank of a linear transformation is the dimension of its image

Properties of the differential :

$$d : C^\infty \to \mathfrak{X}^*(M) \tag{3.28}$$

d is linear, obey the Leibniz rule, chain rule

3.5 Integral curves and vector flow

For any vector field X, we can associate a family of curves γ obeying the differential equation

$$\frac{d}{d\lambda}\gamma(\lambda) = X(\gamma(\lambda)) \tag{3.29}$$

meaning that at parameter λ, and at the point $p = \gamma(\lambda)$, the curve γ will have the tangent vector $X(p)$. Thus, for every point p, there is a curve $\gamma(\lambda)$ going through that point with tangent vector $X(p)$. By the Picard–Lindelöf theorem, if X is uniformly Lipschitz continuous, that curve is locally unique given some initial point p such that $\gamma(0) = p$, $\gamma'(0) = X(p)$.
This allows us to define the flow of a vector field.

Definition 3.12. Given a vector field on $U \subset M$, the *flow* of a point p by the vector field X at t is the point $\gamma(t)$ of the integral curve γ of tangent X for $\gamma(0) = p$. It is noted by the function

$$\begin{aligned} \Phi : M \times \mathfrak{X} \times \mathbb{R} &\to M \\ (p, X, t) &\mapsto \Phi^X_t(p) = \gamma(t) \end{aligned}$$

Flow of a complete vector field X :
such that γ is the maximal integral curve at p with initial tangent vector X_p.

3.6 Lie brackets

Since we can define vector fields as derivatives, it is then possible to chain them by

$$XY(f) = X(Y(f)) \tag{3.30}$$

for X, Y two vector fields on $\mathcal{M}$. Unfortunately, XY does not define a vector field for it does not obey the Leibniz property :

$$\begin{aligned} XY(fg) &= X(Y(fg)) = X(gY(f) + fY(g)) && (3.31)\\ &= X(gY(f)) + X(fY(g)) && (3.32)\\ &= Y(f)X(g) + gX(Y(f)) + fX(Y(g)) + Y(g)X(f) && (3.33)\\ &= (gXY(f) + fXY(g)) + Y(f)X(g) + Y(g)X(f) && (3.34) \end{aligned}$$

But we can notice that conversely,

$$YX(fg) = (gYX(f) + fYX(g)) + Y(f)X(g) + Y(g)X(f) \tag{3.35}$$

meaning that the commutator of two vector fields

$$[X,Y] = XY - YX \tag{3.36}$$

will define itself a vector field. We call this the Lie bracket, corresponding to the map

$$[X,Y](f) = X(Y(f)) - Y(X(f)) \tag{3.37}$$

Alternatively, if we consider vector fields as tangent of curves, we can define the Lie brackets in terms of the flow Φ_λ^X associated with the vector field X, in which case

$$[X,Y](x) = \lim_{t\to\infty} \frac{(d\Phi_{-\lambda}^X) - Y(\Phi_\lambda^X)}{t} \tag{3.38}$$

As the vector field acts linearly on functions, if we express the vectors in the coordinate basis, we have

$$\begin{aligned} [X,Y](f) &= [X^\mu\partial_\mu, Y^\nu\partial_\nu](f) \\ &= X^\mu\partial_\mu(Y^\nu\partial_\nu(f)) - Y^\nu\partial_\nu(X^\mu\partial_\mu(f)) \qquad (3.39) \\ &= (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\mu)\partial_\nu(f) \qquad (3.40) \end{aligned}$$

Since a coordinate vector field has constant components, the Lie brackets of coordinate vector fields are zero.

3.6.1 Properties

Since vector fields are linear, so are Lie brackets

$$[aX + bY, Z] = a[X,Z] + b[Y,Z] \tag{3.41}$$

As with all commutators, Lie brackets anticommute

$$[X,Y] = -[Y,X] \tag{3.42}$$

which implies that $[X,X] = 0$.
Lie brackets obey the Jacobi identity :

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \tag{3.43}$$

which can be proven by expanding it

$$\begin{aligned} [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] &= XYZ - XZY - YZX + ZYX \\ &+ ZXY - ZYX - XYZ + YXZ \\ &+ YZX - ZXY - YXZ + ZYX \end{aligned}$$

It can be checked that every term is cancelled out.
With those properties, vector fields coupled with the Lie bracket form a Lie algebra.

3.7 Dual vectors

Dual vectors are defined as the set of linear functions on vectors, that is, for a dual vector ω belonging to the space of dual vectors at p T_p^*M, it is defined by

$$\begin{aligned} \omega : T_pM &\to \mathbb{R} \\ X &\mapsto \omega(X) \end{aligned} \tag{3.44}$$

obeying the linear property

$$\omega(aX + bY) = a\omega(X) + b\omega(Y) \tag{3.45}$$

Theorem 3.13. The space of dual vectors T_p^*M is a vector space of the same dimension as TM.

Proof. Sum of two dual vectors : The action of $\omega_1 + \omega_2$ on a vector X is $\omega_1(X) + \omega_2(X)$ (still a linear transformation) Product of dual vector with a scalar $a \in \mathbb{R}$: Defined by $a\omega$ acting on X as $a\omega(X)$
As the tangent space itself forms a vector space, vectors can be expressed in a basis e_μ by $X^\mu \partial_\mu$. We define the dual vectors θ^μ by $\theta^\mu(e_\nu) = \delta^\mu_\mu$. Then we can decompose ωX as

$$\omega(X^\mu e_\mu) = X^\mu \omega(e_\mu) \tag{3.46}$$

If each $\omega(e_\mu)$ has the value ω_μ, then we can write ω as

$$\omega = \omega_\mu \theta^\mu \tag{3.47}$$

Then $\{\theta^\mu\}$ is a basis that spans the entire dual vector space. □

Corrolary 3.2. The dual tangent space T_p^*M is of the same dimension as T_pM, and hence as M.

A particularly useful basis for the dual tangent space is the dual of the coordinate basis, noted dx^μ, such that $dx^\mu(\partial_\nu) = \delta^\mu_\nu$

3.8 Tensor algebra

Once we have vector spaces and their duals defined on every point of the manifold, we can construct additional structures from it, first among which are tensors.

3.8.1 The tensor product

The *tensor product* of two vector spaces V, W, noted $V \otimes W$, is defined as an equivalence class on the Cartesian product $V \times W$, with the following vector space structure. If $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, then

$$\begin{aligned} (v_1, w) + (v_2, w) &\sim (v_1 + v_2, w) \\ (v, w_1) + (v, w_2) &\sim (v, w_1 + w_2) \\ c(v, w) &\sim (cv, w) \\ c(v, w) &\sim (v, cw) \end{aligned}$$

In the case of the tensor product of two dual vector spaces, we can apply two vectors to it in the following way. For $(\omega_1, \omega_2) \in V^* \otimes V^*$ and $v_1, v_2 \in V$,

$$(\omega_1, \omega_2)(v_1, v_2) = (\omega_1(v_1), \omega_2(v_2)) \tag{3.48}$$

If we express the vectors in a basis, by bilinearity

$$(V^\mu e_\mu, W^\nu e_\nu) = V^\mu W^\nu (e_\mu, e_\nu) \tag{3.49}$$

3.8.2 The tensor algebra

Tensors on a vector space are defined by the application of the tensor product $\otimes$, which forms the tensor algebra $T(V) = (V, \otimes)$.
Tensor space of rank (r, s) :

$$T^r_s(V) = V^{\otimes r} \otimes V^{*\otimes s} \tag{3.50}$$

$$T^r_s \otimes T^{r'}_{s'} = T^{r+r'}_{s+s'} \tag{3.51}$$

Our tensor algebra at p will be defined with the two vector spaces T_pM and T^*_pM

Proposition 3.14. A tensor of rank (r, s) is a multilinear map from $V^{\otimes r}$ to $V^{*\otimes s}$.

Proposition 3.15. There's an isomorphism between tensors of rank (r, s), $r + s = 2$, and matrices $M \in \mathrm{Mat}_{n\times n}$.

Proof. □

3.9 Quadratic forms

Real quadratic form :

$$\begin{aligned} Q : V \times V &\to \mathbb{R} \\ X, Y &\mapsto Q(X, Y) \end{aligned}$$

Bilinear

Theorem 3.16. For a given basis $\{e_\mu\}$ of V, there is a unique matrix $\mathbf{Q} \in \mathrm{M}_{n\times n}$ associated to a quadratic form such that $Q(X, Y) = X^T \mathbf{Q} Y$.

Proof. X in the basis : $X = \sum_\mu X^\mu$,

$$Q(X, Y) = X^\mu Y^\mu Q(e_\mu, e_\nu) \tag{3.52}$$

Define $Q_{\mu\nu} = Q(e_\mu, e_\nu)$

□

3.9.1 Eigenvectors of a quadratic form

$$\mathbf{Q}X = \lambda X \tag{3.53}$$

$$\det(Q - \lambda I) = 0 \tag{3.54}$$

Theorem 3.17.

3.9.2 Signature of a quadratic form

A quadratic form is said to be of signature (p, q, r) if it has p positive eigenvalues, q negative eigenvalues and r zero eigenvalues.

3.10 Exterior algebra

On any vector space V we can define the exterior algebra defined by $(V, \wedge)$, with $\wedge$ the exterior product defined by

$$\begin{aligned} X \wedge Y &= -Y \wedge X \\ (aX_1 + bX_2) \wedge (cY_1 + dY_2) &= acX_1 \wedge Y_1 + adX_1 \wedge Y_2 \\ &+ bcX_2 \wedge Y_1 + dbX_2 \wedge Y_2 \end{aligned}$$

Interior product :

$$\begin{aligned} i : V \times \Lambda^p &\to \Lambda^{p-1} \\ (v, \omega) &\mapsto i_v\omega \end{aligned}$$

$$(i_v\omega)(v_1, ..., v_{p-1}) = \omega(v, v_1, ..., v_{p-1}) \tag{3.55}$$

4 Lorentz vector spaces

Lorentz vector spaces are the setting used for special relativity.
It will be the structure we will later use for the tangent space $T_p\mathcal{M}$ of every point.
Making those structures global : chapter on fiber bundles and the metric tensor.

4.1 Lorentz inner product

Many of the deep results of general relativity lie in the subtle properties of inner product spaces that allow for negative norms.

Definition 4.1. Let V be an $(n+1)$-dimensional real vector space. A *Lorentzian inner product* on V is a nondegenerate symmetric bilinear map $g : V \times V \to \mathbb{R}$ and vectors $e_0, \ldots e_n$ such that $g(e_0, e_0) = -1$, $g(e_i, e_i) = 1$ for all i, and $g(e_\mu, e_\nu) = \delta_{\mu\nu}$ for all μ, ν. The pair (V, g) is a *Lorentzian vector space.*

From linear algebra it is known that there are $\varepsilon^\mu \in V^*$ such that

$$g = -\varepsilon^0 \otimes \varepsilon^0 + \sum_{i=1}^{n} \varepsilon^i \otimes \varepsilon^i.$$

Since g is nondegenerate, it furnishes an isomorphism $V \to V^*$ by $x \mapsto g(x, \cdot)$. We will commonly denote g by $\langle \cdot, \cdot \rangle$. We define the norm as $|x| = |\langle x, x \rangle|^{1/2}$.
The components of this inner product in the basis $\{e_\mu\}$ are expressed by $\eta_{\mu\nu} = \eta(e_\mu, e_\nu)$, given by the relation

$$\eta(v^\mu e_\mu, w^\nu e_\nu) = v^\mu w^\nu \eta(e_\mu, e_\nu) = v^\mu w^\nu \eta_{\mu\nu} \tag{4.1}$$

4.2 Symmetries of Lorentz vector spaces and the Poincaré group

We will define the group of all symmetries that keep the norm of the metric tensor invariant by the Lorentz group $\mathrm{O}(1, n)$. To construct it, consider first the transformation $\Lambda \in \mathrm{GL}(n, \mathbb{R})$. We want the subgroup obeying

$$x \tag{4.2}$$

$$\Lambda^T g \Lambda = g \tag{4.3}$$

This is equivalent to the invariance of the norm $g(x, y)$

$$(\Lambda x)^T \Lambda^T g \Lambda (\Lambda y) = x^T g y \tag{4.4}$$

$$\eta_{\mu\nu} \to \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{4.5}$$

$$\eta_{\mu\nu} v^\mu v^\nu \to \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\Lambda^\alpha{}_\rho v^\rho)(\Lambda^\beta{}_\sigma v^\sigma) \tag{4.6}$$

The members of the Lorentz group will be defined as the linear transformations that leave the metric invariant.

$$\mathrm{O}(1,n) = \{\boldsymbol{\Lambda} \in \mathrm{Mat}_{n\times n} \,|\boldsymbol{\Lambda}\eta\boldsymbol{\Lambda} = \eta\} \tag{4.7}$$

in which case we have

$$\begin{aligned} \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}(\Lambda^{\alpha}{}_{\rho}v^{\rho})(\Lambda^{\beta}{}_{\sigma}v^{\sigma}) &= \eta_{\alpha\beta}(\Lambda^{\alpha}{}_{\rho}v^{\rho})(\Lambda^{\beta}{}_{\sigma}v^{\sigma}) \\ &= \eta_{\rho\sigma}v^{\rho}v^{\sigma} \end{aligned} \tag{4.8}$$

which shows that it indeed preserves the Lorentzian inner product.

Proposition 4.2. The Lorentz group is a Lie group.

Proof. Something something preimage theorem, subset of $\mathbb{R}^{n^2}$ □

Lie group with Lie algebra of the Lorentz algebra
4 connected components to the Lorentz group : Connected to the identity, time-reversed, space-reversed and time and space reversed

$$\begin{array}{ccc} \mathrm{O}(n,1) & \longrightarrow & \mathrm{SO}(n,1) \\ \downarrow & & \downarrow \\ \mathrm{O}^{\uparrow}(n,1) & \longrightarrow & \mathrm{SO}^{\uparrow}(n,1) \end{array}$$

Figure 5

4.2.1 Lie algebra of the Lorentz group

4.2.2 Representation of the Lorentz group

Connected to the identity :

$$\Lambda^{\mu}{}_{\nu} = e^{i\omega J} \tag{4.9}$$

Complexified basis of the Lie algebra :

$$\begin{aligned} A_i &= \frac{J_i + iK_i}{2} \\ B_i &= \frac{J_i - iK_i}{2} \end{aligned}$$

Proposition 4.3. The algebra basis $\{A_i\}$ and $\{B_i\}$ are each basis of the $\mathfrak{su}(2)$, and commute with each other.

Proof.

$$\begin{aligned} [A_i, A_j] &= \frac{1}{4}\left[J_i + iK_i, J_i + iK_i\right] \\ &= \frac{1}{4}([J_i, J_i] + i\,[K_i, J_i] + i\,[J_i, K_i] - [K_i, K_i]) \end{aligned} \tag{4.10}$$

□

4.2.3 Subgroups of the Lorentz group

Identity, $\{\pm I\}$, discrete rotations and boosts, $\mathrm{O}(k)$, $\mathrm{O}(k, 1)$

4.3 Classification of vectors

Vectors in a Lorentzian vector space can be classified by the sign of their norm.

- timelike if $g(v, v) > 0$
- null if $g(v, v) = 0$
- spacelike if $g(v, v) < 0$

We will also say that a vector is causal if it is either null or timelike, that is, $g(v, v) \geq 0$.

Proposition 4.4. We have the following relations among timelike and null vectors:

1. Two timelike vector are never orthogonal
2. A timelike vector is never orthogonal to a null vector
3. Two null vectors are orthogonal if and only if they are proportional.

Proof. 1. Clear from 1.4 (a).

2. Clear from 1.4 (b).
3. Let l_1 and l_2 be null and orthogonal. Fix a unit timelike vector u; we have

$$l_1 = \lambda_1 u + y_1 \quad \text{and} \quad l_2 = \lambda_2 y + y_2,$$

where $\langle u, y_i \rangle = 0$. Orthogonality of l_1 and l_2 gives $-\lambda_1\lambda_2 + \langle y_1, y_2 \rangle = 0$. Since $\lambda_1^2 = g\langle y_1, y_1 \rangle$ and $\lambda_2^2 = \langle y_2, y_2 \rangle$, we obtain

$$\langle y_1, y_1 \rangle \langle y_2, y_2 \rangle = \langle y_1, y_2 \rangle^2.$$

Since g is positive definite on $\mathbb{R}y_1 \oplus \mathbb{R}y_2$, we have $y_2 = \lambda y_1$. This implies $\lambda_2 = \lambda\lambda_1$, and we are done.

□

4.3.1 Subspaces

As an n-dimensional vector space, a Lorentz vector space admits a number of subspaces. They can be classified by the class of vectors it contain.

Definition 4.5. Let $V \subset W$ be a subspace. We have three possibilities:

1. W is *spacelike* if $g \restriction W$ is positive definite.
2. W is *null* or *lightlike* if $g \restriction W$ is strictly positive semi-definite.
3. W is *timelike* if neither (i) or (ii) hold.

If $W \subset V$ is a subspace, we define $W^\perp$, the orthogonal subspace, to be $\{v \in V : \langle w, v\rangle = 0 \ \forall w \in W\}$.

Proposition 4.6. Let $W \subset V$ be a subspace.

1. W is timelike if and only if $W^\perp$ is spacelike.
2. W is spacelike if and only if $W^\perp$ is timelike.
3. W is null if and only if $W \cap W^\perp \neq \{0\}$, or equivalently, if and only if $W^\perp$ is null.

Proof. 1. W contains a unit timelike vector w and V splits as $S \oplus \mathbb{R}w$. Since g can be negative definite on a subspace of at most dimension 1, we conclude that g is positive definite on S. The forward direction follows. For the converse, suppose $W^\perp$ is spacelike. We have a splitting $V = W \oplus W^\perp$. So if $v \in V$ is timelike then $v = w + w'$ where $w \in W$ and $w' \in W^\perp$. Then $\langle w, w\rangle = \langle v, v\rangle - \langle w', w'\rangle < 0$, so w is timelike and hence W is timelike.

2. Follows from (a) and $W^{\perp\perp} = W$.

3. W lightlike implies W contains a lightlike vector w_0, but no timelike vector. Then for all $a \in \mathbb{R}$ and $w \in W$, $\langle w + aw_0, w + aw_0\rangle = \langle w, w\rangle + 2a\langle w, w_0\rangle \geq 0$. Choosing a negative enough, we see that $\langle w, w_0\rangle = 0$ for all $w \in W$, so $w_0 \in W \cap W^\perp$. On the other hand, if $0 \neq w_0 \in W \cap W^\perp$, then w_0 is null. Since W cannot contain a timelike vector by (a), W is null. The other part follows from $W^{\perp\perp} = W$.

□

4.3.2 Light cones

The timelike vectors of a Lorentz vector space can be split in two categories

Definition 4.7. for a vector v, $v^\perp$ is the set of all orthogonal vectors to v defined by $v^\perp = \{u \in T_p\mathcal{M} | g(u, v) = 0\}$

Theorem 4.8. if u is timelike, the subspace $u^\perp$ is spacelike.

Proof. If we have a timelike vector u and a vector v such that $g(u, v) = 0$, and we switch the basis of $T_p\mathcal{M}$ to (u, v, x, y), the new metric will be $(g(u, u), g(v, v), g(x, x), g(y, y))$. The signature will be correct only if v is spacelike, as Sylvestere's law dictates that a change of basis cannot change the signature. □

Theorem 4.9. There are two disjoint sets of timelike vectors at p.

Proof. If we pick any timelike vector u in $T_p\mathcal{M}$, let's define the future lightcone :

$$\mathfrak{T}^+(u) = \{v \in T | g(u, v) < 0\} \tag{4.11}$$

And the past lightcone

$$\mathfrak{T}^-(u) = \{v \in T | g(u, v) > 0\} \tag{4.12}$$

Obviously $\mathfrak{T}^+(u) \cap \mathfrak{T}^-(u) = \varnothing$, and $g(u, v) = 0$ is impossible since only spacelike vectors are orthogonal to timelike vectors. □

We will call timelike (resp. null, causal) vectors in C^+ future-directed timelike (resp. null, causal) vectors, and past-directed if they are in C^-.

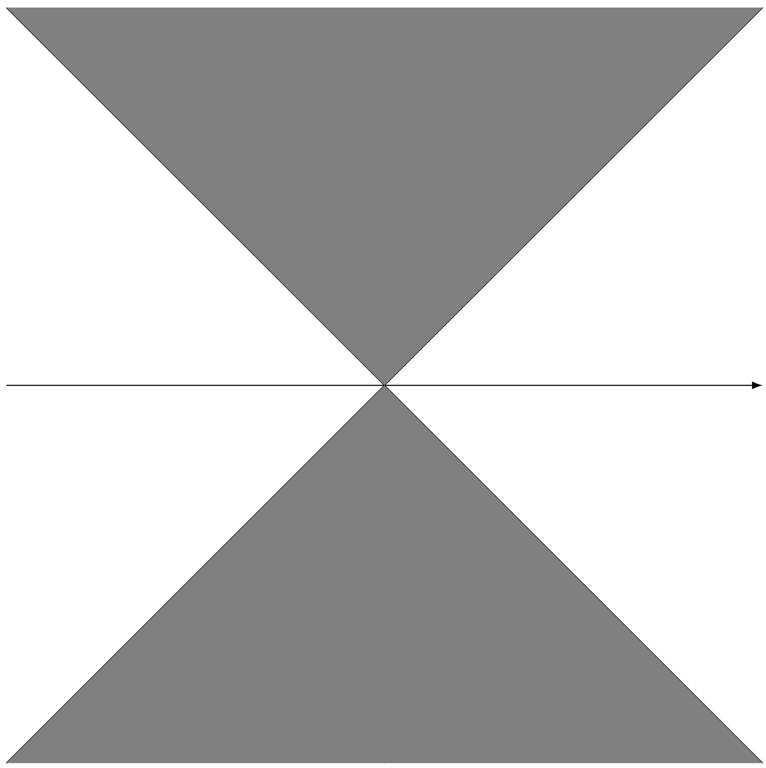


Figure 6: The light cone in $(\mathbb{R}^2, \eta)$

4.3.3 Properties

Proposition 4.10. The sum of two timelike vectors ξ_1, ξ_2 belonging to the same light cone

Definition 4.11. A tensor T is said to be future (resp. past) if $T(u_1, \ldots, u_n) \geq 0$ (resp. ≤ 0) for all future-directed vectors u_i. A causal tensor is a tensor which is either future or past.

Generalizes to T^r_s tensors and tensor fields.

4.4 Basis of the tangent space

As the product of two vectors remain constant under Lorentz transformation, so will the basis remain orthonormal under it. This will allow us to define new more appropriate basis.
We may then define timelike (resp. null, spacelike) vectors in this new orthonormal basis as $v = -|v|\partial_0$ with ∂_0 a timelike vector, $v = |v|\partial_1$ with ∂_1 a spacelike vector, or $v = \partial_u$, with ∂_u a null vector.

$$v = v^0 \partial_0 + \vec{v} \cdot \vec{e} \tag{4.13}$$

For a spacelike vector : Lorentz boost to the $\vec{v}$ direction

$$v \to \tag{4.14}$$

Canonical form of a timelike, null and spacelike vector : $\xi = \pm\partial_t$, $k = \pm(\partial t + \partial_{x_1})$, $s = \partial_{x_1}$ every vector can be expressed in that form up to rotation, dilation and reflection of the basis

4.5 Analysis on vectors

Theorem 4.12. Any null vector k can be expressed as the convergence of a sequence of timelike vectors.

Proof. If we write k in its canonical form $k = \partial_t + \partial_x$, we can use the sequence

$$\xi_n = \partial_t + (1 - n^{-1})\partial_x \tag{4.15}$$

which is always of negative norm $|\xi_n| = -n^{-1}$. It can be easily seen that this sequence of vectors converges to the desired null vector.

$$|\xi_n - k| = |-n^{-1}\partial_x| = n^{-1} \tag{4.16}$$

□

5 Fiber bundles

5.1 Definitions

A bundle is a way to add structures to a manifold M, by means of attaching to every point of the manifold another manifold. This is expressed by the fact that we have a manifold E called the total space (the original manifold plus the structures added to it) which possesses a projection function π, a continuous surjection, defined by

$$\pi : E \to M \tag{5.1}$$

A bundle can thus be expressed by the triplet (E, π, M), generally noted as $\pi : E \to M$. It can also be noted by the total space E, or, since two different bundles can have the same total space and base space, π. As there will be no confusion here, we will simply use E.
A bundle is called a fiber bundle if the bundle is locally homeomorphic to the product of the base manifold and the fiber $\mathcal{F}$, another manifold. In other words, for every point p of the base manifold, there's a neighbourhood $V \ni p$ such that there exists the homeomorphism

$$\psi : V \times \mathcal{F} \to \pi^{-1}(V) \tag{5.2}$$

in addition, the fibers at p :

$$\forall p \in M,\ \pi^{-1}(p) = \mathcal{F}_p \tag{5.3}$$

are all isomorphic to the same manifold $\mathcal{F}$, the typical fiber.

Example 5.1. The simplest example of a fiber bundle, called the trivial bundle, is the cartesian product of the manifold with any space F, $M \times F$, which has the projection function

$$\begin{aligned} \pi : M \times F &\to M \\ (p, f) &\mapsto p \end{aligned} \tag{5.4}$$

and the single local trivialization neighbourhood of $M \times F$, with the identity as its map.

Definition 5.2. For a fiber bundle $\pi : E \to M$, with $\dim E = n$, $\dim M = n$, an *adapted coordinate system* is a map

$$y : U \subset E \to \mathbb{R}^{m+n} \tag{5.5}$$

such that for $a, b \in U$, $\pi(a) = \pi(b) = p$, we have $\pi_1(y(a)) = \pi_1(y(b))$.

In other words, two points in the same fiber share the same adapted coordinates on the base manifold. For a point $a \in E$ where $\pi(a) = p$, $y(a) = (\phi(p), u)$, with

Theorem 5.3. A fiber bundle over a manifold is itself a manifold.

Proof. The adapted coordinates of a fiber bundle give us a natural manifold structure. If we take a coordinate patch $\phi_M : U_M \subset M \to \mathbb{R}^n$ and $\phi_F : U_F \subset F \to \mathbb{R}^m$, for every point $a \in E$, there exists a local trivialization of its neighbourhoods $U_a \ni a$ such that ψ
Trivializations form a cover something □

Example : A quotient manifold $\bar{M} = M/\Gamma$ with a map $\phi : M \to \bar{M}$ is a fiber bundle with base manifold $\bar{M}$, total space M, projection map $\pi = \phi$ and typical fiber Γ.

5.1.1 Bundle morphisms

5.1.2 Sections of a bundle

The sections of a bundle E are assignments of a point in the fiber for every point of the base manifold.

Definition 5.4. A local section of a fiber bundle $\pi : E \to M$ is a map $s : U \to E$, $U \subset M$ and open, such that $\pi \circ s = \mathrm{Id}_U$. A global section is the case where $U = M$.

It is generally not possible to have a global section for a given fiber bundle.
The set of all sections of a bundle E is noted $\Gamma(E)$. We say that a section of a bundle is smooth if the map s is smooth.

Theorem 5.5. If the fiber F is contractible, there exists a global section.

Proof. □

5.1.3 Structure groups

The cover $\{V_\alpha\}$ of the manifold and its associated local trivializations $\psi_\alpha : V_\alpha \times \mathcal{F} \to \pi^{-1}(V)$ is equipped with a group structure, such that, for $V_\alpha \cap V_\beta \neq \varnothing$, the homeomorphism

$$\psi^{-1}_{\beta,p}\psi_{\alpha,p} : \mathcal{F}_p \to \mathcal{F}_p \tag{5.6}$$

is a member of a group G, $g_{\alpha\beta}(p) = \psi^{-1}_{\beta,p}\psi_{\alpha,p}$, where $g(p)$ is continuous. Meaning that, for $p \in V_\alpha \cap V_\beta \cap V_\gamma$,

$$g_{\gamma\beta}(p)g_{\beta\alpha}(p) = g_{\gamma\alpha}(p) \tag{5.7}$$

which means that for $\alpha = \beta = \gamma$, we have

$$g_{\alpha\alpha}(p)g_{\alpha\alpha}(p) = g_{\alpha\alpha}(p) = \mathrm{Id}_G \tag{5.8}$$

And for $\alpha = \gamma$,

$$g_{\alpha\beta}(p)g_{\beta\alpha}(p) = \mathrm{Id}_G \to g_{\beta\alpha}(p) = g^{-1}_{\alpha\beta}(p) \tag{5.9}$$

A fiber bundle will then be fully defined as $(E, \pi, M, \mathcal{F}, G)$, with the fiber bundle E, base manifold M, projection function π, typical fiber $\mathcal{F}$ and structure group G. It will be usually noted

$$\mathcal{F} \to E \xrightarrow{\pi} M \tag{5.10}$$

or, for a shorter notation,

$$\pi : E \to M \tag{5.11}$$

5.2 Pullback bundle

If there's a continuous map $f : M' \to M$ and a bundle $\pi : E \to M$, pullback bundle is f^*E

$$f^*E = \{(p', e) | f(p') = \pi(e)\} \tag{5.12}$$

Pullback bundle diagram

5.3 Algebra bundles

5.4 Vector bundles

One of the most important type of fiber bundles for general relativity is the vector bundle, many structures on spacetime being constructed from them.
A vector bundle is a fiber bundle where the fiber $\mathcal{F}$ is a vector space.
Since all vector spaces are contractible (by the homotopy $f_s(t) = s\gamma(t)$), all vector bundles admit a global section. One of them that is valid for all vector bundles is the zero section $s_0 : p \in M \mapsto 0 \in V$.

Theorem 5.6. The zero section is a global section for all vector bundles.

Proof. Define local zero sections, since the vector bundle has GL as the structure group, the 0 is preserved. □

Basis of the vector bundle
Subspaces of the vector bundle

5.4.1 The line bundle

The simplest vector bundle is the line bundle, where the typical fiber is a field K, typically $\mathbb{R}$ or $\mathbb{C}$.

Proposition 5.7. There is a canonical isomorphism between C^k sections of the trivial line bundle over K and the set of functions $C^k(M) : M \to K$.

5.5 The tangent bundle and tensor bundles

The tangent bundle is a vector bundle of the same dimension as the manifold for which every fiber is the tangent space, that is

$$\pi(T_pM) = p \tag{5.13}$$

Its structure group is $GL(n, \mathbb{R})$, the group of linear transformations.
The construction of the tangent bundle is performed by attaching copies of the tangent space at every point

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{i \in p} \{p\} \times T_pM \tag{5.14}$$

with the projection map

$$\begin{aligned} \pi : TM &\rightarrow M &(5.15)\\ (p, v) &\mapsto p &(5.16) \end{aligned}$$

for $v \in T_pM$. We then endow it with the following topology. Consider an open set $U \in M$. We can define

$$\begin{aligned} \tilde{\psi}_U : \pi^{-1}(U) &\rightarrow \mathbb{R}^n \times \mathbb{R}^n &(5.17)\\ (p, v^\mu \partial_\mu) &\mapsto (\phi^\mu(p), v^\mu) &(5.18) \end{aligned}$$

(show that it's a bijection and that $\tilde{\psi}_U(\pi^{-1}(U))$ is open in $\mathbb{R}^{2n}$)

Theorem 5.8. If a tangent bundle is trivial, the base manifold is orientable.

Proof. □

Theorem 5.9. Smooth sections of the tangent bundle form a $C^\infty(M)$-module.

Theorem 5.10. There is an isomorphism between sections of the tangent bundle and vector fields

Proof. □

5.5.0.1 The complexified tangent bundle

For a few applications, it will be useful to define the complexification of the tangent bundle, which has the tangent bundle as a subbundle but with typical fiber $\mathbb{C} \times T_pM$

5.5.1 The cotangent bundle

Bundle of dual tangent spaces, same topology as tangent bundle

5.5.2 The tensor bundle

$$T^r_p q = \bigcup_{p,q \in \mathbb{N}} T^*M^{\otimes p} \otimes TM^{\otimes q} \tag{5.19}$$

Theorem 5.11. Any tensor field of rank (p, q) can be interpreted as a $(p+q)$ linear map.

Proof. □

C^k differentiable structure for C^k tensors
proof : if we have C^k differentiable structure, then functions at at most C^k, since defined by $f \circ \phi$ C^k tensors defined as mapping C^k vector to C^k functions Hence at most we can have a C^k tensor

5.5.3 Bitensors

For some applications, we will need the concept of bitensors. A bitensor is composed of tensor quantities at two different points of the manifold. For two tensor bundles VM, WM with the same base manifold, $VM \boxtimes WM$ is the exterior tensor product, defined by the vector bundle with typical fiber $V \otimes W$ over the base manifold $M \times M$.
A section of the bitensor bundle will then be a map

$$\begin{aligned} B : M \times M &\to V \times W \\ (p, q) &\mapsto B(p, q) \end{aligned}$$

Coincidence limit with the Synge bracket :

$$B[x] = B(x, x) \tag{5.20}$$

In the coincidence limit, there's a map between bitensors and the tensor product of V and W
Most common bitensor : $\pi : BM \to M \times M$ with typical fiber of a tensor space.

5.5.4 The tensor density bundle

The s-tensor density bundle is a real line bundle over the manifold with typical fiber $\mathbb{R}$

5.5.5 The exterior bundle

Exterior bundle $\Lambda^k M$ Antisymmetrization of the tensor bundle of rank $(0, n)$
The top exterior power has rank 1 and is hence a line bundle. If this line bundle is trivial, then M is orientable.

5.5.6 Vertical and horizontal bundles

A useful bundle for the analysis of most other bundles will be the tangent bundle of that bundle. That is, considering some bundle $\pi_Y : Y \to M$, we'll consider the tangent bundle

$$TY \xrightarrow{\pi_{TY}} Y \xrightarrow{\pi_Y} M \tag{5.21}$$

As the bundle Y has a fairly natural local decomposition as $F_Y \times M$, we can decompose the tangent bundle into subbundles, using the vertical space.

Definition 5.12. For a fiber bundle $\pi : E \to M$, $e \in E$ with $\pi(e) = p$, the *vertical space* $V_e E$ at e is the tangent space $T_e E_p$, that is, the tangent space of the fiber $E_p = \pi^{-1}(p)$.

The vertical space at e will contain the vectors corresponding to changes in the value of the section at p around the value e. It will later be related to the variation of a field.

Example 5.13. The trivial bundle $\pi : S \times S \to S$ will have the tangent space at any point $T_e(S \times S) = \mathbb{R}^2$. Its vertical space will be $V_e(S \times S) = \mathbb{R}$.

Proposition 5.14. The vertical space $V_e E$ is a subspace of $T_e E$, the tangent space of the bundle at E, of the same dimension as the fiber of E.

Proof. Since $E_p \subset E$, we have that $V_e E = T_e E_p \subset T_e E$. For the properties of a subspace :

- $0 \in V_e E$: as it is a vector space, $0 \in V_e E$, which will be the same 0 as $T_e E$, since $\pi_{TE}(0) = e \in E_p$.
- $X, Y \in V_e E$ implies $X + Y \in V_e E$:
- $X \in V_e E$ implies $cX \in V_e E$

□

Proposition 5.15. For a vector bundle E, there is a canonical isomorphism between the vertical space of the origin $V_0 E$

Definition 5.16. The *horizontal space* $H_e E$ is a subspace of $T_e E$ such that $T_e E = V_e E \oplus H_e E$.

Vertical and horizontal bundle : Disjoint union of the vertical and horizontal space, $VE = \ker(\mathrm{d}\pi)$, $\mathrm{d}\pi : TE \to TM$
As $d\pi_e$ is surjective, yields a regular subbundle of TE.

5.5.7 Solder forms

The soldering of a fiber bundle E with fiber F to the manifold M, with $\dim(F) = \dim(M)$, corresponds to the choice of some section $s_E : M \to E$ along with some linear isomorphism

$$\theta : TM \to s_E^* VE \tag{5.22}$$

where $s_E^* VE$ is a pullback of the vertical bundle of E along this section. In essence, what the solder form does is to "solder" (ie identify) the vertical space to the tangent space. This will be useful as things go on to adapt structures of the tangent space to other bundles and vice-versa.
Solder form for vector bundles : if we choose the zero section, then $s_0^* VE \approx E$, so that the solder form is

$$\theta : TM \to E \tag{5.23}$$

The simplest solder form is the one of the tangent bundle to itself,

$$\theta : TM \to TM \tag{5.24}$$

The canonical solder form chosen for it is obviously the identity.

5.6 Principal bundles

Principal bundle : fiber is a Lie group, structure group is the same as the Lie group, left action
Associated bundle to a principal bundle : representation of the Lie group acts upon the vector bundle, structure group is the Lie group of the principal bundle
Action of G on the principal bundle :

$$\phi : G \times P \to P \tag{5.25}$$
$$\phi_g(\xi) \mapsto \xi g \tag{5.26}$$

5.6.1 Reduction of the structure group

Given a subgroup H of G and a principal H-bundle $\pi : P' \to M$, we say that a principal G-bundle $\pi : P \to M$ admits a reduction to the group H if there exists a smooth map $\Phi : P' \to P$ such that it covers the identity on M and is H-equivariant

$$\forall g \in G, h \in h,\ \Phi(g.h) = \Phi(g) \cdot h \tag{5.27}$$

Show it's equivalent to $P' = P/H$ having a section

5.7 The frame bundle

There is a particular bundle of importance, the frame bundle, that is both a vector bundle and a principal bundle, and is linked directly with the structure of the manifold itself and its tangent bundle.
For a real vector bundle $E \to X$, a frame at $x \in X$ is an ordered basis for E_X. Frame is a linear isomorphism $p : \mathbb{R}^k \to E_x$. The set of all frames at x, F_x, has a natural right action by $GL(k, \mathbb{R})$. For $g \in GL(k, \mathbb{R})$, $p \circ g$ is a new frame. F_x is homeomorphic to $GL(k, \mathbb{R})$.

Definition 5.17. A manifold that admits a global section of the frame bundle is called a parallelizable manifold.

Orthonormal frame bundle OM : restriction to the frame bundle with structure group $\mathrm{O}(n)$
if orientable : the oriented orthonormal frame bundle with group $\mathrm{SO}(n)$
Reduction of the frame bundle to the orthonormal frame bundle $\mathrm{GL}(n, \mathbb{R}) \to \mathrm{O}(p, q)$
For orientable manifolds, (orthogonal) frame bundle to oriented (orthogonal) frame bundle $\mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}^+(n, \mathbb{R})$ and $\mathrm{O}(p, q) \to \mathrm{SO}(p, q)$

5.7.1 The solder form

If we define a vector bundle VM with typical fiber $\mathbb{R}^n$, $\dim(M) = n$, the soldering of
Solder 1-forms
Solder form from $FM \times V \to TM$

5.8 Associated bundles and physical fields

In classical physics, matter fields are represented by sections of vector bundles.
Scalar fields : line bundle
Tensor fields : tensor bundle
Spinor fields : associated bundle to the Clifford bundle
Gauge fields : principal bundle
Example : $\mathrm{SU}(3)$ for QCD, $\mathrm{SU}(2) \times \mathrm{U}_1$ for electroweak
Associated bundle to a principal bundle : Fields with gauge symmetry

5.9 The sphere bundle

Fiber is S^{n-1}
Equivalent to the existence of a nowhere vanishing vector field

Definition 5.18. A direction field, also called line element, is a section of the projectivized tangent bundle

5.10 The jet bundle

5.10.1 Jets and jet manifolds

Jets correspond to the bundle equivalent of the Taylor expansion of functions. They will offer us the rigorous justification behind many arguments based on first-order expansions. This is done by equivalence classes on their derivatives.

Definition 5.19. For a bundle (E, π, M), $p \in M$ and two local sections $f, g \in \Gamma_p(\pi)$, we say that f and g are 1-*equivalent* if $f(p) = g(p)$ and, in some bundle coordinate system (x^μ, u^α) around $f(p)$,

$$\frac{\partial f^\alpha}{\partial x^\mu}|_p = \frac{\partial g^\alpha}{\partial x^\mu}|_p \tag{5.28}$$

The equivalence class of functions 1-equivalent to f at p is called the 1-*jet* of f at p, or $j^1_p f$.

Proposition 5.20. For $f, g \in \Gamma(\pi)$ such that $f(p) = g(p)$, we have $j^1_p f = j^1_p g$ if and only if $f_*|_{T_pM} = g_*|_{T_pM}$.

Proof. □

Definition 5.21. The *first jet manifold* of π

Definition 5.22. The *r-th jet manifold* is

$$J^r(\pi) = \left\{ j^r_p \sigma | p \in M, \sigma \in \Gamma(p) \right\} \tag{5.29}$$

Differential equations as jet sections
Infinite jet

6 The metric

The metric is the most important quantity for the study of spacetimes. Its primary use will be to define an inner product on vectors, and through this, a norm as well. It is an inner product of tangent vectors, generally defined as

$$g : TM \times TM \to \mathbb{R} \tag{6.1}$$

unlike for most inner products, we don't require a restriction to $\mathbb{R}^+$, nor for it to be positive definite, although the other conditions of symmetry and bilinearity are kept :

$$\begin{aligned} g(X,Y) &= g(Y,X) \\ g(aX_1 + bX_2, cY_1 + dY_2) &= ac\ g(X_1, Y_1) + ad\ g(X_1, Y_2) \\ &+ bc\ g(X_2, Y_1) + db\ g(X_2, Y_2) \end{aligned}$$

There are two ways of defining the metric : from the definition, we can see that it can be a section of the $(0,2)$ tensor bundle, and we can also define it directly as a bilinear function on the tangent bundle itself, called the bundle metric.

6.1 The metric tensor

The metric tensor is defined as a section of a subbundle of the $(0,2)$ tensor bundle, defined by all tensors being symmetric

$$\forall X, Y \in TM, T(X,Y) = T(Y,X) \tag{6.2}$$

and of signature (p,q). This subbundle is called the $S(p,q)$ bundle, $S_{p,q}M$.

Proposition 6.1. The set of symmetric tensors is open in $T^1_1 M$.

Proof. □

The metric tensor is defined as a section of that bundle

$$g : S(p,q) \subset T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R} \tag{6.3}$$

As with all tensors, we can define the components in some coordinate patch by

$$g(\partial_\mu, \partial_\nu) = g_{\mu\nu} \tag{6.4}$$

And those components will transform covariantly

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{6.5}$$

By linearity, the product of two vectors in this basis will be

$$g(X,Y) = g(X^\mu \partial_\mu, Y^\nu \partial_\nu) = g_{\mu\nu} X^\mu Y^\nu \tag{6.6}$$

By an abuse of terminology, $g_{\mu\nu}$ itself will often be called the metric tensor.

6.1.1 Smoothness

The metric is usually assumed to be smooth, or at least C^2.
Relation between smoothness of manifold and metric???

6.1.2 Signature of the metric

The signature of a metric at a point is noted as (p, q, r), if its associated bilinear form has p negative eigenvalues, q positive eigenvalues and r zero eigenvalues. For almost all applications, we will only consider $r = 0$, and the metric will be of the same signature $(p, q, 0)$, or (p, q), for the entire manifold.

6.1.3 Determinant of the metric

A quantity that will be useful in future chapters will be the determinant of the metric tensor, defined as

$$\det(g) = \frac{1}{n!}\epsilon^{\mu_1\mu_2...\mu_n}\epsilon^{\nu_1\nu_2...\nu_n}g_{\mu_1\nu_1}...g_{\mu_n\nu_n} \tag{6.7}$$

Theorem 6.2. The determinant of the metric g is always of the sign $(-1)^p$.

Proof. Determinant doesn't change sign with coordinates (cf Sylvester's law), and there's a coordinate system where $g = \mathrm{diag}(-1, -1, ..., 1, 1, ...)$, which is of determinant $(-1)^p$ □

In particular, the determinant of a spacetime metric will always be negative. Checking the sign of the determinant is a good method to check whether or not the metric is still Lorentzian when varying a component.

6.2 Frame fields

Another way to consider the metric on a manifold is via the use of frame fields, also known as vierbein or tetrad fields.
A frame field is a section of the frame bundle

6.2.1 Bundle metric

A bundle metric is a metric defined on a vector bundle E by

$$\langle\cdot,\cdot\rangle : E \times E \to M \times \mathbb{R} \tag{6.8}$$

associating for every section of the vector bundle a value to every point of the manifold. The bundle metric that will interest us will be the bundle metric defined on the tangent bundle, defined by

$$\langle\partial_\mu, \partial_\nu\rangle = \eta_{\mu\nu} \tag{6.9}$$

with η the Minkowski metric.

6.2.2 Frame fields

for a section of the frame bundle $\{e_\mu\}$, a vector $V \in TM$ can be decomposed as

$$V = V^\mu e_\mu \tag{6.10}$$

Since the bundle metric is bilinear,

$$\langle V, W\rangle = \langle V^\mu e_\mu, W^\nu e_\nu\rangle = V^\mu W^\nu \langle e_\mu, e_\nu\rangle \tag{6.11}$$

$$\langle e_\mu, e_\nu\rangle = g_{\mu\nu} \tag{6.12}$$

6.2.3 Obstruction

Unlike for the metric tensor, constructing a section of the orthonormal frame bundle is much harder than a section of the metric bundle.

Theorem 6.3. A non-orientable manifold is not parallelizable.

If we try to construct a global section for a non-orientable manifold, overlap has frames of different orientation.

Theorem 6.4. An orientable $3+1$ dimensional spacetime with the structure $\mathbb{R} \times \Sigma$, for Σ a compact manifold, is parallelizable.

Proof. Orientable compact 3-manifolds are parallelizable, same with products of parallelizable manifolds □

Same proof for $(1+1)$ dimensions
Frame field mostly useful locally because of this.

6.3 Topological obstructions to the metric tensor

So far, our definition of spacetimes admitted any of the non-pathological manifolds. In particular any submanifold of R^{2n} is covered by the definition we have used. Unfortunately, not all manifolds admit a section of the metric bundle for arbitrary signatures, although the class of manifolds concerned is of rather little physical interest.
For various applications, we will need a Riemannian metric, so it is of interest to first study the obstructions for them.

Theorem 6.5. Every paracompact manifold admits a Riemannian metric tensor field.

Theorem 6.6. A smooth Hausdorff second-countable manifold admits a smooth metric function g^+.

Proof. If we pick an atlas $\{(U_\alpha, \phi_\alpha)\}$ for the manifold, with a subordinate partition of unity $\{f_\alpha\}$, we can pick the metric function $g^(\alpha)$ acting on $O_\alpha \subset \mathbb{R}^n$ (for instance the canonical metric on $\mathbb{R}^n$)
Then

$$g^+(u, v) = \sum_{a\in A} f_\alpha(x) g^{(\alpha)}(d\phi_\alpha(u), d\phi_\alpha(v)) \tag{6.13}$$

□

Theorem 6.7. The existence of a continuous, nowhere zero direction field is equivalent to the existence of a Lorentzian metric on a manifold.

Proof. Since the class of manifolds we are concerned with always admits a Riemannian metric, we can combine it with the direction field to obtain the following metric :

$$g(x,y) = g^+(x,y) - 2\frac{g^+(\xi,y)g^+(x,\xi)}{g^+(\xi,\xi)}$$

In a coordinate system with the orthonormal basis $\{\xi, x^a\}$, it can be checked that the signature of the metric is indeed $(-,+,+,...)$.
Conversely, if we have a Lorentz metric, □

Theorem 6.8. Every non-compact manifold admits a continuous, nowhere zero vector field.

Proof. Every manifold admits a vector field that is only 0 on a set of measure 0 [PROVE IT] □

Corrolary 6.1. Every non-compact manifold admits a Lorentzian metric

For compact manifold, the proof is more involved, and is only available in Steenrod's "Topology of Fiber Bundles", as every book on general relativity will reference. Here is a sketch of this proof.
Consider the general linear group $\mathrm{GL}(n,\mathbb{R})$ of invertible real matrices of order n, $\mathrm{O}(n)$ the orthogonal group and $\mathrm{SO}(n)$ the rotation group. $\mathrm{S}(n)$ is the subset of $\mathrm{GL}(n,\mathbb{R})$ for symmetric matrices, and $\mathrm{S}(n,k)$ the subset of symmetric matrices of signature k.
Prove that $\mathrm{S}(n,k)$ is open in $\mathrm{S}(n)$ and that $\mathrm{S}(n) = \bigcup_{k=0}^n \mathrm{S}(n,k)$
Metric of signature (p,q) is a global section of $\mathrm{S}(n,q)$
Decomposition of invertible matrices τ as $\sigma \in \mathrm{O}(n)$ and $\alpha \in S(n,0)$, the set of Riemannian metrics, as

$$\begin{aligned} \psi : \mathrm{O}(n) \times \mathrm{S}(n,0) &\to \mathrm{GL}(n,\mathbb{R}) && (6.14)\\ (\sigma,\alpha) &\to \sigma\alpha && (6.15) \end{aligned}$$

[show that $\mathrm{O}(n) \times \mathrm{S}(n,0)$ is a subbundle of the metric bundle]

Theorem 6.9. If $\sigma \in \mathrm{Gr}(n,k)$, $\alpha \in \mathrm{S}(n,0)$ and $\sigma\alpha = \alpha\sigma$, then $\tau \in \mathrm{S}(n,k)$.

Proof. □

Theorem 6.10. A compact manifold admits a section of the Grassmanian line bundle $\mathrm{Gr}(1,k)$ if and only if its Euler characteristic is $\chi(M) = 0$.

Proof. If a manifold ha $\chi(M) = 0$ then it admits a vector field, which can define a section of the Grassmanian line bundle. Converse : If a manifold has a Grassmanian line bundle section, it admits a double cover that admits a nowhere vanishing vector field. Since for double covers $\chi(M') = 2\chi(M)$ QED □

Theorem 6.11. A compact manifold only admits a Lorentzian metric if and only if it admits a continuous nowhere vanishing vector field

Proof. □

Corrolary 6.2. An odd dimensional compact manifold always admits a Lorentzian metric. An even dimensional compact manifold will only admit one if its Euler characteristic is zero.

In two dimensions, this limits the number of compact spacetimes to two : the torus and the Klein bottle.
Equivalent of the theorem for obstruction : frame field is a section of an $\mathrm{O}(n)$ bundle
We will denote the set of all manifolds admitting a Lorentzian metric as $\mathrm{Lor}(\mathcal{M})$. The set of all spacetimes will then be denoted by $\mathrm{Lor}(\mathcal{M})/\,\mathrm{Diff}(\mathcal{M})$. A spacetime will then be the association of a Lorentzian manifold $\mathcal{M} \in \mathrm{Lor}(\mathcal{M})/\,\mathrm{Diff}(\mathcal{M})$ and of a section of the Lorentzian metric bundle $g \in \Gamma(S(n,1)\mathcal{M})$, noted $(\mathcal{M}, g)$.

6.4 Musical isomorphisms and the inverse metric

Having a metric tensor will let us define a canonical map between tangent and cotangent vectors, and more generally between rank (r,s) tensors and rank (r',s') tensors, if $r+s = r'+s'$. By partial application of a vector field X to the metric tensor, we have the function

$$g(X,-) : TM \to \mathbb{R} \tag{6.16}$$
$$Y \mapsto g(X,Y) \tag{6.17}$$

as the metric is bilinear, this quantity is a linear map from vector fields to $\mathbb{R}$, meaning that partial application of vector fields is a subset of T^*M.
From this, we will define the musical isomorphisms.
The musical isomorphisms are isomorphisms between the tangent bundle and the cotangent bundle. They are denoted by $\sharp$ and $\flat$

$$\flat : TM \to T^*M \tag{6.18}$$
$$X \mapsto X^\flat \tag{6.19}$$

$$\sharp : T^*M \to TM \tag{6.20}$$
$$\omega \mapsto \omega^\sharp \tag{6.21}$$

And of course, $\sharp \circ \flat = \mathrm{Id}_{T^*M}$ and $\flat \circ \sharp = \mathrm{Id}_{TM}$.
If the manifold is equipped with a metric, it defines them canonically as

$$X^\flat(Y) = g(X,Y) \tag{6.22}$$
$$g(\omega^\sharp, Y) = \omega(Y) \tag{6.23}$$

Hence we can define $X^\flat$ directly as $g(X,-)$.

Proposition 6.12. $\flat$ is a bijection from the tangent bundle to the cotangent bundle.

Proof. Injection : For $X^\flat = 0$ □

Proposition 6.13. If g is a C^k metric tensor, a smooth vector field X is mapped to a C^k vector field $X^\flat$.

Proof. g maps two smooth vector field X, Y to a C^k function $g(X,Y)$, hence $X^\flat(Y) = [g(X,-)](Y) = g(X,Y)$ maps smooth vectors to C^k. □

To define the $\sharp$ map outside of products, we'll need the inverse of the metric tensor.

Proposition 6.14. There exists a rank $(2,0)$ tensor field g^{-1} such that, for every one-forms $\omega, \omega' \in T^*M$, $g^{-1}(\omega, \omega') = g(\omega^\sharp, \omega'^\sharp)$

Proof. As the metric has a determinant $g > 0$, the matrix $\mathbf{g}$ is invertible at every point, so there exists a matrix $\mathbf{g}^{-1}$. The inverse metric is the section of T^2_0 that associates the matrix $\mathbf{g}^{-1}$ to every point. Since $g(X,Y) = g^{-1}(X^\flat, Y^\flat)$, this is equivalent to, for $\omega = Y^\sharp$,

$$\begin{aligned}
[g^{-1}(\omega, -)](g(X,-)) &= \omega_\mu g^{\mu\nu} X^\sigma g_{\sigma\nu} \\
&= \omega_\mu X^\sigma g_{\sigma\nu} g^{\mu\nu} && (6.24) \\
&= g(X, \omega^\sharp) = \omega(X) = \omega_\mu X^\mu && (6.25)
\end{aligned}$$

Hence $g_{\sigma\nu} g^{\mu\nu} = \delta^\mu_\sigma$. We can then find the coordinate transform rule by

$$\begin{aligned}
g_{\sigma\nu} g^{\mu\nu} &= \delta^\mu_\sigma && (6.26) \\
&= ... && (6.27)
\end{aligned}$$

□

Proposition 6.15. If the metric tensor is a C^k tensor, the inverse metric is a C^k tensor as well.

Proof. Since g is a C^k function, it will map smooth vector fields to C^k functions. Hence

$$\begin{aligned}
g(X,Y) &= f \in C^k(M) \\
&= g^{-1}(X^\flat, Y^\flat) && (6.28)
\end{aligned}$$

[But $X^\flat$ already C^k???] □

The components of the inverse metric tensor are calculated using the relation

$$g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu \tag{6.29}$$

In particular, if $g_{\mu\nu} = \text{diag}(g_{\alpha\alpha})$, we have that $g^{\mu\nu} = \text{diag}(1/g_{\alpha\alpha})$

$$g_{\mu\nu} g^{\mu\nu} = n \tag{6.30}$$

Prove that the inverse metric defines a tensor correctly (check on the intersections of the trivialization)

Proposition 6.16. For any C^k metric tensor g defined on the manifold, there exists an inverse metric tensor g^{-1}, which is C^k as well.

Proof. Since at every point of a coordinate chart, there exists an inverse of the metric at that point, we need to show that the inverse metric on this chart is itself a C^k function. Show that the inverse metric is defined correctly on chart overlaps □

The musical isomorphisms define a solder form on the cotangent bundle

$$\flat : TM \to T^*M \tag{6.31}$$

6.5 Contraction

The contraction of a tensor is a map

$$C : T^r_s M \to T^{r-1}_{s-1} M \tag{6.32}$$

$$\begin{aligned} C : T^1_1 M &\to C^\infty(M) \\ X \otimes \omega &\mapsto C(X \otimes \omega) = \omega(X) \end{aligned} \tag{6.33}$$

If linear,

$$C(X \otimes \omega) = X^\mu \omega_\nu C(\partial_\mu \otimes dx^\nu) = X^\mu \omega_\nu \delta^\nu_\mu \tag{6.34}$$

6.6 Raising and lowering operators

The raising and lowering operators are a generalization of the musical isomorphisms, allowing to map tensors of rank (n, m) to tensors of rank $(n+1, m-1)$ and $(n+1, m-1)$.

$$\uparrow : T^n_m \to T^{n+1}_{m-1} \tag{6.35}$$

Dual of a tensor

$$* : T^p_r M \to T^r_p M \tag{6.36}$$

$$(T^{\alpha\beta\gamma\ldots}{}_{\mu\nu\rho\ldots})^* = T_{\alpha\beta\gamma\ldots}{}^{\mu\nu\rho\ldots} \tag{6.37}$$

This lets us define a norm on tensors

$$\begin{aligned} |\cdot| : T^p_r M &\to \mathbb{R} \\ T &\mapsto T^*(T) \end{aligned}$$

6.6.1 The Hodge star

The metric tensor lets us define the Hodge star, which maps k-forms to their dual

$$\star : \Lambda^k M \to \Lambda^{n-k} M \tag{6.38}$$

For two 1-forms α, β,

$$\alpha \wedge (\star\beta) = g(\alpha, \beta)\varepsilon \tag{6.39}$$

6.7 Products and warped products

For (M_1, g_1) and (M_2, g_2) two Riemannian manifold, their product is $(M_1 \times M_2)$

6.8 Causal class of curves and vector

We can define several class of curves by their tangent vectors, and we will define timelike (resp. null, causal and spacelike) curves to be curves with a tangent vector that is always timelike (resp. null, causal and spacelike). Curves with tangent vectors that switch from one type to the other are fairly rarely used, and so they do not have a specific name.
If a curve is only piecewise C^1, then it is qualified as piecewise timelike (resp. null, causal or spacelike) if it is so on every part of the curve that admits a derivative. Values where the curve isn't defined are called singular points.
If the curve is piecewise causal, and its left and right derivative u^- and u^+ at a singular point p are in two different light cones in $T_p\mathcal{M}$, we call this point an *interior corner* of the curve. If a piecewise timelike (resp. null, causal) curve has no interior corners, we say that it is timelike (resp. null, causal).
[also define exterior corners]

6.9 Isometries

With the addition of the metric tensor, the equality between two spacetimes can no longer solely rely on the diffeomorphism between the two manifolds. To compare two spacetimes $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$, we will need the introduction of isometries.

Definition 6.17. An $\phi : M \to N$ between two manifolds is a diffeomorphism that preserves the metric of the manifold : $\phi^*(g_N) = g_M$

We will then say that two spacetimes are equivalent if there exists an isometry between them.
Examples of isometries : discrete isometries (reflections, discrete rotations, etc)
Isometries of one parameter : translations, rotations, etc
Extensions of a manifold : M' is an extension of M if there exists an inclusion map $M' \hookrightarrow M$ such that there's a subset of M' on which $\hookrightarrow$ is isometric.
Local isometry (cf Manchak)

6.10 Distances

The metric tensor allows us to define a length function on curves γ with tangent vector u by considering the integral

$$l_\gamma = \int_S g(u(\lambda), u(\lambda))^{\frac{1}{2}} d\lambda \tag{6.40}$$

Depending on the signature of the metric and the class of curves, this will give rise to a variety of notions of distances.

6.10.1 Riemannian distance function

if g is Riemannian, this length can define an actual distance function on the manifold, by considering the shortest curve connecting two points.

$$d(p,q) = \min_{\gamma(p,q)} \int_p^q g(u(\lambda), u(\lambda))^{\frac{1}{2}} d\lambda \tag{6.41}$$

where γ belongs to the class of all piecewise smooth curves.

Theorem 6.18. The function $d(p,q)$ is a distance function on the manifold.

Proof. Just show that it obeys the distance axioms

1. $d(p,q) \geq 0$: As the Riemannian metric is itself ≥ 0, this property is verified
2. $d(p,p) = 0$: Consider the trivial curve $\gamma(\lambda) = p$. Its tangent will be 0 for all λ, which will always be the shortest curve in a Riemannian manifold.
3. $d(p,r) \leq d(p,q) + d(q,r)$: by picking a random point q, we have

$$d(p,q) = \min_{\gamma(p,r)} \left[\int_p^q g(u(\lambda), u(\lambda))^{\frac{1}{2}} d\lambda + \int_q^r g(u(\lambda), u(\lambda))^{\frac{1}{2}} d\lambda \right]$$

4. $d(p,q) = 0 \to p = q$: By the Hausdorff property, if $p \neq q$, there is a neighbourhood U_p that does not contain q. Then we can consider a point in U_p on the curve connecting p and q, and by the triangle inequality, $d(p,q) \geq \varepsilon > 0$

□

6.10.2 Lorentzian lengths

As Lorentz metrics have negative norms, as well as zero norms for non-zero vectors, they cannot define a distance function. Instead the length function l_γ has the following properties
For a timelike curve,

$$l_\gamma = -\int_S d\tau \tag{6.42}$$

For a null curve,

$$l_\gamma = 0 \tag{6.43}$$

For a spacelike curve,

$$l_\gamma = \int_S ds \tag{6.44}$$

Theorem 6.19. If two points can be joined by a timelike curve, there is a lower bound to the length of all timelike curves joining them.

Proof. □

7 Derivatives and connections

7.1 Lie derivative

The Lie derivatives measures the change of a tensor field along the flow of a given vector field. It has the benefit of not requiring any extra structures on the manifold to be defined.
It is defined in the usual way that derivatives are defined, by the limit of the difference between two quantities.

Definition 7.1. Given ϕ_t a one-parameter group of diffeomorphisms generated by the flow of a vector field X, The *Lie derivative* of a tensor field T with respect to X is defined by

$$\mathcal{L}_V T = \lim_{t \to 0} \frac{\phi^*_{-t} T - T}{t} \tag{7.1}$$

For scalar fields :

$$\mathcal{L}_X f = X(f) \tag{7.2}$$

For a vector field :

$$\mathcal{L}_X Y = [X, Y] \tag{7.3}$$

Proposition 7.2.

$$\mathcal{L}_{[X,Y]} T = \mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T \tag{7.4}$$

7.2 Exterior derivative

The exterior derivative d is a derivative defined on n-forms which offers the benefit of not requiring any extra-structure on the manifold. It is defined by

$$\mathrm{d} : \Lambda^k M \to \Lambda^{k+1} M \tag{7.5}$$

$$\star : \Lambda^k M \to \Lambda^{n-k} M \tag{7.6}$$

Properties :

$$\mathrm{d}(\mathrm{d}\omega) = 0 \tag{7.7}$$

For a k-form α :

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k (\alpha \wedge \mathrm{d}\beta) \tag{7.8}$$

Poincaré theorem

Theorem 7.3. For a p-form α, $p \geq 1$, if $\mathrm{d}\alpha = 0$

Frobenius theorem

Theorem 7.4. content...

Proof. □

7.3 Covariant derivative

To define the derivative of a tensor in the same way that we would in $\mathbb{R}^n$, we need a way to compare two tensors at different point of a manifold, that is, we need to define a way for the usual derivative

$$\lim_{\varepsilon \to 0} \frac{X(p + \varepsilon ...) - X(p)}{\varepsilon} \tag{7.9}$$

to make sense. Since the manifold does not have a vector space structure, we cannot really make sense of such an expression.

7.3.1 The Ehresmann connection

A common way to get a rather general definition of a connection is the Ehresmann connection, which not only will give us a derivative for tangent vectors but quite a lot more useful tools for general relativity and gauge theory.
As we saw in X, any fiber bundle E, being itself a manifold, admits a tangent bundle TE, which can be split as the vertical bundle VE and a horizontal bundle HM

$$TE = HM \oplus VM \tag{7.10}$$

While the vertical space $V_e E$ is uniquely determined by the bundle structure of E, any vector subspace $H_e M$ such that $V_e E \oplus H_e E = T_e E$ will qualify as a horizontal space.

Definition 7.5. An *Ehresmann connection* on the bundle $\pi : E \to M$ is a collection of vector subspaces $\Gamma = \{H_e E | e \in E\}$ such that

- The map $e \to H_e$ depends smoothly on e
- For each $e \in E$, we have $T_e E = H_e E \oplus V_e E$

We say that a vector field $X \in \Gamma(TE)$ is horizontal if for every e, $X(e) \in H_e E$. Horizontal vectors will correspond to a notion of the vector being orthogonal to the fiber.
Trivial connection : if $E = M \times F$, the trivial connection is $H_e E = T_e M$ for every e
Flat at e if there's a local trivialization such that the horizontal space is trivial at e.

7.3.1.1 Horizontal lifts

For a curve $\gamma(\lambda)$, a lift of γ to E is a curve $\tilde{\gamma}(\lambda)$ in the E bundle such that $\pi_E(\tilde{\gamma}) = \gamma(\lambda)$. We say that $\tilde{\gamma}$ is a horizontal lift if the tangent vector of $\tilde{\gamma}$ is horizontal in E.

Theorem 7.6. For a bundle $\pi : E \to M$ with a connection Γ, and a curve $\gamma(\lambda)$ such that $\gamma(0) = p$. For each choice of lift of p to e, $\pi(e) = p$, there corresponds a unique horizontal lift of γ to $\tilde{\gamma}$.

7.3.1.2 The connection form

7.3.1.3 Holonomy

Given an Ehresmann connection,

7.3.2 Connection on a vector bundle

One of the most important Ehresmann connection is the Ehresmann connection on a vector bundle, in particular the tangent bundle.

Definition 7.7. A connection on a C^∞ vector bundle $\pi : E \to M$ is a map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\to \Gamma(E) \\ (X, s) &\mapsto \nabla_X s \end{aligned} \tag{7.11}$$

such that

- The map is linear in X and s
- It obeys the Leibniz rule such that, for a function $f \in C^\infty$,

  $$\nabla_X(fs) = (X(f))s + f\nabla_X s$$

  or, alternatively,

  $$\nabla_X(fs) = df(X)s + f\nabla_X s$$

Proposition 7.8. Every trivial C^∞ vector bundle $E = M \times V$ admits a connection.

Proof. If E is a trivial vector bundle of rank r, there is a bundle isomorphism

$$\phi : E \to M \times \mathbb{R}^r \tag{7.12}$$

□

Every vector bundle admits a connection. To prove this, we'll need the following lemma :

Lemma 7.1. Any finite linear combination of connections ∇^i

$$\nabla = \sum_i t_i \nabla^i$$

such that

$$\sum_i t_i = 1$$

is itself a connection.

Proof. Since it is simply a sum, the linearity is trivial. By the Leibniz rule, we have

$$\nabla^i_X(fs) = (X(f))s + f\nabla^i_X s \tag{7.13}$$

So that

$$\begin{aligned} \nabla(fs) &= \sum_i t_i(X(f))s + f(\sum_i t_i \nabla^i)s \\ &= (X(f))s + f\nabla s \end{aligned} \tag{7.14}$$

hence the connection obeys the Leibniz rule. □

Theorem 7.9. Every C^∞ vector bundle admits a connection.

Proof. Given a partition of unity $\{\rho_\alpha\}$ over a trivializing open cover U_α, the vector bundle $E|_{U_\alpha}$ □

7.3.3 Principal connection

For a principal G-bundle $\pi : P \to M$ and a point $p \in M$, pick a point in the fiber $\xi \in \pi^{-1}$. The vertical subspace at p, V_p, is a subset of the tangent space at ξ of the principal bundle itself, $T_\xi E$, defined by

$$V_\xi E = T_\xi(P_{\pi(\xi)}) \tag{7.15}$$

The horizontal space H_ξ is the complement such that $T_\xi P = V_\xi P \oplus H_\xi P$. The vertical and horizontal bundle are the disjoint union of every vertical and horizontal space.
A vector field v on P is vertical if $v(\xi) \in V_\xi$ for every point ξ.
The vertical space is the image of the Lie algebra $\mathfrak{g}$ of G under the G action. For $\xi \in P$, the action is

$$\begin{aligned} \alpha_\xi : G &\to P \\ g &\mapsto \xi g \end{aligned} \tag{7.16}$$

Pushforward at the identity gives a ap

$$\begin{aligned} \sigma_\xi : \mathfrak{g} &\to T_\xi P \\ A &\mapsto \frac{d}{dt}(\xi \exp(tA)) \end{aligned} \tag{7.17}$$

Since $\pi(\xi \exp(tA)) = \pi(\xi)$, $\sigma_\xi \in V_\xi$. Since G is free, map is bijective and hence $\sigma_\xi : \mathfrak{g} \to V_\xi$ is an isomorphism.
Take an element a of the Lie algebra $\mathfrak{g}$ of the G-bundle $\mathcal{P}(\mathcal{M}, \mathcal{G})$

$$af(u) = \frac{d}{dt} f(ue^{ta})|_{t=0} \tag{7.18}$$

ue^{ta} is a curve in $\mathcal{P}$ entirely within $\mathcal{G}_p$
A connection is a choice of the horizontal subspace H_ξ such that $(R_g)_* H_\xi = H_{\xi g}$

Definition 7.10. The connection one-form is a $\mathfrak{g}$-valued one-form ω on P defined by $\omega(\xi) = a$ if $\xi = \sigma(a)$ and $\omega(\xi) = 0$ if ξ is horizontal.

The connection one-form defines the choice of H_ξ

Theorem 7.11. The connection one-form obeys

$$\phi_h^*(\omega) = \mathrm{Ad}_{g^{-1}} \circ \omega \tag{7.19}$$

Definition 7.12. A Koszul connection ∇ on a vector bundle $\pi : E \to M$ is a map $\Gamma(E) \to \Lambda^k T^* M \otimes E$ obeying the Leibniz property

$$\forall f \in C^\infty(M), s \in \Gamma(E), \nabla(fs) = df \otimes s + f \nabla s \tag{7.20}$$

In particular, for a vector field X, $\nabla_X : \Gamma(E) \to \Gamma(E)$

$$\begin{aligned} \nabla_{fX}(s) &= f \nabla_X s \\ \nabla_{X+Y}(s) &= \nabla_X(s) + \nabla_Y(s) \\ \nabla_X(fs) &= X(f)s + f \nabla_X s \end{aligned} \tag{7.21}$$

The covariant derivative associated with this principal bundle will be, for an n-form α

$$\nabla\alpha = d\alpha + \rho(\omega) \wedge \alpha \tag{7.22}$$

Curvature form :

$$\nabla^2\alpha = \rho(\Omega) \wedge \alpha \tag{7.23}$$

7.3.4 Soldering and connections

If we have an Ehresmann connection on the frame bundle FM, this will translate to a connection on the tangent bundle TM
then any fiber bundle E equipped with a solder form θ_E will also be equipped with an Ehresmann connection by the mapping

$$... \tag{7.24}$$

Torsion :

$$\Theta = \nabla\theta \tag{7.25}$$

Θ an E-valued 2-form

$$\theta = \theta^i e_i \tag{7.26}$$

$$\Theta = d\theta^i + {\omega^i}_j \wedge \theta^j \tag{7.27}$$

Difference of two connections :

$$C = \nabla_2 - \nabla_1 \tag{7.28}$$

Proposition 7.13. The difference between two Kozsul derivatives is a tensor.

Proof. □

7.3.5 Gauge derivative

Ehresmann derivative of a principal bundle
Of particular use later on will be the Yang-Mills gauge derivative, defined by the principal bundle over the groups SU(N) (or $U(1)$ for $N = 1$)
Electromagnetism : fiber $U(1)$, gauge derivative is

$$D_\mu = \partial_\mu + \tag{7.29}$$

Gauge fixing : choice of a section of the principal bundle
Lorenz gauge
Gribov ambiguity

7.4 The affine connection

The affine connection is the most general connection from the frame bundle FM.

$$\begin{aligned} \nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\nu{}_{\alpha\mu} V^\alpha \\ \nabla_\mu \omega_\nu &= \partial_\mu \omega_\nu - \Gamma^\alpha{}_{\nu\mu} \omega_\alpha \end{aligned}$$

Torsion tensor :

$$S_{\mu\nu}{}^\lambda = \Gamma^\lambda{}_{[\mu\nu]} \tag{7.30}$$

Non-metricity tensor :

$$Q_{\lambda\mu\nu} = -\nabla_\lambda g_{\mu\nu} \tag{7.31}$$

Theorem 7.14. The affine connection is uniquely determined by a choice of metric, torsion and non-metricity tensor.

Proof. □

The Riemann tensor is the curvature of this connection

7.5 The Levi-Civitta connection

The Levi-Civitta connection, also called the covariant derivative, is a type of Kozsul connection with respect to the frame bundle $\pi : FM \to M$, which as seen earlier has structure group $GL(n, \mathbb{R})$
Restriction to the orthonormal frame bundle OM

Definition 7.15. The Levi-Civitta connection is a connection on the orthonormal frame bundle that is :

- Torsion-free :
$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = 0, \nabla_X Y - \nabla_Y X = [X, Y]$$
- Metric :
$$\nabla g = 0$$

This will be the usual derivative that we will use in general relativity, as it can be used to define most other derivatives and is also unique.
Fundamental theorem of Riemannian geometry :

Theorem 7.16. The Levi-Civitta connection is unique.

Proof. x □

Components of the Levi Civita connection :

$$\begin{aligned} X(g(Y, Z)) + Y(g(Z, X)) - Z(g(Y, X)) = & \quad g(\nabla_X Y + \nabla_Y X, Z) \\ & + \; g(\nabla_X Z - \nabla_Z X, Y) \\ & + \; g(\nabla_Y Z - \nabla_Z Y, Y) \end{aligned}$$

Koszul formula :

$$\begin{aligned} 2g(\nabla_V W, X) &= Vg(W,X) + Wg(X,V) - Xg(V,W) \\ &- g(V,[W,X]) + g(W,[X,V]) + g(X,[V,W]) \end{aligned}$$

Definition 7.17. The Christoffel symbols of the Levi-Civitta connection for a chart (U, ϕ_U) are the components defined by

$$\nabla_{\partial_\mu}\partial_\nu = \Gamma^\sigma{}_{\mu\nu}\partial_\sigma \tag{7.32}$$

Applied to the covariant derivative of a vector :

$$\begin{aligned} \nabla_{Y^\mu\partial_\mu} X^\nu\partial_\nu &= Y^\mu(\partial_\mu(X^\nu)\partial_\nu + X^\nu\nabla_{\partial_\mu}\partial_\nu) \\ &= Y^\mu(\partial_\mu(X^\nu)\partial_\nu + X^\nu\Gamma^\sigma{}_{\mu\nu}\partial_\sigma) \end{aligned}$$

To simplify things, we will denote the components of the covariant derivative of a vector simply as

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\rho} X^\rho \tag{7.33}$$

Coordinate transform of the Christoffel symbols
Applied to the Koszul formula :

$$\begin{aligned} 2g(\nabla_{V^\rho\partial_\rho} W^\mu\partial_\mu, X^\nu\partial_\nu) &= V^\rho W^\mu X^\nu g(W,X) + Wg(X,V) - Xg(V,W) \\ &- g(V,[W,X]) + g(W,[X,V]) + g(X,[V,W]) \end{aligned}$$

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}) \tag{7.34}$$

Application to dual vector fields and tensor fields :

$$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\rho_{\mu\nu} X_\rho \tag{7.35}$$

7.5.1 Parallel transport

With the connection, we can now compare two vectors inhabiting different tangent spaces. Parallel transport of a vector V along a curve of tangent u

$$\nabla_u V = 0 \tag{7.36}$$

$$u^\mu \nabla_\mu V^\nu = 0 \tag{7.37}$$

Define the derivative : for a curve γ with tangent u,

$$\frac{dx^\mu}{d\lambda}(\lambda)(\frac{\partial}{\partial x^\mu} V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) \tag{7.38}$$

If we consider the restriction $V^\nu(x^\mu(\lambda))$, by the chain rule,

$$\frac{d}{d\lambda} V^\nu(x^\mu(\lambda)) = \frac{dx^\mu}{d\lambda}\frac{\partial V^\nu}{\partial x^\mu}(x^\mu(\lambda)) \tag{7.39}$$

7.5.2 The geodesic equation

A geodesic will be a curve with a tangent vector that is parallely transported with respect to itself. That is,

$$\nabla_U U = 0 \tag{7.40}$$

or, in a coordinate representation,

$$\nabla_U U = U^\mu \nabla_\mu U^\nu = U^\mu \partial_\mu U^\nu + U^\mu {\Gamma^\nu}_{\mu\rho} U^\rho = 0 \tag{7.41}$$

This will correspond, using the coordinate representation of the curve $x^\mu(\lambda)$, to

$$\frac{d^2 x^\nu}{d\lambda^2} + {\Gamma^\nu}_{\mu\rho} \frac{dx^\rho}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \tag{7.42}$$

We say that a curve is a pre-geodesic if there exists a function $f : I \to \mathbb{R}$ such that

$$\nabla_U U = fU \tag{7.43}$$

Reparametrization : for $\lambda \to a\lambda$, $\gamma(a\lambda)$ with initial u^μ equivalent to $\gamma(\lambda)$ with initial au^μ

7.5.3 Other differential operators

The previously seen differential operators can be defined by the Levi-Civitta connection.

$$\mathrm{d}\omega = \nabla_{[\alpha}\omega_{\mu\nu\rho\ldots]} \tag{7.44}$$

$$\mathcal{L}_X Y = \tag{7.45}$$

We also define a few useful variations on the Levi-Civitta connection
The divergence will be defined by

$$\nabla_\mu T^\mu = \partial_\mu T^\mu + {\Gamma^\mu}_{\mu\nu} T^\nu \tag{7.46}$$

We define

$$\Gamma_\nu = {\Gamma^\mu}_{\mu\nu} \tag{7.47}$$

Curl :

$$(\mathrm{curl}(T))_\mu = \varepsilon_{\mu\nu\rho} \nabla^\nu T^\rho \tag{7.48}$$

Laplace-Beltrami operator on a p-form ω

$$\Box\omega = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g} g^{\mu\nu}\partial_\mu\omega) \tag{7.49}$$

7.5.4 Curvature

Riemann curvature tensor

$$\omega(\mathrm{Riem}(X,Y)) = \nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega \tag{7.50}$$

$$[\nabla_\mu, \nabla_\nu]A_\sigma = \nabla_\mu\nabla_\nu A_\sigma - \nabla_\nu\nabla_\mu A_\sigma = R_{\mu\nu\sigma}{}^\tau A_\tau \tag{7.51}$$

Corresponds to parallel transport around an infinitesimal loop

$$\begin{aligned}
R_{\mu\nu\sigma}{}^\tau A_\tau &= \nabla_\mu(\partial_\nu A_\sigma - \Gamma^\tau_{\nu\sigma}A_\tau) - \nabla_\nu(\partial_\mu A_\sigma - \Gamma^\tau_{\mu\sigma}) \\
&= \partial_\mu\partial_\nu A_\sigma - \Gamma^\tau_{\mu\sigma}\partial_\nu A_\tau - \Gamma^\tau_{\mu\nu}\partial_\tau A_\sigma \\
&- \partial_\mu\Gamma^\tau_{\nu\sigma}A_\tau - ... \\
&- \nabla_\nu\partial_\mu A_\sigma \\
&- \nabla_\nu\Gamma^\tau_{\mu\sigma}
\end{aligned}$$

$$R_{\mu\nu\sigma}{}^\tau = \partial_\nu\Gamma^\tau{}_{\mu\sigma} - \partial_\mu\Gamma^\tau{}_{\nu\sigma} + \Gamma^\alpha{}_{\mu\sigma}\Gamma^\tau{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\sigma}\Gamma^\tau{}_{\alpha\mu} \tag{7.52}$$

Symmetries of the Riemann tensor

$$R_{\mu\nu\sigma}{}^\tau = R_{\mu\nu\sigma}{}^\tau \tag{7.53}$$

Number of components of the Riemann tensor
In coordinate form, the Riemann tensor has at most n^4 components, from its 4 indices, but thanks to its symmetries, that number can be reduced.

$$C_n(\mathrm{Riem}) = \frac{n^2(n^2-1)}{12} \tag{7.54}$$

Ricci tensor

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma \tag{7.55}$$

Ricci scalar

$$R = R_{\mu\nu}g^{\mu\nu} \tag{7.56}$$

Bianchi identity

$$\nabla_{[\rho}R_{\mu\nu]\sigma\rho}{}^\tau = \nabla_\rho R_{\mu\nu\sigma\rho}{}^\tau + \nabla_\nu R_{\rho\mu\sigma\rho}{}^\tau + \nabla_\mu R_{\nu\rho\sigma\rho}{}^\tau = 0 \tag{7.57}$$

Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \tag{7.58}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{7.59}$$

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu R^{\mu\nu} - \frac{1}{2}g_{\mu\nu}\nabla_\mu R \tag{7.60}$$

Decomposition of the Riemann tensor

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma} + G_{\mu\nu\rho\sigma} \tag{7.61}$$

with $C_{\mu\nu\rho\sigma}$ the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} \tag{7.62}$$

$$E_{\mu\nu\rho\sigma} = \tag{7.63}$$

$$G_{\mu\nu\rho\sigma} = \frac{1}{12}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{7.64}$$

7.6 Torsion and non-metricity

It is possible to define a spacetime with non-zero torsion, leading to the Einstein-Cartan theory of relativity. For a general Koszul connection, the derivative is defined by three quantities : the metric tensor, the torsion tensor and the non-metricity tensor.

$$\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = -T^\sigma{}_{\mu\nu}\nabla_\sigma f \tag{7.65}$$

where T is the torsion tensor.

$$\nabla_X g(Y, Z) = N(X, Y, Z) + ... \tag{7.66}$$

$$\nabla_\sigma g_{\mu\nu} = N_{\sigma\mu\nu} \tag{7.67}$$

Proposition 7.18. Any Koszul connection can be decomposed in term of three tensors, the metric tensor, the torsion tensor and the non-metricity tensor.

Proof. If we have two Koszul connections ∇, $\bar{\nabla}$, we need to show that given those three tensors, $(\nabla - \bar{\nabla})T = 0$ for all tensors T.

$$(\nabla - \bar{\nabla})g_X(Y, Z) \tag{7.68}$$

□

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}g^{\sigma\tau}[\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu} - T_{\mu\nu\tau} - T_{\nu\mu\tau} + T_{\tau\mu\nu}] \tag{7.69}$$

$$\{_\mu{}^\sigma{}_\nu\} = \frac{1}{2}g^{\sigma\tau}[\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu}] \tag{7.70}$$

$$K^\sigma{}_{\mu\nu} = \frac{1}{2}g^{\sigma\tau}[-T_{\mu\nu\tau} - T_{\nu\mu\tau} + T_{\tau\mu\nu}] \tag{7.71}$$

K the contortion tensor.
We have that

$$\Gamma^\sigma{}_{(\mu\nu)} = \{_\mu{}^\sigma{}_\nu\} \tag{7.72}$$

Meaning that, for the geodesic equation, we still have

$$\frac{du^\sigma}{d\lambda}(\lambda) + \{_\mu{}^\sigma{}_\nu\} u^\mu(\lambda) u^\nu(\lambda) \tag{7.73}$$

due to the symmetry of μ and ν. This means that geodesics are not affected by torsion, and as such are of limited interest for the scope of this book.

7.7 Spin connection

A similar definition of the covariant derivative exists if, rather than the metric tensor, we use the frame field.

$$\nabla_X e_j = \omega^i{}_j(X) e_i \tag{7.74}$$

$$\nabla_{e_k} e_j = \Gamma^i{}_{kj} e_i = \omega^i{}_j(e_k) e_i \tag{7.75}$$

Ricci rotation coefficients :

$$\omega_{\sigma\mu\nu} = e_\sigma{}^a e_\mu{}^b \nabla_a e_{\nu b} \tag{7.76}$$

Cartan structure equations

$$\Theta^i = d\theta^i + \omega^i{}_j \wedge \theta^j \tag{7.77}$$

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \tag{7.78}$$

Connection is metric if

$$\omega_{ik} + \omega_{ki} = \langle e_i, e_k \rangle \tag{7.79}$$

7.8 The exponential map and normal coordinates

The existence of a connection lets us define a natural set of coordinates stemming from the natural coordinates of the tangent space, as the tangent space itself is simply $\mathbb{R}^n$. To accomplish this, we have to map vectors from the tangent space T_pM to nearby points of p with geodesics.
Since there's no guarantee that any two points can be connected by a geodesic (and indeed it will not always be possible), we first have to define the maximal extension of such a scheme.

Definition 7.19. $\mathcal{D}_p$ is the set of vectors v in $T_p\mathcal{M}$ such that the inextendible geodesic $\gamma(0) = p$ with tangent vector $\dot{\gamma}(p) = v$ is defined on the interval $[0, 1]$.

We only require it to be defined on $[0, 1]$ since a reparametrization can show that if it is defined for $\gamma(\lambda)$, $\lambda > 1$, it will be equivalent to the curve of initial tangent vector λv at $\gamma(1)$.

Theorem 7.20. $\mathcal{D}_p$ is an open set in T_pM, starshaped around 0.

Proof. □

Definition 7.21. The exponential map at p is the function

$$\exp_p : \mathcal{D}_p \to \mathcal{M}$$

such that for $v \in \mathcal{D}_p$, $\exp_p(v) = \gamma_v(1)$

Geodesic equation :

$$\frac{dx^\mu}{d\lambda} = u^\mu \tag{7.80}$$

$$\frac{du^\mu}{d\lambda} = -\Gamma^\mu_{\sigma\rho} u^\rho u^\sigma \tag{7.81}$$

By Picard–Lindelöf theorem [REQUIRES THE METRIC TO BE AT LEAST $C^{1,1}$], locally unique and existing solution for initial conditions $x(0) = x_0$ and $u(0) = u_0$ on the interval $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$

Theorem 7.22. A straight line parametrized by $x^\mu(\lambda) = a^\mu \lambda$ in the tangent space, such that $x^\mu(\lambda) \in \mathcal{D}_p$ for the whole range of λ will map to a geodesic on the manifold.

Proof. Consider the curve mapped by $\exp_p(\lambda a^\mu) = \gamma(\lambda)$. Its tangent vector will be

$$\gamma'(\lambda) = \frac{d}{d\lambda}(\exp_p(\lambda a^\mu)) = \lambda \exp'_p(\lambda a^\mu) \tag{7.82}$$

□

7.9 Convex normal neighbourhood

Definition 7.23. An open set U is convex in M if it is a normal neighbourhood of each of its points $p \in U$.

For every point $p, q \in U$ there exists a unique geodesic
Inverse function theorem : exp continuously differentiable in an open set containing p, $\det exp' \neq 0$, then $\exists V, p \in V, \exists W, \exp(p) \in W$ such that $f^{-1} : W \to V$ which is differentiable and $\forall y \in W$

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$$

Convex normal neighbourhood : For $p \in \mathcal{M}$, the open set U is a convex normal neighbourhood of p if $p \in U$, and $\forall q, r \in U$, there is a unique geodesic connecting q and r that stays entirely in U.
If U is a convex normal neighbourhood of p, the points that can be reached by a timelike (respectively causal) curve are the ones of the form $\exp_q(X)$

Theorem 7.24. For a manifold M, if $p \in M$, there exists a convex normal neighbourhood of p.

Proof. Picking a normal coordinate system x_i around p, $x_i(p) = 0$ and $\Gamma^\mu_{\nu\sigma}(p) = 0$. □

In a convex normal neighbourhood, the parallel transport of a tensor is a bitensor.

$$\gamma : T_pM \to T_qM \tag{7.83}$$

Theorem 7.25. In a convex normal neighbourhood around p, the points that can be reached by p through a timelike (non-spacelike) curves are of the form $\exp_p(X)$, $X \in T_p\mathcal{M}$, with $g(X, X) < 0$ (resp. $g(X, X) \leq 0$)

Proof. x □

The Gauss lemma

Theorem 7.26. For a point $p \in M$, and a vector $X \in T_pM$, $X \neq 0$, if we have two vectors v_X, w_X of the tangent space $T_X(T_pM)$ with v_X radial, then

$$g(d\exp_p(v_X), d\exp_p(w_X)) = g(v_X, w_X) \tag{7.84}$$

Proof. x □

Definition 7.27. A simple region is a convex normal neighbourhood with compact closure in $\mathcal{M}$ that is itself contained in another convex normal neighbourhood.

The tangent space has the natural structure of flat space : an n-dimensional vector space equipped with a symmetric bilinear form $\langle -, - \rangle$ such that in some basis e_μ,

$$\langle x, y \rangle = -\sum_{\mu=0}^{p} (x^\mu - y^\mu)^2 + \sum_{\mu=p+1}^{q} (x^\mu - y^\mu)^2 \tag{7.85}$$

Because of this, we may identify the tangent space of any manifold with the flat space $(\mathbb{R}^n, \eta)$ of the same dimension and signature. In particular,

Theorem 7.28. There's a natural isomorphism between Minkowski space and its tangent space.

Proof. $\omega \in V^*$. We can view it as a function $\omega : V \to \mathbb{R}$. We seek an isomorphism $\phi_v : T_vV \to V$ such that $\omega(\phi_v(w)) = d\omega(w)$ for each $w \in T_vV$ and $\omega \in V^*$. Pick a basis $\{\theta^i\}$ of V^* and a basis $\{e_j\}$ of T_vV. $(\phi_v)_i^j = d\theta^i(e_j)$ is such an isomorphism. For uniqueness : if ψ_v is another, $\omega((\phi_v - \psi_v)w) = 0$ for all $\omega \in V^*$, $w \in T_vV$. It is obvious that this implies $\phi_v - \psi_v = 0$. We can turn the Lorentzian vector space $(V, \langle \cdot, \cdot \rangle)$ into a Lorentzian manifold (V, g) if we define $g_v(w, z) = \langle \phi_v w, \phi_v z \rangle$. □

7.9.1 The diffeomorphism group

As seen previously, the set $\text{Diff}(M)$ of all diffeomorphisms on the manifold forms a group.
Diffeomorphisms on a convex normal neighbourhood
Lie brackets

7.10 Observers and appropriate coordinates

7.11 Frame fields and observers

A curve with an associated frame field along it can be used to represent an observer on spacetime, the curve γ representing the trajectory of this observer through spacetime and the frame field along that curve $e^a{}_\mu(x(\lambda))$ being a system of coordinates defined by the observer. For a physical observer, we will assume that the curve is timelike.

If we take any frame field associated to the metric and an arbitrary curve,
Rotation of a frame on a curve : if the tangent vector $u = u_a e^a$, the rotation of the frame will be

$$\frac{du^\mu}{dt} = -\Omega_{\mu\nu} u_\nu = ()v^b \tag{7.86}$$

$$\nabla_u e_a = -\Omega \cdot e_a \tag{7.87}$$

$$\Omega^{\mu\nu} = a^\mu a^\nu - u^\mu a^\nu + u_\alpha \omega_\beta \varepsilon^{\alpha\beta\mu\nu} \tag{7.88}$$

$$a = \nabla_u u \tag{7.89}$$

$$g(a, u) = g(\omega, u) = 0 \tag{7.90}$$

$\omega = 0$: Fermi-walker transport of the frame $a = \omega = 0$: Geodesic motion

7.11.1 Fermi coordinates

Coordinates locally flat around a geodesic γ
For all $p \in \gamma(S)$, there exists a coordinate patch around p such that

$$g_{\mu\nu}(p) = \eta_{\mu\nu}(p),\ \partial_\rho g_{\mu\nu}(p) = 0 \tag{7.91}$$

and the components of the curve on that patch are

$$x^\mu = (t, 0, 0, ...) \tag{7.92}$$

7.11.2 Riemann normal coordinates

The simplest adapted coordinates are simply the coordinates associated with a Cauchy normal neighbourhood via the exponential map
Coordinates defined to be locally flat at a point p

$$g_{\mu\nu}(p) = \eta_{\mu\nu}(p),\ \partial_\rho g_{\mu\nu}(p) = 0 \tag{7.93}$$

7.11.3 Frenet coordinates

Coordinates along an arbitrary curve $\gamma(\lambda)$ with tangent $u(\lambda)$ and rotation $\omega(\lambda)$. The orthonormal frame associated to the observer obeys

$$\frac{de_a}{d\lambda} = -\Omega e_a \tag{7.94}$$

$$\Omega^{\mu\nu} = a^\mu u^\nu - a^\nu u^\mu + u_\rho \omega_\beta \varepsilon^{\rho\sigma\mu\nu} \tag{7.95}$$

$$a^\mu(\lambda) = \nabla_u u^\mu \tag{7.96}$$

7.12 Properties of curves and geodesics

Theorem 7.29. The length of a curve looks locally like it does in flat spacetime.

Proof. For the curve γ starting at p and ending at q, consider the point p' along that curve that lies within the normal neighbourhood. In the normal coordinates of p,

$$x^\mu(p) = 0,\ g_{\mu\nu}(x) = \delta_{\mu\nu} + \mathcal{O}(|x^2|) \tag{7.97}$$

where δ is the metric of flat space of appropriate signature. We can Taylor expand the coordinates of the curves as

$$x^\mu(\lambda) = x^\mu(0) + tu^\mu + \mathcal{O}(\lambda^2) \tag{7.98}$$

with tangent $u^\mu + \mathcal{O}(\lambda)$. The length of that curve is then

$$s_\gamma = \int_p^{p'} [\delta_{\mu\nu} u^\mu(\lambda) u^\nu(\lambda) + \mathcal{O}(|x^2|)]^{\frac{1}{2}} d\lambda \tag{7.99}$$

□

Theorem 7.30. In a Riemannian manifold, the shortest path between two points is a geodesic.

Proof. □

Properties of geodesics as minimal curves for Lorentz metric
It can be useful to define a more general definition of causal curves.

Definition 7.31. A C^0 curve γ is causal (resp. timelike) if, for all $p \in \gamma$, there is a convex neighbourhood $U \ni p$ such that for any $q \neq p$, $q \in \gamma \cap U$, we can connect p and q by a C^1 causal (resp. timelike) curve contained in U.

Proposition 7.32. For a point $p \in \mathcal{M}$, there is a convex coordinate neighbourhood $U \ni p$ with coordinates x^μ such that for any causal curve $\gamma \subset U$, γ can be parametrized by x^0 and for $k > 0$, we have

$$[\sum_\mu (x^\mu(t) - x^\mu(s))^2]^{\frac{1}{2}} \leq k|t - s| \tag{7.100}$$

for all $t, s \in I$.

Proof. (cf Kriele) □

7.13 The geodetic interval

The geodetic interval, also called the world function, assigns a positive semi-definite length to geodesics connecting two points. That is, for a geodesic γ connecting p and q, consider the quantity s_γ defined by

- γ timelike :

$$s_\gamma(p, q) = -l_\gamma$$

- γ null :

$$s_\gamma(p,q) = 0$$

- γ spacelike :

$$s_\gamma(p,q) = l_\gamma$$

The geodetic interval is then defined by

$$\begin{aligned} \sigma_\gamma(p,q) &= \pm\frac{1}{2}[s_\gamma(p,q)]^2 \\ &= \frac{1}{2}(\lambda_q - \lambda_p)\int_{\lambda_p}^{\lambda_q} g(u(\lambda),u(\lambda))d\lambda \end{aligned} \tag{7.101}$$

If geodesics between p and q are unique, it can be considered as a bitensor field of rank $(0,0)$.
Value independant of the parametrization
$g(u(\lambda),u(\lambda))$ is constant along γ so we can write

$$\sigma_\gamma(p,q) = \frac{1}{2}(\lambda_q - \lambda_p)^2 g(\gamma'(\lambda),\gamma'(\lambda)) \tag{7.102}$$

If parametrized so that $\gamma(0) = p$, $\gamma(1) = q$,

$$\Omega_\gamma(p,q) = \frac{1}{2}g(\gamma'(\lambda),\gamma'(\lambda)) \tag{7.103}$$

$$\Omega_\gamma(p,q) = \frac{1}{2}\varepsilon(\int_p^q ds)^2 \tag{7.104}$$

$\varepsilon = \pm 1, 0$ if the curve is spacelike, timelike or null.
For γ a geodesic, we define the geodetic distance as the world function on γ normalized, that is, for a timelike geodesic,

$$s_\gamma(p,q) = \int_p^q d\tau \tag{7.105}$$

for a spacelike geodesic

$$s_\gamma(p,q) = \int_p^q ds \tag{7.106}$$

and of course, for a null curve, geodesic or not, we always have $s_\gamma(p,q) = 0$.
Unlike Riemannian manifolds, the minimal
If the geodesic isn't timelike,

$$\nabla^x_\mu \sigma_\gamma(x,y) = \pm s_\gamma(x,y)\nabla^x_\mu s_\gamma(x,y) \tag{7.107}$$

We define the unit tangent vector of our geodesic from y to x by

$$t^\mu(x,\gamma,x \leftarrow y) = \pm g^{\mu\nu}\nabla^x_\nu s_\gamma(x,y) \tag{7.108}$$

$$\nabla^x_\mu \sigma_\gamma(x,y) = s_\gamma(x,y) t^\mu(x,\gamma,x \leftarrow y) \tag{7.109}$$

If the geodesic is null, $s_\gamma(x,y) = 0$, so that if we use the previous expression, it will be 0. The covariant derivative in this case will be proportional to some null vector k

$$\nabla^x_\mu \sigma_\gamma(x,y) = \zeta_\gamma(x,y) k_\mu(x,\gamma,x \leftarrow y) \tag{7.110}$$

We then have to introduce a canonical observer by a timelike vector ξ^μ at x that we will parallel transport

$$(g^{\mu\nu}\nabla_\mu \sigma_\gamma)\nabla_\nu \xi^\rho = 0 \tag{7.111}$$

Or, in other words,

$$l^\mu(x,\gamma,x \leftarrow y)\nabla_\nu \xi^\rho = 0 \tag{7.112}$$

If we define it to be of the same time orientation as l,

$$l^\mu(x,\gamma,x \leftarrow y)\xi_\mu = -1 \tag{7.113}$$

$$\zeta_\gamma(x,y) = -V^\mu \nabla^x_\mu \sigma_\gamma(x,y) \tag{7.114}$$

ζ_γ : distance along the null geodesic as measured by the canonical observer.
We also define $m = 2\xi - l$

$$\begin{aligned} m_\mu \xi^\mu &= -1 \\ l_\mu m^\mu &= -2 \end{aligned}$$

$$\nabla^x_\mu \zeta_\gamma(x,y) = -\frac{1}{2} l_\mu(x,\gamma,x \leftarrow y) \tag{7.115}$$

Ex from Visser : prove ζ is a parametrization of γ, show $l\nabla l = 0$, hint : $l\nabla l = fl$, $l\nabla\xi = 0$, $l\xi = -1$, compute f, show $l\nabla m = 0$ show $l\nabla\zeta = 0$, $m\nabla\zeta = 0$

7.13.1 The van Vleck determinant

$$\Delta_\gamma(x,y) = (-1)^n \frac{\det(\nabla^x_\mu \nabla^y_\nu \sigma_\gamma(x,y))}{\sqrt{g(x)g(y)}} \tag{7.116}$$

Motivation : In the case where geodesics between two points are unique, there are two ways to specify a geodesic : by either specifying an initial point p and tangent vector u, or by specifying two points p and q. The Jacobian associated with this change of variables is

$$\begin{aligned} J_\gamma(x,y) &= \frac{1}{\sqrt{g(x)g(y)}} \det\left(\frac{\partial(x,u)}{\partial(x,y)}\right) && (7.117) \\ &= \frac{1}{\sqrt{g(x)g(y)}} \det\left(\frac{\partial u}{\partial y}\right) && (7.118) \end{aligned}$$

8 Spacetime submanifolds

We briefly saw earlier what a submanifold is, and we will now see in more details the various kinds of submanifolds one can find as well as what happens to the manifold structures on them.
All submanifolds are defined in the same manner : a manifold N is a submanifold of the manifold M if there exists a map $f : N \to M$. The subset $f(N) \subset M$ is then called the submanifold. The map f is not generally required to be injective, as will be seen with some examples.

8.1 Immersions, submersions and embeddings

Definition 8.1. A smooth map $f : M \to N$ is an *immersion* if $df(p)$ is a bijection for all $p \in M$.

Equivalent : Jacobian $d\phi_p$ has rank m relative to some coordinate system, a coordinate system $\{y^\mu\}$ of N there are integers such that $\{y^{i(\nu)}\}_{1\leq\nu\leq n}$ is a coordinate system on a neighbourhood $p \in M$.
Examples : regular curves ($\gamma' \neq 0$) are immersions.

Definition 8.2. An embedding of a manifold P in M is an immersion $\phi : P \to M$ such that ϕ is a homeomorphism.

The submanifold topology is the same as the subspace topology
Example : embedding of $(a_1, a_2, ...a_m) \to (a_1, a_2, ...a_m, 0, 0, ...)$

Definition 8.3. A smooth map $f : M \to N$ is a submersion if

Immersion, submersion, embedding, submanifold
Injection maps $\iota : M \hookrightarrow M'$, diffeomorphic on $\iota^{-1}(M')$

Definition 8.4. A subset N is a k-dimensional submanifold of M, $k \leq n$, if there is an atlas of N $\{(U_\alpha, \phi_\alpha)\}$ such that for all α where $U_\alpha \cap N \neq 0$,

Immersion, embedding
Example of manifold from submanifolds : The n-sphere S^n as a submanifold of $\mathbb{R}^{n+1}$
Whitney embedding theorem :

Theorem 8.5. Any smooth, Hausdorff and second-countable manifold of dimension n can be smoothly embedded in $\mathbb{R}^{2n}$.

8.2 Induced structures on submanifolds

If we consider the submanifold M of a manifold M', with inclusion map $\iota : M \hookrightarrow M'$
A vector field X on M passed through the inclusion map is a M' vector field on M.
Tangent space T_pM is a non-degenerate subspace of T_pM'

$$T_pM' = T_pM \oplus T_pM^\perp \tag{8.1}$$

Vectors of $T_pM^\perp$ are called normal vectors to M
Induced connection :

Induced metric
timelike, spacelike and null submanifolds
Gauss-Codazzi equation

8.3 Hypersurfaces

A hypersurface is an $(n-1)$ dimensional submanifold. For an $(n-1)$-manifold Σ and an embedding $\theta : \Sigma \to M$, then we say that $\theta(\Sigma)$ is a hypersurface in M.
induced metric $\theta^* g$ on Σ where $X, Y \in T_p M$, $\theta^* g(X,Y)|_p = g(\theta_* X, \theta_* Y)|_{\theta(p)}$

Theorem 8.6. The intersection of an n-dimensional timelike submanifold and a spacelike hypersurface is an $n-1$ spacelike submanifold.

8.3.1 Hyperquadrics

In the flat manifold $\mathbb{R}^n$:
Take the function q in $\mathbb{R}^{p+q}$

$$q = -\sum_{i=1}^{p} (x^i)^2 + \sum_{i=p+1}^{n+1} (x^i)^2 \tag{8.2}$$

$q(x) = \langle x, x \rangle$, hence $\operatorname{grad} q = 2x$

$$\langle \operatorname{grad} q, X \rangle = X(q) = X\langle x, x \rangle = 2\rangle D_X x, x\rangle = 2\langle X, x \rangle \tag{8.3}$$

8.4 Foliations

Definition 8.7. A *foliation* of a manifold M is the decomposition of M into a disjoint union of connected subsets L_α, called leaves, such that they cover the manifold

$$M = \bigcup_\alpha L_\alpha \tag{8.4}$$

and there is an atlas (U_β, ϕ_β) such that the intersection of a leaf L_α wuth U_β

Foliations of interest in GR : foliation by spacelike hypersurfaces, foliations by timelike curves.
Froebinius theorem

9 Singularities

Singularities play an important role in the study of spacetime, as they are a common feature of many realistic spacetimes, so it may be of interest to have the proper definitions and classification for them.
On flat spacetime, field singularities are generally defined by divergences in quantities at defined points. For instance, the singularity of the electric field of a static charge in electromagnetism

$$\vec{E}(\vec{x}) = k_e \frac{q}{r^2} \vec{e}_r \tag{9.1}$$

which presents a singularity at $r = 0$, as the field is unbounded around this value. One might think that the definition of a singularity in general relativity is similar (as it usually appear in similar forms in such famous examples such as the Schwarzchild metric), but unlike classical physics, singularities in general relativity are not trivial to define due to a few obstacles :

- Points can be (and for singularities, are) removed from the manifold, in which case if a quantity diverges at p on $\mathcal{M}$, we need to define what this means on $\mathcal{M} \setminus \{p\}$.
- For tensor components, the values of a field depend on the atlas we use, and this can lead to divergences in one coordinate system but not in another, such as the horizon in the Schwarzschild metric.
- For some choice of coordinates, a divergence at a given coordinate can correspond to a quantity which is "at infinity", and as such not really problematic.

9.1 Regular boundary points

The simplest singularities arise when a well-behaved point is removed from a manifold. While they may appear as singularities in all the following definitions, unlike other more pathological singularities, they do not necessarily form a boundary of the manifold. For instance, for the Minkowski half-space $\{(t, x) | t > 0\}$, there is a singularity formed by $(t, x) | t = 0$, but it is possible to extend (in a non-unique way) the spacetime.
To differentiate regular boundary points from more pathological ones, we first try to extend the spacetime in a way that remove them.

Definition 9.1. An extension $\mathcal{M}'$ of a spacetime is a spacetime $\mathcal{M}'$ such that there exists an isometric embedding $\mu : \mathcal{M} \to \mathcal{M}'$.

Definition 9.2. A boundary point is a *regular boundary point* if there exists an extension of the spacetime such that the boundary point is not present in the extension.

Such singularities, while often used for examples, are often considered unlikely to be physical.

9.2 m-completeness and g-completeness

If a manifold is equipped with a Riemannian metric, we can define the notion of completeness of the manifold by its status as a metric space. We then say that the manifold is m-complete if every Cauchy sequence converges to a point in the manifold, where the norm of the Cauchy sequence is the metric d defined by the Riemann tensor. That is, for a sequence p_n of points of the manifold

$$\exists p \in \mathcal{M},\ \forall \varepsilon,\ \exists N,\ \forall n > N, \tag{9.2}$$

Unfortunately, Lorentzian metrics do not permit the definition of a metric on the manifold.
g-complete : Every inextendible half-geodesic is defined for arbitrarily large values of its affine parameter.
The definition is limited to geodesics due to the fact that any spacetime may admit curves of finite affine length by a proper choice of the parametrization

Theorem 9.3. Even benign spacetimes may admit half-curves of finite affine length.

Proof. Consider a curve in Minkowski space with unbounded acceleration in finite time

$$\ddot{x}^\mu(\lambda) = (0, \tan(\lambda)) \tag{9.3}$$

That curve is both inextendible and of finite affine length. □

This corresponds to a curve of unbounded acceleration.
Unfortunately, the notion of g-completeness is not sufficient to fully capture the concept of singularities.

Theorem 9.4. There exists geodesically incomplete compact spacetimes.

Proof. cf Misner 63, O'neill p. 193 □

The notion of g-boundary gives rise to three classes of singularities, depending on the type of curve considered. These are naturally timelike singularities, spacelike singularity and null singularity. Spacetimes with those types of singularities are qualified as timelike incomplete, spacelike incomplete or null incomplete, or g_t, g_s and g_n incomplete.
Unfortunately, there is no obvious relation between those types of incompleteness, as it can be shown that they can all occur independently of each other.

9.2.0.1 Spacetime which is spacelike and null complete but timelike incomplete

2D Conformal Minkowski space $g = \Omega\eta$, $\Omega = 1$ if $|x| > 1$, conformal factor is space reflection symmetric : $\Omega(x,t) = \Omega(-x,t)$, Ω decays fast as $t \to \infty$ (for instance t^6). This conformal factor could be built easily enough with bump functions.
Geodesic on $t = 0$ is timelike and finite length. Every spacelike or null geodesic will leave the region $(-1, 1)$ and hence be complete.

9.2.0.2 Spacetime which is spacelike complete but timelike and null incomplete

Metric due to Kundt

$$ds^2 = -f^{-2}(x)dt^2 - dx^2 \tag{9.4}$$

for $x \in (0, \infty)$, $t \in \mathbb{R}$
Geodesics :

$$s - s_0 = \int (f^2(x) + \epsilon)^{-\frac{1}{2}} dx \tag{9.5}$$

ϵ being -1 for timelike geodesics, 1 for spacelike geodesics and 0 for null geodesics.
for $f(x) = x$

9.2.0.3 Spacetime which is timelike complete but null and spacelike incomplete

[cf Kundt]

9.2.0.4 Spacetime which is timelike, spacelike and null incomplete

9.2.0.5 Spacetime which is timelike and null complete but spacelike incomplete

9.2.0.6 Spacetime which is spacelike and timelike complete but null incomplete

9.2.0.7 Spacetimes which are complete for all geodesics but not for timelike curves of bounded acceleration

Example of a spacetime which is geodesically complete but with non-terminating timelike curves
Covering space of two-dimensional anti de Sitter space

$$ds^2 = -(1 + x^2)dt^2 + (1 + x^2)^{-1}dx^2 \tag{9.6}$$

Construct a spacetime with "geodesic traps" ended by a singularity
Blocking sets

9.2.1 Naked singularities

A particularly important class of singularities in general relativity is the naked singularity

Definition 9.5. A spacetime is *nakedly singular* if there is a point $p \in \mathcal{M}$ and a future-incomplete timelike or null geodesic γ such that $\gamma \subset I^-(p)$

(definition extended to causal curves of bounded accelerations maybe)
By this definition, it is easy to see that a spacetime that is both timelike and null complete is never nakedly singular, and so spacetimes that are only

9.3 b-completeness

The notion of completeness that fulfills best the broadest notion of singularity is b-completeness, or bundle completeness, where we define a Riemannian metric on the bundle of linear frames $L\mathcal{M}$
An inextendible curve in $\mathcal{M}$ defines a point of the b-boundary $\partial\mathcal{M}$ if and only if its euclidian length measured in a parallelly propagated frame is finite.

$$g_b(X,Y) = \sum_i \theta^i(X)\theta^i(Y) + \sum_{i,j} \omega^i_k(X)\omega^i_k(Y) \tag{9.7}$$

b-boundary : Completion of the orthonormal bundle $\pi : O \to \mathcal{M}$ with the metric e on the bundle. The b-boundary

9.4 Classification of singularities

The b-boundary gives us the boundary points $\partial\mathcal{M}$ of the spacetime. These singularities can then be classified by the behaviour of the Riemann tensor near them.
First, as we have defined previously, singularities in $\partial\mathcal{M}$ which are actual points of a manifold $\mathcal{M}'$, where $\mathcal{M}'$ is an extension of $\mathcal{M}$, are called regular boundary points. If for one of those extensions $\mathcal{M}'$, the Riemann tensor is a C^k tensor, and k is the highest value of any of those extensions, we say that it is a C^k regular boundary point. If there is no extension where the Riemann tensor is C^k, we say that this point is a C^k singular point. If the metric is itself not smooth, the Riemann tensor might be itself C^k itself at some points, even discontinuous or worse (if we allow for weak derivatives). Although not part of the b-boundary, this will be of a similar behaviour to C^k regular boundary points. We may then get points as badly behaved as C^0 regular boundary points, for instance in the thin shell formalism or with gravitational shockwaves.
(manifold + b-boundary is a topological space)

9.4.1 Regular boundary points

As we saw previously, some boundary points are simply the result of spacetime not being fully extended.

Theorem 9.6. If a spacetime admits an extension, it contains regular boundary points.

Proof. Consider a spacetime $\mathcal{M}$ with an injection map $\iota : M \hookrightarrow M'$ into an extension M' such that $\operatorname{Im}\iota \subset M'$. The image of the inclusion will then have a non-empty boundary ∂M. As they are part of the manifold $\mathcal{M}'$, there exists Cauchy sequences $(e_i)_{i\in\mathbb{N}}$ in the bundle of linear frame $L\mathcal{M}'$, $e_i \in L\mathcal{M}$ for all i, such that

$$\pi(\lim_{i\to\infty} e_i) \in \partial \operatorname{Im}\iota \tag{9.8}$$

As M does not contain any point in ∂M, those sequences of $LM'|_M$ do not converge in the bundle itself and are boundary points. □

9.4.2 Quasi-regular singularities

A boundary point $p \in \partial M$ is a quasi-regular Quasiregular singularities if it is not possible but the Riemann tensor can be extended in a parallel frame

Definition 9.7. A singular boundary point b is called a *quasi-regular singularity* if, for every curve γ that ends on the boundary point, the frame field $\{e_a\}$ and all its derivatives on those curves remains bounded.

Local extensibility
Example : conical singularity
Take Minkowski space, express in cylindrical coordinates

$$ds^2 = -dt^2 + d\rho + \rho d\theta^2 + dz^2 \tag{9.9}$$

remove the timelike plane $\{(t, \rho, \theta, z) | \rho = 0\}$
Effects of quasi-regular singularities : affects the parallel transport of curves

9.4.3 Curvature singularity

Curvature singularity : Riemann tensor cannot be extended in a parallel frame

Definition 9.8. A boundary point b is called a *curvature singularity* if, for every curve γ that ends on the boundary point, the frame field $\{e_a\}$ or one of its derivatives on those curves is unbounded.

The curvature singularity is the most common example of singularity. It is for instance the type we may find in the interior Schwarzschild solution $r < 2M$

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

Non-scalar singularity : no curvature scalar is badly behaved
Scalar singularity : badly behaved curvature scalar

10 Distributions on manifolds

For many applications, such as classical field theory, point particle sources, quantum field theory or singular spacetimes, it will be useful to define distributions of various types on the spacetime manifold.

10.1 Schwartz distributions

The simplest class of distributions are the Schwartz distributions, simply defined with respect to scalar test functions, or just simply test functions.

Definition 10.1. A *test function* f is a smooth function of compact support on M, that is, $f \in C^\infty(M)$, $\operatorname{supp} f \subset\subset M$.

As we've seen with bump functions, such functions do exist on any Hausdorff manifold. The set of all test functions on M is noted $\mathscr{D}(M)$. From this we can define Schwartz distributions as

Definition 10.2. A *distribution* is a linear functional on the space of test functions

$$\Phi : \mathscr{D}(M) \to \mathbb{R} \tag{10.1}$$

which means that, for $\Phi \in \mathscr{D}'(M)$ and $f, g \in \mathscr{D}(M)$, we have the property

$$\Phi[af + bg] = a\Phi[f] + b\Phi[g] \tag{10.2}$$

Embedding of L^1 functions : If a function $\Phi \in L^1(D)$, then there is an embedding of that function in $\mathscr{D}'(D)$, as

$$\Phi(f) = \int_D \Phi f d\mu[g] \tag{10.3}$$

Definition 10.3. The *delta distribution* is the class of distribution defined by a point $p \in M$ such that $\delta_p[f] = f(p)$.

Definition 10.4. The *singular support* of a distribution sing supp(ϕ) is a subset of the manifold such that $p \notin$ sing supp(ϕ) if there exists a neighbourhood $U \ni p$ such that we can define a smooth function $\phi(p) \in C^\infty(U)$ such that, for any test function with support in U, we have

$$\phi[f] = \int \phi(x) f(x) d\mu[g] \tag{10.4}$$

The singular support expresses at which point a distribution fails to behave like a smooth function.
Example : sing supp(δ) = $\{0\}$

Theorem 10.5. A distribution with an empty singular support is equivalent to a smooth function.

Proof. □

Derivatives of distributions :

$$\phi'[f] = -\phi[f'] \tag{10.5}$$

Proposition 10.6. The weak derivative of a distribution is equivalent to the standard definition of derivatives for smooth functions.

Proof. By integration by parts, we have

$$\phi'[f] = \int_D \phi'(x) f(x) d\mu[g] = \int_{\partial D} \phi(x) f(x) - \int_D \phi(x) f(x) d\mu[g] \tag{10.6}$$

□

10.1.1 Distributions as sequences of test functions

Another definition of distributions is to consider the space of sequences of smooth functions $\{\phi_n(x)\} \in \mathbb{N} \times C^\infty(M)$ such that, for a test function f,

$$\lim_{n\to\infty} \int \phi_n(x) f(x) d\mu[g] \in \mathbb{R} \tag{10.7}$$

If the limit exists, the sequence is said to converges weakly. A distribution is then defined as the equivalence class of sequences converging to the same values for the same test functions.

Proposition 10.7. The set of weakly converging function sequences is equivalent to the set of distributions.

Delta distribution as sequence of smooth functions : for any coordinate chart centered on p with $x(p) = 0$, we have

$$\delta_p = \{\varepsilon \phi_n(\frac{x}{\varepsilon})\}_n \tag{10.8}$$

10.1.2 Product of distributions

Theorem 10.8. There is no differential algebra of distributions that includes the unit function and obeys the usual multiplication on smooth functions.

Proof. Cf. Schwartz's paper, with the distribution $1/x$ as a counterexample □

Possible products :
Product with smooth functions $C^\infty(M) \times \mathscr{D}'(M) \to \mathscr{D}'(M)$
products of two distributions with disjoint singular supports

10.1.3 The wavefront set

Distribution products using the Fourier transform :

$$\widehat{f^2 w}(k) = (\widehat{fu} \star \widehat{fv})(k) = \tag{10.9}$$

absolutely convergent
Wavefront set : Set on the cotangent bundle $T^*M \setminus M \times \{0\}$ defined by the fast decrease of the Fourier transform on an open cone around (x, k)

10.2 Schwartz functions and tempered distributions

In $\mathbb{R}^n$, the space of Schwartz functions is the set of all fast decreasing functions, which are functions where the n-th derivative decays faster with distance than a polynomial.

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(M) | \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_\beta f| < \infty\} \tag{10.10}$$

where α and β are multi-indices, that is

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i} \tag{10.11}$$

$$\partial^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_n^{\beta_n} \tag{10.12}$$

For manifolds, we will require that, if there exists a boundary, the derivatives of the function mapped to the conformal compactification of the spacetime vanish on it.

$$\mathscr{S}(\mathcal{M}) = \{f \in C^\infty(M) | \forall \alpha \in \mathbb{N}^n, \nabla_\alpha f(\mathscr{I}) = 0\} \tag{10.13}$$

Tempered distributions : $\mathscr{S}'$, space of linear functionals $f : \mathscr{S} \to \mathbb{R}$

10.3 Tensor distributions

Distributions can be defined on tensors in a similar way, by first defining tensor test functions.

Definition 10.9. A *test field* $\mathfrak{t}$ on an orientable manifold $\mathcal{M}$ is a smooth tensor density of weight -1 with compact support.

The set of test fields from tensors of rank (r, s) will be noted $\mathscr{T}_s^r(M)$.

Definition 10.10. A *distributional tensor* is a linear functional on $\mathscr{T}_s^r(M)$, noted $\mathscr{T}_s'^r(M)$.

As with distributions on scalar fields, there exists a mapping from tensor fields to distributions : given a tensor T of rank (r, s), we can define a distribution from it with the action

$$T[\mathfrak{t}] = \int T_{\mu\nu...}{}^{\alpha\beta...}\mathfrak{t}^{\mu\nu...}{}_{\alpha\beta...} \tag{10.14}$$

where $\mathfrak{t}$ is a test field of rank (s, r). Hence the map is from T_s^r to $\mathscr{T}_r^s$. This integral is always well defined since $_mathfrakt$ is of compact support.

Definition 10.11. The contraction of a tensor distribution is defined [in coordinates] as

$$T^{abc...p...}{}_{a'b'c'...p...}[\mathfrak{t}^{\mathfrak{a}'\mathfrak{b}'\mathfrak{c}'...}{}_{abc...}] = T^{abc...p...}{}_{a'b'c'...p'...}[\mathfrak{t}^{\mathfrak{a}'\mathfrak{b}'\mathfrak{c}'...}{}_{abc...}\delta^{p'}{}_p] \tag{10.15}$$

Similarly to scalar field distributions, we can define derivatives on tensor distributions.

Definition 10.12. For any derivative operator D_μ on tensor fields, we define the derivative of a tensor distribution by

$$D_\mu T^{abc...p...}{}_{a'b'c'...p...}[\mathfrak{t}^{\mathfrak{a}'\mathfrak{b}'\mathfrak{c}'...}{}_{abc...}] = -T^{abc...p...}{}_{a'b'c'...p...}[D_\mu\mathfrak{t}^{\mathfrak{a}'\mathfrak{b}'\mathfrak{c}'...}{}_{abc...}] \tag{10.16}$$

10.3.1 The Geroch-Traschen class of metrics

The Geroch-Traschen class of metrics (or gt-regular metrics) are metric tensor fields (not purely distributions) with well-defined curvatures as distributions.

- The inverse of g exists everywhere and the metric and its inverse are locally bounded.
- The first weak derivative of the metric exists and is locally square-integrable.

Theorem 10.13. For a d-dimensional submanifold S of the n-dimensional spacetime $\mathcal{M}$, let $\alpha \in \mathscr{T}_s'^r$ be a non-zero distribution with support in S and is a sum of a distribution from a locally integrable tensor field and the derivative of a distribution from a locally square-integrable field, that is, for a test field $\tau \in \mathscr{T}_s^r$,

$$\alpha[\mathfrak{t}] = \int_{\mathcal{M}} \mu\mathfrak{t} + \beta\nabla\mathfrak{t} \tag{10.17}$$

where $\mu \in T_r^s$ is locally integrable and $\beta \in T_r^{s+1}$ is locally square-integrable. Then $d = n - 1$.

Proof. We define a Riemannian metric g^+ on M and pick some $\varepsilon > 0$ for a neighbourhood U_ε composed of open balls of radius ε on S. For some h_ε a smooth, non-negative function on $\mathcal{M}$ that vanishes on some neighbourhood of S, such that $h_\varepsilon(\mathcal{M} \setminus U_\varepsilon) = 0$ and such that its gradient has the g^+ norm inferior or equal to $2/\varepsilon$ in the support of $\mathfrak{t}$. Then we have

$$\left| \int_M (\mu\mathfrak{t} + v\nabla\mathfrak{t}) \right| \quad =$$

□

10.4 Colombeau algebras

Some distribution-valued stress-energy tensors still do not make sense for the Einstein equations within the Geroch-Traschen class of metrics.
For some applications, such as the study of shockwaves or the thin-shell formalism, it may be useful to consider tensor-valued distributions rather than tensor fields. As most quantities will require second derivatives of the metric tensor at most, this will happen if the metric tensor is weaker than C^2. Unlike for other theories like electromagnetism, though, the usual theory of distributions cannot be applied directly since general relativity is not a linear theory. Since distributions do not form an algebra, they cannot be used directly for general relativity. This is encapsulated in Schwartz's impossibility theorem : Defining generalized functions by sequences of smooth functions $(u_\varepsilon)_\varepsilon$ with a regularization ε.
For $(u_\varepsilon)_\varepsilon \in C^\infty(\mathcal{M})$, space of sequences of moderate growth :

$$\mathcal{E}_M(\mathcal{M}) = \{(u_\varepsilon)_\varepsilon | \forall \in \mathbb{N}, \forall N > 0, \forall \xi_i \in T\mathcal{M}, \sup_{p \in K} |L_{\xi_1} ... L_{\xi_k} u_\varepsilon(p)| = \mathcal{O}(\varepsilon^{(-N)})\}$$

with U a subset of $\mathcal{M}$ with compact closure. The set of all sequences of functions with derivatives that are at most divergent as ε^N.

Set of negligible functions :

$$\mathcal{N}\{(u_\varepsilon)_\varepsilon | \forall \in \mathbb{N}, \forall N > 0, \forall \xi_i \in T\mathcal{M}, \sup_{p\in K} |L_{\xi_1}...L_{\xi_k} u_\varepsilon(p)| = \mathcal{O}(\varepsilon^N)\} \tag{10.18}$$

Mollifier of distributions

Definition 10.14. A mollifier is a function such that bla bla bla

Product of two mollified distribution
Generalized functions

10.5 Generalized sections of tensor bundle

11 The Weyl transformation and conformal spacetimes

11.1 The Weyl transformation

The Weyl transformation is the local rescaling of the metric tensor by a nowhere vanishing, positive continuous function Ω.

$$g(-,-) \to \Omega(x) g(-,-) \tag{11.1}$$

That is, we have two diffeomorphic spacetimes $(\mathcal{M}, g)$ and $(\mathcal{M}, g')$, where $g' = \Omega g$, or in components, $g'_{\mu\nu} = \Omega(x) g_{\mu\nu}$. Unless $\Omega(x) = 1$, this map is not an isometry.
(It would be possible to also include a change of coordinates between the two, by considering (M, g) and (N, g') with a diffeomorphism $f : M \to N$, but this will help to avoid confusion with the conformal invariance. As it is, we're only considering the diffeomorphism $f = \mathrm{Id}_M$, so that $f^* g = g$)

$$g_{\mu\nu} \to \Omega(x) g_{\mu\nu} \tag{11.2}$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \to g^{\mu\nu}(\Omega(x) g_{\nu\rho}) = \Omega(x) \delta^\mu_\rho \tag{11.3}$$

meaning that the Weyl transform of the inverse metric is $\Omega^{-1} g^{\mu\nu}$.

$$\det(g) \to \Omega^n \det(g) \tag{11.4}$$

Derivatives of the metric :

$$\partial_\rho g_{\mu\nu} \to \Omega(\partial_\rho g_{\mu\nu}) + (\partial_\rho \Omega) g_{\mu\nu} \tag{11.5}$$

Christoffel symbols :

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} \tag{11.6}$$

11.2 The conformal transformation

A conformal transformation is a diffeomorphism f from $(\mathcal{M}, g)$ to $(\mathcal{M}', g')$ such that the metric is of the form

$$f^* g' = \Omega g \tag{11.7}$$

Or, in coordinate form,

$$J^{\mu'}_\mu J^{\nu'}_\nu g_{\mu'\nu'}(x') = \Omega(x) g_{\mu\nu}(x) \tag{11.8}$$

The components of the metric in the old coordinate system are related to the components in the new coordinate system by a factor $\Omega(x) > 0$. The difference between the Weyl transformation and the conformal transformation should be noted (the terms used can be ambiguous in the literature) : a Weyl transformation is not an isometry, it changes the components of the metric even in the same coordinate system. On the other hand, the conformal transformation is simply a change of coordinates, and because of this, it is

an isometry. The change in the metric tensor will be compensated by the change in the definition of the coordinates on the vectors.
For instance, given the dilation $x^\mu \mapsto ax^\mu$, the Jacobian will be $J^{\mu'}_{\mu} = a\delta^{\mu'}_{\mu}$, $J^{\mu}_{\mu'} = a^{-1}\delta^{\mu}_{\mu'}$, transforming the metric tensor as $g \to a^{-2}g$ and vectors as $X \to aX$. By linearity it can be easily checked that this is invariant. On the other hand, a Weyl scaling $g \to a^{-2}g$ will only affect the metric and $g(X,Y) \to a^{-2}g(X,Y)$.
By the fact that it is generated by strictly positive functions, it is not too difficult to show that the conformal transformation has a group structure.
The Poincaré group is a subgroup : for $\Omega(x) = 1$,

$$J^{\mu'}_{\mu} J^{\nu'}_{\nu} g_{\mu'\nu'}(x') = g_{\mu\nu}(x) \tag{11.9}$$

which corresponds to the Poincaré group.
For an infinitesmal transformation :

$$x'^\mu = x^\mu + \epsilon^\mu(x) \tag{11.10}$$

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{11.11}$$

Since $g'(x') = \Omega(x)g$, we should have that

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = (1 - \Omega(x))g_{\mu\nu} \tag{11.12}$$

In other words, there is some function f such that $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = f(x)g_{\mu\nu}$. By taking the trace :

$$f(x) = \frac{n}{2}\partial_\rho \epsilon_\rho \tag{11.13}$$

Set of all conformal transformations :

- Translations : $x'^\mu = x^\mu + a^\mu$
- Rigid rotations : $x'^\mu = \Lambda^\mu_\nu x^\nu$
- Dilation : $x'^\mu = \alpha x^\mu$
- Special conformal transformation :

$$x'^\mu = \frac{x^\mu - b^\mu x^\nu x_\nu}{1 - 2b^\nu x_\nu + b^\nu b_\nu x^\rho x_\rho}$$

Translations and rotations : $\Lambda = 1$
Dilations : $\Lambda = \alpha^2$
SCT : $\Lambda = (1 - 2b_\mu x^\mu + b^2x^2)^2$
Generators :

- Translation : $P_\mu = -i\partial_\mu$
- Dilation : $D = -ix^\mu \partial_\mu$
- Rotation : $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
- SCT : $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

Commutators of generators

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\mu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})
\end{aligned}$$

$$J_{\mu\nu} = L_{\mu\nu} \tag{11.14}$$

$$J_{-1,0} = D \tag{11.15}$$

$$J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{11.16}$$

$$J_{0,\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{11.17}$$

$J_{ab},\ a, b \in \{-1, 0, 1, ..., n\}$
$\eta_{ab} = \mathrm{diag}(-1, 1, 1, ..., 1)$

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{11.18}$$

Algebra of SO$(n+1, 1)$
Poincaré + dilations : subgroup of the conformal group

12 Spacetime orientability, spin structures and spinors

12.1 Orientability

As we have seen in chapter 2, orientability can be defined directly on manifolds by the positivity of all its Jacobian determinants. With the structures we defined, it is also possible to define it equivalently as

Proposition 12.1. A spacetime is an orientable manifold if and only if there exists a global section of the $\Lambda^n\mathcal{M}$ bundle, a volume form.

Proof. Consider an orienting atlas $\{U_\alpha, \phi_\alpha\}$ on $\mathcal{M}$, and a partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$. On each coordinate patch U_α, we can define a nowhere-vanishing n-form by

$$\omega_\alpha = \bigwedge_{\mu=0}^{n} dx^\mu_\alpha \tag{12.1}$$

On every overlap $U_\alpha \cap U_\beta \neq \varnothing$, consider the new coordinates $x^\mu_\beta = \phi_\beta \circ \phi_\alpha^{-1}$. The form transforms as
If we now define the form

$$\omega = \sum_\alpha \psi_\alpha \omega_\alpha \tag{12.2}$$

□

Proposition 12.2. If (M, g) is orientable, there exists a unique n-form, the volume form ε, defined by those properties :

- For every positively oriented orthonormal basis $\{e_\mu\}$ at every point p,

$$\varepsilon(e_1, e_2, ..., e_n) = 1$$

- For every orientation-preserving local chart (U, ϕ),

$$[\phi^{-1*}\varepsilon](p) = \sqrt{\det(g(p)} dx_1 \wedge ... \wedge dx_n$$

12.2 Integrals on manifolds

With the metric tensor, it is possible to define integrals on the manifold, at least locally (the discussion of global integrals will wait until the chapter on orientations).

Proposition 12.3. Every coordinate chart U allows the definition of a nowhere-vanishing n-form ω_U.

Proof. On a single chart we can simply consider the set of basis vectors of the cotangent bundle dx^μ, and any strictly positive function $f(p)$ defined on U. Then the n-form

$$\omega_U = \bigwedge_\mu dx^\mu \tag{12.3}$$

is nowhere zero. □

We will define the integral on the domain U of that chart as a map from n-forms to $\mathbb{R}$.

$$\int_U : \Gamma(\Lambda^n M)_c(\Omega) \to \mathbb{R} \tag{12.4}$$

with $\Gamma(\Lambda^n M)_c(\Omega)$ the set of sections of the exterior bundle of compact support on the subset U. The integration is done by projecting this n-form on $\mathbb{R}^n$ using the coordinate chart and using the usual Lebesgue integral.
For a chart (U, ϕ_U), the n-form $(\phi_U^{-1})^*\omega$ is an n-form on $O \subset \mathbb{R}^n$
If $\mathrm{supp}(\omega) \subset U$, we can define the integral

$$\int_O (\phi^{-1})^*\omega \tag{12.5}$$

Proof. □

Stoke's theorem :

Theorem 12.4. For an orientable manifold, given a k-form ω,

$$\int_{\partial D} \omega = \int_D d\omega \tag{12.6}$$

12.3 Space orientability

Theorem 12.5. If there exists a global section of $\Lambda^n \mathcal{M}$, noted ε, the manifold is orientable.

Theorem 12.6. If the manifold isn't orientable, there exists a closed curve such that the parallel propagation of a volume form isn't consistent.

12.4 Time orientability

Time orientability is the notion that, globally speaking, there is the possibility of defining a future and a past direction for all causal vectors. This is always possible locally (cf. theorems below), but as some examples will show, this may fail to hold up globally.

12.4.1 Local time orientation

As all spacetimes are locally equivalent to Minkowski space, it is always possible to classify causal vectors into two distinct sets, corresponding to the two halves of the light cone. First, defining the set of vectors orthogonal to a vector,

Definition 12.7. Two causal vectors a^μ, b^μ have the same time orientation if $g(a, b) < 0$

A choice of a timelike vector v at p will then define a time orientation at p. A causal vector is said to be future directed if it has the same orientation as v, and past directed otherwise.
If we are able to define a smooth, non-vanishing timelike vector field, then this time orientation is promoted to a global one. Much like a choice of orientation on a manifold, this is not always possible. A spacetime where it is possible to define such an orientation

is called time-orientable. A spacetime associated with an orientation $(\mathcal{M}, \tau)$ with τ the vector field is called time oriented.
If not time oriented, there is a smooth, non-vanishing timelike line element (ie, an ordered pair of a causal vector and its opposite $(V^a, -V^a)$)

Theorem 12.8. A spacetime is time-orientable if and only if every closed curved through a point p is time-preserving.

Proof. If we consider a time-orientation τ and a curve γ, take τ_p to be the vector field generated by the parallel propagation of τ at a point p on the curve. By definition, $g(\tau_p, \tau)$ will be negative at p. Since τ_p will remain timelike on the entire curve, $g(\tau_p, \tau)$ will always be $\neq 0$. By continuity, τ_p will always have the same time orientation as τ.
Converse : Choose a time orientation at p, define a time orientation at q by carrying the time orientation from p to q. Since the curve $pq - qp$ is time preserving, there is a time orientation defined at every point of the manifold. □

Theorem 12.9. A spacetime is time-orientable if and only if every homotopically equivalent closed curved through a point p is time-preserving.

Proof. □

Since the trivial loop is trivially time-preserving, we also have that

Corrolary 12.1. Every simply connected spacetime is time orientable.

Theorem 12.10. Any manifold that admits a Lorentz metric also admits a time-orientable Lorentz metric.

Proof. As we have seen earlier, the conditions for the existence of a section of the metric bundle are the same as the conditions for the existence of a nowhere vanishing vector field. Since we can pick that vector field to define a metric in which it is timelike, this vector field can define a time orientation. □

Once we have a global time orientation defined on a spacetime, we can define the notions of past and future.

Definition 12.11. A causal vector v is called *future-pointing* if $g(v, \tau) < 0$. It is called *past-pointing* if $g(v, \tau) > 0$.

which will translate into the same notions for causal curves.

Definition 12.12. A causal curve is called *future-oriented* if its tangent vector is future-pointing everywhere it is defined. It is called *past-oriented* if it is past-pointing.

While most spacetimes admit a time orientation, it is possible to find spacetimes that do not. The most common example being the following metric on the two dimensional cylinder :

$$\begin{aligned} ds^2 &= -2\sin(\pi x)\cos(\pi x)dt^2 + 2\sin(\pi x)\cos(\pi x)dx^2 \\ &+ (\sin^2(\pi x) - \cos^2(\pi x))dxdt \end{aligned} \tag{12.7}$$

with the identification $(x, t) = (x + 1, t)$.

Inverse metric :
The Christoffel symbols in this coordinate system are

$$\Gamma^t_{tt} = -\frac{1}{2} g^{tx} g_{tt,x} \tag{12.8}$$

If we consider the closed coordinate curve of x, with coordinates

$$t(\lambda) = 0$$
$$x(\lambda) = \lambda$$

its tangent vector U^μ will be

$$\dot{t}(\lambda) = 0$$
$$\dot{x}(\lambda) = 1$$

Picking the timelike vector $X^\mu = (1, 1)$ at $(0, 0)$ and propagating it around this curve, we find that

$$U^\mu \nabla_\mu X^\nu = \partial_x X^\nu + \Gamma^\nu{}_{x\rho} X^\rho = 0 \tag{12.9}$$

System of equations :

$$\partial_x X^t + \Gamma^t{}_{xx} X^x + \Gamma^t{}_{xt} X^t = 0$$
$$\partial_x X^x + \Gamma^x{}_{xx} X^x + \Gamma^x{}_{xt} X^t = 0$$

Theorem 12.13. The double cover of a non-time orientable spacetime is time-orientable.

Proof. □

For most non-time orientable spacetimes, we will often do the calculations on the double cover before pulling back on the original spacetime.

12.4.1.1 Examples of non-time orientable spacetimes

A wide class of non-time orientable spacetimes can be generated by the following type of manifolds

$$\mathcal{M} = (\mathbb{R} \times \Sigma)/(T \times I) \tag{12.10}$$

Where $\mathbb{R}$ is a timelike coordinate of the original spacetime $\mathbb{R} \times \Sigma$, T is the time reversal operator, where for $(t, x) \in \mathbb{R} \times \Sigma$, $T(t, x) = (-t, x)$, and I is an involution of Σ with no fixed points (that is, $I(I(p)) = p$ and there is no point p such that $I(p) = p$), with a metric where $\pm\partial_t$ is timelike. A common method for this is to pick antipodal points on some n-sphere, such as $I(\theta) = (\theta + \pi)$ or $I(\theta, \varphi) = (\theta + \pi, -\varphi)$.

Proposition 12.14. The spacetime $\mathcal{M} = (\mathbb{R} \times \Sigma)/(T \times I)$ is not time-orientable.

Proof. Consider a closed curve on $\mathbb{R} \times \Sigma$ that passes through two points, p and p' such that $p \sim p'$ under the transformation $T \times I$. For any timelike field, the transport around that curve will preserve its orientation. For an initially future-oriented vector, we have $g(X, \partial_t) < 0$.
After identification, the curve from p to p' will be itself a closed curve, as $p' = I(T(p))$. [Something something the vector transported gets flipped] □

The simplest example of such a spacetime is the two-dimensional flat spacelike cylinder $\mathbb{R} \times S$, with the identification $(t, \theta) \to (-t, \theta + \pi)$.
There is also quite a wide variety of non-time orientable quotients of de Sitter space, called elliptic de Sitter spaces. $\mathbf{dS}/\mathbb{Z}_2$
Lorentzian universe from nothing
De Sitter space with antipodal points identified
basic de Sitter space topology : $\mathbb{R}^1 \times S^{(n-1)}$

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(\alpha^{-1}t)[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2(\theta)d\varphi^2)] \tag{12.11}$$

2D :

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(\alpha^{-1}t)d\chi^2 \tag{12.12}$$

Identification of antipodal points ($t \to -t$, $\chi \to \chi + \pi$)
Geodesic : Simplest geodesic is $U^\mu = (1, 0)$, $x^\mu = (\lambda, 0)$
Timelike geodesic coming from timelike past infinity, crosses $t = 0$, goes back the same way with $\chi = \pi$

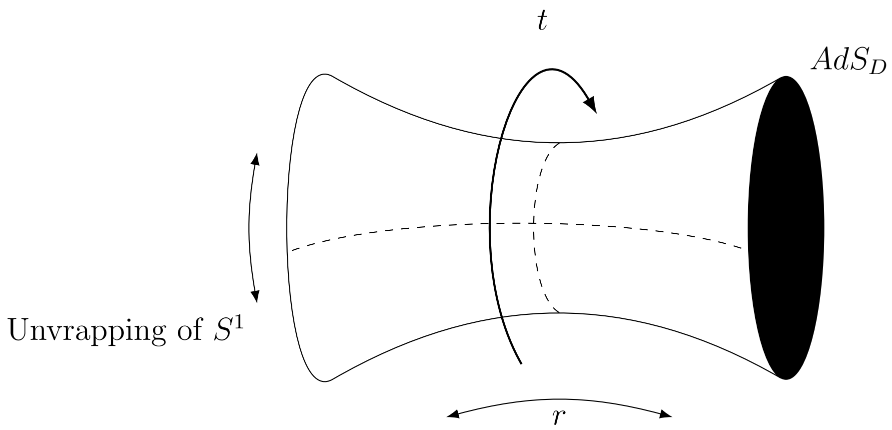


Figure 7: The topological structure of anti de- Sitter.

Calabi Markus example : De Sitter generated by $-t^2 + x^2 + y^2 = 1$, identification like $(t, x, y) \to (-t, -y, x)$
"For a large class of these spacetimes, one can always choose metrics without CTCs; time nonorientability is then their only causal pathology."

12.5 Spacetime orientability

A spacetime is called spacetime orientable if there exists an n-form ε which admits a global section.

Proposition 12.15. If a spacetime is orientable in the topological sense, it is spacetime orientable.

Proof. x □

Theorem 12.16. A spacetime is spacetime orientable if and only if it is time orientable and space orientable.

Proof. x □

12.6 Clifford algebras

We previously constructed both the tensor algebra and the exterior algebra for the tangent space, and used those to construct the associated tensor bundle and exterior bundle. In addition to those algebras, we can construct a more general version of the exterior algebra called the Clifford algebra, which will be used to build the space for spinors.
If we consider some vector space V equipped with a quadratic form $q : V \to \mathbb{R}$

$$q(av) = a^2 q(v) \tag{12.13}$$

The map $(v, w) \mapsto q(v + w) - q(v) - q(w)$ is linear in v and w
We can verify that the bundle metric on the tangent plane T_pM forms a quadratic form

$$q(v) = \langle v, v\rangle \tag{12.14}$$

Exterior algebra on the vector space
antisymmetrized product of two vectors :

$$v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v) \tag{12.15}$$

inner product defines musical isomorphisms between V and V^*, which extends to $\bigwedge V$ and $\bigwedge V^*$, a hodge duality between the exterior algebras and the Clifford algebra

12.6.1 Multivectors

From this, we can define multivectors, which are made from sums of members of the exterior algebras $\bigwedge^p$.

Definition 12.17. The set of *multivectors* is the direct sum of the exterior algebras of the vector space

$$\bigwedge V = \bigoplus_{p=0}^{n} \bigwedge^{p} V \tag{12.16}$$

We will note a multivector A as

$$A = \sum_{0}^{n} A_i = A_0 + A_1 + ... + A_n \tag{12.17}$$

with $A_i \in \bigwedge^i V$. A Polyvector of definite order is called homogeneous.

Definition 12.18. The projector $\langle \cdot \rangle_r$ projects a multivector on its homogeneous part of grade r.

$$\langle A \rangle_r = A_r \tag{12.18}$$

Proposition 12.19. The projector has the following properties :

1. $\langle A + B \rangle_r = \langle A \rangle_r + \langle B \rangle_r$
2. $\langle \lambda A \rangle_r = \lambda \langle A \rangle_r$
3. $\langle \langle A \rangle_r \rangle_s = \langle A \rangle_r \delta_{rs}$

Proof. 1.

$$A + B = \sum_{i=0}^{n} (A_i + B_i) \to \langle A + B \rangle_r = A_r + B_r = \langle A \rangle_r + \langle B \rangle_r$$

2.

$$\langle \lambda A \rangle_r = \langle \sum_{i=0}^{n} \lambda A_i \rangle_r = \lambda A_r = \lambda \langle A \rangle_r$$

3. x

□

A multivector that only has non-zero components of grade 0 and 1 is called a *paravector*. $A = A^0 + A^i e_i$, $A^0 = \langle A \rangle_0$, $A^i e_i = \langle A \rangle_1$

If we note the inner product (v, w) as $v \cdot w$, we define the Clifford product of two vectors as

$$uv = u \cdot v + u \wedge v \tag{12.19}$$

This is a multivector $uv \in \bigwedge V$, with the sum of a scalar and bivector.

This has the property

$$\begin{aligned} u \cdot v &= \frac{1}{2}(uv + vu) \\ u \wedge v &= \frac{1}{2}(uv - vu) \end{aligned}$$

The wedge product is generalized easily enough, simply by using

$$A \wedge B = \sum_{i,j}^{n} \langle A \rangle_i \wedge \langle B \rangle_j \tag{12.20}$$

If we pick an orthonormal basis $\{e_i\}$ for V, then, as $e_i \cdot e_j = 0$, it is not too hard to show that

$$e_i e_j = e_i \wedge e_j \tag{12.21}$$

which will help to extend the Clifford product to all multivectors.

Proposition 12.20. The Clifford product is distributive and associative.

Proof.

$$\begin{aligned}(a+b)c &= (a+b).c+(a+b)\wedge c \\ &= a.c+b.c+a\wedge c+b\wedge c \\ &= ac+bc\end{aligned} \tag{12.22}$$

$$(ab)c = \tag{12.23}$$

□

12.6.2 The Clifford algebra

Clifford algebra $C\ell(V)$ is the set of multivectors $\bigwedge V$ equipped with the Clifford product.

Definition 12.21. The *Clifford algebra* $C\ell(V,q)$ of a vector space V with quadratic form q is an associative unital algebra such that, given the ideal $\mathscr{I}_q$

Definition 12.22. The Clifford algebra has the following properties, for $A,B,C\in C\ell(V)$:

1. $AB\in C\ell(V)$
2. $1A=A1=A$
3. $A(BC)=(AB)C$
4. $A(B+C)=AB+AC$
5. $(B+C)A=BA+CA$
6. For $a\in V$

$C\ell^k(V)$ is the vector space of polyvectors of grade k

$$C\ell(V)=\bigoplus_{k=0}^{n} C\ell^k(V) \tag{12.24}$$

$C\ell^0=\mathbb{R}$, $C\ell^1(V)=V$

$$\mathbb{R}\oplus V=C\ell^0(V)\oplus C\ell^1(V)\subset C\ell(V) \tag{12.25}$$

Even and odd Clifford algebras :

$$C\ell^{\text{even}}(V)=\bigoplus_{k\in 2\mathbb{N}} C\ell^k(V) \tag{12.26}$$

$$C\ell^{\text{odd}}(V)=\bigoplus_{k\in 2\mathbb{N}+1} C\ell^k(V) \tag{12.27}$$

Both with dimension 2^{n-1}

$C\ell^{\text{even}}(V)$ is a subalgebra of $C\ell(V)$
Pseudoscalars : n-multivector. Orientation operator :

$$\varepsilon = e_1 \wedge ... \wedge e_n \tag{12.28}$$

$$\varepsilon^2 = (-1)^{\frac{n(n-1)}{2}+s} \tag{12.29}$$

Multiples of ε are pseudoscalars.
If n is odd, ε commutes with all multivectors. Otherwise, it commutes with even grade multivectors and anticommutes with odd graded ones

$$\varepsilon P_r = (-1)^{r(n-1)} P_r \varepsilon \tag{12.30}$$

The center (set of elements that commute with all) of $C\ell(V)$ is $C\ell^0(V)$ if n even, $C\ell^0(V) \oplus C\ell^n(V)$ if odd
Talk about Grassman algebras, even/odd Clifford algebras, Pin group

12.6.3 Automorphisms

Reversion : Transformation from $C\ell$ to $C\ell$, such that for all polyvectors, the factors are reversed

$$(v_1 ... v_k)^T = v_k ... v_1 \tag{12.31}$$

Main involution : acts on basis vectors as $e_i^* = -e_i$.

$$a \mapsto (-1)^n \varepsilon a \varepsilon^{-1} \tag{12.32}$$

Factor of 1 for even graded elements, -1 for odd graded.
Clifford conjugation : $\bar{R} = (R^*)^T$
Scalar product on multivector : $\langle A, B \rangle = A \cdot B = \langle A^T B \rangle_0$

$$A \cdot B = \langle A \rangle_0 \cdot \langle B \rangle_0 + \langle A \rangle_1 \cdot \langle B \rangle_1 + ... + \langle A \rangle_n \cdot \langle B \rangle_n \tag{12.33}$$

Hodge duality :

$$\star : \Lambda^p \rightarrow \Lambda^{n-p} \tag{12.34}$$

$$A_p \mapsto \star A_p \tag{12.35}$$

Frame $\{e_i\}$ on V defines a frame on $\bigwedge V$ by the multivector $e_I = e_{i_1} \wedge ... \wedge e_{i_k}$, $I = \{i_1, ..., i_k\}$, $e_\varnothing = e_0 = 1$. Multivectors e_I gives a basis to $\bigwedge V$, and then to $C\ell(V)$, I goes from 1 to 2^n.

$$A = A^I e_I = A_0 + A^i e_i + A^{ij} e_{\{ij\}} + ... + A^{i_1 i_2 ... i_n} e_{\{i_1, i_2, ..., i_n\}} \tag{12.36}$$

For an orthonormal basis $e_i \cdot e_j = \eta_{ij} = \pm \delta_{ij}$, we can define $\eta_{IJ} = e_I \cdot e_J$, then

$$A \cdot B = \eta_{IJ} A^I B^J \tag{12.37}$$

Complexified Clifford algebra : $\mathbb{C}\ell(V) = \mathbb{C} \times C\ell(V)$

12.6.4 Clifford groups

The Clifford group $\Gamma(V)$ is the restriction of the Clifford algebra to elements that admit an inverse, that is, for $x \in \Gamma(V)$, there exists an $x^{-1} \in \Gamma(V)$ such that $x^{-1}x = xx^{-1} = 1$. The group action on an element of the Clifford algebra is

$$\begin{aligned} \Gamma(V) \times C\ell(V) &\rightarrow C\ell(V) \\ (x, a) &\mapsto -xax^{-1} \end{aligned} \tag{12.38}$$

We will also ask that this group action preserves the vector space, so that for $v \in V$,

$$-xvx^{-1} \in V \tag{12.39}$$

Those conditions define the Clifford group

$$\Gamma(V) = \{x \in C\ell(V) | \exists x^{-1},\ x^{-1}x = xx^{-1} = 1, \forall v \in V,\ -xvx^{-1} \in V\} \tag{12.40}$$

The existence of an inverse and the fact that $x1 = 1x = x$ proves that it has a group structure, since the Clifford product is associative.
Representation of $\Gamma(V)$:

$$\begin{aligned} \rho : \Gamma(V) &\rightarrow \mathrm{GL}(V) \\ x &\rightarrow \rho(x) \end{aligned}$$

such that $\rho(x)v = -xvx^{-1}$

Theorem 12.23. ρ is a surjective homomorphism from $\Gamma(V)$ to $\mathrm{O}(p, q)$

Theorem 12.24. ρ is a surjective homomorphism from $\Gamma^{\text{even}}(V)$ to $\mathrm{SO}(p, q)$

Even and odd subsets of the Clifford groups :

$$\begin{aligned} \Gamma^{\text{even}}(V) &= \Gamma(V) \cap C\ell^{\text{even}}(V) \\ \Gamma^{\text{odd}}(V) &= \Gamma(V) \cap C\ell^{\text{odd}}(V) \end{aligned}$$

Since the product of two even elements is even, $\Gamma^{\text{even}}(V)$ is a subgroup of $\Gamma(V)$. $\Gamma^{\text{odd}}(V)$ is not since the product of two odd elements is not odd.
if $x \in V$:

$$-xvx^{-1} = v - 2\frac{v \cdot x}{x \cdot x}x \tag{12.41}$$

Reflection of v with respect to the hyperplane orthogonal to x.
Rotation is the product of two reflections : a rotation will be

$$(xy)v(xy)^{-1} = xyvy^{-1}x^{-1} = RvR \tag{12.42}$$

(xy) is a product of two vectors : $A_0 + A_2$.
Rotation : $R \in \Gamma^{\text{even}}$ such that RvR^{-1}, $\bar{R}R = \pm 1$

The Pin group $\mathrm{Pin}(p,q)$ is the subgroup of $\Gamma(V)$

$$\mathrm{Pin}(p,q) = \{s \in C\ell(V) | s_i \in V,\ s = s_1 s_2 ... s_k, \bar{s}s = \pm 1\} \tag{12.43}$$

The Spin group $\mathrm{Spin}(p,q)$ is the subgroup of $\Gamma(V)$

$$\mathrm{Spin}(p,q) = \{s \in C\ell(V) | s_i \in V,\ s = s_1 s_2 ... s_{2k}, \bar{s}s = \pm 1\} \tag{12.44}$$

$\mathrm{Spin}(p,q)$ is a subgroup of $\mathrm{Pin}(p,q)$ and $C\ell^{\mathrm{even}}(V)$

$$\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap C\ell^{\mathrm{even}}(V) \tag{12.45}$$

$$\begin{aligned} \mathrm{O}(p,q) &= \mathrm{Pin}(p,q)/\mathbb{Z}_2 \\ \mathrm{SO}(p,q) &= \mathrm{Spin}(p,q)/\mathbb{Z}_2 \\ \mathrm{SO}^{\uparrow}(p,q) &= \mathrm{Spin}^{\uparrow}(p,q)/\mathbb{Z}_2 \end{aligned}$$

$\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$, not true for $\mathrm{Pin}(p,q)$ and $\mathrm{Pin}(q,p)$

12.7 Specific algebras

12.7.1 Algebra of real space

A few simple examples are Clifford algebras of signature $(p,0)$ for low dimensions.
For $(1,0) :\ C\ell(\mathbb{R}) = \mathbb{C} :\ z = a + be_1$,

$$(b_1 e_1)(b_2 e_1) = \tag{12.46}$$

$$z_1 z_2 = a_1 a_2 + \tag{12.47}$$

12.7.2 The spacetime algebra and the Dirac algebra

The case that will interest us will be for the case where the vector space will be the tangent space of the manifold, in which case the vector space will be $\mathbb{R}^n$ and the inner product will be either the bundle metric $\langle \cdot, \cdot \rangle$ or the metric tensor g (we will see later that this will lead us to the same thing).
We will denote this as $C\ell(\mathbb{R}^{1,n})$, called the spacetime algebra. Its complexification will be $\mathbb{C}\ell(\mathbb{R}^{1,n})$, called the Dirac algebra.
$\mathbb{C}\ell(\mathbb{R}^{1,n})$ isometric to $C\ell(\mathbb{R}^{2,n})$?
Dirac algebra : Clifford group $\mathbb{C}\ell(\mathbb{R}^{1,3}) \approx \mathrm{SL}(2,\mathbb{C})$

12.7.3 Constructing the Clifford bundle

Clifford bundle of multivector fields :

$$C\ell(T\mathcal{M}) = \sqcup_p C\ell(T_p\mathcal{M}) \tag{12.48}$$

Clifford bundle of differential multiforms :

$$C\ell(T^*\mathcal{M}) = \sqcup_p C\ell(T_p^*\mathcal{M}) \tag{12.49}$$

Both have typical fiber the spacetime algebra $C\ell(\mathbb{R}^{1,n-1})$
Isomorphism between $C\ell(T\mathcal{M})$ and $C\ell(T^*\mathcal{M})$ by extension of the musical isomorphisms.
Action of the connection on the Clifford bundle

12.7.4 Associated bundles and pinor bundles

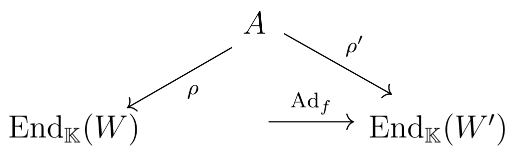
Pinor : vector spaces acted upon by irreducible representation of a Clifford algebra
Real representation of the algebra $C\ell(V)$:

$$\rho : C\ell(V) \to \mathrm{End}_{\mathbb{R}}(W) \tag{12.50}$$

complex rep :

$$\rho : C\ell(V) \to \mathrm{End}_{\mathbb{C}}(W) \tag{12.51}$$

Endomorphisms on some $\mathbb{K}$-vector space W.
Representations are equivalent if



$$\mathrm{Ad}f : \mathrm{End}_{\mathbb{K}}(W) \to \mathrm{End}_{\mathbb{K}}(W') \tag{12.52}$$

Definition 12.25. A pinor representation of $\mathrm{Pin}(V)$ is the restriction of an irreducible representation of $C\ell(V)$.

Spinor bundle $\pi_S : S \to \mathcal{M}$
Notation : components of a spinor

$$\xi = \xi^A e_A \tag{12.53}$$

dual spinors, in the cospinor bundle $\pi_{S^*} : S^* \to \mathcal{M}$

$$\psi(\xi) = \psi_A \xi^a \tag{12.54}$$

Complex conjugate : $\xi \to \bar{\xi} = \bar{\xi}^A e_A$

12.7.5 Solder form and spinor metric

Solder form

$$\gamma : \tag{12.55}$$

One benefit of using frame fields rather than the metric tensor is that it permits to define the usual tensor operations on spinor fields naturally.

$$\gamma^\mu = \gamma^a e^\mu{}_a \tag{12.56}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{ab}(e^\mu)_a(e^\nu)_b I = 2g^{\mu\nu} I \tag{12.57}$$

12.8 Orientability and spin structure

Definition 12.26. A manifold is said to admit a spin structure if it possesses a principal Spin bundle $\pi : P \to \mathcal{M}$ with the universal cover

$$F : P \to SO\mathcal{M} \tag{12.58}$$

Theorem 12.27. If the oriented orthonormal frame bundle exists and is trivial, there exists a spin structure on the manifold.

Pin group $\mathrm{Pin}(p, q)$: double cover of the orthogonal group $\mathrm{O}(p, q)$
if manifold is space orientable but not time orientable : sinors
if manifold is spacetime orientable : spinors
Spinors : group $\mathrm{SL}(2, \mathbb{C})$
Spin structure
Rokhlin theorem

Definition 12.28. A manifold is called a spin manifold if it admits a spin structure.

12.9 Associated vector bundles

Spinor bundle S : Complex vector bundle assocated to the principal bundle of spin frames over $\mathcal{M}$
If the manifold admits a spin structure, we can define

$$\nabla_a \psi^A = \partial_a \psi^A - (\Gamma_a)^A_B \psi^B \tag{12.59}$$

$$(\Gamma_a)^A_B = -\frac{1}{4}\omega_{abc}(\gamma^b)^A_C(\gamma^c)^C_B \tag{12.60}$$

12.10 Discrete symmetries

CPT symmetry

13 Killing vectors and symmetries

Killing vectors are a generalization of the early notions of the symmetries of the metric tensor, as the simple notion of the symmetry with respect to a coordinate x if the components fulfil $\partial_x g_{\mu\nu} = 0$ fails to be coordinate independent.
A Killing vector corresponds to a vector field which, if the metric tensor is pushed along its flow, will leave it invariant.
For a Killing vector field K,

$$\mathcal{L}_K g = 0 \tag{13.1}$$

For all X, Y

$$g(\nabla_X K, Y) + g(X, \nabla_Y K) = 0 \tag{13.2}$$

Killing equation :

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{13.3}$$

Proposition 13.1. If we have a nowhere vanishing Killing vector field K on some neighbourhood, then there exists a coordinate patch with a coordinate x for which $K = \partial_x$ and $\partial_x g_{\mu\nu} = 0$

Proof. As with all nowhere-vanishing vector fields, such a coordinate exist. In which case we have

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu \quad = \tag{13.4}$$

□

Theorem 13.2. The largest number of Killing vector a 4-dimensional spacetime can have without being maximally symmetric is 7.

Theorem 13.3. For $n > 5$, the largest number of Killing vector a spacetime can have without being maximally symmetric is $\frac{1}{2}n(n-1) + 2$.

13.1 Stationary spacetimes and spacetimes of a single spacelike Killing vector

If there is a single spacelike or timelike Killing vector [7], we may
Consider a Killing vector field K that is everywhere timelike or everywhere spacelike. Take S the collection of all trajectories of K, $\gamma \in S$ means γ is an inextendible curve with tangent K. Define a mapping $\psi : \mathcal{M} \to S$, for every point $p \in M$, $\psi(p)$ is the trajectory of K passing through p. [PROVE SOMEWHERE THAT THE TRAJECTORIES ARE UNIQUE FOR SUCH A FIELD]
We assume that S has the structure of a smooth 3-manifold such that ψ is a smooth mapping, to avoid the case where a trajectory of K passes arbitrarily near itself.
S is not necessarily the hypersurface, it is just a quotient space of M with covering map ψ

13.1.1 Stationary spacetimes

If K is everywhere timelike, the spacetime is called stationary.

$$h_{ab} = g_{ab} - \frac{K_a K_b}{K_a K^a} \tag{13.5}$$

$$ds^2 = K_a K^a (dt + A_\mu dx^\mu)^2 + h_{\mu\nu} dx^\mu dx^\nu \tag{13.6}$$

If, in addition of being timelike, the Killing vector field is orthogonal to the spacelike hypersurfaces, the spacetime is said to be static.

$$K_{(\mu,\nu)} = 0,\ K_{[\mu} K_{\nu;\sigma} = 0 \tag{13.7}$$

13.2 Axisymmetric spacetimes

If there is a single isometry group $G = S$

13.3 Cylindrically symmetric spacetimes

2 Killing vectors : one with isometries acting on S^1, the other on $\mathbb{R}^1$, second one spacelike.

13.4 Spherically symmetric spacetimes

Spherical symmetry : Group of motion acting on a spacelike S^2

$$ds^2 = -e^{2\nu(r,t)} dt^2 + e^{2\lambda(r,t)} + Y^2(r,t)(d\theta^2 + \sin^2(\theta) d\varphi^2) \tag{13.8}$$

If a spherically symmetric spacetime is also stationary,

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} + Y^2(r)(d\theta^2 + \sin^2(\theta) d\varphi^2) \tag{13.9}$$

[SHOW IT TRUE FOR ANY TIMELIKE KILLING FIELD]

13.5 Homogeneous spacetimes

$n - 1$ spacelike Killing vectors

$$ds^2 = -dt^2 + a(t) g^R_{ab} dx^a dx^b \tag{13.10}$$

13.6 Isotropic spacetimes

Spacetime isotropic at a point p if there exists an isometry with the group structure of SO$(n-1)$ with spacelike Killing vectors.

13.7 Maximally symmetric spacetimes

A Riemannian space is maximally symmetric if and only if it admits a group G_r of motion with $r = \frac{1}{2}n(n+1)$

$$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{(1 + \frac{1}{4}K\eta_{\mu\nu}x^\mu x^\nu)} \tag{13.11}$$

In $(3+1)$ dimensions and spherical coordinates

$$ds^2 = -(1 - Kr^2)dt^2 + \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{13.12}$$

$$R_{\mu\nu\sigma\tau} = \frac{R}{n(n-1)}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma}) \tag{13.13}$$

13.8 Null Killing vector fields

13.9 Killing spinors

14 Petrov and Segre classification

14.1 Subspaces of the tangent plane

n-dimensional vector subspaces of $T_p\mathcal{M}$: S_n, T_n, N_n
The subspace S_n is a spacelike subspace : its generators are all spacelike
The subspace T_n is a timelike subspace : its orthogonal space $T_n^\perp$ is spacelike
S_1, T_1, N_1 : subspace generated by a tangent vector that is spacelike, resp. timelike, null.
Jordan normal form :

Theorem 14.1. For a finite-dimensional vector space V and any linear operator $A : V \to V$, there exists a decomposition of V into a direct sum of invariant subspaces of A

$$V = V_1 \oplus V_2 \oplus ... \oplus V_k \tag{14.1}$$

and a basis $\{e_j^{(i)}\}_{1\leq j\leq n_i}$ of each V_i such that

$$\begin{aligned}(A - \lambda_i \,\mathrm{Id})e_1^{(i)} &= 0\\ (A - \lambda_i \,\mathrm{Id})e_2^{(i)} &= e_1^{(i)}\\ &...\\ (A - \lambda_i \,\mathrm{Id})e_{n_i}^{(i)} &= e_{n_{i-1}}^{(i)}\end{aligned}$$

Decompose a matrix into block diagonal matrix

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix} \tag{14.2}$$

with J_i upper diagonal matrices

$$J = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 1\\ 0 & 0 & \lambda_1\end{matrix}} & & & \\ & \lambda_2 & & \\ & & \begin{matrix}\lambda_3 & 1\\ 0 & \lambda_3\end{matrix} & \\ & & & \ddots \end{pmatrix} \tag{14.3}$$

14.2 Petrov classification

Classification of spacetimes according to the eigenvalues of the Weyl tensor with eigenbivector $X^\mu\nu$

$$\frac{1}{2}C_{\mu\nu\rho\sigma}X^{\rho\sigma} = \lambda X^{\mu\nu} \tag{14.4}$$

given a unit timelike vector, take $X^\mu = X^{\mu\nu}u_\nu$
Petrov types :

- Type I
- Type D
- Type II
- Type N
- Type III
- Type O

14.3 Segre classification

Classification of rank 2 symmetric tensors
Rank $(1,1)$ tensors define a map $T : T_pM \to T_pM$, $T(X) = Y$
Eigenvalue problem on the symmetric tensor S

$$S^\mu{}_\nu X^\nu = \lambda v^\mu \tag{14.5}$$

$$X^\nu (S^\mu{}_\nu - \lambda \delta^\mu_\nu) = 0 \tag{14.6}$$

Segré notation : between brackets, orders of different Jordan blocks of a matrix. If multiple Jordan blocks for the same eigenvalue, written between parenthesis. Complex conjugate eigenvalues : Z and $\bar{Z}$
For a matrix with k real eigenvalues and l pairs of complex conjugates eigenvalues :

$$\{(p_{11}...p_{1r_1})...(p_{k1}...p_{kr_k})Z_1\bar{Z}_1...Z_l\bar{Z}_l\} \tag{14.7}$$

Example :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{14.8}$$

Type [1111]

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \tag{14.9}$$

Type [31] if $\lambda_1 \neq \lambda_2$, [(31)] otherwise.
Possible types : partitions of n + replacement of 2, 11 by $Z\bar{Z}$

$$\begin{aligned} 2 &\to [2], [11], [Z\bar{Z}] \\ 3 &\to [3], [21], [Z\bar{Z}1], [111] \\ 4 &\to [4], [31], [22], [2Z\bar{Z}], [Z\bar{Z}Z\bar{Z}], [211], [1111] \end{aligned}$$

Peblański notation

14.3.1 Segre classification of the Ricci tensor

$$R_{\mu\nu}X^{\mu\nu} = \lambda X^{\mu\nu} \tag{14.10}$$

4 basic Segre types :
A_1 :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{14.11}$$

A_2 :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & Z & 1 \\ 0 & 0 & 0 & \bar{Z} \end{pmatrix} \tag{14.12}$$

A_3 :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \tag{14.13}$$

B :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \tag{14.14}$$

Type	Segre notation	Plebański notation
A_1	$[111,1]$	$[S_1 - S_2 - S_3 - T]_{(1111)}$
	$[11(1,1)]$	$[S_1 - S_2 - 2T]_{(111)}$
	$[(11)1,1]$	$[2S_1 - S_2 - T]_{(111)}$
	$[(11)(1,1)]$	$[2S - 2T]_{(11)}$
	$[1(11,1)]$	$[S - 3T]_{(11)}$
	$[(111),1]$	$[3S - T]_{(11)}$
	$[(111,1)]$	$[4T]_{(1)}$
A_2	$[11,Z\bar{Z}]$	$[S_1 - S_2 - Z - \bar{Z}]_{(1111)}$
	$[(11),Z\bar{Z}]$	$[2S - Z - \bar{Z}]_{(111)}$
A_3	$[11,2]$	$[S_1 - S - 2N]_{(112)}$
	$[1(1,2)]$	$[S - 3N]_{(12)}$
	$[(11),2]$	$[2S - 2N]_{(12)}$
	$[(11,2)]$	$[4N]_{(2)}$
B	$[1,3]$	$[S - 3N]_{(13)}$
	$[(1,3)]$	$[4N]_{(3)}$

15 The causal relations

A Lorentzian metric naturally define a causal structure on the manifold, corresponding to the regions that a causal curve can access, or physically, the spacetime events that can influence each others. This will be expressed by the causal relations between each points of the manifold.
For most of this chapter, the spacetimes will be assumed to be time-orientable, as most of the important concepts require the existence of a well-defined past and future direction for causal curves. If the spacetime is not time-orientable, some of these results can be salvaged by going to the time-orientable double cover.

15.1 Definition

Once we have a manifold equipped with a metric and a time orientation, it becomes possible to define some ordering relations on the manifold.

15.1.1 Relations

First, a brief reminder on relations.

Definition 15.1. A *relation* R on a set X is a subset $R \subset X \times X$. If $(p, q) \in R$, we note it by pRq.

A relation is *reflexive* if for all p, pRp, *symmetric* if for all p, q, pRq implies qRp, and *transitive* if for all p, q, r, pRq and qRr implies pRr. It is also *antisymmetric* if pRq and qRp implies $p = q$.
A relation is a preorder if it is reflexive and transitive is a *preorder*. If it is in addition symmetric, it is an *equivalence relation*. If a preorder is antisymmetric, it is a *partial order*. If a relation is irreflexive (pRp is never true) and transitive, it is a *strict preorder*. A strict partial order that is asymmetric (pRq implies that qRp is never true) is a *strict partial order*.
If for two points $p, q \in X$ and a (strict) partial order R, we have pRq or qRp, we say that p and q are *comparable*. If this is true for all points of X, R is a (strict) total order.

Definition 15.2. A relation R on two sets A, B holds, noted ARB, if for every point $p \in A$, $q \in B$, pRq holds.

In particular, we can write all the previous relations as relations on sets, by the equivalence

$$pRq \leftrightarrow \{p\}R\{q\} \tag{15.1}$$

If a relation R_1 is a subset of a relation R_2, $R_1 \subset R_2$, we have that for all $p, q \in X$

$$pR_1q \implies pR_2q \tag{15.2}$$

15.1.2 Causal relations

The important relations on a spacetime are the chronological, strictly causal, causal and horismos relations.

- p chronologically precedes q if there is a future-directed timelike curve connecting p to q, noted $p \ll q$.
- p strictly causally precedes q if there is a future-directed causal curve connecting p to q, $p \neq q$, noted $p < q$.
- p causally precedes q if there is a future-directed causal curve connecting p to q, with possibly $p = q$, noted $p \leq q$.
- p horismos q if $p \leq q$ and $p \not\ll q$, noted $p \nearrow q$.

Those causal relations will form the basis of the causal structure. We can also define their negation in the following way

Definition 15.3. A point p is *causally independent*, noted $p \| q$, if there is no causal curve connecting p and q.

Definition 15.4. A point p is *chronologically independent*, noted $p|q$, if there is no timelike curve connecting p and q.

As with relations in general, those can also be defined on subsets of $\mathcal{M}$.
An important notion for causality will be the notion of a vicious point

Definition 15.5. A point p is *vicious* if $p \leq p$. A point that isn't vicious is said to be a *virtuous point.*

A point being vicious implies the existence of a *closed causal curve*, which is a continuous mapping of S^1 to the manifold that is everywhere causal. If it is additionally everywhere timelike, we say that it is a *closed timelike curve.*
As the degenerate curve of length 0 is not timelike, and neither is the constant curve $\forall \lambda,\ \gamma(\lambda) = p$, $p \ll p$ and $p < p$ implies the existence of a second point on this curve, which implies that there exists a point $q = \gamma(\lambda_q)$, $\lambda_q \neq \lambda_p$, such that $p \ll q$ and $q \ll p$. The same applies to strict causality. This will be of use for further proofs.

15.2 Properties of causal relations

To prove a lot of properties of causal relations, it will be practical to approximate them by a series of piecewise geodesic curves, in the following way :

Definition 15.6. A trip from p to q is a curve piecewise composed of future-oriented geodesics such that its past endpoint is p and future endpoint is q. There is a series of points (x_i), $i = 1, ..., n$, such that $x_0 = p$ and $x_n = q$, and such that x_i and x_{i+1} are linked by a timelike geodesic. The tangent vectors at endpoints are required to have the same time orientation : $g(u^+(x_i),^- (x_i)) < 0$

We can then prove the following [24] :

Theorem 15.7. $p \ll q$ is equivalent to the existence of a trip going from p to q.

Proof. Cover the timelike geodesic by a finite number of convex normal neighbourhood N_i (this is possible due to the curve between two points being compact and hence having a finite subcover). The first point of the trip is $x_0 = p$, in the convex normal neighbourhood N_{i_0}. x_1 is the future endpoint of $\gamma \cap \bar{N}_{i_0}$. Since it is a convex normal neighbourhood, there exists a future-oriented timelike geodesic connecting x_0 and x_1. If $x_1 = q$, it is proven. Otherwise, $x_1 \in N_{i_1}$, and the argument repeats.
Converse : Let α be the trip from p to q. μ and λ are two consecutive geodesic segments of α, with r the future endpoint of λ and the past endpoint of μ. Consider some convex normal neighbourhood of r. For some basis in $T_r\mathcal{M}$, $exp_r^{-1}(\mu)$ has coordinates of the form $(\tau, \tau\tan(\chi), 0, 0)$, while $exp_r^{-1}(\lambda)$ has coordinates of the form $(-\tau, \tau\tan(\chi), 0, 0)$ ($\tau > 0$, $\chi \in [0, \pi/4)$). This corresponds to coordinates for a future-oriented and past-oriented geodesic starting at r.
Connect the two segments $(\tau_0, \tau_0\tan(\chi), 0, 0)$ and $(-\tau_0, \tau_0\tan(\chi), 0, 0)$, for some small enough value of τ_0, by a smooth curve η in $T_r\mathcal{M}$ that is everywhere timelike. For instance, by switching the plane (t, x) to polar coordinates

$$F(R,\theta) = R\cos(\frac{\theta\pi}{\pi - 2\chi}) - \exp(R^2\sin^2(\frac{\theta\pi}{\pi - 2\chi}) - 1)^{-1} = 0 \tag{15.3}$$

This is a deformation of the standard bump function.
[INTERSECTION OF SMOOTHING AND CURVES : $\theta = \pi/(2-\chi)$???]
[Something something composition of the bump function in polar coordinates + coordinate change to $\theta \to \theta\frac{\pi}{\pi - 2\chi}$, smooth transformation so smooth joint]
Tangent vector of this curve in $T(TM)$:

$$\eta' = (\partial_R F, \partial_\theta F) \tag{15.4}$$

The slope of η never reaches the null cone. For a small enough neighbourhood (small enough τ_0), $\exp_r(\eta)$ is timelike in $\mathcal{M}$

□

Theorem 15.8. If γ_1 is a null geodesic from p to q and γ_2 is a null geodesic from q to r, then either $p \ll r$ or $\gamma_1 \cup \gamma_2$ is a null geodesic from p to r.

Proof. If $\gamma_1 \cup \gamma_2$ is not a single null geodesic, this is due to a discontinuity in the tangent vector at q (a "joint"). As previously seen, we can take □

Corrolary 15.1. $p \leq q$ is equivalent to the existence of a causal trip between p and q. Also, if

Equivalence of chronological/causal/strictly causal/horismos relation with piecewise geodesic curves : cf Penrose
Horismos equivalent to piecewise null geodesic proof : zig zags also "this always works because the convex normal neighborhoods form an open cover"
With the equivalence to causal trips, we can then prove the following properties :

Proposition 15.9.

1. $p \ll q$ implies $p \leq q$.
2. $p \ll q$ and $q \ll r$ implies $p \ll r$.
3. $p \leq q$ and $q \leq r$ implies $p \leq r$.

4. If $p \ll q$ and $q \leq r$, $p \ll r$.

5. If $p \leq q$ and $q \ll r$, $p \ll r$.

Proof.

1. By definition.

2. Identifying the future endpoint of the trip $p \ll q$ with the past endpoint of $q \ll r$.

3. Identifying the future endpoint of the causal trip $p < q$ with the past endpoint of $q < r$.

4. Let α be a trip from a to b, γ a causal trip from b to c. Cover the trip γ from q to r with a finite number of convex normal neighbourhoods $N1, ..., N_r$ (because causal trips are compact). We can assume no loops in γ because the redundant parts can be deleted. $x_0 = q \in N_{i_0}$. x_1 the future endpoint of the connected component of $\gamma \cap \bar{N}_{i_0}$. $y_1 \in N_{i_0}$ on the final segment of α, $y_1 \neq x_0$.

□

Property 1 implies that the chronological order relation is a subset of the causal order relation, $\ll \subset \leq$. This is also true for the horismos order relation, $\nearrow \subset \leq$.

Proposition 15.10. If $p \leq q$, then we have either $p \ll q$, $p \nearrow q$, $p = q$, or both $p = q$ and either $p \ll q$ or $p \nearrow q$.

15.2.1 Inverse relations

From the basic causal relations we can easily define their inverse. We define the inverse relations $p \gg q$, $p \geq q$, $p \nwarrow q$ to signify the existence of a past-directed timelike, causal, causal but not timelike curve connecting p to q.

Proposition 15.11.

1. If $p \ll q$, then $q \gg p$.

2. If $p \leq q$, then $q \geq p$.

3. If $p \nearrow q$, then $q \nwarrow p$.

Proof. This can easily be constructed by taking the curve γ connecting p to q and then taking the curve $\gamma(-\lambda)$. □

15.3 Restricted causal relations

We can define causal relations on subsets of the manifold. Given a subset $U \subset \mathcal{M}$, we say that the restriction of $\ll$, $\leq$, $\nearrow$ to U, noted $\underset{U}{\ll}$, $\underset{U}{\leq}$, $\underset{U}{\nearrow}$, is defined by

Definition 15.12. We write that $p \underset{U}{\ll} q$, $p \underset{U}{\leq} q$, $p \underset{U}{\nearrow} q$ if there exists a future-directed timelike, causal, causal but not timelike curve connecting p and q lying entirely within U.

We can in fact write every causal relation in this form, since pRq is simply equivalent to $p \underset{\mathcal{M}}{R} q$.

15.3.1 Properties

Proposition 15.13. For two subsets U, V of $\mathcal{M}$ such that $U \subset V$, we have

- $\underset{U}{\ll} \subset \underset{V}{\ll}$
- $\underset{U}{\gg} \subset \underset{V}{\gg}$
- $\underset{U}{\leq} \subset \underset{V}{\leq}$
- $\underset{U}{\geq} \subset \underset{V}{\geq}$

Proposition 15.14. $\underset{U}{\ll} \subset \underset{U}{\leq}$ and $\underset{U}{\gg} \subset \underset{U}{\geq}$ remains true in a restricted relation.

Proposition 15.15. If $S \subset U$, then $S \underset{U}{\leq} S$ and $S \underset{U}{\geq} S$

15.4 Causality and maps

For a map f between two spacetimes $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$, we say that it is

- chronal preserving if $p \ll q$ implies $f(p) \ll f(q)$
- causal preserving if $p \leq q$ implies $f(p) \leq f(q)$
- a chronal isomorphism if f is bijective and f, f^{-1} are chronal preserving
- a causal isomorphism if f is bijective and f, f^{-1} are causal preserving

It can be shown that these relations hold under a Weyl transformation.

Proposition 15.16. If we consider the sets of relations on a spacetime $(\mathcal{M}, g)$, the conformal map

$$f : (\mathcal{M}, g) \to (\mathcal{M}, \Omega g) \tag{15.5}$$

is a causal isomorphism.

Proof. if $p \ll q$ in the spacetime $(\mathcal{M}, g)$, there exists a curve connecting p and q such that $g(\gamma', \gamma') < 0$. Under a Weyl transform, this curve will have the inner product $\Omega g(\gamma', \gamma')$. As $\Omega > 0$, this curve will always be timelike. The same applies to all other relations. □

Thanks to this, we can define the causal structure as follows

Definition 15.17. The *causal structure* of a spacetime $(\mathcal{M}, g)$ is the set of all conformally equivalent spacetimes.

$$\operatorname{coset}(\mathcal{M}, g) = \{(\mathcal{M}, \tilde{g}) | \tilde{g} = \Omega g,\ \forall p,\ \Omega(p) > 0\} \tag{15.6}$$

This will allow us to study the causal structure of the spacetime without taking into consideration more complicated structures such as the distances between points.

16 Causal sets

From those causal relations, several causal sets can be defined for regions of spacetime.

- The chronological future of a point p is the set of all points that can be reached by a future-directed timelike curve from p. It is noted $I^+(p) = \{q \in \mathcal{M} \mid p \ll q\}$. The same definition applies for the chronological past $I^-(p)$ with past-directed curves.
- The causal future of a point p is the set of all points that can be reached by a future-directed causal curve from p. It is noted $J^+(p) = \{q \in \mathcal{M} \mid p \leq q\}$. The same definition applies for the causal past $J^-(p)$ with past-directed curves. Unlike the chronological future and past, $\forall p,\ p \in J^\pm(p)$
- The future and past horismos of a point p is the substraction of the chronological past/future from the causal past/future, the set of points that can only be reached by a null curve. It can be defined as $E^+(p) = \{q \in \mathcal{M} \mid p \nearrow q\}$, or $E^+(p) = J^+(p) \setminus I^+(p)$, with similar definitions for $E^-(p)$

We will starting here mostly only talk about I^+, J^+ and E^+, the generalization to I^-, J^- and E^- being trivial in most cases.
Those causal sets can be generalized to be defined on a set of spacetime points by

$$I^+(S) = \bigcup_{p \in S} I^+(p) \tag{16.1}$$

and identically for $J^+(S)$ and $E^+(S)$. This leads to the following useful identity by commutativity of the union

$$I^\pm(A \cup B) = I^\pm(A) \cup I^\pm(B) \tag{16.2}$$

The intersection of $I^+(p)$ and $I^-(q)$, also called the causal diamond of p and q, is useful enough to merit its own notation. It will be noted $\langle p, q \rangle$. The intersection of $J^+(p)$ and $J^-(q)$ will be noted as $\langle\langle p, q \rangle\rangle$.

Definition 16.1. The *chronology violating set* (resp. *causality violating region*) V is the set of all points $p \in \mathcal{M}$ such that $p \in I(p)$ (resp. there exists a non-trivial causal curve frop p to itself).

If we replace the causal relations in the definition by their restriction to an open set U, we have their equivalent, noted

$$\begin{aligned}
I^+(S,U) &= \bigcup_{p \in S} \{q \in U | p \underset{U}{\ll} q\} \\
J^+(S,U) &= \bigcup_{p \in S} \{q \in U | p \underset{U}{\leq} q\} \\
E^+(S,U) &= \bigcup_{p \in S} \{q \in U | p \underset{U}{\nearrow} q\}
\end{aligned} \tag{16.3}$$

As with restricted causal relations, we have immediatly that $I^+(S, \mathcal{M}) = I^+(S)$, and identically for J and E.
Some immediate consequences of those definitions and the properties of the causal relations are

Proposition 16.2.

1. $I^+(S,U) \subset J^+(S,U)$
2. $I^+(I^+(S,U),U) = I^+(S,U)$
3. $J^+(J^+(S,U),U) = J^+(S,U)$
4. $J^+(I^+(S,U),U) = I^+(J^+(S,U),U) = I^+(S,U)$

Proposition 16.3. The restricted causal sets on a geodesically convex set U are the intersection of those sets with U.

Proof. NOT SURE IF TRUE □

16.0.1 Past and future sets

Past and future sets P, F are sets that will contain all their own past and future points, that is, if $p \in P$, all points $q \ll p$ belong to P, and similarly with $p \ll q$ with F. This is efined by

Definition 16.4. A *future set* F obeys the property $I^+(F) = F$. Similarly, a *past set* P obeys $I^-(P) = P$.

16.0.2 Common past and future

Definition 16.5. The *chronological common past* and *future* of a set $U \subset \mathcal{M}$ is the set of points

$$\begin{aligned} \downarrow U &= I^-(\{p \in \mathcal{M} | \forall q \in U,\ p \ll q\}) \\ \uparrow U &= I^+(\{p \in \mathcal{M} | \forall q \in U,\ p \gg q\}) \end{aligned} \quad (16.4)$$

This means that first we consider the set of points
Set of all past and future sets : $\mathcal{P}$, $\mathcal{F}$

Definition 16.6. A *hull pair* is an element $(P, F) \in \mathcal{P} \times \mathcal{F}$ such that $P = \downarrow F$ and $F = \uparrow P$.

16.1 Minkowski space

As a lot of proofs will rely on the causal behaviour of objects in the tangent space $T_p\mathcal{M}$, it will be interesting to first study the causal structure of Minkowski space.
We consider Minkowski space as the spacetime $(\mathbb{R}^n, \eta)$, with the time orientation $\tau = (1, \vec{0})$.
In Minkowski space, the geodesic equation being simply $\ddot{x}(\lambda) = 0$, a geodesic between the points a and b can be written as

$$\vec{ab} = a + (b - a)\lambda \quad (16.5)$$

for $\lambda \in [0, 1]$. That geodesic is timelike/null if $|\vec{ab}| > 0$, $= 0$. Those are future/past oriented if $(b_t - a_t) > 0$.

Theorem 16.7. If $a \ll b$, then $b \not\ll a$, and if $a \leq b$, $b \not\leq a$.

Lemma 16.1. The metric tensor η has range $\mathbb{R}$.

Proof. Consider the timelike vector $(t, 0, 0, ...)$ and the spacelike vector $(0, x, 0, ...)$. Their norm will be respectively $-t^2$ and x^2. Along with any null vector, this will span all of $\mathbb{R}$. □

Theorem 16.8. $I^+(p)$ is open in Minkowski space, while $J^+(p)$ is closed.

Proof. $I^+(p)$ is the pre-image of the open interval $(-\infty, 0)$ with g, while $J^+(p)$ is the pre-image of the closed interval $(-\infty, 0]$. g being continuous, those pre-images are respectively open and closed. □

16.2 Properties in a convex neighbourhood

The causal sets possess a lot of interesting properties in convex neighbourhood, which as we will happen to be the causal properties of Minkowski space. For this section, we will consider a point p in $\mathcal{M}$, with a convex neighbourhood U around p, and a point q such that $q \in U$ and $p \neq q$.

Theorem 16.9. $q \in I^+(p, U)$ (resp. $J^+(p, U)$, $E^+(p, U)$) is equivalent to the geodesic between p and q being timelike (resp causal, null) and future pointing.

Proof. Since $q \in I^+(p, U)$, there exists a timelike curve γ connecting p with q. If we consider the pullback of that curve in the tangent space $T_p\mathcal{M}$ via the exponential map $\exp^{-1}(\gamma)$, since it is a timelike curve by theorem X, it will lie entirely within the future light cone. Since D_p is convex, it is possible to define a segment from $\exp_p^{-1}(p)$ to $\exp_p^{-1}(q)$, lying in the future light cone. The image of this segment is a future pointing timelike geodesic. □

Theorem 16.10. The image of $I^+(0) \cap D_p$ (resp. $J^+(0) \cap D_p$, $E^+(0) \cap D_p$) in the tangent space at p by the exponential map is $I^+(p, U)$, resp. $J^+(p, U)$, $E^+(p, U)$.

Proof. □

Theorem 16.11. $I^+(p, U)$ is open in U and $\mathcal{M}$, while $J^+(p, U)$ is closed.

Proof. Since $I^+(0)$ is open in Minkowski space and the domain of the exponential map is itself open, $I^+(0) \cap exp_p^{-1}(U)$ is open. $\exp_p$ being a homeomorphism , the image of that set will be open, which is $I^+(p, U)$. □

Theorem 16.12. $J^+(p, U)$ is the closure in U of $I^+(p, U)$

Proof. Since $\exp_p$ is a homeomorphism over the convex normal neighbourhood, $J^+(p, U) = \exp_p(J^+(0)) = \exp_p(\bar{I}^+(0)) = \overline{\exp_p(I^+(0))} = \overline{I}^+(p)$ □

Theorem 16.13. Causal curves in a compact subset of U have two endpoints.

Proof. □

16.3 Global properties

Proposition 16.14. The causal and chronological future of any set obeys the following properties

1. $\text{int}J^+(S) = I^+(S)$.

2. $J^+(S) \subset \overline{I^+(S)}$, and $J^+(S) = \overline{I^+(S)}$ if $J^+(S)$ is closed.

3. $\bar{J}^+ = \bar{I}^+$.

Proof.

1. Since $I^+(S)$ is open and $I^+ \subset J^+$, $I^+(S) \subset \text{int}(J^+(S))$ follow. If $p \in \text{int}(J^+(S))$, since $\text{int}(J^+(S))$ is open, there exists a $q \in I^-(p) \cap$ J$^+$(S). So $p \in I^+(q) \subset I^+(J^+(S)) = I^+(S)$. Since this is true for every point p, $\text{int}(J^+(S)) \subset I^+(S)$, and the equality is verified.

2. For any $q \in J^+(S)$, there will be a causal curve γ from $p \in S$ to q ...

3. $I^+ \subset J^+$, so $\bar{I}^+ \subset \bar{J}^+$. [USE PROP 1]

□

Let $(p, q) \in \bar{J}^+$, U and V neighbourhoods of p and q.
U, V open subsets of $\mathcal{M}$, $V \subset U$. V is causally convex in U if any causal curve in U with endpoints in V is entirely contained in V
if $U = \mathcal{M}$, V is called causally convex
$\leq_U$ is the causal relation when U is treated as a spacetime. If V is causally convex in U, then $\leq_U$ restricted to V is identical to $\leq_V$
Converse is false : example : $U = \mathcal{L}^2, V = \{(t, x) \in \mathbb{R} | |t|, |x| < 1\}$
Totally vicious spacetimes are such that only $V = \mathcal{M}$ is causally convex in $\mathcal{M}$
If $V \subset W \subset U$, V is causally convex in W

Proposition 16.15. $I^+(p)$ is open.

Proof. If $q \in I^+(p)$, then there is a trip α from p to q. For some convex neighbourhood V of q, let's pick a point q^- in between p and q, and p^+ in a convex neighbourhood U of p such that p^+ is between p and q. □

Proposition 16.16. The chronological future of every point forms a cover of the manifold : $\bigcup_{p \in \mathcal{M}} I^+(p) = \mathcal{M}$.

Proof. For any point $p \in \mathcal{M}$, we have a convex normal neighbourhood U_p. Since D_p is an open set around 0 in $T_p\mathcal{M}$, we have a past-directed geodesic to some point q in $I^-(p, U_p)$. The point p itself will be in $I^+(q, U_q) \subset I^+(q)$. Hence every point is in the chronological future of some point of the manifold. □

Proposition 16.17. $J^+(p)$ is not necessarily closed.

Proof. The standard counterexample for this is Minkowski space with a point removed. □

Proposition 16.18. $J^+(p)$ is closed for all p if and only if the spacetime contains no naked singularities (???)

Proof. If the spacetime contains a naked singularity, there exists a point q and a future-directed timelike or null incomplete geodesic $\gamma \subset I^-(p)$ converging to p in the b-boundary of the spacetime. Since for all $p_i \in \gamma$, $p_i \leq p$ and $p_i \leq q$, $p \leq q$, so that $p \in E^-$
If $J^+(p)$ is closed, any sequence of points $\{r_i\}_{i\in\mathbb{N}}$ such that for all i, $r_i \in J^+(p)$.
Consider a point $q \in J^+(p)$. If there is a naked singularity, there will be a pair of points p, q such that there is a
Pick a point to the future of all q_n??? No causal incomplete curve - All sequences converge??? □

Proposition 16.19. For $i = \{1, 2, ...\}$, the chronology violating set (resp. causal) corresponds to connected components of the form $V_i = I^+(p_i) \cap I^-(p_i)$ (resp. $J^+(p_i) \cap J^-(p_i)$). (cf Kriele)

Proof. For a connected component $V_i \subset V$, since V_i is connected, for every $p, q \in V_i$ there exists a continuous path $\gamma \subset V_i$ connecting them. If we consider for all $z \in \gamma$, $I^+(z)$ is a neighbourhood of z, since $z \in I^+(z)$ and $I^+(z)$ open. Then since γ is compact, we can cover it by $I^+(z_j)$, for a finite collection $z_j \in \gamma$, meaning that there is a future-directed timelike curve from p to q, and the same argument applies with $I^-(p)$, so that $V_i = I^+(p) \cap I^-(p)$. □

16.4 Causal ordering

Definition 16.20. $g_1 < g_2$: the causal cone of g_1 is in the timelike cone of g_2

n-degree causal relations :

$$\begin{aligned} \overset{1}{<} \mathscr{S}) &= ... \\ (\mathscr{S} \overset{1}{>} &= ... \end{aligned} \tag{16.6}$$

theorem : $p \in \mathcal{M}$, neighbourhood $U \ni p$. $\exists V \ni p, V \subset U$ such that there are two metrics on V, g^+ and g^-, such that $g^- < g < g^+$

16.5 Causally convex sets

Definition 16.21. An open set U is causally convex if and only if every causal curve γ that intersects it is such that $\gamma \cap U$ is connected.

$\mathcal{M}$ and $\varnothing$ are causally convex sets.
Causal diamond $\langle p, q \rangle$ is causally convex.

Definition 16.22. A point $p \in \mathcal{M}$ is strongly causal if for any neighbourhood $U \ni p$, there is a causally convex subset $V \subset U$.

16.6 Achronal sets

Definition 16.23. A subset S of $\mathcal{M}$ is achronal if for every point $p, q \in S$, there is no points such that $q \in I^+(p)$, or $I^+(S) \cap S = \varnothing$

Definition 16.24. The edge of an achronal set S is the set of all the points $p \in \bar{S}$ such that every neighbourhood U of p contains a timelike curve from $I^-(p, U)$ to $I^+(p, U)$ that does not intersect S. [WARNING DIFFERENT DEFINITION IN PENROSE AND O'NEILL, CHECK IT]

Definition 16.25. For a closed, achronal set S, its past (resp. future) domain of dependence $D^+(S)$ (resp. $D^-(S)$) is the set of points p such that every future (resp. past) inextendible causal curve going through p intersects S. The total domain of dependance $D(S)$ of dependance is the union of the two.

Definition 16.26. The future (resp. past) Cauchy horizon $H^+(S)$ (resp. $H^-(S)$) of an achronal closed set S is the set of points $p \in D^+(S)$ (resp. $D^-(S)$) such that the chronological future (resp. past) does not intersect with $D^+(S)$ (resp. $D^-(S)$). The total Cauchy horizon $H(S)$ is the union of the two.

$$\begin{aligned} H^+(S) &= \{p | p \in D^+(S), I^+(p) \cap D^+(S) = \varnothing\} && (16.7)\\ H^-(S) &= \{p | p \in D^-(S), I^+(p) \cap D^-(S) = \varnothing\} && (16.8)\\ H(S) &= H^+(S) \cup H^-(S) && (16.9) \end{aligned}$$

$$H^\pm(S) = D^\pm \setminus I^\mp[D^\pm(S)] \tag{16.10}$$

Properties :
for $S \subset \mathcal{M}$ a closed achronal set :

1. $D^+(S)$ is closed
2. $H^+(S)$ is achronal and closed
3. $S \subset D^+(S)$
4. $p \in D^+(S)$ implies $I^-(p) \cap J^+[S] \subset D^+(S)$
5. $\partial D^+(S) = H^+(S) \cup S$
6. $\partial D(S) = H(S)$
7. $I^+[H^+(S)] = I^+(S) \setminus D^+(S)$
8. $\text{int} D^+(S) = I^+[S] \cap I^-[D^+(S)]$

(proof as exercize in Penrose, gotta prove it)

Definition 16.27. A future Cauchy horizon is compactly generated if its null generators remain in a compact region K for $\lambda < \lambda 0$

Definition 16.28. The base set $\mathcal{B}$ of a compactly generated future Cauchy horizon is the set of all points $p \in H^+(S)$ such that there is a null generator γ such that p is a past terminal accumulation point of γ.

Either a closed timelike curve going through $p \in \mathcal{B}$

Theorem 16.29. For a compactly generated Cauchy horizon $H^+(S)$, the *base set* $\mathcal{B} \subset K$ is always non-empty. All null generators of $H^+(S)$ approach $\mathcal{B}$ asymptotically : for every past-directed generator and every open neighbourhood U of $\mathcal{B}$, there exists $\lambda_0 \in I$ such that for every $\lambda > \lambda_0$, $\gamma(\lambda) \in U$ [?].

Proof. Consider a past directed null geodesic γ in $H^+(S)$ and a monotone increasing sequence $\{\lambda_i\}$ without limit in I. There are infinitely many points $\gamma(\lambda_i)$ in K [show it]. Then there must be an accumulation point p of $\{\gamma(\lambda_i)\}$ that is within $H^+(S)$, since it is a closed subset of $\mathcal{M}$. Then $p \in \mathcal{B}$, and all of those accumulation points are in K by compactness (by contradiciton something something cf Kay). □

Theorem 16.30. $\mathcal{B}$ is composed of null geodesic generators entirely contained within $\mathcal{B}$ and are past and future inextendible.

Proof. Riemannian metric h on $\mathcal{M}$, parametrize all curves on $\mathcal{M}$ by arc length s with respect to this metric. For $p \in \mathcal{B}$, there is a null generator of $H^+(S)$ such that p is a past accumulation point. Parametrize γ so that s so that s is increasing in the past direction. Since γ is past inextendible and K compact, s extends to infinite values even if λ doesn't. So there is a sequence $\{s_i\}$ diverging to infinity such that $\gamma(s_i)$ converges to p. $(k_i)^a$ the tangent of γ at s_i in the arc length parametrization, with unit norm wrt h. Since the subset of TM such that $\pi(TM) = K$ and with vectors of unit norm is compact, there must exists a tangent vector k^α at p such that $\{(\gamma(s_i), (k_i)^\alpha)\}$ converges to (p, k^α). Since each $(k_i)^\alpha$ is null, by continuity so is k^α.
For γ a maximally extended null curve with initial condition (p, k^α), parametrized by arc length, $\gamma(0) = p$. For $q \in \gamma$ and $\gamma(s) = q$. Since $\{s_i\}$ diverges, for large i $(s + s_i)$ will be in the interval of γ. Since $\{(\gamma(s_i), (k_i)^\alpha\}$ converges to (p, k^α), by continuity of exp and s (wrt g and h), $\{\gamma(s + s_i)\}$ converges to q. Thus $q \in \mathcal{B}$ □

Closed null geodesics of $\mathcal{B}$: fountains
There are cases where the null geodesics aren't closed (non-strongly causal spacetimes)

Theorem 16.31. Strong causality is violated at every $p \in \mathcal{B}$.

Theorem 16.32. $H^+(S)$, $\mathcal{B}$ its base set. For $p \in \mathcal{B}$ and U a globally hyperbolic neighbourhood of p, there exists $q, r \in U \cap \mathrm{int} D^+(S)$ such that q and r are connected by a null geodesic, but not a causal curve in U.

16.7 Cauchy hypersurface

Cauchy hypersurfaces will be the model on which the notion of a specific moment in time is built for a spacetime.

Definition 16.33. A *Cauchy hypersurface* is a subset $S \subset \mathcal{M}$ that intersects every inextendible timelike curves exactly once.

Theorem 16.34. A Cauchy hypersurface S is a closed achronal hypersurface.

Proof. Since every inextendible timelike curve intersects S and every point in $\mathcal{M}$ contains a timelike curve, we have that $\mathcal{M} = I^+(S) \cup S \cup I^-(S)$. □

Example of a closed boundaryless achronal hypersurface that isn't a Cauchy hypersurface (Hyperbole in Minkowski space)

Theorem 16.35. A Cauchy hypersurface S intersects every inextendible causal curve.

Proof. □

Theorem 16.36. A closed achronal set Σ is a Cauchy surface if and only if $D(\Sigma) = \mathcal{M}$

Theorem 16.37. A closed achronal set Σ is a Cauchy surface if and only if $H(\Sigma) = \varnothing$.

Theorem 16.38. If Σ is a closed, achronal and edgeless subset of the spacetime, then it is a Cauchy surface if and only if every inextendible null geodesic intersects Σ, enters $I^+(\Sigma)$ and $I^-(\Sigma)$.

Proof. x □

Theorem 16.39. A nakedly singular spacetime does not admit a Cauchy surface (Geroch and Horowitz 1979)

Definition 16.40. A partial Cauchy surface is a closed boundaryless achronal surface such that every causal curve intersects it at most once.

As not all spacetimes admit a Cauchy surface, but may be somewhat well-behaved up to a point, we can also define the notion of a partial Cauchy surface.

Definition 16.41. A *partial Cauchy surface* is a closed achronal hypersurface that intersects every causal curve at most once.

From this definition, we can immediatly see that any Cauchy hypersurface is also a partial Cauchy hypersurface.

Proposition 16.42. If a spacetime contains a partial Cauchy hypersurface, there is a submanifold containing it where it is a Cauchy hypersurface.

17 Causal properties

17.1 Time separation

For $p, q \in \mathcal{M}$, with $\mathcal{C}_t^+(p,q)$ the set of all future directed timelike curves connecting p to q. The time separation $d : \mathcal{M} \times \mathcal{M} \to [0, +\infty]$ is the longest proper time connecting p to q

$$d(p,q) = \begin{cases} 0 & \text{if } \mathcal{C}_t^+(p,q) = \varnothing \\ \sup[L(\alpha), \alpha \in \mathcal{C}_t^+(p,q)] \end{cases} \tag{17.1}$$

Properties :

1. $d(p,q) > 0$ if and only if $p \in I^-(q)$
2. If there's a closed timelike curve through p, then $d(p,p) = \infty$, otherwise $d(p,p) = 0$
3. $d(p,q) \in (0, +\infty)$ implies $d(q,p) = 0$
4. Triangle inequality : $p \leq q \leq r$ implies $d(p,q) + d(p,r) \leq d(p,r)$

Theorem 17.1. *d is lower semi-continuous. For $p_m, q_m \in \mathcal{M}$, with $\lim p_m = p$, $\lim q_m = q$, we have*

$$\liminf_{m\to\infty} d(p_m, q_m) \geq d(p,q) \tag{17.2}$$

Example where d is not upper semi-continuous : remove line from spacetime, q on the light cone of that line's boundary, p_n just to the right of that line, converges to just under the line's edge. Then $\limsup_{m\to\infty} d(p_n, q) = 1$ while $d(p,q) = 0$

Proposition 17.2. *If $p \ll p$, $d(p,p) = \infty$*

Proof. Since $p \ll p$, there is a closed timelike curve γ, $\gamma(0) = \gamma(1) = p$. The length of that curve is positive. It is then possible to construct a piecewise timelike curve by making identical copies of this curve and connecting their endpoints, using $\gamma(k) = p$, $k \in \mathbb{Z}$. For n copies, the length of this curve will be n time the original length. As this process does not converge, $d(p,p) = \infty$. □

Something about timelike singularities and convergence???

17.2 Time functions

A time function on a spacetime is a
Borel measure m on the spacetime such that

- m is finite : $m(\mathcal{M}) < \infty$
- For a non-empty open subset U, $m(U) > 0$

For an orientable spacetime, pick an orientation and ε be the oriented volume element associated to the metric. Take an open covering of $\mathcal{M}$ such that every open set has measure by ε inferior to 1. Take a partition of unity $\{\rho_n\}$ surbordinate to this covering. The measure m is the one associated to the volume element

$$\varepsilon_m = \sum_{n=1}^{\infty} 2^{-n} \rho_n \varepsilon \tag{17.3}$$

If the spacetime isn't orientable, go to the double cover and pullback, take half the measure

Definition 17.3. The future and past volume functions, t^+ and t^- are functions $t^\pm : \mathcal{M} \to R$ such that

$$t^\pm(p) = \mp m(I^\pm(p)) \tag{17.4}$$

Proposition 17.4. $t^\pm$ is non-decreasing on any future-directed causal curve.

Proof. If $p \leq q$, then $I^+(q) \subset I^+(p)$ and $I^-(p) \subset I^-(q)$. Since m is a measure, $-m(I^+(p)) < -m(I^+(q))$ and $m(I^-(p)) < m(I^-(q))$. □

Definition 17.5. A generalized time function is a function $t : \mathcal{M} \to \mathbb{R}$ that is strictly increasing on all future-directed causal curves. It is called a time function if it is continuous as well, and a temporal function if it is smooth with a past-directed timelike gradient ∇t.

Proposition 17.6. A spacetime with closed timelike curves admits no generalized time function. A spacetime with closed null curves admits no time function.

Proof. If a function is strictly increasing on a closed timelike curve, we have $f(p) > f(p)$. Hence there can be no such function. If the spacetime includes a closed null curve but no closed timelike curve, we can represent this curve as the limit of a sequence of future-directed timelike curves $(\gamma_n)_{n\in\mathbb{N}}$. If a generalized time function exists, then for every γ_n, $f(\gamma_n(0)) < f(\gamma_n(1))$, meaning that if f is continuous, the limit should be that $f(p) < f(p)$. □

17.3 Spacetime topologies

Beyond the manifold topology of the spacetime itself, we may define topologies on it by its causal structure.

17.3.1 The path topology

Definition 17.7. The *path topology*, or Zeeman topology, is the topology in which a subset $U \subset \mathcal{M}$ is open if for every timelike curve γ, there is a subset $V \subset \mathcal{M}$ in the manifold topology such that

$$U \cap \gamma = V \cap \gamma \tag{17.5}$$

Strictly finer than the manifold topology since trivially true for any open set U in the manifold topology.
Prove that it's a topology :
Since it's finer than the manifold topology, $\varnothing$ and $\mathcal{M}$ are in it

Intersection of two open sets $U_1 \cap U_2$: if $U_1 \cap \gamma = V_1 \cap \gamma$ and $U_2 \cap \gamma = V_2 \cap \gamma$, then

$$(U_1 \cap U_2) \cap \gamma = U_2 \cap (V_1 \cap \gamma) = V_1 \cap (U_2 \cap \gamma) = (V_1 \cap V_2) \tag{17.6}$$

Union of a collection of open sets $\bigcup U_i$: if for all i, $U_i \cap \gamma = V_i \cap \gamma$, then

$$\begin{aligned}(\bigcup_i U_i) \cap \gamma &= \bigcup_i (U_i \cap \gamma) \\ &= \bigcup_i V_i \end{aligned} \tag{17.7}$$

Basis for the topology : light cones $I^+(p, U) \cup p \cup I^-(p, U)$ for a convex normal neighbourhood U.

17.3.2 The Alexandrov topology

In general, the Alexandrov topology is a topology defined by a partial order on a set

Definition 17.8. The *Alexandrov topology* on a set X is a topology where the intersection of arbitrarily many open sets is open.

This is in contrast with the usual definition of topology where only a finite intersection is required to be open.
[SHOULD ACTUALLY BE INTERVAL TOPOLOGY]
The topology of interest in general relativity is the one generated by the basis $\langle p, q \rangle$, for all p and q.

Proposition 17.9. The basis $\langle p, q \rangle$ generates an Alexandrov topology.

Since this will be the only Alexandrov topology of interest, it will usually be referred to as the Alexandrov topology of a spacetime.
Show that the Alexandrov topology is a topology : any point p is in a causal diamond, so M is a union of the base. Empty set is the union of no basis element.
Intersection of two causal diamonds is a causal diamond.
Union of causal diamonds???

Proposition 17.10. The Alexandrov topology is at least as coarse as the manifold topology.

Proof. This only requires us to prove that any open set of the Alexandrov topology is an open set of the manifold topology, which is trivially true since we have shown that $I^\pm$ is always an open set in the manifold topology. □

Very coarse Alexandrov topology : timelike cylinder, which has for all $p, q \in \mathcal{M}$

$$I^+(p) \cap I^-(q) = \mathcal{M} \tag{17.8}$$

It is the trivial topology.

17.4 Timelike homotopies

Similarly to the general notion of curve homotopies if there exists a continuous map mapping one curve to another, it can be useful to have a notion of homotopies specifically for timelike curves.
There are several ways to deal with this. The first one is due to Smith.

Definition 17.11. A *q-loop* based at p , $q \in \mathbb{N}$, is a piecewise closed timelike curve with exactly q corners.

The timelike homotopies of q-loops of arbitrary q would be too large, and indeed reduce to the usual homotopy group (we can approximate any loop with some q-loop zig-zagging in a very small neighbourhood of the original loop), but on the other hand, the product of q-loops of fixed q does not form a group, as the trivial curve is not a q-loop, and if we have a timelike curve γ, the product $\gamma * \gamma^{-1}$ will have $(2q + 2)$ corners. To prevent this, we define, in addition to q-loops, stings.

Definition 17.12. A *sting* based at p is a curve composed from an arbitrary finite curve γ such that the sting will be the path $\gamma * \gamma^{-1}$, with $\gamma(0) = p$.

And then we define the set of generalized q-loops.

Definition 17.13. The p-factorization of a curve γ, with $p \in \gamma$, is the decomposition $\gamma = \gamma_+ * \gamma_-$, where $\gamma_+(1) = p$.

Definition 17.14. For a curve ζ starting at p, the curve

$$\gamma^* = \gamma_+ * \zeta * \zeta^{-1} * \gamma_- \tag{17.9}$$

is said to be obtained fom γ by the insertion of a sting. Likewise, γ is obtained from γ^* by the deletion of a sting.

Definition 17.15. A *generalized q-loop* is a curve made of a q-loop by a finite number of insertions and deletions of stings. The constant curve e_p will also be considered a generalized q-loop based at p.

Proposition 17.16. Any q-loop can be turned into a q'-loop, $q' < q$, by the insertion of a sting.

Proof. By inserting a purely spacelike sting at singular points, we can remove any singular point, as one of the left or right tangent vector will fail to be causal at that point. □

Definition 17.17. equality modulo stings

Corrolary 17.1. Given a q-loop γ, there is a q-loop equivalent modulo stings to $\gamma * \gamma^{-1}$

Proposition 17.18. The set of generalized q-loops with the path product $*$ and equivalence relation modulo stings $\doteq$ forms a group. .

Proof. If we consider the □

The group of q-loops and stings at p will be noted $\tau_q(L, x)$ under the operation

$$\gamma_1(\lambda) * \gamma_2(\lambda) = \gamma_{12} \tag{17.10}$$

Proposition 17.19. If a spacetime isn't time-orientable, all of its odd τ_q groups will be trivial.

Theorem 17.20. All $\tau_q(L, x)$ groups are trivial for Minkowski space.

Proof. □

Universal coverings
Def : Covering space of a topological space X is a space C with a continuous surjective map $p : C \to X$, $\forall x \in X$, there exists a neighbourhood $U \ni x$ such that $p^{-1}(U)$ is a union of disjoint open sets in C which are mapped homeomorphically onto U by p.
Universal cover : simply connected cover
Ideas : check if all contractible CTCs -¿ causal cover
If CTC is contractible → quasiregular singularities ("unwrapping CTCs") Otherwise no problem
Definition : timelike connected, timelike contractible.

18 Causal hierarchy

It is possible to classify spacetimes by their causal pathologies. There are many possible schemes and many possible classes of such spacetimes, but the principal classification is the causal ladder, which classifies spacetimes, from most to least pathological, as

- Any spacetime
- Non-totally vicious
- Chronological
- Causal
- Distinguishing
- Strongly causal
- Stably causal
- Causally continuous
- Causally simple
- Globally hyperbolic

Each rung of the causal ladder implies the one above it. That is, if a spacetime is distinguishing, then it is also causal, chronological and non-totally vicious.

18.1 Totally vicious spacetime

While not directly part of the causal ladder, the lowest rung on it will be defined in opposition to this type of spacetime, which has the worst causal pathologies. A totally vicious spacetime is a spacetime for which every point is vicious.

Definition 18.1. A *totally vicious spacetime* is a spacetime in which all $p \in \mathcal{M}$ obeys $p \ll p$.

Equivalently, the chronology violating region V is the entire spacetime.

18.1.1 Properties

Proposition 18.2. A spacetime is totally vicious if and only if for all $p, q \in \mathcal{M}$, $d(p, q) = \infty$.

Proof. By the triangle inequality, $d(p, q) \geq d(p, p) + d(p, q)$. Since $d(p, p) = \infty$ if $p \ll p$, $d(p, q) = \infty$. Conversely, if $d(p, q) = \infty$ for all points, then $d(p, p) = \infty > 0$, meaning there is always a future-directed timelike curve between them, so $p \ll p$. □

Proposition 18.3. A spacetime is totally vicious if and only if $I^+(p) = I^-(p) = \mathcal{M}$.

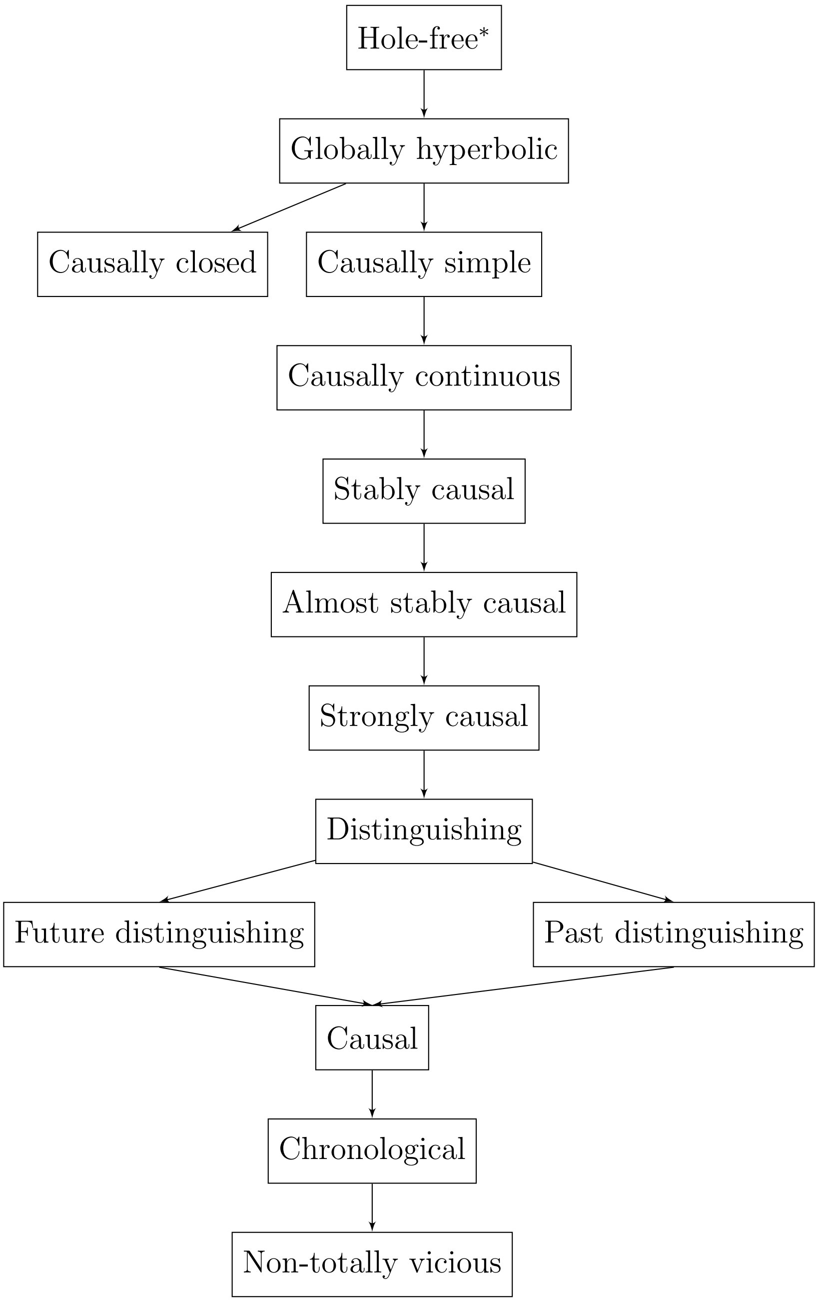


Figure 8: The causal ladder

Proof. Since $p \in \mathcal{M}$, if $I^\pm(p) = \mathcal{M}$, $p \in I^\pm(p)$, or $p \ll p$.
Conversely, if we take $d(p,q) = \infty$ for all points per the previous property, for any point r, $d(p,r) > 0$ and $d(r,p) > 0$, meaning that for all points $p, r \in \mathcal{M}$, $r \in I^\pm(p)$, so $I^\pm(p) = \mathcal{M}$. □

Proposition 18.4. A totally vicious spacetime has trivial Alexandrov topology, $\tau = \{\varnothing, \mathcal{M}\}$.

Proof. For every basis set of the topology $\langle p, q \rangle$,

$$I^+(p) \cap I^-(q) = \mathcal{M} \cap \mathcal{M} = \mathcal{M} \tag{18.1}$$

Their intersections and unions will only be $\mathcal{M}$, with $\varnothing$ being the empty union of no member of the basis. □

Proposition 18.5. Totally vicious spacetime admits no generalized time function, and its future and past volume functions are constants.

Proof. Since it contains closed timelike curves, there is no generalized time function. Its future and past volume functions are

$$t^\pm(p) = \mp m(I^\pm(p)) = \mp m(\mathcal{M}) \tag{18.2}$$

□

Theorem 18.6. All compact spacetimes with a timelike killing vector are totally vicious.

This relies on a few other theorems

Proof. For a compact spacetime with timelike Killing vector K^μ, if it is static, K is parallel for the conformal metric

$$\hat{g} = \frac{1}{g(K,K)} g \tag{18.3}$$

the 1-form $\omega = -\hat{g}(K, -)$ is closed so if $\mathcal{M}$ is compact it cannot be simply connected. [TO FINISH] □

Timelike loop at every $p \in \mathcal{M}$ equivalent to timelike vector field with periodic integral curves

18.1.2 Examples

A few examples of totally vicious spacetimes include the Gödel spacetime, variations on the timelike cylinder $S^1_t \times \Sigma$ or the van Stockum dust.

18.2 Non-totally vicious spacetime

Opposite to totally

Definition 18.7. A spacetime is non-totally vicious if it is not totally vicious, ie, there is a point p such that $p \not\ll p$.

18.3 Chronological spacetime

A spacetime is chronological if it contains no closed timelike curves.

Definition 18.8. A spacetime is chronological if there is no point $p \in \mathcal{M}$ such that $p \ll p$.

Equivalently, it is chronological if its chronology violating region is empty : $V = \varnothing$.
By definition, if a spacetime is chronological, it is also non-totally vicious.

18.3.1 Properties

Proposition 18.9. For all $p \in \mathcal{M}$, $d(p,p) = 0$.

Theorem 18.10. A spacetime is chronological if and only if $t^{\pm}$ is strictly increasing on any future-directed timelike curves.

Proof. □

Theorem 18.11. No compact spacetime is chronological.

Proof. Since $I^+(p)$ for all p forms an open cover of the manifold, if the manifold is compact, then this cover has a finite subcover, of the form $\{I^+(p_1), I^+(p_2), ..., I^+(p_N)\}$. If there were no closed causal curves, p_1 would not be in $I^+(p_1)$, and hence must belong to $I^+(p_i)$, $i \neq 1$. But if $p_1 \in I^+(p_i)$, then $I^+(p_1) \in I^+(p_i)$, and then the cover can be reduced to $\{I^+(p_2), ..., I^+(p_N)\}$. The process can be repeated until only one chronological future forms the entire cover of the manifold, meaning that p has to be within $I^+(p)$. □

Remark 18.12. The original proof of this theorem was a little more involved,

Original proof by Markus in "Remark on cosmological models"
Example of a non-totally vicious spacetime that is not chronological : any spacetime with compact chronology violating region

Proposition 18.13. A non-totally vicious spacetime may not be chronological.

Proof. Example : Toroidal spacetime, Killing vector field ∂_t, everywhere timelike except two points □

Proposition 18.14. A spacetime is chronological if and only if $t^{\pm}$ is strictly increasing on any future-directed timelike curve.

Proof. Since $t^{\pm}$ is constant on any closed timelike curve, being strictly increasing means that there is no point such that $p \ll p$. Conversely, if $p \ll q$ but $t^-(p) = t^-(q)$ □

18.3.2 Examples

18.4 Causal spacetime

Definition 18.15. A spacetime is causal if there is no point $p \in \mathcal{M}$ such that $p < p$.

By definition, if a spacetime is causal, it is chronological.

Theorem 18.16. A chronological but non-causal spacetime has a closed null geodesic.

Proof. If the spacetime isn't causal, then there's a point p such that $p < p$. Since it is chronological, that curve cannot be a timelike curve. So by theorem X, it must be a geodesic. □

Proposition 18.17. Not all chronological spacetimes are causal.

Proof. □

18.5 Distinguishing spacetime

Definition 18.18. A spacetime is future (resp. past) *distinguishing* if the chronological future (resp. past) of two different point is not the same. If it is both future and past distinguishing, we say it is distinguishing.

$$I^{\pm}(p) = I^{\pm}(q) \to p = q \tag{18.4}$$

Theorem 18.19. If a spacetime is either past or future distinguishing, then it is causal.

Proof. If a spacetime is non-causal, then there is a closed causal curve, meaning that we have two points $p, q \in \mathcal{M}$ with the property $p \ll q \ll p$. Which means that we have both $I^+(p) \subset I^+(q)$ and $I^+(q) \subset I^+(p)$, in other words, $I^+(p) = I^+(q)$. Hence it cannot be future distinguishing. The same proof applies for past distinguishing spacetimes. □

Theorem 18.20. For two spacetimes $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$, $(\mathcal{M}_1, g_1)$ distinguishing, and $f : \mathcal{M}_1 \to \mathcal{M}_2$ a diffeomorphism such that $p \leq q \to f(p) \leq f(q)$, then $(\mathcal{M}_2, g_2)$ is distinguishing and $g_1 = \Omega f^* g_2$.

Proof. □

Proposition 18.21. A causal spacetime may not be distinguishing.

Proof. Punctured cylinder example □

Proposition 18.22. A past-distinguishing spacetime may not be future-distinguishing, and vice-versa.

Proof. Punctured cylinder - a strip example □

18.6 Strongly causal spacetime

Definition 18.23. A spacetime is *strongly causal* if for every neighbourhood U of every point $p \in \mathcal{M}$, there is a strongly causal neighbourhood $V \subset U$.

Proposition 18.24. Strongly causal spacetimes are non-imprisoning.

Proposition 18.25. A spacetime is strongly causal if and only if the Alexandrov topology is equivalent to the manifold topology.

Example of a distinguishing spacetime that is not strongly causal : Carter spacetime $\mathbb{R}^3$ with the identifications

$$\begin{aligned} (t,x,y) &\sim (t,x,y+1) \\ (t,x,y) &\sim (t,x+1,y+z) \end{aligned} \tag{18.5}$$

With $a \in (0,1) \cap (\mathbb{R} \setminus \mathbb{Q})$

$$ds^2 = -c(t)(dt^2 - dx^2) - 2dtdy + dz^2 \tag{18.6}$$

18.7 Almost stably causal spacetime

[paper by E. Howard Causal Stability Conditions for General Relativistic Spacetimes]

- There's at least one open set Q_g such that under small variations of g, large enough to introduce closed causal curves on $\mathcal{M}$, stable causality is still holding outside of Q_g
- The spacetime is globally causally stable and it's possible to widen the light cones on an open region in order to produce closed causal curves.
- Light cones can be open within the causal stable spacetime at a local and macroscopic level

18.8 Reflecting spacetime

Definition 18.26. A spacetime is past (resp. future) reflecting if $I^-(q) \subset I^-(p)$ implies $I^+(p) \subset I^-(q)$ (resp. $I^+(q) \subset I^+(p)$ implies $I^+(p) \subset I^+(q)$). It is reflecting if it is both past and future reflecting.

Proposition 18.27. In a reflecting spacetime, $(I^-(p), I^+(p))$ is a hull pair.

18.9 Stably causal spacetime

Definition 18.28. A spacetime is stably causal if there exists a metric $g' \in \mathrm{Con}(\mathcal{M})$ such that $g < g'$ and g' is causal.

equivalent to : There exists a timelike vector field ξ^μ such that $(\mathcal{M}, g + \xi^\flat \otimes \xi^\flat)$ is chronological
Spacetime is stably causal iff it admits a global time function.

18.10 Causally continuous

$I^+(p) \subset I^+(q) \equiv I^-(q) \subset I^-(p)$
close points have roughly the same past and future
Stably causal spacetime that isn't causally continuous : Minkowski space minus a non-empty closed set

18.11 Causally simple

Definition 18.29. A spacetime is causally simple if it is ccausal and $J^{\pm}(p)$ is closed for every point p.

Theorem 18.30. A causally simple spacetime is distinguishing.

Proof. □

18.12 Globally hyperbolic spacetime

A globally hyperbolic spacetime is the most well-behaved category of spacetimes regarding causality, and most reasonable spacetimes are assumed to be globally hyperbolic. It is roughly defined as having no closed timelike curves or naked singularities (although not all spacetimes with such characteristics are globally hyperbolic). It has a few equivalent definitions :

Definition 18.31. A spacetime is globally hyperbolic if it is causal and $\forall p, q \in \mathcal{M},\ J^+(p) \cap J^-(q)$ is compact.

$J^+(p)$ and $J^-(p)$ are closed for all p
They can also be defined with respect to a Cauchy surface,

Theorem 18.32. A spacetime is globally hyperbolic if and only if it admits a Cauchy hypersurface Σ.

Proof. □

Theorem 18.33. A globally hyperbolic spacetime admits a Cauchy time function.

Proof. □

Geroch's theorem

Theorem 18.34. A globally hyperbolic manifold has the topology $\mathcal{M} \approx \mathbb{R} \times \Sigma$, where Σ is the topology of one of its Cauchy surface.

Proof. □

Theorem 18.35. In a globally hyperbolic spacetime, any points $p, q \in \mathcal{M}$ such that $p \leq q$ can be joined by a causal geodesic with length equal to their time-separation.

18.12.1 Hole-freeness

While global hyperbolicity is sometimes defined informally as corresponding to the absence of naked singularities and closed causal curves, this is not quite the case. First because a spacetime can obey those conditions and still fail to be globally hyperbolic (as we will see, de Sitter space is the most common example), but also because there exists globally hyperbolic spacetimes that do not quite correspond to the intuitive notion of the

Definition 18.36. A spacetime is called hole-free if for any spacelike hypersurface S and for any metric-preserving embedding $\theta : D(S) \to \mathcal{M}'$ into another spacetime $(\mathcal{M}', g')$, $\theta(D(S)) = D(\theta(S))$

18.12.2 Causal closedness

Definition 18.37. A spacetime is causally closed if for every $p \in \mathcal{M}$, $J^{\pm}(p)$ is closed.

Doesn't imply even chronology : Gödel is causally closed

19 Abstract causal spaces

It is possible to describe the causality of a spacetime without any reference to any manifold structure or topology. Those are called causal spaces. There is a number of different models for them.

19.1 Etiological space

Definition 19.1. An *etiological space* is a topological space X equipped with two binary relations $<$ and $\ll$, obeying the axioms that for $p, q \in X$,

1. $p < p$.
2. $<$ and $\ll$ are transitive.
3. if $p \ll q$, then $p < q$.
4. if $p \ll p$, then there exists a q such that $p \neq q$ and $p \ll q \ll p$.
5. The topology of X is a refinement of the Alexandrov topology from $\ll$.

Proposition 19.2. Any time-orientable spacetime is an etiological space.

Proof. If a spacetime is time-orientable, the relations $<$ and $\ll$ are well-defined on it. Then axioms $1, 2, 3$ are from the properties of the causal and chronological ordering of points. Axiom 4 stems from $p \ll p$ implying a closed timelike curve, on which will lie at least one point. Axiom 5 is from property X. □

Definition 19.3. A *causal space* is a quadruple $(X, <, \ll, \nearrow)$ of a set X and three relations on that set, with the axioms that for three points $p, q, r \in X$,

1. $p < p$.
2. if $p < q$ and $q < r$, then $p < r$.
3. if $p \ll q$, then $p < q$.
4. if $p < q$ and $q \ll r$, then $p \ll r$.
5. if $p \ll q$ and $q < r$, then $p \ll r$.
6. $p \nearrow q$ is equivalent to $p < q$ and $p \not\ll q$.

Those are precisely the rules that we derived previously for $<$ and $\ll$. We can then study their properties abstractly in this setting without worrying about the manifold structure behind it.
In addition, we may also add the axioms of causality for specific causal spaces

1. if $p < q$ and $q < p$, then $p = q$.
2. $p \not\ll p$

A few immediate consequences of the axioms of a causal space are

- $<$ is a partial order
- $\ll$ is anti-reflexive and transitive

Proposition 19.4. If $p < q < r$ and $p \nearrow r$, then $p \nearrow q \nearrow r$.

Define inverses $>, \gg, \nwarrow$
Relations on subsets $Y \subset X$, $<_Y$, $\ll_Y$, $\nearrow_Y$.

Definition 19.5. For a finite chain of N points $G = (p_i)_{1 \leq i \leq N}$ ordered by $<$ $(g_i < g_{i+1})$, $N \geq 3$, G is called a girder if for every i, $p_i \nearrow p_{i+2}$.

Girders, hypergirders, etc

19.2 Reinterpretation on causal anomalies

Reichenbach on causal chains, similar vs identical events (p. 141)
If CTCs are multiply connected, going to covering space
Otherwise, what happens with unwrapping
Going to double cover for non-time orientable
Going from GR to Pauli-Fierz $\to$ exchanging CTCs for tachyonic effects

19.3 Theorems on causality

Lemma 19.1. For a closed curve γ in $\mathbb{R}^n$, for every component U^μ of the tangent vector in a coordinate basis, there exists a point $\gamma(\lambda)$ such that $U^\mu = 0$.

Proof. This is a fairly trivial application of Rolle's theorem. If we pick the trivial chart of $\mathbb{R}^n$, then the coordinates of the loop for a component μ just a map $\phi^\mu \circ \gamma : I \to \mathbb{R}$, with $\phi^\mu \circ \gamma(0) = \phi^\mu \circ \gamma(1)$. By Rolle's theorem, there exists a $\lambda \in I$ such that

$$U^\mu = \frac{d}{d\lambda}(\phi^\mu \circ \gamma(\lambda)) = 0 \tag{19.1}$$

□

Theorem 19.6. A static spacetime with trivial topology $\mathcal{M} = \mathbb{R}^n$ is always causal.

Proof. Since the metric of a static spacetime can be expressed as

$$ds^2 = -\alpha(x)dt^2 + g_{ij}(x)dx^i dx^j \tag{19.2}$$

with g_{ij} a Riemannian metric and the coordinate t such that ∂_t is the Killing vector, then by the previous lemma, there exists a point in any closed curve where $U^t = 0$, meaning that at that point, the tangent vector will be

$$g(U, U) = g_{ij} U^i U^j \geq 0 \tag{19.3}$$

Since the Riemannian metric is only 0 for $U = 0$, any non-degenerate curve will have spacelike tangent vectors. □

Corrolary 19.1. Minkowski space is causal.

A related theorem is

Theorem 19.7. A spacetime with trivial topology $\mathcal{M} = \mathbb{R}^n$ and a diagonal metric is always causal.

Proof. Using a similar proof, we have

$$ds^2 = -\alpha(x)dt^2 + \sum_{i=1}^{N} g_{ii}(x)(dx^i)^2 \tag{19.4}$$

where $g_{ii} \geq 0$ to preserve the signature of the metric. Then again for loops, there exists a point where

$$g(U, U) = \sum_{i=1}^{N} g_{ii}(U^i)^2 \geq 0 \tag{19.5}$$

which again implies that no loop can be timelike at every point. □

Lemma 19.2. Any null-homotopic curve γ can be included within a single chart homeomorphic to $\mathbb{R}^2$.

Proof. I STILL DON'T KNOW □

Theorem 19.8. A static spacetime only has non-null homotopic closed causal curves.

Find a proof first that contractible closed timelike curves have to be contained within a single simply co
□

Corrolary 19.2. All static spacetimes have a causal cover.

Theorem 19.9. If a compact spacetime admits a covering space with a compact Cauchy surface, then it contains a closed timelike geodesic. (Tipler)

"Some spacetimes, such as Godel spacetime, do not admit any global time slices. This is a consequence of three features: it is time orientable; a CTC passes through each point; and it is simply connected. The edge of an achronal surface S is the set of points p such that every open neighborhood $p \in O$ includes points in I+(p) and $I^-(p)$ that can be connected by a timelike curve that does not cross S."
Proof : if the space is simply connected, then a boundaryless spacelike hypersurface Σ would split the spacetime into two disconnected pieces $\mathcal{M} \setminus \Sigma$ (Jordan Brouwer theorem, generalized). Components are D_+ (chronological future of Σ and D_- (chronological past). $\mathcal{M}$ is time oriented : timelike vector field everywhere continuous $\to$ timelike vector field everywhere oriented the same way on the surface wrt the normal vector
If $p \in \Sigma$ and γ a future-oriented CTC, then at p, $\dot{\gamma}$ points to D_+. When γ intersects with Σ again, $\dot{\gamma}$ points the way opposite to the time orientation vector field, which contradicts that the spacetime is time oriented.

Theorem 19.10. For $n > 2$, two points of an n-dimensional Lorentz manifold can always be connected by a spacelike curve.

Proof. Helicoidal curve between the two points in the tangent space □

20 Spacetime boundaries and asymptotic behaviours

20.1 Conformal compactification

Conformal compactification : Embedding of a non-compact spacetime into a compact one as a dense open subset, such that the embedding is a conformal map
The conformal compactification of a spacetime $(\mathcal{M}, g)$ is an embedding of a conformal spacetime $(\mathcal{M}, \Omega g)$ into a larger spacetime $(\hat{\mathcal{M}}, \hat{g})$ such that $\mathcal{M}$ is an open submanifold of $\hat{\mathcal{M}}$, with a boundary $\partial\mathcal{M} = \mathscr{I}$, and the conformal factor can be smoothly extended in $\hat{\mathcal{M}}$ with $\Omega(\mathscr{I}) = 0$, $d\Omega(\mathscr{I}) \neq 0$

20.1.1 Conformal structure of Minkowski space

The basic example of this is to consider 4-dimensional Minkowski space. The easiest way to get its structure at infinity is to switch to spherical coordinates, which will allow us to consider the limit $r, t \to \infty$.

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega \tag{20.1}$$

with the use of null coordinates, $u = t - r$ and $v = t + r$

$$ds^2 = -dudv + (\frac{v-u}{2})^2 d\Omega \tag{20.2}$$

Take the conformal metric

$$\hat{g} = \frac{g}{(1+u^2)(1+v^2)} \tag{20.3}$$

Coordinate transformation $u = \tan(p)$, $v = \tan(q)$, $p, q \in [-\pi/2, \pi/2]$

$$ds^2 = -dudv + (\frac{\sin(p-q)}{2})^2 d\Omega \tag{20.4}$$

Then finally $T = (p+q)/2$ and $\rho = q - p$, $\rho \in [-\pi, \pi]$

$$d\hat{s} = -dT^2 + \frac{1}{4}[d\rho^2 + \sin^{n-2}(\rho) d\Omega] \tag{20.5}$$

If we maximally extend this spacetime, we get $\mathbb{R}_t \times S^3$.
By suppressing the angular coordinates, we can represent this spacetime as a diagram. This is called a Penrose or conformal diagram. i^0, i^+ and i^- represent spatial, future temporal and past temporal infinity, $\mathscr{I}^+$ and $\mathscr{I}^-$ are future and past null infinity.

20.2 Spacetime boundaries

If we perform a conformal compactification on a non-compact spacetime, we'll end up with a few different points at infinity in the boundary $\mathscr{I}$. We may also wish to include the points corresponding to singularities in the boundary, as we did with the g-boundary and b-boundary. All those points will be referred to as *ideal points*.
There are many different method to classify ideal points and give them a topology, each with their own benefits.

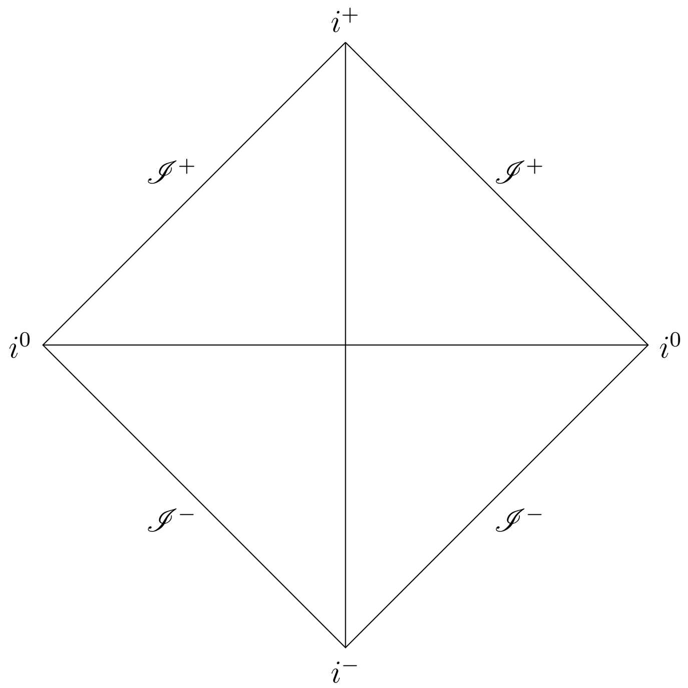


Figure 9: Conformal diagram of Minkowski space

20.2.1 The GKP construction

The Geroch-Kronheimer-Penrose boundary construction is performed by identifying points of spacetime with causal sets, which form a larger set than the spacetime itself, the remaining points forming the boundary. The basis for this construction is to associate each point with its chronological future and past, which will only be well-defined if the spacetime is distinguishing, so that if $p \neq q$, then $I^\pm(p) \neq I^\pm(q)$.
The relevant properties to extend this are that

- $I^\pm(p)$ is open.
- $I^\pm(p)$ is a past/future set.
- It cannot be expressed by the union of two proper subsets with the same properties, since for any proper open subset $S \subset I^\pm(p)$, $I^\pm(S) \subset I^\pm(p)$.

We then define the generalized sets with those properties. As usual, any proof relating to past versions of sets will generalize to the future verisons by switching time orientation.

Definition 20.1. A future (resp. past) set is called *irreducible* if it is not empty and is not the union of two proper subsets which are themselves future (resp. past) sets. They are noted respectively IF and IP.

The set of all IP on $\mathcal{M}$ will be noted $\hat{\mathcal{M}}$, while the set of all IF will be noted $\check{\mathcal{M}}$.

Proposition 20.2. A subset $S \subset \mathcal{M}$ is an IP if and only if there exists a future-directed timelike curve γ such that $S = I^-(\gamma)$.

Proof. If we have a timelike curve γ, suppose that $I^-(\gamma) = A \cup B$, with A, B past sets. $I^-(\gamma)$ will be an IP if for all A, B, either $A \subset B$ or $B \subset A$, otherwise we will have both

points $q \in A \setminus B$ and $r \in B \setminus A$. Since $q, r \in I^-(\gamma)$, there are points $q', r' \in \gamma$ such that $q \in I^-(q')$ and $r \in I^-(r')$. Since our spacetime is distinguishing and γ is a timelike curve, we must have either $q' \ll r'$, $r' \ll q'$ or $q' = r'$. Which means that if A or B contains the futuremost point of q' and r', ît will also contain both q and r. By contradiction, $I^-(\gamma)$ is an IP.
Conversely, for P an irreducible past set, consider $p \in P$. Since P is a past set, we have

$$P = I^-(P \cap I^+(p)) \cup I^-(P \setminus I^+(p)) \tag{20.6}$$

Since it is irreducible, either $P = I^-(P \cap I^+(p))$ or $P = I^-(P \setminus I^+(p))$, but since $p \notin I^-(P \setminus I^+(p))$,

$$P = I^-(P \cap I^+(p)) \tag{20.7}$$

meaning that if we pick a point $q \in P$, then $q \in I^-(P \cap I^+(p))$, so that there exists a point $r \in P \cap I^+(p)$ such that $q \ll r$, which cannot be p or q since our spacetime is distinguishing. As we have $q \ll r$ and $p \ll r$, this means that for any two points in P, there is a point which lies to the future of both.
If we pick a countably dense family of points $p_i \in P$, and $q_0 \in P$ such that $p_0 \ll q_0$. We can then pick a point q_1 to the future of both q_0 and p_1, and then iteratively a point q_i to the future of both q_{i-1} and p_i. As $\{q_i\}$ is ordered by $\ll$, we can form a timelike curve γ with appropriate smoothing. The set $P \cap I^+(p_i)$ will be open in P, and since they are dense, there exists some $k \in \mathbb{N}$ such that $p_k \in P \cap I^+(p_i)$. $p_k \in I^-(q_k)$, so $p_i \in I^-(\gamma)$. Which means that $P \in I^-(\gamma)$. □

Once we have irreducible sets, we can classify them between the ones that will correspond to spacetime points and the ones that will correspond to ideal points.

Definition 20.3. Future (resp. past) sets are classified as either *proper irreducible future* (resp. past) sets if they can be expressed by $I^\pm(p)$, or *terminal irreducible future* (resp. past) sets if they cannot. They are noted as PIF, PIP, TIF and TIP.

Proposition 20.4. A set S is a TIP if and only if there is a future-inextendible timelike curve γ such that $I^-(\gamma) = S$. It is a PIP if and only if there is a future timelike curve γ with endpoint p such that $S = I^-(\gamma)$.

Proof. If there is a curve γ such that $I^-(\gamma) = S$ □

In Minkowski space :

Proposition 20.5. Minkowski space has two classes of TIP/TIF : the ones generated by inextendible curves of bounded acceleration and the ones generated by inextendible curves of unbounded acceleration.

Proof. Show that curves of bounded accelerations have $I^\pm(\gamma) = \mathcal{M}$, but this is not true for unbounded accelerations. □

The set of TIP/TIF of curves of constant acceleration will be denoted $\mathscr{I}^-/\mathscr{I}^+$, called past null infinity and future null infinity, while the TIP/TIF composed of the entire manifold will be denoted i^-/i^+, called past (timelike) infinity and future (timelike) infinity.
The chronological past and future form an injection from the spacetime to the set of past and future irreducible sets.

$$
\begin{aligned}
I^+ &: \mathcal{M} \to \check{\mathcal{M}} \\
I^- &: \mathcal{M} \to \hat{\mathcal{M}}
\end{aligned} \tag{20.8}
$$

the set of PIF is $I^+(\mathcal{M})$, the set of PIP is $I^-(\mathcal{M})$.
With irreducible sets, we can form the completion of the manifold $\bar{\mathcal{M}}$, such that $\mathcal{M} \subset \bar{\mathcal{M}}$. The basis of this completion is to consider the set $\check{\mathcal{M}} \cup \hat{\mathcal{M}}$. Since each element of M corresponds to two elements of $\check{\mathcal{M}} \cup \hat{\mathcal{M}}$, $I^+(p)$ and $I^-(p)$, we have to perform some identifications. The most obvious one is that $\check{p} \sim \hat{p}$ if $\check{p} = I^-(p)$ and $\hat{p} = I^+(p)$, but things are more complicated for terminal sets, as
$\mathcal{M}^\natural = (\check{\mathcal{M}} \cup \hat{\mathcal{M}})/\sim$
Example where two terminal sets correspond to the same ideal point
If $\mathcal{M}$ is in addition strongly causal, the manifold topology is the Alexandrov topology, so that we can extend the Alexandrov topology

Definition 20.6. The *extended Alexandrov topology* is the coarsest topology such that the following sets are always open for $A \in \check{\mathcal{M}}$, $B \in \hat{\mathcal{M}}$:

$$
\begin{aligned}
A^{\text{int}} &= \{P^* \in M^\natural | P \in \hat{\mathcal{M}}, P \cap A \neq \varnothing\} \\
A^{\text{ext}} &= \{P^* \in M^\natural | P \in \hat{\mathcal{M}}, \forall S \subset \mathcal{M}, P = I^-(S) \to I^+(S) \not\subset A\} \\
B^{\text{int}} &= \{P^* \in M^\natural | P \in \check{\mathcal{M}}, P \cap B \neq \varnothing\} \\
B^{\text{ext}} &= \{P^* \in M^\natural | P \in \check{\mathcal{M}}, \forall S \subset \mathcal{M}, P = I^+(S) \to I^-(S) \not\subset B\}
\end{aligned} \tag{20.9}
$$

$\mathcal{M}^\natural$ with the extended Alexandrov topology has a topology that isn't necessarily Hausdorff. Equivalence relation R_H by the intersection of all equivalence relations $\subset \mathcal{M}^\natural \times \mathcal{M}^\natural$ such that $\mathcal{M}^\natural / R$ is Hausdorff. Then we have

$$
\bar{M} = M^\natural / R \tag{20.10}
$$

Proposition 20.7. If $\mathcal{M}$ is strongly causal, the image of $p \in \mathcal{M}$ in $\mathcal{M}^\natural$ will never be identified with the image of another point $q \in \mathcal{M}$.

c-boundary : $\bar{\mathcal{M}} \setminus \mathcal{M}$

20.2.1.1 Budic and Sachs construction

Since for the set $\check{\mathcal{M}} \cup \hat{\mathcal{M}}$, we want to identify at least $I^+(p)$ with $I^-(p)$, it will be interesting to consider reflecting spacetimes, for which the pair $(I^-(p), I^+(p))$ forms a hull set

20.3 Asymptotic flatness

Once we have the boundaries of our spacetime, it is possible to define its asymptotic behaviour by observing how it behaves on the boundary.
A spacetime is asymptotically simple if every null geodesic in $\mathcal{M}$ has future and past endpoints on the boundary of its conformal compactification $\hat{\mathcal{M}}$.

A spacetime is weakly asymptotically simple if there is an open set $U \subset \mathcal{M}$ isometric to a neighbourhood of the boundary of the conformal compactification $\hat{\mathcal{M}}$ of an asymptotically simple spacetime.
A spacetime is asymptotically flat if it is weakly asymptotically simple and its Ricci tensor vanishes in some neighbourhood of the boundary of $\hat{\mathcal{M}}$.
Differences between asymptotic flatness at null infinity, spatial infinity or both
Coordinate version :
A globally hyperbolic spacetime is asymptotically flat if there is a contractible compact set C
outside of C, $g = \eta + h$

$$\begin{aligned}
\lim_{r\to\infty} h_{\mu\nu} &= \mathcal{O}(r^{-1}) \\
\lim_{r\to\infty} h_{\mu\nu,\rho} &= \mathcal{O}(r^{-2}) \\
\lim_{r\to\infty} h_{\mu\nu,\rho\sigma} &= \mathcal{O}(r^{-3})
\end{aligned}$$

20.4 More general asymptotic behaviour

Since quite a wide class of physically important spacetimes are not asymptotically flat, and the question of isolated objects within the is important, we broaden the definition of asymptotic behaviours to include more asymptotic spacetimes.
Asymptotically anti de Sitter
Asymptotically FRW

21 Horizons and trapped surfaces

Apparent horizons, trapped surfaces, event horizons, particle horizons, Hubble horizon, marginally trapped tubes, trapping horizons, future outer trapping horizons, dynamical horizons, etc

21.1 Horizons

The general notion of a horizon, as defined by Rindler [26], is a frontier between observables and unobservable things.
Horizons are generally defined with respect to observers with future-directed timelike curves γ, to correspond to the notion of a physical observer. This means that horizons are not entirely defined with respect to the spacetime itself, and even the most benign spacetimes such as Minkowski space possess families of observers with horizons.

21.1.1 Event horizons

GKP : Event horizons are $\approx \partial TIP$ while the particle horizon is $\approx \partial TIF$.
In a time-oriented spacetime, the (future) event horizon of an observer with timelike curve γ is the boundary beyond which any point will never be observed by that observer. This means that no causal curve from any point q beyond the horizon will be able to reach any point of γ.
Hence to define the horizon, we will first consider the set of all points that can be observed, all points q such that $J^+(q) \cap \gamma \neq \varnothing$, or in other words, $J^-(\gamma)$. The horizon will then be the boundary of that set,

$$\mathrm{EvtHor}^+(\gamma) = \partial J^-(\gamma) \tag{21.1}$$

In a similar manner, we will define the past event horizon, comprised of all points that can never be influenced by the observer γ. It can be easily shown to be

$$\mathrm{EvtHor}^-(\gamma) = \partial J^+(\gamma) \tag{21.2}$$

The set of future and past event horizons may be disconnected, in which case we say that every connected component of that set is an individual horizon.

Proposition 21.1. If for two timelike curves γ_1, γ_2, we have that for any $p \in \gamma_1$ and $q \in \gamma_2$, $I^+(p) \cap \gamma_2 \neq \varnothing$ and $I^+(q) \cap \gamma_1 \neq \varnothing$, then

$$\mathrm{EvtHor}(\gamma_1) = \mathrm{EvtHor}(\gamma_2) \tag{21.3}$$

Proof. It suffices to show that the two sets defined by $\{q | J^+(q) \cap \gamma_{1,2} \neq \varnothing\}$ are identical. If this is true for the first observer, then there is a future-directed causal curve from q that intersects γ_1. If we call this intersection point p, by our assumptions, we have that there exists a future-directed timelike curve from p to a point r of γ_2. From the properties of timelike curves,

$$q \leq p,\ p \ll r \to p \leq r \tag{21.4}$$

Hence the two observers share the same event horizon. □

This will allow us to define the event horizon of entire subsets of the spacetime for some given class of observers.

Example 21.2. Rindler observers in Minkowski space possess an event horizon, called the Rindler horizon.

Proposition 21.3. In an asymptotically flat spacetime, the horizon for an observer going from i^- to i^+ is $\partial I^-(\mathscr{I}^+)$

show that if a particle passes a future horizon, it never leaves it, and if it passes a past horizon, it has never been outside of it.

21.1.2 Black holes

Definition 21.4. In an asymptotically flat spacetime, a *black hole* is the remainder of the spacetime minus the past of future null infinity

$$\mathcal{M} \setminus I^-(\mathscr{I}^+) \tag{21.5}$$

A black hole corresponds to a region where no particle can escape to infinity.
We have similarly the definition of a white hole

Definition 21.5. In an asymptotically flat spacetime, a *white hole* is the remainder of the spacetime minus the future of past null infinity

$$\mathcal{M} \setminus I^+(\mathscr{I}^-) \tag{21.6}$$

Boundary of black and white holes are horizons???

21.1.3 Particle horizon

21.1.4 Non-local horizons

Particle horizon : $E^+(p)$ for an observer at p

21.2 Trapped surfaces

spacelike surface of co-dimension two
Embedding

$$\Phi : S \to \mathcal{M} \tag{21.7}$$

in coordinates :

$$x^\mu = \Phi^\mu(y^a) \tag{21.8}$$

x coordinates on $\mathcal{M}$, y coordinates on S.
Pushforward of $\partial_a \in TS$ to vectors of $T\mathcal{M}$

$$e_a = \Phi_*(\partial_a) \to e^\mu{}_a = \frac{\partial \Phi^\mu}{\partial \lambda^a} \tag{21.9}$$

First fundamental form γ:

$$\gamma = \Phi^* g \to \gamma_{ab}(\lambda) = g|_S(e_A, e_B) = g_{\mu\nu}(\Phi(\lambda)) e^\mu{}_a e^\nu{}_b \tag{21.10}$$

S spacelike : γ Riemannian.

$$T_p\mathcal{M} = T_pS \oplus T_pS^\perp \tag{21.11}$$

Definition 21.6. A future (resp. past) closed trapped surface is a closed spacelike surface with an area that decreases locally along any possible future (resp. past) direction.

Causal orientation of $\vec{H}$:
$\downarrow$: past-pointing timelike $\searrow$ or $\swarrow$: past-pointing null ($\vec{k}^{\pm}$) $\leftarrow$ or $\rightarrow$: spacelike $\cdot$: vanishes $\nearrow$ or $\nwarrow$: future-pointing null $\uparrow$: future-pointing timelike

22 Topology change and Lorentz cobordism

Topology change corresponds to the notion of the topology of space changing between two moments. The basic idea being that we have a topology change if we have two boundaryless achronal spacelike hypersurfaces that are not diffeomorphic. To make this rigorous, we will have to use the notion of cobordisms.

22.1 Cobordisms

A cobordism is an n-dimensional manifold with boundaries with two $(n-1)$-dimensional manifolds as boundaries Σ_1 and Σ_2.
Lorentz cobordism : $(\mathcal{M}, v)$ is a Lorentzian manifold with boundaries $\mathcal{M}$ and a vector field v such that $\partial M = \Sigma_1 \sqcup \Sigma_2$ and v is interior normal on Σ_1 and exterior normal on Σ_2.
M_1, M_2 are spacelike hypersurfaces
$\mathcal{M}$ is a "cutout" of a larger spacetime along Σ_1 and Σ_2. We can extend $\mathcal{M}$ into a manifold without boundaries
Example of non-trivial Lorentz cobordisms :
Trouser spacetime : topology $S^2 \setminus (D^2 \sqcup D^2 \sqcup D^2)$, cobordism $M_1 = S^1$, $M_2 = S^1 \sqcup S^2$

23 Continuous metrics

For most of this part, we have assumed the metric to be smooth, or at least C^2.
Theorems if the metric is C^0.
No convex normal neighbourhood if not $C^{1,1}$

23.1 Causality

Definition 23.1. A curve is said to be locally uniformly timelike if there is a smooth Lorentz metric $\check{g} < g$ such that $\check{g}(\gamma', \gamma') < 0$

Definition 23.2. The chronological future $\check{I}_g^+(U, V)$ is the set of locally uniformly timelike curves starting at $p \in U$ and ending at $q \in V$.

Causal bubbles
Causally plain

24 Exotic spacetimes

As we've seen in the chapter on manifold, it is possible for a single manifold to have more than one smooth structure. This will lead to important differences in the properties of a spacetime.
While it is not a perfectly rigorous definition, we will say that a spacetime is exotic if its smooth structure is not the standard one (the lack of rigor is that we cannot specify for every manifold which one is standard). In other words, if we have a family of homeomorphic manifolds with non-diffeomorphic smooth structures, if one of them can be called the standard smooth structure, the others will be exotic.

Proposition 24.1. There are no exotic spacetimes for $n < 4$

24.1 Exotic $\mathbb{R}^4$

The most well-known spacetime example is to use one of the so-called fake $\mathbb{R}^4$, noted $\mathbb{R}^4_\Theta$, as it is the only example of $\mathbb{R}^n$ with more than one smooth structure and also happens to be of the dimension of physical space.

24.1.1 Construction of exotic $\mathbb{R}^4$

Theorem 24.2. Any open 4-manifold M with $\pi_1(M) = 0$, $H_2(M) = 0$ and end collared (topologically) by $S^3 \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.

24.1.2 Properties

Definition 24.3. A manifold $\mathbb{R}^4_\Theta$ is called *small* if it can be smoothly embedded as an open subset of $\mathbb{R}^4$. It is called *large* otherwise.

Example : Donaldson-Freeman $\mathbb{R}^4$, noted $\mathbb{R}^4_{DF}$

Proposition 24.4. $\mathbb{R}^4_\Theta$ has the tangent bundle $\mathbb{R}^8$.

Proposition 24.5. $\mathbb{R}^4_\Theta$ is the product $\mathbb{R} \times \mathbb{R}^3$ without the product topology.

To convey the different topology, we will write it as $\mathbb{R} \times_\Theta \mathbb{R}^3$

Proposition 24.6. There's no geodesically complete Riemannian metric on $\mathbb{R}^4_\Theta$ with $R \leq 0$.

25 Non-Hausdorff manifolds

For various reasons, pertaining to either extensions of spacetimes, causality, quantum mechanics or metaphysics, spacetime manifolds are sometimes allowed to not obey the Hausdorff property.

25.1 Properties of non-Hausdorff manifolds

[REMEMBER TO CHECK ALBERTO S. CATTANEO FOR VECTOR FIELDS]
[If M is Hausdorff and second-countable, so is TM If an atlas is second countable Hausdroff so is its maximal atlas] [example of a non-Hausdorff manifold with a non-closed compact set] [a non-Hausdorff manifold Q need not admit any nonconstant C1 functions or one-forms (Wazewski [59]; [34]).] [A one dimensional manifold of class C^r is regular if any germ of a C^r function at any point q is the germ at q of a global C^r function. Every Hausdorff manifold is regular.
From Haefliger and Reeb : A regular, simply connected, one dimensional manifold of class C^r admits a C^r function with nowhere vanishing differential.] [An oriented one dimensional manifold of class C^r which is regular and has at most finitely many double branch points (and no multiple branch points) admits a nowhere vanishing differential one-form of class C^{r-1}.
Proof: Fix an orientation, let p_i, q_i all pairs of branch points (p_i, q_i). For every i we choose an open neighbourhood U_i of p_i and an orientation preserving C^r diffeomorphism $h_i : U_i \to I = (-1, 1)$. By regularity, we can assume that h_i extends to a C^r function $h_i : M \to \mathbb{R}$, the extended function has non-zero derivative at q_i. Thus $\theta_i = dh_i$ is a non-vanishing one-form of class C^{r-1} in an open connected neighbourhood V_i of the pair (p_i, q_i). We may assume that the closures $\bar{V}_i$ are pairwise disjoint.
Choose a smaller compact neighbourhood $E_j \subset\subset V_j$ of (p_i, q_i) such that $V_i \subset E_i$ is a union of finitely many segments, none of them relatively compact in V_i. Set $V = \bigcup V_i$, $E = \bigcup E_i$, then $M_0 = M \backslash E$ is an open, one dimensional, paracompact, oriented Hausdorff manifold, hence a union of open segments and one-spheres. Any one-sohere in M_0 is a connected component of M. Choosing a non-vanishing one-form on it (in the correct orientation class) does not affect the choices that we shall make on the rest of the set. We do the same on any open segment of M_0 which is a connected component of M.
Open segments of M_0 which intersect at least one of the sets $V_i \backslash E_i$. Choose such a segment J and an orientation preserving parametrization $\phi : I \to J$. $I' = \{t \in I | \phi(t) \in V\}$, then I' consists of either one or two subintervals of I, each of them having an endpoint at -1 or $+1$. Each of these subintervals is mapped by ϕ onto a segment in some $V_j \setminus E_i$ (other possibilities would require a branch point of M_0 but it is Hausdorff)
Consider the case where $I' = I_0 \cup I_1$ where $I_0 = (-1, a)$, $I_1 = (b, 1)$ for a pair of points $-1 < a \leq b < 1$. Let j and l be chosen such that $\phi(I_0) \subset V_j$, $\phi(I_1) \subset V_l$ (we might have $j = l$). Note that $\phi(a)$ is an endpoint of $V(j)$ and $\phi(b)$ is an endpoint of V_l. Then $\phi^*\theta_j = d(h_j \circ \phi)$ resp $\phi^*\theta_l = d(h_l \circ \phi)$ are non-vanishing one-forms on I_0 resp. I_1, both positive with respect to the standard orientation of $\mathbb{R}$. Choose a one-form τ on $(-1, 1)$ which agrees with the above forms near the respective end points -1 and $+1$ (obviously such a τ exists). Then $(\phi^{-1})^*\tau$ is a one-form on $J = \phi(I) \subset M_0$ which agrees with θ_j in a neighbourhood of E_j and with θ_l in a neighbourhood of E_l. Similarly we deal with the case that J intersects only one of the set V_j. Performing this construction for each of the

finitely many segments $J \subset M_0$ which intersect V we obtain a non-vanishing one-form on M.
]

25.2 A few non-Hausdorff manifolds

Since the Hausdorff property is preserved under homeomorphism, and $\mathbb{R}^n$ is itself Hausdorff, any coordinate chart $U_\alpha = \phi_\alpha^{-1}(O_\alpha)$ of a manifold will itself be Hausdorff. Meaning that any non-Hausdorff manifold will get this property from its transition maps (any non-Hausdorff manifold will then have at least two coordinate charts).
A common method to generate non-Hausdorff manifolds from this is the use of a gluing function, where given two manifolds M_1 and M_2, we define the gluing function

$$\phi : A \overset{o}{\subset} M_1 \to B \overset{o}{\subset} M_2 \tag{25.1}$$

Proposition 25.1. If the gluing function ϕ is a C^k homeomorphism, the atlas $\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_2$ defines a C^k manifold, with the transition functions between $\mathcal{A}_1$ and $\mathcal{A}_2$ being

$$\tau_{\alpha_1\alpha_2} = \tag{25.2}$$

Proposition 25.2. The glued manifold is equivalent to $M = (M_1 \sqcup M_2)/\sim$, with $p_1 \sim p_2$ if $p_2 = \phi(p_1)$.

In particular, we can take the disjoint unions of identical copies of the same manifold

$$M^A = \bigsqcup_{i\in A} M_i = \bigcup_{i\in A} \{i\} \times M \tag{25.3}$$

And then glue them along an open set $S \subset M$ with the identity function $\{i, p\} = \phi_{ij}(\{j, p\})$. We then get

Theorem 25.3. The manifold $M' = M^A/\sim_S$ is not Hausdorff if $S \neq M$.

Proof. If we consider the points on the boundaries ∂S_i, then □

Gluing a manifold to itself
With those methods, we can define the following useful examples.

25.2.0.1 The line with two origins

The line with two origins corresponds to the case where $A = \{1, 2\}$ and $S = \mathbb{R} \setminus \{0\}$, with boundary point $\partial S = \{0\}$. It will be noted $\mathbb{R}_{\div}$ in this book. It has the manifold structure given by the charts $(\mathbb{R}_1, \phi_1 = \mathrm{Id}), (\mathbb{R}_2, \phi_2 = \mathrm{Id})$, with the overlap $\mathbb{R}_1 \cap \mathbb{R}_2 = \mathbb{R} \setminus \{0\}$ and transition map $\tau_{12} = \mathrm{Id}_{\mathbb{R}_1 \cap \mathbb{R}_2}$.

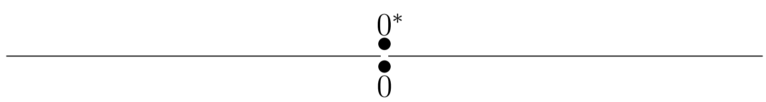


Figure 10: Representation of the line with two origins

The two adjacent points, $\{1\} \times \{0\} \in \mathbb{R}_1$ and $\{2\} \times \{0\} \in \mathbb{R}_2$, are generally written as 0 and 0^*.
The identification of adjacent points just give us the usual real line $\mathbb{R}$.

25.2.0.2 The splitting real line

The splitting real line is the gluing of two copies of the real line, with $S = \{x | x < 0\}$, and boundary point $\partial S = \{0\}$. It will be noted $\mathbb{R}_Y$ in this book. It has the topology given by the atlas $(\mathbb{R}_1, \mathrm{Id}_1), (\mathbb{R}_2, \mathrm{Id}_2)$
The identification of adjacent points give us a Y-shaped topological space, that is Hausdorff but not a manifold.

25.2.0.3 The noose

The noose, noted $\mathbb{R}_{\varrho}$, is the gluing of the real line with itself, with the identification $x \sim y$ if $x = -y$ for $x, y \in (\infty, -1) \cup (1, \infty)$.
It is described by the coordinate patches $O_1 = (-1, \infty)$ and $O_2 = (-\infty, 1)$, with the overlap $\varphi_1(U_1 \cap U_2) = (1, \infty)$ and $\varphi_2(U_1 \cap U_2) = (-\infty, -1)$, along with the transition maps $\tau_{12} = \tau_{21} = -x$ on the overlap.

Proposition 25.4. The noose manifold isn't orientable.

Proof. Since we only have two charts, then we just need the Jacobian of τ, which is simply -1. □

The lack of orientability can be seen fairly easily by trying to define a volume form on the manifold, since any nowhere vanishing vector field will fail to properly join up on the overlap. This is a peculiarity of non-Hausdorff manifolds : they may fail to be orientable even in one dimension. This gives us the occasion of seeing how the lack of orientability affects the structure of the tangent bundle, since it is just a 2-manifold. As we know, the tangent bundle fails to be trivial if the manifold isn't orientable. For the noose, the tangent bundle is the manifold defined by the topology

$$x \tag{25.4}$$

25.2.0.4 The feather

The feather is constructed from the disjoint union of a continuum of copies of the real like

$$\bigsqcup_{x \in \mathbb{R}} \{x, \mathbb{R}\} \tag{25.5}$$

with the identification $p_{x_1} \sim p_{x_2}$ if for $p_1 = \{x_1, y_1\}$, $p_2 = \{x_2, y_2\}$, we have
Real line splitting at every point
It should be noted that splitting manifolds can be represented as actual splits, but it should be remembered that at branching points, there are no neighbourhoods containing points from both branches, unlike for the wedge sum of two lines.

25.2.1 Topological properties

Despite not being separated, non-Hausdorff manifolds still inherit from their locally $\mathbb{R}^n$ structure the structure of a T_1 space, that is, for $p, q \in M$, there exists a neighbourhood U_p, U_q with $U_p \cap U_q \neq \varnothing$.

Theorem 25.5. Every manifold M is T_1 (from "Non metrizable manifold").

Proof. Take any $p, q \in M$, $p \neq q$, with a coordinate patch $U \ni p$. If $q \notin U$, then we are done. If $q \in U$, consider the points $\phi(p), \phi(q)$ in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is itself T^1, there is an open set $O \subset \mathbb{R}^n$ that contains $\phi(p)$ but not $\phi(q)$. Hence $\phi^{-1}(O)$ contains p but not q. □

The main difference between Hausdorff manifolds and non-Hausdorff manifolds is the existence of points with overlapping open sets. This is expressed by the notion of adjacency.

Definition 25.6. For a cardinal number α, $p \in M$ is *α-adjacent* to $q \in M$ (noted $p \curlyvee^\alpha q$) if for each neighbourhood U and V of p and q, we have $\mathrm{Card}(U \cap V) \geq \alpha$. If $\alpha = 1$, we also say that p is *adjacent* to q, and it is noted $p \curlyvee q$.

$p \curlyvee q$ means that there is at least a single point in common with every open neighbourhood of both p and q. If p is not adjacent to q, we say that p is apart from q.

Proposition 25.7. $\curlyvee$ is reflexive and symmetric, but not transitive.

Proof.

- Reflexive : In two open neighbourhoods $U_1, U_2 \ni p$, we always have $p \in U_1 \cap U_2$, hence $p \curlyvee p$.
- Symmetric : by the commutativity of the intersection.
- Not transitive : If $p \curlyvee q$ and $q \curlyvee r$, we have for all open neighbourhoods $U_p \ni p$, $U_q \ni q$, $U_r \ni r$, $U_p \cap U_q = U_{pr}$ and $U_q \cap U_r = U_{qr}$, both non-empty sets.

□

Hence $\curlyvee$ is a tolerance relation.
A point is called regular if there are no adjacent points besides itself. Otherwise it is singular. The number of points adjacent to a singular point is the singularity number.
Y_M^N is the set of all points $x \in M$ such that for $y \in N$, $x \curlyvee y$.

Proposition 25.8. If $p \curlyvee q$, then there is no coordinate patch containing p and q.

Proof. As the Hausdorff property is preserved by homeomorphism, and $\mathbb{R}^n$ is Hausdorff, $U_\alpha = \phi_\alpha^{-1}(O_\alpha)$ is Hausdorff, so no points in it are adjacent. □

Theorem 25.9. For a gluing of identical copies of a manifold along a set S, the set of boundary points ∂S of S are all adjacent in $M^A / \sim_S$.

Proof. □

Corrolary 25.1. If S has a non-empty boundary, the gluing manifold will be non-Hausdorff.

It is possible to construct a paracompact non-Hausdorff manifold that isn't second-countable, simply by considering a disjoint union of an uncountable number of copies of $\mathbb{R}$ before identifying $x \in \mathbb{R}, x \neq 0$. Any basis will require at least one open set around each 0, which will require at a basis with the same cardinality.

Proposition 25.10. A convergent sequence (x_n) in a Hausdorff space has a unique limit. This isn't necessarily true for non-Hausdorff manifolds

Proof. Suppose we have

$$\begin{aligned}\lim_{n\to\infty} x_n &= p \\ \lim_{n\to\infty} x_n &= q\end{aligned}$$

with $p \neq q$. By the Hausdorff property, there are two neighbourhoods U and V such that $p \in U$ and $q \in V$ and $U \cap V = \varnothing$.
By the definition of convergence, for a N large enough, it should be that for all $n \geq N$, $x_n \in U$ and $x_n \in V$, which contradicts the fact that their intersection is empty. □

Theorem 25.11. If a manifold constructed by gluing Hausdorff manifolds has a continuously extendable gluing, it has non-unique limits.

While not all non-Hausdorff manifolds violate the uniqueness of sequence limits, the ones we will be concerned with will, making it so that analysis cannot be defined globally on those manifolds. Many other properties usually taken for granted for manifolds will likewise fail (in particular, they are not metrizable).
"Every manifold that is defined as a subset of $\mathbb{R}^n$ by the implicit function theorem inherits from $\mathbb{R}^n$ the property of being Hausdorff and second countable."
"Let $F : N \to M$ be an injective immersion. If M is Hausdorff, then every point p in N has an open neighborhood U such that $F|_U$ is an embedding."

Proposition 25.12. If $p \curlyvee q$, then for any sequence $\{p_i\}$, $\lim p_i = p$, we have $\lim p_i = q$.

Proof. □

Theorem 25.13. For any continuous function f on a non-Hausdorff manifold M, if $p \curlyvee q$ then $f(p) = f(q)$

Proof. By the definition of continuity, we have that for every point $p \in M$,

$$\lim_{x\to p} f(x) = f(p)$$

Since we also have that any sequence convering to q will also converge to q,

$$\lim_{x\to p} f(x) = \lim_{x\to q} f(x) = f(p) = f(q) \tag{25.6}$$

□

Definition 25.14. A *bifurcate curve* is a family of curves γ_i defined on I such that either

$$\gamma_i([0,c)) = \gamma_j([c,1]),\ \gamma_i([0,c]) \neq \gamma_j((c,1]) \tag{25.7}$$

or

$$\gamma_i([0,c]) = \gamma_j((c,1]),\ \gamma_i([0,c)) \neq \gamma_j((c,1]) \tag{25.8}$$

25.2.2 Structures on non-Hausdorff manifolds

Theorem 25.15. Non-Hausdorff manifolds are not metrizable.

Proof. By the Nagata-Smirnov metrization theorem, a topological space X is only metrizable □

Lemma 25.1. Non-Hausdorff manifolds do not have bump functions for every open neighbourhoods.

Proof. If we consider two adjacent points p, q, there exists two different open sets in the manifold atlas U_p, U_q such that $q \notin U_p$ and $p \notin U_q$. For any bump function f defined on U_p, we have by continuity that $f(p) = f(q)$, hence the function does not have compact support in U_p. □

Theorem 25.16. Non-Hausdorff manifolds do not admit a partition of unity.

Proof. □

Lemma 25.2. A manifold M is Hausdorff if and only if the diagonal

$$\Delta_M = \{(p,p) \in M \times M\} \tag{25.9}$$

is closed in $M \times M$.

Proposition 25.17. The vector flow of a vector field is not unique for a given vector field X.

Proof. Any curve passing via a singular point of the manifold can go through any of the adjacent point If $\gamma : I \to M$ and $\gamma' : J \to M$ are two curves, then the set

$$K = \{\lambda \in I \cap J | \gamma(\lambda) = \gamma'(\lambda)\} \tag{25.10}$$

set where the two curves agree is closed in $I \cap J$ if M is Hausdorff, as it is the preimage of Δ_M under the map $\lambda \to (\gamma(\lambda), \gamma'(\lambda))$.
If γ, γ' are integral curves of X, then K is open in $I \cap J$. K is a clopen interval, so it must be all of $I \cap J$, so uniqueness. □

Example 25.18. For $\mathbb{R}_{:}$, we have two coordinate charts $\phi_0(\{0,x\}) = x$ and $\phi_1(\{1,x\}) = x$, with $\phi_0 \circ \phi_1^{-1} = \mathrm{Id}_{\mathbb{R}\setminus\{0\}}$. The constant vector field ∂_x defines a vector field on M, and $\gamma_0(\lambda) = \phi_0^{-1}(t+1)$ and $\gamma_1(\lambda) = \phi_1^{-1}(t+1)$ are both integral curves of X, with $\gamma_0(0) = 0$ and $\gamma_1(0) = 0^*$.

Proposition 25.19. The distance function generated by a Riemannian metric tensor is a pseudometric, and it does not generate the manifold topology.

Proof. Pseudometric : if $p \curlyvee q$ show that $d(p,q) = 0$ Not generate the topology : for two adjacent points, the distance is identical : always part of the same open ball, but not part of the same manifold chart □

Bifurcate curves, continuous extendible spacetimes

25.2.3 Y-manifolds and H-submanifolds

To work with non-Hausdorff spacetimes, we will generally have to use Hausdorff submanifolds, called H-manifolds :

Definition 25.20. An open submanifold V of a Y-manifold W is called a H-manifold if V is Hausdorff and it is not the proper subset of any open Hausdorff submanifold of W

Any H-submanifold of W is paracompact and metrizable.

Theorem 25.21. The set $\mathscr{H}$ of all H-submanifolds of a Y-manifold W is an open covering of W.

Proof. Take Ω the set of all open Hausdorff submanifolds of W. We define the order relation $<$ on $\Omega \times \Omega$ to be, for $U, V \in \Omega$, $U < V$ if U is a proper subset of V. Then $<$ obeys the conditions that either $U < V$ or $V < U$, and if $U < V$ and $V < X$, then $U < X$. $<$ is then a strict partial order on Ω.
If we take a point $p \in W$, and some Hausdorff neighbourhood of p $U \in \Omega$, $\{U\}$ is a non-empty subset of Ω. By the maximal principle, there's □

Theorem 25.22. if $p, q \in M$ and $p \curlyvee q$, then for every sequence (x_n) such that $\lim_{n\to\infty} x_n = p$, we also have $\lim_{n\to\infty} x_n = q$

Proof. if the sequence (x_n) converges to p, this means that in an open set U of the chart containing p, there is an N such that for all $n > N$, $\phi_U(x_n)$ will be within an open ball of diameter ε for any $\varepsilon > 0$. As the intersection of any neighbourhood of p and q will never be empty, we can pick the smallest open ball that will fit around both p and q (since the balls define the $\mathbb{R}^n$ topology). [PROVE THAT IF ε FOR U_p THEN THE BALL WILL BE SMALLER OR EQUAL IN U_q] [MAYBE USE $\lim_{x\to p} f(x) = y$ for every sequence (x_n), $\lim(x_n) = p \to \lim f(x_n) = y$] □

Theorem 25.23. Non-Hausdorff manifolds do not admit bump functions with compact support in every open set.

Proof. The line with two origins : open set $(-1, 0) \cup \{0\} \cup (0, 1)$, by continuity of the bump function $f(0) = f(0^*)$ and $0^* \notin \operatorname{supp}(f)$ □

Maybe try proving equivalence of non-Hausdorff manifolds and branching manifolds under the equivalence relation of their adjacent points

Definition 25.24. A branching manifold of class C^k is a metrizable space K with the following conditions :

- There's a covering of K by closed subsets $\{U_i\}$, $\bigcup_i \operatorname{Int}(U_i) = K$
- For every U_i, there's a finite collection of closed subsets of U_i, $\{D_{ij}\}$, such that $\bigcup_j D_{ij} = U_i$
- For each i, there's a map $\pi_i : U_i \to D_i^n$, D_i^n a closed n-disk of class C^k, with $\pi_i|_{D_{ij}}$ a homeomorphism onto D_i^n.
- There's a collection of diffeomorphisms $\{a_{i_1 i_2}\}$ of class C^k such that $\pi_{i_1} = \alpha_{i_1 i_2} \circ \pi_{i_2}$ on $U_{i_1} \cap U_{i_2}$. They obey the cocycle conditions, $a_{i_1 i_2} a_{i_2 i_3} = a_{i_1 i_3}$ and $a_{ii} = \mathrm{Id}$

Non-Hausdorff, completely separable, n-dimensional, C^k manifold : Y-manifold

25.3 Non-Hausdorff spacetimes

Define tensors and such by defining them on open Hausdorff subsets of the manifold, then enforcing that they agree on their intersections (Same condition as the local trivialization?)
Maximal extension of the Taub-NUT spacetime
extension of Rindler space

25.4 Branching spacetimes

The idea of branching spacetime is to have a spacetime manifold with branching points at individual events. To perform this in a causal way (such that the branching is "caused" by a single point), for a point $p \in \mathcal{M}$, we cut $\bar{J}^+(p)$ out of the manifold and glue in its place n copies of $\bar{J}^+(p)$. Hence for such a manifold, all points in each copy of $\partial J^+(p)$ are adjacent.

Proposition 25.25. Every branching spacetime contains bifurcate timelike curves.

Proof. □

Definition 25.26. $(W, \leq)$ is a branching spacetime model if, for a non-empty set W and a partial order $\leq$, we have

1. $\leq$ is dense
2. $\leq$ has no maximal element
3. Every lower bounded chain in W has an infimum in W.
4. Every upper bounded chain in W has a supremum in every history that contains it.
5. For any lower bounded chain $O \in h_1 - h_2$, there exists a point $e \in W$ such that e is maximal in $h_1 \cap h_2$ and for all $e' \in O$, $e < e'$.

Part II

General relativity

Now that we have defined what a spacetime exactly is, we can study the interaction of this spacetime with matter and with observers.
recapitulate what a spacetime is with what we have learned so far :

- A Lorentzian manifold, which is connected, Hausdorff, paracompact, second countable , of finite dimension $n \geq 2$ and if of even dimension and compact, of Euler characteristic 0.
- A smooth structure $\mathcal{A}$.
- A Lorentz metric g defined on that manifold.
- A connection ∇, usually assumed torsion-free and metric-compatible.
- If the spacetime is orientable, a nowhere-vanishing n-form
- If the spacetime is time-orientable, a non-vanishing timelike vector field τ to define a time-orientation.
- If the spacetime is space-orientable, a non-vanishing spacelike $(n-1)$-form.

defined up to equivalence by diffeomorphisms that preserve the orientations. This will be the basis that all metric theories of gravity will use.

26 The Einstein field equations

The Einstein field equations connect the spacetime metric with the matter fields upon it. It can be expressed in its simplest form in terms of the Einstein tensor $G_{\mu\nu}$ by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{26.1}$$

or, expressed in terms of the Ricci tensor,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{26.2}$$

with κ the Einstein constant,

$$\kappa = \frac{8\pi G}{c^4} \tag{26.3}$$

$$\approx 2.076\,579 \times 10^{-43}\,\mathrm{s}^2\,\mathrm{m}^{-1}\,\mathrm{kg}^{-1} \tag{26.4}$$

The tensor T is a rank $(0,2)$ tensor called the stress-energy tensor, or energy-momentum tensor, which depends on the matter fields $\phi_i(x)$ and their derivatives $\nabla\phi_i(x)$.
Taking the trace of the equation gives us the following equality

$$\frac{1}{2} R + \frac{n}{2-n} \Lambda = \frac{1}{2-n} (\kappa T) \tag{26.5}$$

By multiplying it by $g_{\mu\nu}$ and adding it to the Einstein field equation, we obtain

$$R_{\mu\nu} + \frac{2}{2-n} \Lambda g_{\mu\nu} = \kappa (T_{\mu\nu} + \frac{1}{2-n} T) \tag{26.6}$$

In particular, for $n = 4$, this gives

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa (T_{\mu\nu} - \frac{1}{2} T) \tag{26.7}$$

This variant of the EFE has the benefit of showing easily that, in the absence of a cosmological constant, a vacuum solution $T_{\mu\nu} = 0$ implies immediatly that $R_{\mu\nu} = 0$, and trivially $R = 0$.
The content of the stress energy tensor will be explored later on, where we will give two different ways to find a form of it.

26.0.1 Local conservation of energy

One important reason behind the form of the Einstein field equation is that the local conservation of energy is built in the theory, as the second Bianchi identity implies that

$$\nabla_\mu G^{\mu\nu} = 0 \tag{26.8}$$

which, along with the metric-compatible connection for the cosmological term, implies

$$\nabla_\mu T^{\mu\nu} = 0 \tag{26.9}$$

This corresponds to the local conservation of the stress-energy tensor, as is to be expected.

26.1 Coordinate fixing

The Einstein field equations give us $(n^2 + n)/2$ equations, but as the energy conservation imposes that $\nabla_\mu(G^{\mu\nu} + \Lambda g^{\mu\nu}) = 0$, imposing n relations on them, only $(n^2 - n)/2$ are truly independent. Since we have $(n^2 + n)/2$ metric components to solve for, the equations are underdetermined.

This is due to the diffeomorphism invariance of general relativity. If we use the vacuum Einstein field equations without coordinates, $G = 0$, even for the same initial conditions, we can have the same metric tensor expressed in different coordinates, giving rise to different components. It can easily be seen by considering a diffeomorphism of the form $f = \mathrm{Id}$ for $t < t_0$ and $f \neq \mathrm{Id}$ for $t > t_0$, in which case for some given identical initial conditions for $t < t_0$, we will have different possible evolutions of the metric components. To fix this down, we need to impose some conditions on the coordinates of the manifold. Fix the diffeomorphism and the tetrad rotation

26.1.1 Harmonic gauge

A commonly used coordinate gauge is the harmonic gauge, due to its simplicity.

$$\Gamma^\rho{}_{\mu\nu} g^{\mu\nu} = 0 \tag{26.10}$$

$$\partial_\mu (g^{\mu\nu} \sqrt{-g}) = 0 \tag{26.11}$$

26.2 Uniqueness of the field equations

One of the motivation for the Einstein field equations is to consider the most general form of equations that we would get for a set of properties that would fit the theory.

At its core, for a metric theory of gravity, we want a differential equation of the metric tensor g. We will consider two main terms of that equation, the "kinetic" term $D_{\mu\nu}[g]$, and a dynamic term that will link it to other matter fields $T_{\mu\nu}[g]$. The equation will be

$$D_{\mu\nu}[g] = T_{\mu\nu}[g] \tag{26.12}$$

The conditions we want to obey generally are

- The differential equation will depend at most on the second derivatives of g.
- The stress energy tensor obeys local conservation of energy $\nabla_\nu T^{\mu\nu} = 0$

The second condition gives us immediatly

$$\nabla_\nu T^{\mu\nu} = \nabla_\nu D^{\mu\nu} = 0 \tag{26.13}$$

If we restrict our attention for now to $n = 4$, we can get Lovelock's theorem

Theorem 26.1. If we want the tensor $D_{\mu\nu}$ to only contains derivatives of $g_{\mu\nu}$ up to second order and obey the local conservation of energy, then in four dimensions, the class of tensors will be

$$D_{\mu\nu}[g] = \alpha G_{\mu\nu} + \beta g_{\mu\nu} \tag{26.14}$$

Proof. □

27 The Lagrangian formalism

27.1 Lagrangian mechanics

Lagrangian mechanics is the fundamental principle at the core of most classical physics, and will still be fairly important in quantum theory later on.
The basis of the Lagrangian formalism is the action, a map from the physical configuration of the system to the reals.

Definition 27.1. An action functional is a functional from the jet bundle of fields $J^\infty E$ over a region of spacetime $U \in \tau(\mathcal{M})$ to the real numbers

$$S : J^\infty E \times \tau(\mathcal{M}) \to \mathbb{R} \tag{27.1}$$

For most realistic systems, this will actually just be a map from $J^1 E$ to $\mathbb{R}$, as the action typically doesn't include any field derivatives beyond the first. In a more physical way, we will usually write the action for a family of fields ϕ_i over an open set U as $S[\phi_i, \nabla\phi_i; U]$.
Maps a region of spacetime and section of a vector bundle to $\mathbb{R}$
The action functional is usually expressed as the integral over the domain U of a 0-form, the Lagrangian density $\mathscr{L}$.

$$S[\phi_i, \nabla\phi_i; U] = \int_U \mathscr{L}(\phi_i, \nabla\phi_i) d\mu(g) \tag{27.2}$$

We can also define the n-form Lagrangian L as

$$L = \mathscr{L}\varepsilon \tag{27.3}$$

with ε the volume n-form, in which case the action can simply be expressed as

$$S[\phi_i, \nabla\phi_i; U] = \int_U L(\phi_i(x), \nabla\phi_i(x)) \tag{27.4}$$

The fundamental equation of Lagrangian mechanics is given by Hamilton's principle ,

$$\frac{\delta S}{\delta \phi_i}[\phi_i, \nabla\phi_i] = 0 \tag{27.5}$$

Proposition 27.2. If an action functional has vanishing surface terms, it will be invariant under the addition of a divergence

$$\int_D \mathscr{L} d\mu[g] = \int_D (\mathscr{L} + \nabla_\mu X^\mu) d\mu[g] \tag{27.6}$$

Proof.

$$\int_D \nabla_\mu X^\mu d\mu[g] = \tag{27.7}$$

□

27.1.1 The Euler-Lagrange equations

Hamilton's principle applied to an action functional given by an integral of some n-form will give us

$$\begin{aligned}\frac{\delta S[\phi_i,\partial_\mu\phi_i]}{\delta\phi_i}[f] &= \int_D [\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial\phi}(\phi,\partial_\mu\phi) - \partial_\mu\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial(\partial_\mu\phi)}(\phi,\partial_\mu\phi)]f(x)d^nx \\ &+ \int_{\partial D}\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial(\partial_\mu\phi)}d^{n-1}x \end{aligned} \tag{27.8}$$

We will generally want the boundary term to vanish, either by constraining the class of solutions that we consider, the domain of integration or by the addition of counterterms. In this case, the Hamilton principle gives us

$$\int_D [\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial\phi}(\phi,\partial_\mu\phi) - \partial_\mu\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial(\partial_\mu\phi)}(\phi,\partial_\mu\phi)]f(x)d^nx = 0 \tag{27.9}$$

which allows us to define the variation of the action as a function

$$\frac{\delta S}{\delta\phi}[\phi,\partial_\mu\phi](x) = \frac{\partial(\mathscr{L}\sqrt{-g})}{\partial\phi}(\phi,\partial_\mu\phi) - \partial_\mu\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial(\partial_\mu\phi)}(\phi,\partial_\mu\phi) \tag{27.10}$$

Hamilton's principle will then be true for all test functions if the variation of the action always vanishes.

$$\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial\phi}(\phi,\partial_\mu\phi) - \partial_\mu\frac{\partial(\mathscr{L}\sqrt{-g})}{\partial(\partial_\mu\phi)}(\phi,\partial_\mu\phi) = 0 \tag{27.11}$$

To shorten the notation somewhat, we'll define the Lagrangian scalar as $L = \mathscr{L}\sqrt{-g}$, we'll denote $\partial_\mu\phi$ by $\phi_{,\mu}$ and drop the arguments of the Lagrangian.

$$\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial\phi_{,\mu}} = 0 \tag{27.12}$$

This is the Euler-Lagrange equation. If in particular, ϕ is not related to the metric components, we will have

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial\phi_{,\mu}} = 0 \tag{27.13}$$

27.2 Constraints

By the chain rule, the Euler-Lagrange equations gives us

$$(\partial_\mu\partial_\nu\phi)\,\frac{\partial L}{\partial\phi_{,\mu}\partial\phi_{,\nu}} = \frac{\partial L}{\partial\phi} - \partial_\mu\phi\frac{\partial L}{\partial\phi\partial\phi_{,\mu}} \tag{27.14}$$

$$\det\frac{\partial L}{\partial\phi_{,\mu}\partial\phi_{,\nu}} = 0 \tag{27.15}$$

27.3 Common functional derivatives

Some terms will crops up quite frequently for the Lagrangians of various theories of gravitation and matter. Here are a few of them.

27.3.1 Derivatives with respect to the metric

27.3.1.1 Constant terms

$$S[g] = \int_D C\sqrt{-g}d^n x \tag{27.16}$$

As this does not depend on the derivatives of the metric, there will be no surface terms, and we can consider directly the functional derivative as a function.

$$\frac{\delta S[g]}{\delta g_{\mu\nu}} = C\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}} = \frac{C}{\sqrt{-g}}\frac{\partial \det(g)}{\partial g_{\mu\nu}} \tag{27.17}$$

27.3.1.2 Terms depending only on the metric tensor

$$\frac{\delta}{\delta g_{\mu\nu}}\int d^n x T^{\mu\nu} g_{\mu\nu}\sqrt{-g} \tag{27.18}$$

27.3.1.3 Terms depending on the Ricci scalar

There are two important variations of the Ricci scalar that we need. First is the Ricci scalar as a function of the metric tensor

$$\frac{\delta}{\delta g_{\mu\nu}}\int d^n x R[g]\sqrt{-g} \tag{27.19}$$

27.3.2 Derivatives with respect to a scalar field

27.4 The Einstein-Hilbert action

The full form of the gravitational action to obtain the Einstein field equations is the

$$\begin{aligned} S[g] &= S_{EH} + S_{GHY} + S_M \\ &= \frac{1}{2\kappa}\int_D R d\mu(g) - \frac{1}{\kappa}\int_{\partial D}\mathrm{Tr}(\bar{K}) Vol(\partial D) + \int_{\partial D}\bar{\omega}^i_j \wedge \eta^j_i + S_M[g] \end{aligned}$$

S_{EH} being the Einstein-Hilbert action, which encapsulates the dynamic of the gravitational field, S_{GHY} the Gibbons-Hawking-York term, to deal with boundary effects, and S_M the action for the matter components.
The integral is only defined on a specified domain D because unlike for some fields, it is not guaranteed that S_{EH} converges on the entire spacetime, and the integral may fail to be defined if the spacetime isn't orientable. Hence we will use the various counter-terms to make the action well-defined with respect to boundary terms.
By Hamilton's principle, we want to have

$$\frac{\delta S[g]}{\delta g} = 0 \tag{27.20}$$

for any domain D the integral is defined on.

27.5 The Noether theorems

A global symmetry of a Lagrangian theory is defined as a transformation that will leave the action invariant. That is, for an action defined by $S[\mathscr{L}(\phi_i)]$, where ϕ_i is any quantity the Lagrangian may depend on (fields, point particles, coordinates, etc), then a set of functions f_i will be a symmetry if we have

$$S[\mathscr{L}(f_i(\phi_i))] = S[\mathscr{L}(\phi_i)] \tag{27.21}$$

Since S is an integral of the Lagrangian, a general transformation can leave it invariant if it adds a divergence to the Lagrangian.

$$\mathscr{L}(f_i(\phi_i)) = \mathscr{L}(\phi_i) + \partial_\mu f \tag{27.22}$$

If the symmetry of the action is a Lie group,

Theorem 27.3. Noether's first theorem :

$$\phi_i \to \phi_i + \varepsilon_{ij}\phi_j \tag{27.23}$$

$$\mathscr{L} \to \mathscr{L} + \varepsilon_{ij}\partial_\mu \Lambda^\mu_{ij} \tag{27.24}$$

$$j^\nu_{ij} = \Lambda^\nu_{ij} - \frac{\partial \mathscr{L}}{\partial \phi_{i,\nu}}\phi_j \tag{27.25}$$

Proof. □

28 Alternative formulations of general relativity

Within general relativity itself, there are a few different formalism variant which can be useful for solving the field equations in specific circumstances.

28.1 Palatini formulation and second order formulation

The Palatini formulation is not strictly a simple alternative formalism (it is the basis for the metric-affine theories of gravity), but with the appropriate constraint, it is equivalent to it. Its action is

$$S[e,\omega] = \frac{1}{2\kappa}\int \varepsilon_{abcd}(e^a{}_\mu \wedge e^b{}_\nu \wedge R^{bc} - \frac{1}{12}\Lambda) \tag{28.1}$$

28.2 Newman-Penrose formalism

The Newman-Penrose formalism is one adapted for general relativity in four dimensions where the equations are projected on a set of "null" vectors.
Pick a frame field, Pick two null vectors l, n such that $l^a n_a = -1$ (same orientation). Complex null vector m from two orthogonal spacelike vectors x, y

$$m = \frac{1}{\sqrt{2}}(x + iy) \tag{28.2}$$

such that $m^a \bar{m}_a$. Then we consider the tetrad of vectors $\{l, n, m, \bar{m}\}$

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0 \tag{28.3}$$

$$l^a n_a = l_a n^a = -1,\ m^a \bar{m}_a = m_a \bar{m}^a = 1 \tag{28.4}$$

$$l_a m^a = l_a \bar{m}^a = n_a m^a = n_a \bar{m}^a = 0 \tag{28.5}$$

$$g_{\mu\nu} = \eta_a e^a{}_\mu e^b{}_\nu \tag{28.6}$$

28.2.1 Derivatives

With that local basis, we can also define the covariant derivatives

$$D = \nabla_l,\ \Delta = \nabla_n,\ \delta = \nabla_m,\ \bar{\delta} = \nabla_{\bar{m}} \tag{28.7}$$

This lead us to defining the spin coefficients

28.2.2 The field equations

28.3 Self-dual formalism

Given a rank 2 tensor T, we define its dual as

$$*T = \frac{1}{2}(T) \tag{28.8}$$

Ashtekar variables

28.4 Plebański formalism

28.5 Spinor general relativity

Define the EFE on the spinor tensor bundle
It is possible to express spinor general relativity in the language of quaternions as well [27] thanks to the relation between quaternions and the Clifford algebra.

29 The stress-energy tensor

From the Lagrangian formulation of general relativity, we have that the stress-energy tensor has the general form

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{29.1}$$

Properties from the Einstein tensor : $T_{\mu\nu} = T_{\nu\mu}$, conservation of energy momentum (local)

$$\nabla_\mu T^{\mu\nu} = 0 \tag{29.2}$$

Scalar field

$$T_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}(g^{\sigma\tau}\nabla_\sigma\varphi\nabla_\tau\varphi + \frac{m^2}{\hbar}\varphi^2) \tag{29.3}$$

EM field

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\sigma}F_{\nu\tau}g^{\sigma\tau} - \frac{1}{4}g_{\mu\nu}F_{\sigma\tau}F_{\alpha\beta}g^{\alpha\sigma}g^{\beta\tau}) \tag{29.4}$$

29.1 The canonical stress-energy tensor

The canonical stress-energy tensor is the Noether current associated with the variation of the action under translation.
That is, for the translation $x^\mu \to x^\mu + a^\mu$, the field will transform as

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi(x) + \mathcal{O}(a^2) \tag{29.5}$$

and the action, which is invariant under translation, transforms as

$$S \to S = \int \mathscr{L}(x+a)d^nx = \int(\mathscr{L}(x) + a^\mu\partial_\mu\mathscr{L} + \mathcal{O}(a^2)d^nx \tag{29.6}$$

Noether current :

$$T^\mu{}_\nu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu \tag{29.7}$$

The various Noether currents are referred to as the energy tensity $\rho = T_{00}$, the pressure $p_i = T_{ii}$, the momentum density T_{0i} and the shear stress $\varepsilon^{ijk}T_{ij}$.
Belifante tensor, etc etc

30 Geodesic deviation and congruence

The effects of gravity from the curvature of spacetime can be best understood by looking at the effect it has on the motion of geodesics and curves, as this will be kind of effect we can most readily observe physically.

30.0.1 Geodesic deviation

Geodesic deviation considers the separation of a family of geodesics. This will be helpful to understand tidal forces in general relativity. Two or more points initially separated by a distance evolve on the spacetime and we analyse the evolution of their distance as time goes on.
To represent this situation, we consider a one-parameter family of geodesic curves, $\gamma_s(\lambda)$. A choice of n value of s will give us the geodesic representing the trajectories to look at. As usual, the tangent vector is

$$U^\mu = \frac{d}{d\lambda} x^\mu_s(\lambda) \tag{30.1}$$

But we will also be interested by the deviation vector :

$$S^\mu = \frac{d}{ds} x^\mu_s(\lambda) \tag{30.2}$$

Relative velocity of geodesics :

$$V^\mu = U^\rho \nabla_\rho S^\mu \tag{30.3}$$

Relative acceleration :

$$A^\mu = U^\rho \nabla_\rho V^\mu \tag{30.4}$$

$$A^\mu = R^\mu_{\nu\sigma\tau} T^\nu T^\rho S^\sigma \tag{30.5}$$

30.0.2 The Raychaudhuri equation

A more general method to consider tidal forces is to consider a family of geodesics going through every point of space.
Open subset $U \subset \mathcal{M}$, a is a family of curves such that for every point $p \in U$, there is one curve in the congruence passing through p. Tangents of a congruence defines a vector field in U and the integral curves of a vector field in U defines a congruence. Smooth congruence if the vector field is smooth.
Timelike congruence of geodesics, parametrized by proper time τ so that the tangents ξ^μ are normalized $\xi^\mu \xi_\mu = -1$. The tensor field $B_{\mu\nu}$

$$B_{\mu\nu} = \nabla_\nu \xi_\mu \tag{30.6}$$

is "spatial", ie

$$B_{\mu\nu} \xi^\mu = B_{\mu\nu} \xi^\nu = 0 \tag{30.7}$$

For a one-parameter subfamily $\gamma_s(\lambda)$ of the geodesic congruence, with S^μ the deviation vector, then

$$\mathcal{L}_\xi S^\mu = 0 \tag{30.8}$$

Hence,

$$\xi^\mu \nabla_\mu S^\nu = S^\mu \nabla_\mu \xi^\nu = B^\nu_\mu S^\mu \tag{30.9}$$

B is the failure of S to be parallely transported. It measures the spread, twist and compression of the congruence of geodesics along their path.
Projection of the metric

$$h_{\mu\nu} = g_{\mu\nu} + \xi_\mu \xi_\nu \tag{30.10}$$

Expansion :

$$\theta = B^{\mu\nu} h_{\mu\nu} \tag{30.11}$$

Shear :

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu} \tag{30.12}$$

Twist :

$$\omega_{\mu\nu} = B_{[\mu\nu]} \tag{30.13}$$

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{30.14}$$

By Frobenius theorem and B being purely spatial, the congruence is hypersurface orthogonal iff $\omega_{\mu\nu} = 0$

31 Energy conditions

The Einstein field equations by themselves does not place any restriction on the class of metrics allowed, including metrics that are generally considered to be unlikely to be physical. While the simplest restriction would be to only consider metrics stemming from actual matter fields, it is rarely clear what class of metrics this would allow overall. Instead, general conditions on the stress energy tensor itself are considered.
To avoid any complications of the notation, we will consider the cosmological constant to be part of the stress energy tensor.

$$\kappa T \to \kappa T - \Lambda g_{\mu\nu} \tag{31.1}$$

This is completely equivalent to the usual Einstein field equations, with the simple addition of a term of the form $T_{\mu\nu} = \Lambda g_{\mu\nu}$ (the sign was changed but since Λ is a priori of arbitrary sign that is not much of an issue), called a lambda vacuum term.

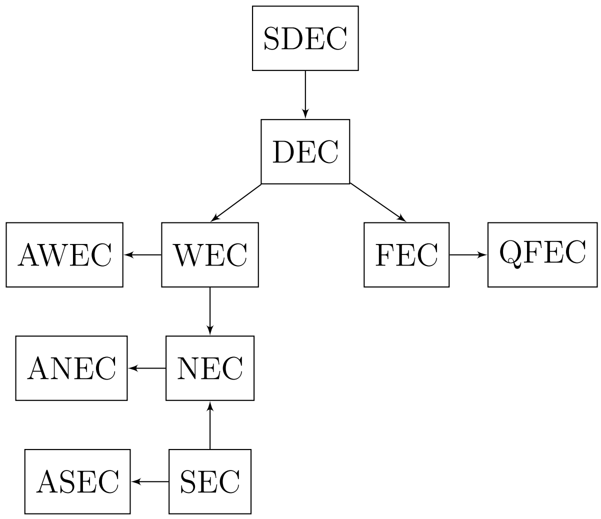


Figure 11: Relations of the energy conditions

31.1 Segre classification of stress-energy tensors

Much in the same way that we can classify the Ricci tensor,
Segre classification [cf Martin-Moruno and Visser]
As we have seen previously, it is possible to classify the Ricci tensor into 4 different categories in the Segre classification. This in turns gives us a classification of the stress-energy tensor.
Express T at p with respect to an orthonormal basis $\{e_a\}$
Type I : corresponds to the A_1 Segre type

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix} \tag{31.2}$$

$$T^{ab} = \rho u^a u^b + p_1 s_1^a s_1^b + p_2 s_2^a s_2^b + p_3 s_3^a s_3^b \tag{31.3}$$

Eigenvalues $\{\rho, p_1, p_2, p_3\}$
Type II :

$$\begin{pmatrix} \mu + f & f & 0 & 0 \\ f & -\mu + f & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix} \tag{31.4}$$

$$T^{ab} = f k^a k^b - \mu \eta_2^{ab} + p_2 s_2^a s_2^b + p_3 s_3^a s_3^b \tag{31.5}$$

Type III :

$$\begin{pmatrix} \rho & 0 & f & 0 \\ 0 & -\rho & f & 0 \\ f & f & -\rho & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix} \tag{31.6}$$

$$T^{ab} = -\rho \eta_3^{ab} + f(s_2^a k^b + k^a s_2^b) + p_3 s_3^a s_3^b \tag{31.7}$$

Type IV :

$$\begin{pmatrix} \rho & f & 0 & 0 \\ f & -\rho & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{pmatrix} \tag{31.8}$$

Eigenvalues : $\{-\rho + if, -\rho - if, p_1, p_2\}$

31.2 A few examples

Fields to consider : free scalar field, free EM field, free spinor field, scalar field with potential, with curvature coupling

31.3 Classical energy conditions

31.3.1 Null Energy Condition

The null energy condition (or NEC) is the weakest of the classical energy conditions, just requiring that for every null tensor k, we have

$$T_{\mu\nu} k^\mu k^\nu \geq 0 \tag{31.9}$$

Type I : $\rho + p_i \geq 0$ Type II : $\mu + p_i \geq 0$, $f > 0$

31.3.1.1 Violations of the NEC

The NEC is not easily violated by classical field, except by constructing ad-hoc ones such as phantom fields
It is much more easily violated by quantum effects, such as the Casimir effect, squeezed states or along domain walls.

31.3.2 Weak Energy Condition

The weak energy condition : any observer with a timelike tangent ξ measures the energy of the stress-energy tensor as non-negative.

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \tag{31.10}$$

$$\rho \geq 0,\ \ \rho + p_j \geq 0 \tag{31.11}$$

The WEC used to be considered the weakest of all energy conditions, as the requirement that the energy be positive was considered the minimum anyone could ask for.

Theorem 31.1. The WEC implies the NEC.

Proof. Consider a sequence of timelike vectors converging to a null vector $\{\xi_n\}$, $\lim \xi_n = k$. Then we have that if the stress-energy tensor is continuous, $\lim T(\xi_n, \xi_n) = T(k, k) \geq 0$. □

By the perfect fluid expression, a simple counterexample of the NEC implying the WEC will be a cosmological stress energy tensor $T = \Lambda g$ in Minkowski space, for which $T(\partial_t, \partial_t) = -\Lambda$.

Theorem 31.2. For any energy condition such that $T_{\mu\nu}\xi^\mu\xi^\nu \geq -b$ for some positive b,

Violation of the WEC :

31.3.3 Ubiquitous Energy Condition

The ubiquitous energy condition (or UEC) corresponds to the requirement that for every causal vectors k,

$$T_{\mu\nu}k^\mu k^\nu > 0 \tag{31.12}$$

It trivially implies the weak energy condition, just forbidding the existence of vacuum spacetimes

$$\rho > 0,\ \ \rho + p > 0 \tag{31.13}$$

31.3.4 Strong Energy Conditions

The strong energy condition is defined by what is also referred to as the convergence condition,

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \tag{31.14}$$

So called because it will cause all geodesics to eventually converge (cf. chapter on geodesic congruences). In terms of the stress energy tensor, this corresponds to

$$(T_{\mu\nu} + \frac{1}{2-n}Tg_{\mu\nu})t^\mu t^\nu \geq 0 \tag{31.15}$$

or, in 4 dimensions,

$$(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) t^\mu t^\nu \geq 0 \quad (31.16)$$

$$\forall j,\ \rho + p_j \geq 0,\ \rho + \sum_i p_i \geq 0 \quad (31.17)$$

Theorem 31.3. The SEC implies the NEC.

Proof. By the same process as with the WEC, we take $\{t_n\}$, $\lim t = k$, for which, using the fact that $g(k,k) = 0$, we have

$$\lim T_{\mu\nu} t_n^\mu t_n^\nu = T_{\mu\nu} k^\mu k^\nu = R_{\mu\nu} k^\mu k^\nu \quad (31.18)$$

which is □

Theorem 31.4. If the SEC is obeyed, gravity is always attractive.

Proof. cf. Raychaudurah equation □

Violation of the SEC :
Classical scalar field :

$$\mathscr{L}_M = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + \frac{1}{2} m \phi^2(x) \quad (31.19)$$

Stress energy tensor :

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi - \frac{1}{2} g_{\mu\nu} m^2 \phi^2 \quad (31.20)$$

31.3.5 Dominant Energy Conditions

The Dominant Energy condition (or DEC) corresponds to the notion that the stress energy tensor is "well behaved", as it both has the flow energy constrained to remain within the light cone, and obeys the weak energy condition.
This is expressed by the usual condition of the weak energy condition, and that for any timelike vector ξ^μ, $T_{\mu\nu}\xi^\mu$ is a causal vector. Or, in a more compact form, for any two timelike vectors ξ, χ of the same (local) time orientation,

$$T_{\mu\nu} \xi^\mu \chi^\nu \geq 0 \quad (31.21)$$

Theorem 31.5. If the stress-energy tensor vanishes on a region S and it obeys the DEC, then it also vanishes on $D^+(S)$.

Proof. cf hawking □

Violation of the DEC :

31.3.6 Strengthened Dominant Energy Condition

The Strengthened Dominant Energy Condition (or SDEC) is the limiting case of the DEC where the flux of energy cannot be null, that is, all energy flow is within the lightcone itself, and never on its boundary.

31.3.7 The generic condition

The generic condition is satisfied if every causal geodesic has a point p such that the tangent u at p satisfies

$$u^\mu u^\nu u_{[\rho} R_{\sigma]\mu\nu[\alpha} u_{\beta]} \neq 0 \tag{31.22}$$

This condition implies that the tidal force is non-zero at p on a causal geodesic, meaning that every geodesic feels a tidal force at some point in its history, which is generally considered like a realistic condition in our universe.

31.3.8 Trace Energy Conditions

The trace energy condition, or TEC, corresponds to the requirement of a non-positive trace for the stress energy tensor

$$T_{\mu\nu} g^{\mu\nu} \leq 0 \tag{31.23}$$

Or in other words,

$$p \leq \frac{1}{3}\rho \tag{31.24}$$

While it was historically firmly believed to hold for any realistic matter field, it was proven [cf Novikov] that this condition can be violated for instance with the matter in neutron stars.
Argument of von Neumann in Chandrasekhar's paper "Stellar configurations with degenerate cores". Proof probably due to considering systems of free particles.

$$T^\mu{}_\mu = (1 - \frac{n}{2})R \tag{31.25}$$

so the TEC translates, for every dimension (for $n > 2$), to $R \geq 0$
Subdominant energy condition (cf Bekenstein paper):

$$|T_{\mu\nu}(g^{\mu\nu} + t^\mu t^\nu)| < T_{\mu\nu} t^\mu t^\nu \tag{31.26}$$

From the SEC :

$$(T_{\mu\nu} + \frac{1}{2-n} T g_{\mu\nu}) t^\mu t^\nu \geq 0 \tag{31.27}$$

31.4 Quasilocal energy conditions

cf. paper of Geoff Hayward

31.5 The averaged energy conditions

Energy conditions averaged on a causal curve
ANEC :

$$\int_\gamma T_{\mu\nu} k^\mu k^\nu d\lambda \geq 0 \tag{31.28}$$

31.6 The quantum inequalities

$$\int \langle T_{\mu\nu} k^\mu k^\nu \rangle g(t) dt \geq f(t) \tag{31.29}$$

31.7 Nonlinear energy conditions

None of the classical energy conditions hold in general for the quantum case, even for realistic scenarios.

31.7.1 The Flux Energy Condition

The Flux Energy Condition, or FEC, is a weaker version of the DEC, where the WEC is not assumed. In other words, for the flux $F^\mu = -T^{\mu\nu}\xi^\rho g_{\nu\rho}$ for any timelike vector ξ, we have

$$F^\mu F_\mu \leq 0 \tag{31.30}$$

31.7.2 The Determinant Energy Condition

DETEC

$$\det(T^{\mu\nu}) \geq 0 \tag{31.31}$$

31.7.3 Trace-of-square Energy Condition

TOSEC

$$T^{\mu\nu} T_{\mu\nu} \geq 0 \tag{31.32}$$

$$(\langle T^{\mu\nu} \rangle \xi_\nu)(\langle T^{\sigma\tau} \rangle \xi_\tau) g_{\mu\sigma} \geq 0 \tag{31.33}$$

32 Hamiltonian formulation

32.1 Hamiltonian mechanics and general relativity

Given a foliation by Cauchy hypersurfaces Σ_t of normal n^μ and a time function t
Symplectic structure
The purely fiber bundle interpretation of the Hamiltonian for field theory is rather complex, involving the Legendre bundle which, for a bundle $Y \to \mathcal{M}$, corresponds to

$$\Pi = V^*Y \wedge (\bigwedge^{n-1} T^*\mathcal{M}) \tag{32.1}$$

To simplify matters somewhat we will stay with the "classical" interpretation of the Hamiltonian

32.1.1 The Legendre-Fenchet transform

Legendre transform : for a function $f : X \to \mathbb{R}$ a convex function, $X \subset \mathbb{R}^n$ a convex subset, the Legendre transform $f^* : X^* \to \mathbb{R}$ is

$$f^*(x^*) = \sup_{x\in X}(\langle x^*, x\rangle - f(x)) \tag{32.2}$$

$(X^* = \{x^* \in \mathbb{R}^n | \sup_{x\in X}(\langle x^*, x\rangle - f(x)) < \infty\})$
We will note x^* as p
Properties :
if $x(p)$ maximizes $\langle p, x\rangle - f(x)$, then $f^*(p) = px(p) - f(x(p))$,

$$\frac{df}{dx} = p \tag{32.3}$$

32.1.2 The Hamiltonian

Legendre transform of the Lagrangian $\mathscr{L}$:
We will denote for any quantity that $\dot{X} = n^\mu \partial_\mu X$
For a matter field with Lagrangian $\mathscr{L}$, the canonical momentum is usually defined as :

$$\pi = \frac{\partial \mathscr{L}}{\partial(\dot{\phi})} \tag{32.4}$$

$$\mathscr{H}(\phi_i, \pi_i, t, x) = \sum_i \dot{\phi}_i \pi_i - \mathscr{L}(\phi_i, \nabla\phi_i, x) \tag{32.5}$$

The action is then

$$S[\phi_i, \pi_i] = \int_D [\sum_i \dot{\phi}_i \pi_i - \mathcal{H}(\phi_i, \pi_i, t, x)] dt d^{n-1}x \tag{32.6}$$

$$H = \int \mathscr{H} d^{n-1}x \tag{32.7}$$

Hamiltonian field equations :

$$\dot{\phi}_i = \frac{\delta H}{\delta \pi_i},\ \dot{\pi}_i = -\frac{\delta H}{\delta \varphi_i} \tag{32.8}$$

[DO FUNCTIONAL DERIVATIVES MAKE SENSE HERE]
Lie brackets

32.1.3 Hamiltonian constraint

The Lagrangian may not obey the necessary conditions to perform a Legendre transform. That is,

$$\frac{\partial^2 \mathscr{L}}{\partial \dot{\phi}_i \partial \dot{\phi}_j} \neq 0 \tag{32.9}$$

Degenerate or singular Lagrangian

$$\beta_k(p, q) = 0 \tag{32.10}$$

$$\mathscr{H} = \mathscr{H}_0 + \sum_{k=1}^{\alpha} v_k \beta(k) \tag{32.11}$$

$$\mathscr{H}_0 = \sum \dot{\phi}_i \pi_i - \mathscr{L} \tag{32.12}$$

$$\begin{aligned} \dot{\phi}_i &= \frac{\partial \mathscr{H}_0}{\partial \pi_i} + \sum v_k \frac{\partial \beta_k}{\partial \pi_i} \\ \dot{\pi}_i &= -\frac{\partial \mathscr{H}_0}{\partial \phi_i} - \sum v_k \frac{\partial \beta_k}{\partial \phi_i} \end{aligned}$$

Or

$$\begin{aligned} \dot{\phi}_i &= \frac{\partial \mathscr{H}}{\partial \pi_i} \\ \dot{\pi}_i &= -\frac{\partial \mathscr{H}}{\partial \phi_i} \end{aligned}$$

Constraints on the primary restrictions β_k :

$$\dot{\beta}_k = 0 \tag{32.13}$$

32.1.4 Gauge stuff

If the field is associated to a symmetry

$$S[\phi] = S[f(\phi)] \tag{32.14}$$

$$\mathscr{L}(\phi) = \mathscr{L}(f(\phi)) + \partial_\mu \alpha^\mu \tag{32.15}$$

$$\frac{\partial \mathscr{L}(\phi)}{\partial \phi} \tag{32.16}$$

32.2 Hamiltonian of fields on spacetime

32.3 The ADM formalism

The ADM formalism (Arnowitt-Deser-Misner) is the application of the Hamiltonian formalism to general relativity.
Link between foliations $X_t : \Sigma \to \mathcal{M}$, $X_t(x) = X(t,x)$, diffeomorphism $X : \mathbb{R} \times \Sigma \to \mathcal{M}$
Any diffeomorphism $f \in \mathrm{Diff}(\mathcal{M})$ is of the form $f = X' \circ X^{-1}$, any two foliations are related by $X' = f \circ X$
Diffeomorphism invariance is the freedom in foliation choice
If the spacetime is globally hyperbolic with a foliation Σ_t, unit orthogonal vector to σ_t is n, time function t with timelike vector field t^μ such that $t^\mu \nabla_\mu t = 1$
Foliation required to be timelike everywhere : $-N^2 + g_{\mu\nu} N^\nu N^\mu < 0$
Lapse function is nowhere vanishing
Lapse function > 0 : future-directed foliation
Volume element :

$$^{(3)}\epsilon_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma} n^\sigma \tag{32.17}$$

Gauss-Codazzi equation
Einstein tensor :

$$\begin{aligned} G_{\mu\nu} n^\mu n^\nu &= R_{\mu\nu} n^\mu n^\nu - \frac{1}{2} R g_{\mu\nu} n^\mu n^\nu \\ &= R_{\mu\nu} n^\mu n^\nu + \frac{1}{2} R \end{aligned}$$

giving us the relation

$$R = 2(G_{\mu\nu} n^\mu n^\nu - R_{\mu\nu} n^\mu n^\nu) \tag{32.18}$$

$$^{(3)}g = g|_{T\Sigma} \tag{32.19}$$

$$\mathscr{L} = \int_{t_1}^{t_2} \int_M [\beta(^{(3)}R + |K|^2 + \alpha)] d\mu[^{(3)}g] dt \tag{32.20}$$

$$P^{ab} = \frac{\delta S}{\delta^{(3)} g_{ab}} = \tag{32.21}$$

33 Mass in general relativity

Mass not generally defined in general relativity due to the lack of conservation of energy
Mass classically :

$$M = \int_D \rho(x) d^3x = -\frac{1}{4\pi G} \int_D \Delta\phi d^3x \tag{33.1}$$

If we try to apply it directly, most general case for some spacelike hypersurface Σ and two timelike vector fields ξ, χ :

$$M = \int_\Sigma T(\xi, \chi) d\mu[^{(3)}g] \tag{33.2}$$

If we assume a foliation by spacelike hypersurfaces :

$$M(t) = \int_{\Sigma_t} T(\xi, \chi) d\mu[^{(3)}g] \tag{33.3}$$

Conservation of mass :

$$\partial_t M(t) = \tag{33.4}$$

This is due to the general invalidity of Noether's theorem, as metrics will not generally be time-invariant.

33.1 Bondi mass

33.2 Komar mass

For a stationary asymptotically flat spacetime, with Killing vector ξ, the Komar mass is

$$M = \int_D (2T(u, \xi) - Tg(u, \xi)) d\mu[g] \tag{33.5}$$

As a surface integral

$$M = -\frac{1}{8\pi G} \int_S \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma \tag{33.6}$$

33.3 ADM mass

33.4 Positive mass theorem

Theorem 33.1. If the dominant energy condition holds, the total ADM mass of an asymptotically flat spacetime is non-negative. If its mass is zero, it can only be Minkowski space.

Paper on why a spacetime with the Casimir effect still has positive mass.
"Bondi mass cannot become negative in higher dimension"

34 Causality, topology and general relativity

As the matter content will define the metric on the manifold for general relativity and related theories, this means that the behaviour of matter will affect the causal structure and possibly the topology itself.

34.1 Chronology and the energy conditions

Violation of the chronology condition is generally considered to be unphysical. Because of this, many theorems were developped to study under which physical conditions we can be sure that it is preserved.
One of the first such theorem was Tipler's theorem, trying to show that for a physically reasonable matter content, the resulting spacetime was not

Theorem 34.1. An asymptotically flat spacetime cannot be null geodesically complete if

1. $R_{ab}k^a k^b \geq 0$ for all null vectors k
2. The generic condition is satisfied
3. It possesses a partial Cauchy surface S
4. The chronology condition is violated on $J^+(S) \cap J^-(g^+)$

Proof. □

Hawking's theorem :

Theorem 34.2. A spacetime with a compactly generated Cauchy horizon will violate the weak energy condition if the partial Cauchy surface is non-compact. If it is compact, then at best $R_{ab}l^a l^b = 0$.

Theorems on CTCs : Hawking on compactly generated CTCs, that one about compact Cauchy horizons (rigidity theorem)
"At first blush the suggested approach to identifying the operation of a time machine by means of the Potency Condition is threatened by a result of Krasnikov (2002) showing that every time oriented spacetimeM, gab without CTCs has as an extension a maximal time oriented spacetime such that any CTC in the extension lies to the chronological past of the image of M in the extension."

34.2 Singularities and general relativity

Singularity theorems

Theorem 34.3. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and there exists a compact slice are timelike or null geodesically incomplete.

Theorem 34.4. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and there exists a trapped surface are timelike or null geodesically incomplete.

Theorem 34.5. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and there exists a contracting region (cf Ellis 2007) are timelike or null geodesically incomplete.

Theorem 34.6. Spacetime that satisfies chronology, convergence condition ($R_{\mu\nu}\xi^\mu\xi^\nu > 0$), the generic condition and stable causality is not satisfied are timelike or null geodesically incomplete.

Cosmic censorship hypothesis

34.3 Topological restrictions

Energy conditions and topology change

34.3.1 Topological censorship

While not directly related to the restriction of topology, the topological censorship theorem gives some constraints on whether or not some topological features are measurable.

Theorem 34.7. Every causal curve extending from past null infinity to future null infinity can be continuously deformed to a curve near infinity.

35 General relativity in n dimensions

Despite the similar equations, the number of spacelike dimensions has rather drastic effects on how general relativity behaves.
The number of dimension is noted as $(p+q)$ dimensions for a spacetime with signature (p,q).

35.1 $(1+1)$ dimensions

$1+1$ dimensional spacetimes, also called Lorentz surfaces, possess a great number of attractive features for analysis. Their topologies can be exactly classified, many solutions of PDEs are known exactly in two dimensions.
But as we will see later on, it suffers from its simplicity by missing on most of the properties of four dimensional spacetimes, making them of limited use.

35.1.1 Lorentzian geometry in $(1+1)$ dimensions

In two dimensions, the Riemann tensor has $2^2(2^2-1)/12 = 1$ independent component, meaning that with its symmetries, it can be expressed as

$$R_{\mu\nu\rho\sigma} = C\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma} \tag{35.1}$$

for some component C, which can be expressed as

$$R_{\mu\nu\rho\sigma} = C(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \tag{35.2}$$

$$R_{\nu\sigma} = R_{\mu\nu\rho\sigma}g^{\mu\rho} = -Cg_{\nu\sigma} \tag{35.3}$$

By contraction, it's easy to see that $C = -R/2$, meaning that the Ricci tensor can be expressed as

$$R_{\mu\nu} = \frac{R}{2}g_{\mu\nu} \tag{35.4}$$

This means that the Einstein tensor G is identically 0 for all metrics, making the field equations

$$\Lambda g_{\mu\nu} = T_{\mu\nu} \tag{35.5}$$

35.1.2 Angles in the Lorentz plane

We will need to define the notion of angles on Lorentz surfaces. There are several possible definitions for angles, we will be using the one of Birman and Nomizu.
For an oriented, time oriented 2 dimensional Lorentzian manifold, existence of a global unit timelike vector field τ
In the tangent plane we define the frame defined by the basis vectors τ and $\tau^\perp$, the unique unit spacelike vector such that $g(\tau,\tau^\perp)$
Allowable coordinate chart if $(1,0)$ is τ and $(0,1)$ $\tau^\perp$
For X, Y two timelike vectors with the same orientation, the angle u is defined by

$$\frac{1}{g(X,Y)}\begin{pmatrix}\cosh(u) & \sinh(u)\\ \sinh(u) & \cosh(u)\end{pmatrix}\begin{pmatrix}X_t\\ X_x\end{pmatrix}=\begin{pmatrix}Y_t\\ Y_x\end{pmatrix} \tag{35.6}$$

u is independant of the allowable coordinate system. Angle between X and Y : (X,Y)
If X and Y with different time orientations : X and $-Y$ have the same time orientation : $(X,-Y)=u$ so we define $(X,Y)=-u$
Properties :

1. $(X,Y)=-(Y,X)$
2. $(X,X)=0$
3. $(X,-X)=0$
4. $(X,Y)+(Y,Z)=(X,Z)$
5. $(-X,Y)=(X,Y)$
6. $(X,-Y)=(X,Y)$

For a normalized smooth timelike curve γ, take the unit tangent vector T and unit normal vector $T^\perp$. Geodesic curvature k_g is $k_g=g(\nabla_T T,T^\perp)$. Then $\nabla_T T=k_g T^\perp$, and $\nabla_T T^\perp=k_g T$, and $k_g=-g(\nabla_T T^\perp,T)$
Z a unit timelike vector field parallely transported by γ, since the manifold is time orientable, Z preserves orientation. $Z^\perp$ is also parallely transported. Consider the angle $\alpha(\lambda)=(T,Z)$

Theorem 35.1.

$$\frac{d\alpha}{d\lambda}=-k_g \tag{35.7}$$

Proof. If T is future pointing,

$$\begin{aligned}T &= \cosh(\alpha)Z-\sinh(\alpha)Z^\perp\\ T^\perp &= -\sinh(\alpha)Z+\cosh(\alpha)Z^\perp\end{aligned}$$

if past-pointing

$$\begin{aligned}T &= -\cosh(\alpha)Z+\sinh(\alpha)Z^\perp\\ T^\perp &= \sinh(\alpha)Z-\cosh(\alpha)Z^\perp\end{aligned}$$

□

For a region D bounded by Γ, a Lorentz polygon made of timelike curves Γ_i, with

Theorem 35.2. A timelike Lorentz polygon in the Lorentz plane with tangent vectors X_i for the sides Γ_i obeys

$$\sum_i \theta_i=(X_1,X_2)+(X_2,X_3)+...+(X_{k-1},X_k)+(X_k,X_1)=0 \tag{35.8}$$

Gauss-Bonnet theorem :

Theorem 35.3. In a 1+1 dimensional Lorentzian manifold, for a domain D with compact closure with a boundary Γ that is a piecewise smooth timelike curve, then the following equality holds

$$\int_\Gamma k_g ds + \sum_i \theta_i - \int_D R dA = 0 \tag{35.9}$$

with θ_i the exterior angles formed by the boundary.

Proof. □

35.1.3 Triviality of the Einstein field equations

The Lorentzian Gauss-Bonnet theorem has rather important consequences for general relativity in $1+1$ dimensions, as we can guess from the appearance of the Einstein-Hilbert action in it.

Theorem 35.4. The gravitational action in $(1+1)$ dimension is constant with respect to the metric, up to a volume term.

Proof. If we consider a spacetime orientable subset of the Lorentz surface, and then consider the closed domain defined by timelike curves (the existence of such a domain is guaranteed by looking at the image of a diamond via the exponential map, for instance). We can now define the gravitational action inside this domain

$$\frac{1}{2\kappa}\int_D R dA + \frac{1}{\kappa}\int_\Gamma K^{\frac{1}{2}}(\pm h) dl = 0 \tag{35.10}$$

□

Theorem 35.5. The stress-energy tensor is directly proportional to the cosmological constant.

Proof. Since the gravitational action reduces to Λg,

$$(\int_D (\Lambda + \mathscr{L}_M) d\mu[g]) + \tag{35.11}$$

□

35.1.4 Classification of $(1+1)$ Lorentz manifolds

For the study of spacetimes in $(1+1)$ dimensions, we have the benefit of every manifold being classifiable in 2 dimensions.
For compact manifolds :

Theorem 35.6. Every compact 2-manifold is homeomorphic to either the sphere, a connected sum of toruses, or a connected sum of real projective planes.

Proof. □

The sum of g toruses has Euler characteristic $\chi(\mathcal{M}) = -2g$, and the sum of 2 projective planes has $\chi(\mathcal{M}) = 0$ (this is the Klein bottle K). The connected sum of K and P will always be $\chi(\mathcal{M}) < 0$
There are only two compact spacetimes : the torus and the Klein bottle.

Theorem 35.7. Every 2-manifold is homeomorphic to the sphere S^2 from which we remove a closed totally disconnected set X, before removing an infinite sequence of non-overlapping disks $\{D_i\}$ before identifying the boundaries of those disks in pairs (possibly with themselves).

For non-compact manifolds, this will correspond roughly to the plane with some identifications along boundaries, with the addition of (possibly non-orientable) handles and the removal of closed sets.

Definition 35.8. A Lorentz surface is a equivalence class of $(1+1)$ dimensional Lorentz $(\mathcal{M}, [g])$ such that two Lorentz manifolds $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ describe the same Lorentz surface if $\mathcal{M}_1 \approx \mathcal{M}_2$ and (under some diffeomorphism) $g_1 = \Omega g_2$

For Riemannian two dimensional manifolds, there exists a theorem relating conformally equivalent metrics, the uniformization theorem

Theorem 35.9. Every Riemann surface is the quotient of a manifold that is diffeomorphic and conformally equivalent to either the sphere, plane or hyperbolic disk.

In particular, compact manifolds are of positive curvature for $g = 0$, zero for $g = 1$, and negative for $g > 1$.
For Lorentzian manifolds, the situation is more complex, as the metric can be constructed from both a Riemannian metric and a non-vanishing vector field.

35.2 $(1+2)$ dimensions

The Riemann tensor is a function of the Ricci tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})R$$

Hence the curvature depends directly on the matter content, no propagation of gravity
Weyl tensor is identically zero
Classification of globally hyperbolic $(1+2)$ spacetimes : σ is gonna be one of the 2D Riemannian manifolds
Every Riemannian manifold will be $g = \Omega f^* \bar{g}$, $f \in \mathrm{Diff}(\Sigma)$, $\bar{g}$ is the metric of the plane, sphere or hyperbolic plane.

35.3 $(1+3)$ dimensions

General relativity in $(1+3)$ dimensions is the most important case as it corresponds with the apparent dimension of our own universe (this can be verified by such phenomenons as the dropoff of solutions of the Laplace equation or the number of degrees of freedom of monoatomic gases in statistical mechanics).
Unlike for the two previous cases, $(1+3)$ dimensional spacetimes are the first example where the Riemann tensor does not depend directly on the stress-energy tensor, allowing for gravity to act at a distance.
Non-zero Weyl tensor generally

35.4 Higher dimensions

Non-spherical horizons

36 Conceptual notions

36.1 Diffeomorphism invariance

For a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, there corresponds a metric g' in the original coordinates such that $f_* g \approx g'$ pointwise

36.1.1 The hole argument

One of the important departure of general relativity compared to classical mechanics is the abandonment of the notion of absolute space and time.
Consider a spacetime with for which we know the

37 Linearized gravity

For a wide variety of reasons, it may be useful to decompose the metric into a fixed background metric g^0 and a deformation of that metric h.

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \tag{37.1}$$

Decomposition of the inverse metric tensor :
The binomial inverse theorem :

$$(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{I}+\mathbf{B}\mathbf{A}^{-1})^{-1}\mathbf{B}\mathbf{A}^{-1} \tag{37.2}$$

$$g^{\mu\nu} = (\eta_{\mu\nu} + g^0_{\mu\nu})^{-1} = (g^0)^{\mu\nu} - (g^0)^{\mu\alpha}[(\delta + h(g^0)^{-1})^{-1}]^\beta_\alpha h_{\beta\gamma}\eta^{\gamma\nu} \tag{37.3}$$

This gives us a recursive definition of the inverse metric.

$$g^{\mu\nu} = \eta^{\mu\nu} - h_{\alpha\beta}\eta^{\alpha\mu}\eta^{\beta\nu} + \mathcal{O}(h_{\mu\nu}) \tag{37.4}$$

From now on we will write $h^{\mu\nu}$ to mean $h_{\alpha\beta}\eta^{\alpha\mu}\eta^{\beta\nu}$, and the raising and lowering of indexes will be done with the Minkowski metric.

$$\begin{aligned}
\Gamma^\sigma{}_{\mu\nu} &= ((g^0)^{\rho\sigma} - h^{\rho\sigma} + \mathcal{O}(h^2_{\mu\nu}))[(g^0_{\mu\rho,\nu} + g^0_{\nu\rho,\mu} - g^0_{\mu\nu,\rho}) + (h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho})] \\
&= (\Gamma^0)^\sigma{}_{\mu\nu} - h^{\rho\sigma}\Gamma^0_{\rho\mu\nu} + (g^0)^{\rho\sigma}(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}) + \mathcal{O}(h_{\mu\nu}h_{\alpha\beta,\gamma}) + \mathcal{O}(h^2_{\mu\nu})
\end{aligned}$$

$$\begin{aligned}
R_{\mu\nu\sigma}{}^\tau = \quad & \partial_\nu(\Gamma^0)^\tau{}_{\mu\sigma} - \partial_\nu(h^{\rho\tau}\Gamma^0_{\rho\mu\sigma}) + \partial_\nu((g^0)^{\rho\tau}(h_{\mu\rho,\sigma} + h_{\sigma\rho,\mu} - h_{\mu\sigma,\rho})) \\
- \quad & \partial_\mu(\Gamma^0)^\tau{}_{\nu\sigma} + \partial_\mu(h^{\rho\tau}\Gamma^0_{\rho\nu\sigma}) - \partial_\mu((g^0)^{\rho\tau}(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho})) \\
+ \quad & [(\Gamma^0)^\alpha{}_{\mu\sigma} - h^{\rho\alpha}\Gamma^0_{\rho\mu\sigma} + (g^0)^{\rho\alpha}(h_{\mu\rho,\sigma} + h_{\sigma\rho,\mu} - h_{\mu\sigma,\rho})] \\
& [(\Gamma^0)^\tau{}_{\alpha\nu} - h^{\rho\tau}\Gamma^0_{\rho\alpha\nu} + (g^0)^{\rho\tau}(h_{\alpha\rho,\nu} + h_{\nu\rho,\alpha} - h_{\alpha\nu,\rho})] \\
- \quad & [(\Gamma^0)^\alpha{}_{\nu\sigma} - h^{\rho\alpha}\Gamma^0_{\rho\nu\sigma} + (g^0)^{\rho\alpha}(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho})] \\
& [(\Gamma^0)^\tau{}_{\alpha\mu} - h^{\rho\tau}\Gamma^0_{\rho\alpha\mu} + (g^0)^{\rho\tau}(h_{\alpha\rho,\mu} + h_{\mu\rho,\alpha} - h_{\alpha\mu,\rho})] \\
+ \quad & \mathcal{O}(h_{\mu\nu}h_{\alpha\beta,\gamma}) + \mathcal{O}(h^2_{\mu\nu})
\end{aligned}$$

$$\begin{aligned}
R_{\mu\nu\sigma}{}^\tau = \quad & R^0_{\mu\nu\sigma}{}^\tau \\
+ \quad & \partial_\mu(h^{\rho\tau}\Gamma^0_{\rho\nu\sigma}) - \partial_\nu(h^{\rho\tau}\Gamma^0_{\rho\mu\sigma}) + (\Gamma^0)^\alpha{}_{\nu\sigma}h^{\rho\tau}\Gamma^0_{\rho\alpha\mu} - (\Gamma^0)^\alpha{}_{\mu\sigma}h^{\rho\tau}\Gamma^0_{\rho\alpha\nu} \\
- \quad & h^{\rho\alpha}\Gamma^0_{\rho\mu\sigma}(\Gamma^0)^\tau{}_{\alpha\nu} + h^{\rho\alpha}\Gamma^0_{\rho\nu\sigma}(\Gamma^0)^\tau{}_{\alpha\mu} \\
+ \quad & (\partial_\nu(g^0)^{\rho\tau})(h_{\mu\rho,\sigma} + h_{\sigma\rho,\mu} - h_{\mu\sigma,\rho}) \\
- \quad & (\partial_\mu(g^0)^{\rho\tau})(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho}) \\
+ \quad & (\Gamma^0)^\alpha{}_{\mu\sigma}(g^0)^{\rho\tau}(h_{\alpha\rho,\nu} + h_{\nu\rho,\alpha} - h_{\alpha\nu,\rho}) \\
+ \quad & (g^0)^{\rho\alpha}(h_{\mu\rho,\sigma} + h_{\sigma\rho,\mu} - h_{\mu\sigma,\rho})(\Gamma^0)^\tau{}_{\alpha\nu} \\
- \quad & (\Gamma^0)^\alpha{}_{\nu\sigma}(g^0)^{\rho\tau}(h_{\alpha\rho,\mu} + h_{\mu\rho,\alpha} - h_{\alpha\mu,\rho}) \\
- \quad & (g^0)^{\rho\alpha}(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho})(\Gamma^0)^\tau{}_{\alpha\mu} \\
+ \quad & (g^0)^{\rho\tau}\partial_\nu(h_{\mu\rho,\sigma} + h_{\sigma\rho,\mu} - h_{\mu\sigma,\rho}) - (g^0)^{\rho\tau}\partial_\mu(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho}) \\
+ \quad & \mathcal{O}(h_{\mu\nu,\sigma}h_{\alpha\beta,\gamma}) + \mathcal{O}(h_{\mu\nu}h_{\alpha\beta,\gamma}) + \mathcal{O}(h_{\mu\nu}h_{\alpha\beta})
\end{aligned} \tag{37.5}$$

If we take the case of Minkowski space as the background metric, then things simplify immensely, with $g^0_{\mu\nu} = \eta_{\mu\nu}$, $g^0_{\mu\nu,\rho} = 0$, $(\Gamma^0)^\rho{}_{\mu\nu} = 0$ and $R^0_{\mu\nu\sigma}{}^\tau = 0$. If we choose to ignore every products of components of the perturbation and its derivatives, we have

$$R_{\mu\nu\sigma}{}^\tau = \eta^{\rho\tau}[\partial_\nu(h_{\mu\rho,\sigma} + h_{\sigma\rho,\mu} - h_{\mu\sigma,\rho}) - \partial_\mu(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho})]$$

By contraction we obtain the Ricci tensor

$$R_{\mu\sigma} = \lambda[(h_\mu{}^\nu{}_{,\nu\sigma} + h_\sigma{}^\nu{}_{,\mu\nu} - \Box h_{\mu\sigma}) - (h_\nu{}^\nu{}_{,\mu\sigma} + h_\sigma{}^\nu{}_{,\mu\nu} - h_{\nu\sigma,\mu}{}^\nu)] \tag{37.6}$$

By defining the trace of the field $h = h_{\mu\nu}\eta^{\mu\nu}$, we can rewrite this in the simplified form

$$R_{\mu\sigma} = \lambda[h_\mu{}^\nu{}_{,\nu\sigma} + h_{\nu\sigma,\mu}{}^\nu - \Box h_{\mu\sigma} - h_{,\mu\sigma}] \tag{37.7}$$

37.1 Coordinate gauge

Since the metric is fixed, we have to consider what happens in the case of a diffeomorphism on the manifold, corresponding to a coordinate change

$$y^\mu = f(x^\mu) \tag{37.8}$$

The metric itself transforms with the Jacobian $J^\mu_{\mu'}$

$$g_{\mu'\nu'} = J^\mu_{\mu'} J^\nu_{\nu'} g_{\mu\nu} = J^\mu_{\mu'} J^\nu_{\nu'} (\eta_{\mu\nu} + h_{\mu\nu}) \tag{37.9}$$

Hilbert gauge :
De Donder gauge :

$$\Box \bar{h}_{\mu\nu} = \kappa T_{\mu\nu} \tag{37.10}$$

37.2 Solution

Linearized gravity in the Hilbert gauge offers a fairly obvious comparison with electromagnetism, and we may then use similar methods to find its solutions if we choose to consider the case where T does not depend on h.
Green function :

$$\Box_x G(x, y) = \delta(x - y) \tag{37.11}$$

The Green function will be

$$\bar{h}_{\mu\nu} = \kappa(G * T_{\mu\nu}) \tag{37.12}$$

37.3 Linearized gravitational waves

Even for vacuum solutions $\Box\bar{h}_{\mu\nu} = 0$, we still get wavelike solutions

$$\bar{h}_{\mu\nu} = \int d^{n-1}p f_{\mu\nu}(p) e^{ix^\alpha p_\alpha} \tag{37.13}$$

$f_{\mu\nu}(p)$: polarization of the wave of momentum p
We get Minkowski space for the choice $f_{\mu\nu}(p) = 0$
Generation of gravitational waves by quadrupolar momentum
We will see fully non-linear gravitational waves later on.

38 Distributional general relativity

Stress-energy tensors that generate tensors only defined in the sense of distributions.

38.1 Thin shell formalism

If we allow the metric to be C^0, we can describe spacetimes $(\mathcal{M}, g)$ where there exists a region $D \subset \mathcal{M}$ for which $g|_D = g^+$ and $g|_{M\backslash D} = g^-$, with g^+, g^- arbitrary Lorentzian metrics with the only condition that, on the boundary, the limit of both coincide.

$$g^+(\partial D) = g^-(\partial D) \tag{38.1}$$

If we define a function $\eta(x)$ such that $\eta(\partial D) = 0$, then

$$g(x) = \Theta(\eta(x))g^+(x) + \Theta(-\eta(x))g^-(x) \tag{38.2}$$

Normal vector to the shell : $n_\mu = \nabla^\pm_\mu \eta$, $n^\mu n_\mu = 1$
Israel's Junction condition
Consider a smooth hypersurface Σ dividing the spacetime $\mathcal{M}$ in two parts $\mathcal{M}^+$ and $\mathcal{M}^-$, such that $\partial M^+ = \partial M^- = \Sigma$. The function $f : \mathcal{M} \to \mathbb{R}$ defining Σ by $f(\Sigma) = \{0\}$ is defined to be positive for $f(M^+)$ and negative for $f(M^-)$. Then we can define the characteristic functions

$$\begin{aligned}\chi^+ &= \chi_{\mathcal{M}^+} = f \circ \theta \\ \chi^- &= \chi_{\mathcal{M}^-} = 1 - \chi^+\end{aligned} \tag{38.3}$$

Up to the values on Σ, any function $h : \mathcal{M} \to V$ which has at most simple discontinuities in Σ can be written as

$$h = h^+\chi^+ + h^-\chi^- \tag{38.4}$$

with $h^\pm = h$ on the restriction to $\mathcal{M}^\pm$. For a point $p \in \Sigma$ and a sequence of points $q_n \in \mathcal{M}^+$, $r_n \in \mathcal{M}^-$, we define

$$\begin{aligned}[h](p) &= \lim_{q\to p} h^+ - \lim_{r\to p} h^- \\ h|(p) &= \frac{1}{2}(\lim_{q\to p} h^+ + \lim_{r\to p} h^-)\end{aligned} \tag{38.5}$$

which correspond to the discontinuity jump and the averaged value at Σ.
From algebra distribution properties :

$$\begin{aligned}\chi^+ \cdot \chi^+ &= \chi^+ \\ \chi^- \cdot \chi^- &= \chi^- \\ \chi^+ \cdot \chi^- &= 0 \\ \chi^+ \cdot \delta_\Sigma &= \frac{1}{2}\delta_\Sigma\end{aligned} \tag{38.6}$$

$$
\begin{aligned}
[h_1 + h_2] &= [h_1 + h_2] && (38.7)\\
[h_1 h_2] &= h_1|[h2] \text{ (if } [h_1] = 0) && (38.8)\\
(h_1 + h_2)| &= h_1| + h_2| && (38.9)\\
(h_1 h_2) &= h_1|h_2| \text{ (if } [h_1] = 0 \text{ or } [h_2] = 0) && (38.10)
\end{aligned}
$$

Part III

Behaviour of matter on spacetime

39 Classical mechanics

39.1 Point particles

Point particles are simply defined as curves in general relativity, with the class of curve determining the sign of the particle's mass, as the square of the mass is defined as being of the opposite sign of the norm of tangent vector.

$$\mathrm{sgn}(u^\mu u_\mu) = \mathrm{sgn}(-m^2) \tag{39.1}$$

which means that timelike curves correspond to particles with $m^2 > 0$ (also called tardyons), null curves to $m^2 = 0$ (or luxons), and spacelike curves to $m^2 < 0$ (or tachyons). This is based on the signature of the metric $(-,+,+,+)$, though it is also common to see $(+,-,-,-)$, in which case the sign will be flipped.

39.1.1 The Nambu-Goto action

The basic action for positive mass particles is proportional to the proper time of the curve

$$S = -m \int_{\lambda_1}^{\lambda_2} (g(u(\lambda), u(\lambda)))^{\frac{1}{2}} d\lambda \tag{39.2}$$

This definition isn't suitable for the action as a functional on $\mathcal{D}(\mathcal{M})$, so we will redefine it, with some abuse of notation, as

$$S = -m \int_{\lambda_1}^{\lambda_2} \int_\Sigma \sqrt{g(\dot{x}, \dot{x})} \delta^n(x - x(\lambda)) d^n x d\lambda \tag{39.3}$$

The action is a distribution that will map test functions to

$$S[\phi] = -m \int_{\lambda_1}^{\lambda_2} \sqrt{g(\dot{x}, \dot{x})} \phi(x(\lambda)) d\lambda \tag{39.4}$$

Hence we cannot use directly the Euler-Lagrange equation, as our action is not directly an integral, and we will have to use the Gâteaux derivative

$$\frac{\delta S[g_{\mu\nu}]}{\delta g_{\mu\nu}}[f] = \lim_{\varepsilon \to 0} \frac{S[g_{\mu\nu} + \varepsilon f] - S[g_{\mu\nu}]}{\varepsilon} \tag{39.5}$$

The difference between those two distributions is the distribution

$$\delta S[\phi] = -m \int_{\lambda_1}^{\lambda_2} \sum_{\mu,\nu} [((g_{\mu\nu} + \varepsilon f) u^\mu u^\nu)^{\frac{1}{2}} \phi(x(\lambda)) - (g_{\mu\nu} u^\mu u^\nu)^{\frac{1}{2}} \phi(x(\lambda))] d\lambda \tag{39.6}$$

This definition of the action
[...]

$$T^{\mu\nu} = m u^\mu u^\nu \delta(\vec{x} - \vec{x}(\lambda)) \tag{39.7}$$

Total energy :

$$E = \int_\Sigma T_{\mu\nu} \xi^\mu \xi^\nu = m g(u, \xi)^2 \delta(x - x(\lambda)) \tag{39.8}$$

In Minkowski space : $E = \gamma m$
If we pick the Fermi coordinates for the curves,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = m \xi_\mu \xi_\nu \delta(\vec{x}) \tag{39.9}$$

Show that it is spherically symmetric and static
The stress-energy tensor of a point particle is a source for a Schwarzschild black hole in the distributional sense.

39.1.2 General Lagrangian for point particles

This action has the drawback of only being defined for timelike curves. For a more general action that will accept particles of more general trajectories, we switch to the Polyakov action, where

$$S = \int \tag{39.10}$$

39.1.3 The Hamiltonian formalism for point particles

While the bundle explanation of the Hamiltonian formalism was skipped for the general case, it becomes fairly simple in the case of the point particle. While the Lagrangian is a function on the tangent bundle $(x^\mu, v^\mu) \in T\mathcal{M}$, the Hamiltonian becomes a function on the cotangent bundle (x^μ, p_μ).

$$X_0 = v_\mu(x^\mu, p_\mu)\partial_\mu + f_\mu(x^\mu, p_\mu) \tag{39.11}$$

$$\theta_0 = p_\mu dx^\mu \tag{39.12}$$

One-particle phase space : cotangent bundle $T^*\mathcal{M}$, symplectic 2-form

$$\omega = dx^\mu \wedge dp_\mu \tag{39.13}$$

Metric defines a solder form between the symplectic manifold and the tangent bundle

$$(x^\mu, p^\alpha) \mapsto (x^\mu, g_{\alpha\beta} p^\alpha) \tag{39.14}$$

Symplectic form :

$$\omega_g = dx^\mu \wedge d(g_{\mu\nu} p^\nu) \tag{39.15}$$

Hamiltonian vector field :

$$i_{X_g} \omega_g = dL \tag{39.16}$$

$$X_g = p^\mu \partial_\mu - \Gamma^\mu{}_{\rho\sigma} p^\rho p^\sigma \partial_\mu \tag{39.17}$$

39.2 Extended objects

39.2.1 The Polyakov action

Much like how we used the Polyakov action to represent point particles as an embedding of a 1-submanifold, we can generalize it as the embedding of a k-submanifold.
Consider the embedding $X : S \to \mathcal{M}$, the polyakov action for this is

$$S[X, \gamma] = \int d\mu[\gamma] \det(X^* g) \tag{39.18}$$

$(X^*(g))(u, v) = g(dX(u), dX(v))$
In a coordinate system on $\mathcal{M}$: $\phi^\mu \circ X(\sigma) = X^\mu(\sigma)$
Coordinates on the tangent bundle :

$$dX_p(\partial_a|_p) = \partial_a X^\mu \tag{39.19}$$

Pullback metric : $G_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$
[CHECK IT]

$$S[X, \gamma] = -M \int d\mu[\gamma] \gamma^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{39.20}$$

This is the general Polyakov action for extended objects, which can be used to approximate such things as topological defects in field theories or strings in string theory. The associated equation of motion is then

$$\frac{\delta S}{\delta \gamma^{ab}} = \tag{39.21}$$

$$\frac{\delta S}{\delta X^\mu} = \tag{39.22}$$

In Minkowski space :

$$T^{ab} = -\frac{1}{\alpha'}(G_{ab} - \frac{1}{2} h_{ab} h^{cd} G_{cd}) \tag{39.23}$$

EoM : $T_{ab} = 0$

39.2.1.1 Strings

The simplest Polyakov action for a dimension > 1 is the relativistic string, modelled by a $1+1$-dimensional submanifold.. It can be used to approximate the action of a cosmic string, as well as being the basis for string theory.
2 possible timelike submanifolds : sheet $\mathbb{R} \times I$ and tube $\mathbb{R} \times S$
Boundary conditions to get rid of the boundary terms :
Periodic boundary condition : $X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma)$ Neumann boundary condition : $\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0$ Dirichlet boundary conditions : $X^\mu(\tau, 0) = X^\mu(\tau, \pi)$, $\partial^\rho X^\mu(\tau, 0) = \partial^\rho X^\mu(\tau, \pi)$, $\gamma(\tau, 0) = \tau(\tau, \pi)$

39.2.1.2 Surfaces

The next model of higher dimensionality is surfaces, a $2+1$-dimensional submanifold. It can be used to approximate the action of domain walls.

40 Thermodynamics

40.1 The n-body problem

For a system of n point particles on a spacetime with coordinates x_i^μ for the i-th particle, system is

$$u_i^\mu \nabla_\mu u_i^\nu = m^{-1} F_i^\nu (x_j) \tag{40.1}$$

F : interactions between the point particles
Liouville equation

40.2 Vlasov matter

While point particles furnish a possible model for this, we want to keep it as general as possible. To allow this, we will instead consider particles as distributions
Number of particles :

$$\rho(p) = \int_{P_p\mathcal{M}} f\omega_p \tag{40.2}$$

$$N = \int_\Sigma f i_X \theta \tag{40.3}$$

Equation of N free particles : Lioville-Vlasov equations

$$p^\mu = \frac{dx^\mu}{d\lambda},\ \frac{dp^\mu}{d\lambda} = -\Gamma^\mu{}_{\rho\sigma} p^\rho p^\sigma \tag{40.4}$$

$$T^{\mu\nu}(x) = \int_{P_p\mathcal{M}} f(x,p) p^\mu p^\nu d\mu_p \tag{40.5}$$

40.3 Boltzmann distribution in curved spacetime

40.4 Entropy

Tipler and Boltzmann recurrence time
Poincaré theorem :

Theorem 40.1. For any second-countable, Hausdorff measure space (X, Σ, μ) with a one-parameter map T_t that preserves the measure on X, then for $A \in \Sigma$ with $\mu(A) > 0$, the set of points for there exists no $t \in \mathbb{R}$ such that $T_t(x)$ is not in an arbitrarily small neighbourhood of x has measure 0.

Proof. Consider a basis of open sets $\{U_n\}_{n\in\mathbb{N}}$ for X. For every $n \in \mathbb{N}$, we define

$$U'_n = \{x \in U_n,\ \forall n \geq 1,\ f^n(x) \notin U_n\} \tag{40.6}$$

□

41 Classical field theory

41.1 Scalar field

Section of the real or complex line bundle, transforms under the $(0,0)$ representation of the Lorentz group
Real :

$$\mathscr{L} = \frac{1}{2} g(\nabla\varphi, \nabla\varphi) - m^2\varphi^2 - \xi R\phi - V(\phi) \tag{41.1}$$

Complex :

$$\mathscr{L} = \frac{1}{2} g(\nabla\varphi, (\nabla\varphi)^*) - m^2|\varphi|^2 - \xi R|\phi| - V(|\phi|) \tag{41.2}$$

In coordinates :

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - m^2\varphi^2(x) - \xi R(x)\phi(x) - V(\phi(x)) \tag{41.3}$$

Klein-Gordon equation

$$(\Box + m^2)\varphi + \xi R + V(\phi) = 0 \tag{41.4}$$

Canonical momentum for a :

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \xi^\mu \partial_\mu \phi \tag{41.5}$$

Hamiltonian density :

$$\mathscr{H} = \tag{41.6}$$

Important class of potentials :
$V(\phi) = 0$: free field $V(\phi) = g\phi^3$: cubic interaction $V(\phi) = g\phi^4$: quartic interaction $V(\phi) = g\sin(\phi)$: Sine-Gordon

41.2 Spinor fields

Section of the associated bundle to the Clifford bundle
Representations : $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$$({e_a}^\mu \gamma^a D_\mu + m)\psi + V(\psi) = 0 \tag{41.7}$$

Potentials :
$V(\psi) = 0$: Free field $V(\psi) = (\bar{\psi}\psi)^2$: 4-point interaction $V(\psi) =$ (the axial current one)
Majorana equation

41.3 Electromagnetic fields

Connection on the U(1) principal bundle
Curvature $F = dA$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{41.8}$$

$$\mathscr{L} = \frac{1}{4\mu_0}\mathrm{Tr}(F \wedge F^*) = \frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} \tag{41.9}$$

Maxwell equation :

$$\nabla_\mu F^{\mu\nu} = j^\nu \tag{41.10}$$

$$\nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) = j^\nu \tag{41.11}$$

$$\Box A^\nu - \nabla_\mu \nabla^\nu A^\mu = j^\nu \tag{41.12}$$

Gauge invariance : invariant under the action of U(1) on the connection
For the Lorentz gauge :

$$\nabla_\mu A^\mu = 0 \tag{41.13}$$

$$\Box A^\nu + [\nabla^\nu, \nabla_\mu] A^\mu - \nabla^\nu \nabla_\mu A^\mu = j^\nu \tag{41.14}$$

$$\Box A^\nu - R^\nu{}_\rho A^\rho = j^\nu \tag{41.15}$$

Dual electromagnetic tensor

41.3.1 The Einstein-Maxwell equations

Combining the Einstein field equations and the Maxwell equations gives us a set of partial differential equations called the Einstein-Maxwell equations, which describe spacetimes with sourceless electromagnetic fields, also called electrovacuum spacetimes. The equations are then of course

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa(F_{\mu\sigma}F_{\nu\rho}g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho}F^{\sigma\rho}) \tag{41.16}$$

$$\nabla_\mu F^{\mu\nu} = 0,\ \nabla_{[\mu} F_{\mu\nu]} = 0 \tag{41.17}$$

Properties
Rainich conditions :

$$T^\mu{}_\mu = (F_{\mu\sigma}F^{\mu\sigma} - \frac{n}{4} F_{\sigma\rho}F^{\sigma\rho}) \tag{41.18}$$

For $n = 4$, $T = 0$, implying $R = 0$
Penrose-Newman
Kaluza-Klein

41.4 Yang-Mills fields

Connection on the principal bundle SU(N) (or U(1) for $N = 1$).
We will only consider here $N > 1$ since the case of U(1) was already done in the previous section.
Generators of SU(N) : $N - 1$ dimensions so $N - 1$ generators T_a of the algebra $\mathfrak{su}(N)$ of traceless hermitian complex $N \times N$ matrices

$$\{T_a, T_b\} = \frac{1}{N}\delta_{ab} I_n + \sum_{c=1}^{N^2-1} d_{abc} T_c \tag{41.19}$$

$$[T_a, T_b] = i \sum_{c=1}^{N^2-1} f_{abc} T_c \tag{41.20}$$

$$content... \tag{41.21}$$

Part IV

The Cauchy problem

42 Classical fields and particles

42.1 The Cauchy problem of classical fields and particles on a globally hyperbolic spacetime

Evolution of a set of fields (sections of vector bundles) with values known in a subset of spacetime to the entirity of spacetime, for a fixed spacetime $(\mathcal{M}, g)$
Initial conditions on hypersurface S : $(\phi_i|_S, \nabla\phi_i|_S)$
Second order linear PDEs :

$$A_{\mu\nu}\frac{\partial^2 u(t,x_i)}{\partial x_\mu \partial x_\nu} + B_\mu \frac{\partial u(t,x_i)}{\partial x_\mu} + C(t,x_i)u(t,x_i) = D(t,x_i) \tag{42.1}$$

If derivatives commute : $A_{\mu\nu}$ is symmetric

Theorem 42.1. The second order terms can be rewritten as $A_\mu u_{\mu\mu}$ with an appropriate change of coordinates.

Canonical form :

$$A_\mu u_{\mu\mu} + B_\mu u_\mu + Cu = D \tag{42.2}$$

Quasilinear hyperbolic equations

Definition 42.2. A PDE is said to be :

- *elliptic* if all the eigenvalues of $A_{\mu\nu}$ are positive or all are negative
- *hyperbolic* if none are zero and one has the opposite sign of the $(n-1)$ others
- *parabolic* if one is zero and all others are of the same sign
- Ultrahyperbolic if none are zero and the same number are positive and negative.

In the canonical form :
Elliptic :

$$\sum_{i=1}^{n} u_{ii} \tag{42.3}$$

Hyperbolic :

$$-u_{tt} + \sum_{i=2}^{n} u_{ii} \tag{42.4}$$

Parabolic :

$$\sum_{i=2}^{n} u_{ii} \tag{42.5}$$

Ultra hyperbolic :

$$\sum_{i=1}^{n/2} u_{ii} - \sum_{i=n/2+1}^{n} \tag{42.6}$$

Constraint equations

42.1.1 Point particles

Cauchy problem for causal point particles → uniqueness of solutions for causal curves
Solution is unique for globally hyperbolic spacetimes
Show tachyons don't have a well defined Cauchy problem in general

42.1.2 Classical fields

That paper Valter Moretti linked on the wave equation in GR

42.2 The Cauchy problem of classical fields and particles on non-globally hyperbolic spacetime

Definition 42.3. An open set $U \subset \mathcal{M}$ is causally regular if for every $\varphi \in C(U)$ that satisfies $\Box\varphi = 0$, there is a function $\varphi' \in C(\mathcal{M})$ that satisfies $\Box\varphi$ and $\varphi'|_U = \varphi$

Definition of Yurtsever :

Definition 42.4. An open set U is causally regular if its closure $\bar{U}$ contains an open neighbourhood U' such that for every $(n/2-1)$-form $\omega \in \Gamma(\Lambda^{n/2-1}U')$ satisfying $\Box\omega$ on U, there exists a smooth form ω' on $\mathcal{M}$ such that $\omega'|_U = \omega$ and $\Box\omega' = 0$.

With this definition, $\mathcal{M}$ itself is obviously causally regular.
Causally regular : a local solution can be extended to a global solution

Definition 42.5. A point $p \in \mathcal{M}$ is causally regular if every neighbourhood U such that $p \in U$ contains a causally regular neighbourhood.

(Uniqueness of solution?)
(Does it apply for fields that are hyperbolic but not free scalars)

Definition 42.6. A spacetime is causally benign if for every open set U, there is an open set $U' \subset U$ which is causally regular.

43 The Cauchy problem in general relativity

In a manner similar to the Cauchy problem for fields, the Cauchy problem for spacetime itself will involve a set of initial data for the spacetime to evolve.

Definition 43.1. A set of initial data for a spacetime is a triplet $(\Sigma, \bar{g}, K)$ such that Σ is a hypersurface of codimension 1, $\bar{g}$ is a Riemannian metric and K is a rank $(0, 2)$ tensor on Σ.

From given initial data find a solution (ideally unique) for future times.

Definition 43.2. A development of a set of initial data is a triplet $(\mathcal{M}, g, \sigma)$ where $(\mathcal{M}, g)$ is a Lorentz manifold and σ is a diffeomorphism $\sigma : \Sigma \to \mathcal{M}$

In other words, we have an initial spacelike hypersurface (which we can consider as some spacelike neighbourhood at a given time), with the values of the metric and its derivatives on it.

43.1 The Cauchy problem for globally hyperbolic vacuum solutions

Uniqueness of equations : $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ + some gauge condition
Splitting via ADM formalism

43.2 The Cauchy problem for globally hyperbolic solutions with matter fields

cf Hawking

43.3 The Cauchy problem for non-globally hyperbolic solutions

Once we drop the requirement of global hyperbolicity, the situation becomes much more complicated, as can be seen simply by considering the examples of taking Minkowski space $(\mathbb{R}^n, \eta)$ and Minkowski space minus the point $\{(t, \vec{x}) | t > 0,\ \vec{x} = 0\}$. Not only is the development of some surfae at $t = 0$ non-unique (in fact, without any further conditions, there are infinitely many developments), but we are not even required to have the same topology of the spacelike hypersurface.
Non-unique development from topology (cf Krasnikov)

43.3.1 Cauchy problem and singularities

Boundary conditions on singularities
Example : Kerr extremal black hole

43.3.2 Cauchy problem and closed causal curves

Krasnikov on the non-uniqueness of development for non-causal spacetimes
Non-unique matter evolution
Benign closed timelike curves

Part V

Alternative theories of relativity

44 Theories of more general connections

If we drop the constraints on the Koszul connection of the frame bundle, we will get a theory with an independent metric and connection term. It is described by the Palatini action that we saw earlier.

$$S[e,\omega] = \frac{1}{2\kappa}\int \varepsilon_{abcd}(e^a{}_\mu \wedge e^b{}_\nu \wedge R^{bc} - \frac{1}{12}\Lambda) \tag{44.1}$$

Or just by using the Einstein-Hilbert action with independent terms

$$S[g^\sharp,\Gamma] = \frac{1}{2\kappa}\int d\mu[g](g^{\mu\nu}R_{\mu\nu}(\Gamma) + \Lambda) \tag{44.2}$$

$$\frac{\delta S}{\delta e} = \tag{44.3}$$

$$\frac{\delta S}{\delta \omega} = \tag{44.4}$$

$$G^{\mu\nu} = \kappa T^{\mu\nu} = \tag{44.5}$$

3 parameters : curvature C, torsion T, non-metricity N
Metric-affine gravity CTN Weitzenböck gravity (teleparallel gravity) T Einstein Cartan CT General relativity C

44.1 Metric-affine gravity

Curvature, torsion, non-metricity
Source of torsion and non-metricity : hypermomentum, defined by

$$\frac{\delta S_M}{\delta \omega} \tag{44.6}$$

44.2 Einstein-Cartan-Sciama-Kibble theory

The Einstein-Cartan-Sciama-Kibble theory (or Einstein-Cartan theory) is the case of a theory with a connection that displays both curvature and torsion, but is still metric.

$$x \tag{44.7}$$

Source of torsion : the spin density tensor, mostly sourced by fermion fields

44.3 Teleparallel gravity

No curvature, only torsion
Same results as Einstein general relativity

45 Scalar and vector theories

The first theories attempted for relativistic theories of gravity were generally scalar theories.

45.1 Einstein-Nordstrøm theory

$$\Box\phi = -4\pi GT \tag{45.1}$$

$$\mathscr{L} = \frac{1}{8\pi}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \rho\phi \tag{45.2}$$

Equivalent to $g_{\mu\nu} = \phi^2\eta_{\mu\nu}$, or $e^a{}_\mu = \phi\delta^a{}_\mu$, with curvature

$$R = -\frac{6\Box\phi}{\phi^3} \tag{45.3}$$

$$R = 24\pi T \tag{45.4}$$

45.2 Einstein's scalar theory

$$T_g^{\mu\nu} = \frac{1}{4\pi G}[\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\rho\phi\partial^\rho] \tag{45.5}$$

$$T_m^{\mu\nu} = \rho\phi u^\mu u^\nu \tag{45.6}$$

46 $f(R)$ gravity

$$\mathscr{L} = \frac{1}{2\kappa} f(R) \tag{46.1}$$

$$\frac{\partial f(R)}{\partial R} R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + [g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu] \frac{\partial f(R)}{\partial R} = \kappa T_{\mu\nu} \tag{46.2}$$

47 Brans-Dicke gravity

$$S = \int d\mu[g] \frac{1}{16\pi}(\phi R - \omega \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi} + \mathscr{L}_M) \tag{47.1}$$

$$\Box \phi = \frac{8\pi}{3 + 2\omega} T \tag{47.2}$$

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2}(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi) + \frac{1}{\phi}(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi)$$

48 Pauli-Fierz gravity

General relativity needs not necessarily be described by a theory of dynamic geometry for spacetime. If the problem is considered from the point of view of a field theory, then by its basic properties, we obtain a field theory that is massless (due to its r^{-2} decay), of integer spin (due to [static???]), of even spin (due to being universally attractive) and that cannot be of spin 0, since such a theory does not predict the observed deflection of light. As spins superior to 2 tend to have pathological properties, let's consider the theory of spin 2 (it is possible also to have a theory of spin 2 and 0 which will be equivalent to a Brans-Dicke type theory).
The free theory of spin 2 is uniquely determined by a symmetric tensor field $h_{\mu\nu}$ with Lagrangian

$$\mathscr{L} = -\frac{1}{4} h_{\mu\nu,\sigma} h^{\mu\nu,\sigma} + \frac{1}{2} h_{\mu\nu,\sigma} h_{\sigma\nu,\mu} + \frac{1}{4} (\partial_\sigma h)(\partial^\sigma h) - \frac{1}{2} (\partial_\sigma h)(\partial_\nu h^{\nu\sigma}) \tag{48.1}$$

with $h = h_{\mu\nu} \eta^{\mu\nu}$
This is the same form as the linearized gravity that we have seen earlier for a flat background metric. And indeed, we will see that in its final form, the Pauli-Fierz theory is the equivalent of the substitution $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.
Issues with coupling : for

$$\mathscr{L} = \mathscr{L}_{PF} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \tag{48.2}$$

Equation of motion : $\Box \bar{h}_{\mu\nu} = 0$, $\Box \phi = 0$
Noether current
Decomposition of the determinant :

$$\det(g) = \exp(\mathrm{Tr}(\ln(g))) \tag{48.3}$$

$$\sqrt{\det(g)} = \exp(\frac{1}{2} \mathrm{Tr}(\ln(\eta + h))) \tag{48.4}$$

While it should be classically equivalent to general relativity, it will later on be useful for the quantization of gravity.

49 Massive gravity

Pauli Fierz with a mass term

50 Kaluza-Klein

The Kaluza-Klein theory was one of the first attempt at a unification of general relativity with other forces.
Manifold of $n+1$ dimensions. For $n = 4$:

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & g_{\mu 5} \\ g_{5\nu} & g_{55} \end{pmatrix} \tag{50.1}$$

Topology : $\hat{\mathcal{M}} = \mathcal{M} \times S$, the 5th dimension is compactified, characteristic dimension of S is very small

$$\begin{aligned} B_\mu &= \frac{g_{\mu 5}}{g_{55}} \\ \Phi &= g_{55} \end{aligned}$$

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + \Phi B_\mu B_\nu & \Phi B_\mu \\ \Phi B_\nu & \Phi \end{pmatrix} \tag{50.2}$$

51 Supergravity

Extension of symmetry groups from the Poincaré group to symmetries between different reps.

51.1 Grassmann algebras

51.2 Supermanifolds

Supermanifolds are defined in the same manner as manifolds, but the space is no longer locally diffeomorphic to Euclidian space, but to the product $\mathbb{R}^n_c \times \mathbb{R}^m_a$, of the usual n-dimensional Euclidian space and anticommuting numbers.
The definitions will be somewhat similar to those of a manifold, with a few nuances.

Definition 51.1. A supermanifold of dimension (n, m) is a topological space M with a complete atlas (U_α, ϕ_α) of maps

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n_c \times \mathbb{R}^m_a \tag{51.1}$$

Canonical isomorphism between the tangent bundles of Lorentz spin manifolds and supermanifolds

Theorem 51.2. For a spin manifold $(\mathcal{M}_1, g)$ with a spinor bundle $\pi_S : S \to \mathcal{M}_1$ and a supermanifold $(\mathcal{M}_2, g)$, for every $p \in \mathcal{M}_1$, there is a canonical isomorphism of $\mathbb{Z}_2$-graded vector spaces

$$\iota : T_p\mathcal{M}_1 + S^*_p \to T_p\mathcal{M}_2 \tag{51.2}$$

Proof. cf Killing spinors are Killing vector fields in Riemannian Supergeometry □

Supersymmetric groups : Graded $\mathbb{Z}_2$ Lie algebra : Lie superalgebra
Supersymmetric Poincaré algebra : additional generators that transform as undotted spinors Q^I_A or dotted $\bar{Q}^I_{\dot{A}}$

$$\begin{aligned}
[P_\mu, Q^I_A] = [P_\mu, \bar{Q}^I_{\dot{A}}] &= 0 \\
[M_{\mu\nu}, Q^I_A] &= i(\sigma_{\mu\nu})_A{}^B Q^I_B \\
[M_{\mu\nu}, \bar{Q}^{I\dot{A}}]] &= i(\bar{\sigma}_{\mu\nu})^{\dot{A}}{}_{\dot{B}} Q^{I\dot{B}} \\
\{Q^I_A, \bar{Q}^J_{\dot{B}}\} &= 2\sigma^\mu_{A\dot{B}} P_\mu \delta^{IJ} \\
\{Q^I_A, Q^J_B\} &= \varepsilon_{AB} Z^{IJ} \\
\{\bar{Q}^I_{\dot{A}}, \bar{Q}^J_{\dot{B}}\} &= \varepsilon_{\dot{A}\dot{B}} Z^{IJ}
\end{aligned} \tag{51.3}$$

Superspace : space with coordinates (x^μ, θ^A), with fermionic θ
Supermanifold :
$\{\theta, \theta\} = \theta^2 = 0$

$$S_{EH} = \frac{1}{2\kappa} \int d\mu [e] R^{ab}_{\mu\nu}(\omega)(e^{-1})^\mu_a (e^{-1})^\nu_b \tag{51.4}$$

$$S_{EH} = \frac{1}{2\kappa} \int d^n x \varepsilon_{abcd} \varepsilon^{\mu\nu\rho\sigma} R^{cd}_{\rho\sigma}(\omega) e^a_\mu e^b_\nu \tag{51.5}$$

$$S_{EH} = \frac{1}{2\kappa} \int d^n x \varepsilon_{abcd} e^a \wedge e^b \wedge R^{cd}(\omega) \tag{51.6}$$

Rarita-Schwinger field :

$$S_{RS} = -\frac{i}{2} \int d^n x \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma \tag{51.7}$$

$\mathcal{N} = 1$ supergravity :
multiplet $\{e^a_\mu, \psi^a_\mu\}$

52 Regge calculus

52.1 Simplicial manifolds

Define spacetime as a graph of spacetime points linked together by edges, metric is defined by the length of the edges or their angles
Definition : The lattice definition of a spacetime is a graph (v_i, e_i) with spacetime points for vertices and the edges defining the distances between two points. To every edge between points v_i and v_j is associated a real number which is the length between the two points.
It is a simplicial complex.
Curvature on the hinges : in an n-dimensional spacetime, the lattice is made of n-simplices, with the curvature concentrated on subsimplices of dimension $n-2$, aka hinges.
Deficit angle : for a point v_i, consider the angle made by every simplicial n-polytope around v_i. The deficit angle

$$S = \frac{1}{2\kappa} \int R d\mu[g] \to S_R = \sum_i |\sigma^i| \varepsilon_i \tag{52.1}$$

52.2 In two dimensions

$$S_R = \sum_i l^i \varepsilon_i \tag{52.2}$$

53 Theories of different signatures

While many theories of a spacetime with a non-Lorentzian metric are not physically relevant, they can be interesting to study.

53.1 Signature conventions

One important issue regarding the signature of spacetime is that there are two signature with apparent similarity, the Lorentzian signature $-+++$ and its inverse $+---$.

53.2 Riemannian spacetime

As all Hausdorff, paracompact manifolds admit a Riemannian metric
The most important difference of a Riemannian spacetime is that the lack of a timelike dimension means that there are no classification of vectors or curves. Hence we cannot pick a particular class of curves as observers.
If we just pick any curve as possible observers, then every observer carries its own time with him. If we have a set of curves with roughly the same tangent, we can define some sort of common time orientation for those observers. This idea was explored by Greg Egan.
Riemannian manifold is a metric space.

53.2.1 Riemanninan black hole

$$ds^2 = (1 + \frac{2M}{r})dt^2 + (1 + \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{53.1}$$

Distance function :

$$d(0, r) = \sqrt{2(r + 2M)} - 2M \operatorname{arcsinh}(\sqrt{r/2M}) \tag{53.2}$$

Surface at r : $4\pi r^2$, $d(0, r) < r$, because negative curvature

53.3 Ultrahyperbolic spacetime

For a spacetime of dimension $n > 2$, it is possible to have a signature where the number of positive and negative eigenvalues are both superior to 1. Such a spacetime, of signature (p, q), $p, q > 1$, are called ultrahyperbolic spacetimes, so called because the wave equation on such a spacetime is an ultrahyperbolic PDE, that is

$$\sum_{i,j} a_{ij}\partial_i\partial_j f + \sum_i b_i \partial_i f + cf = d \tag{53.3}$$

where a_{ij} itself has a signature (p, q).
Stability, Cauchy, etc

53.4 Degenerate signatures

(p, q, r), singular points

53.5 Dynamic signature

signature $f(x) + ++$

54 Misc. Theories

Some theories rely on the replacement of spacetime manifolds by more general structures
Manifolds with boundaries : charts are on $\phi : U \to \mathcal{O} \subset \mathbb{H}^n$, the half space
Conifolds

54.1 Hoyle's theory of gravity

Hoyle's theory of gravity was a modification of the Einstein field equations to allow a (globally) stationary universe to still remain possible with cosmological observations. It is obtained by the addition of a tensor field C called the creation field

$$G_{\mu\nu} + C_{\mu\nu} = \kappa T_{\mu\nu} \tag{54.1}$$

The creation field is so called because in general, it will lead to a break of the local conservation of energy, as can be easily seen from the Bianchi identity

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu C^{\mu\nu} \tag{54.2}$$

The creation field is obtained by considering a timelike vector field ξ, in which case

$$C_{\mu\nu} = \nabla_\nu \xi_\nu \tag{54.3}$$

If $\xi_\mu = C(1, 0, 0, 0)$, $C_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu}\xi_\sigma$, which is symmetrical.
For the FRW spacetime :

$$C_{ij} = -CR\dot{R}\delta_{ij} \tag{54.4}$$

Field equations :

$$\begin{aligned} 2R\ddot{R} + \dot{R}^2 - CR\dot{R} &= 0 \\ 3\dot{R}^2 &= \kappa\rho R^2 \end{aligned} \tag{54.5}$$

For $R = 1$ at $t = 0$,

$$R = e^{t\frac{Ct}{3}} \tag{54.6}$$

Used to prevent the singularity in the FRW model

Part VI

Quantum theory and general relativity

55 Quantum theory in general relativity

55.1 Quantum theories

For the most part, all quantum theories share the following axioms :

1. Physical states are represented by rays in a Hilbert space, that is, for $\psi \in \mathcal{H} \setminus \{0\}$, a state is represented by $\psi / \|\psi\|$.
2. Observables are represented by self-adjoint operators on this Hilbert space.
3. For a system in a state ψ, an experiment to measure a quantity associated with the observable $\hat{A}$, with eigenvectors $\{\psi_n\}$ and eigenvalues A_n, the probability of measuring that system in a state ψ_k (with A_k as a result of this measurement) is

$$P(X = A_k) = \langle \psi, \psi_k \rangle$$

55.2 Quantization

While we will want to define quantum theories independently of their classical counterparts, it will be useful to generate some quantum theories from the current existing theories we have as a starting point. This is done by a process called quantization.
There are several ways to perform quantization. We will only see three such process : canonical quantization, path integral quantization and deformation quantization.

55.2.1 Canonical quantization

Canonical quantization is the most common and the first historically of the quantization methods. Given a classical theory with a set of observable quantities $\{A_i\}$, we put each of those quantities in correspondance with a linear hermitian operator acting on a projective Hilbert space $\mathscr{H}$.
Transformation of Lie brackets to commutators. For two classical observables A_c, B_c, converted to linear operators on a Hilbert space A_q, B_q

$$\{A_c, B_c\} \to \frac{1}{i\hbar}[A_q, B_q] \tag{55.1}$$

If we obtain the product of two non-commuting operators A, B for a given quantity, any operator of the form $\alpha AB + \beta BA$ will be a valid quantization. The correct order will usually be implied by constraints of symmetry and self-adjointness.

55.2.2 Path integral quantization

A rather simple if hard to define method of quantization is the path integral, for which we need to define a measure on the set of possible configuration, giving us the transition between two configurations as

$$K(C_1, C_2) = \int_{C_1}^{C_2} d\mu[C] \tag{55.2}$$

with the measure being usually defined on some variation of the Gaussian measure

$$d\mu = \lim_{n\to\infty} \exp(i \int \mathscr{L} d^n x) \prod_{i=0}^{n} \frac{dC_i}{A} \tag{55.3}$$

Measurable quantities will then be defined in a similar manner by considering the transition

$$A(C_1, C_2) = \int_{C_1}^{C_2} A(C) d\mu[C] \tag{55.4}$$

55.2.3 Deformation quantization

The idea of deformation quantization is to keep all quantities classical but to reproduce the results of quantum theory by the choice of a specific product, the Moyal product $*$. Since we want that in the classical limit $\hbar \to 0$, the results align with the classical theory, this imposes the following conditions :

$$\begin{aligned} f * g &= fg + \mathcal{O}(\hbar) \\ [f, g] &= i\hbar\{f, g\} + \mathcal{O}(\hbar^2) \end{aligned}$$

55.2.4 BRST quantization

A lot of quantum systems involve constrained systems, for which the configuration space is too large : many states are physically equivalent. When quantized, this causes a variety of problem due to a too large Hilbert space, in particular the quantization procedure produces either states for which the inner product isn't positive definite or the path integral diverges.

55.3 Symmetries

55.3.1 Symmetries

We need to define how symmetries act on the Hilbert space of quantum theories, as we may need to change things such as gauge or coordinates. Generally speaking this will map states by some transformation

$$S : \mathcal{H} \to \mathcal{H} \tag{55.5}$$

Hence a physical state ψ will get mapped to $S(\psi)$. We will ask that this map be a bijection so that two different states do not get mapped onto the same one, and we also require that the probabilities of measurements do not change. In other words, for a measurement of $\hat{A}$ with eigenvectors ψ_n and eigenvalues A_n, we want

$$P(X = A_k) = P(X = A'_k) = \langle \psi, \psi_k \rangle = \langle S(\psi), S(\psi_k) \rangle \tag{55.6}$$

where A'_k is the image of the eigenvalue. To make things more general, we can also use the more general condition

$$|\langle \psi, \phi \rangle| = |\langle S(\psi), S(\phi) \rangle| \tag{55.7}$$

Definition 55.1. A symmetry transformation on a Hilbert space is a bijection $S : \mathcal{H} \to \mathcal{H}$ such that

$$|\langle\psi, \phi\rangle| = |\langle S(\psi), S(\phi)\rangle| \tag{55.8}$$

Proposition 55.2. Symmetry transformations have a group structure.

Proof.

- Composition :
- Unity : The identity map $\mathrm{id}_{\mathcal{H}}$ is a symmetry transformation.
- Inverse :
- Associativity :

□

Wigner's theorem :

Theorem 55.3. For a symmetry transformation S, there exists a transformation U which is compatible with S such that U is either unitary or antiunitary.

55.4 Whatever

55.4.1 Hegerfeldt's theorem

There are many interpretations and formalisms for quantum theory, but we will take the standard couple where the quantum theory is defined by states in a Hilbert space (or any equivalent formalism) and use some variant of the Copenhagen interpretation.
If we try to construct the usual construction of the quantum theory of a single particle, we can probably assume the following basic structures :

- A separable Hilbert space $\mathcal{H}$
- A projection P_U on $\mathcal{H}$ for every open set U
- Invariance of measurements under diffeomorphism??? (in malament : unitary rep of the translation group)

P_U : the possibility of measurement of a particle in U if a measurement is performed.
Constraints from Malament :

- Translation covariance : for a translation a, $PU + a = U(a)P_U U(-a)$ [Replace by a diffeomorphism?]
- Energy : for a future directed timelike vector ξ such that $U(t\xi) = e^{itH(\xi-)}$ (GR equivalent?)
- Localizability : if $U_1 \cap U_2 = \varnothing$ in the same achronal spacelike hyperplane, $P_{U_1}P_{U_2} = P_{U_2}P_{U_1} = 0$
- Locality : If U_1, U_2 are spacelike related, $P_{U_1}P_{U_2} = P_{U_2}P_{U_1}$

Localizability : particle can't be in two places at once at the same time

Theorem 55.4. If a quantum theory satisfies all outlined conditions, $P_U = 0$ for every open set $U \subset \mathcal{M}$.

This means that any quantum theory of single particles on a spacetime will be the trivial theory of no particles.
(This theorem does not seem to apply to Bohmian quantum mechanics)

55.5 The relativistic point particles

The simplest possible quantum system in general relativity is the quantization of the relativistic point particle, as we've seen earlier. While not the most realistic quantum system for relativistic particles (except in circumstances such as the worldline formalism), it remains a possible one. We can do it in two ways, using the standard action or the Polyakov action.

55.5.1 Point particle action

As seen previously, the action of a massive particle can be expressed by

$$S = -\int_{\tau_i}^{\tau_f} m\sqrt{g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{55.9}$$

with the Poisson brackets

$$\{x^a, p_b\} = \delta^a_b \tag{55.10}$$

The quantization of those variables as operators is straightforwardly the same as in classical mechanics, that is,

$$[\hat{x}^a, \hat{p}_b] = i\hbar\delta^a_b \tag{55.11}$$

Both $\hat{x}^a$ and $\hat{p}_a$ are unbounded operators, which will lead to some difficulties for the use of theorems pertaining to linear operators on Hilbert spaces and the definition of derivatives. Instead we will switch to the Weyl operators, where for a given $t, s \in \mathbb{R}$,

$$\begin{aligned} Q &\to U = e^{itx} \\ P &\to V = e^{isp} \end{aligned} \tag{55.12}$$

Proposition 55.5. The Weyl operators obey the *Weyl form of the CCR*

$$U(t)V(s) = e^{-ist}V(s)U(t) \tag{55.13}$$

Proof. □

Stone-von Neumann theorem :

Theorem 55.6. Up to unitary equivalence, there is only one representation for $\mathcal{H} = L^2(\mathbb{R}^n, d^n x)$ of the finite-dimensional commutation relation

$$[\hat{x}^a, \hat{p}_b] = i\hbar\delta^a_b \tag{55.14}$$

For any $\psi \in L^2(\mathbb{R}^n, d^n x)$,

$$\begin{aligned} \hat{x}^a \psi(x) &= x^a \psi(x) \\ \hat{p}_a \psi(x) &= -i\hbar \partial_a \psi(x) \end{aligned} \tag{55.15}$$

Weyl version (rigorous) :

$$- \tag{55.16}$$

Something something rigged Hilbert space
Modes :

$$\Box \psi(x) = 0 \tag{55.17}$$

55.5.2 Canonical quantization

55.5.3 Polyakov quantization

55.5.4 BRST quantization of the point particle

55.6 Quantization of extended objects

For later chapters, it will be useful to also perform quantization of the Polyakov action of other extended objects, primarily the quantization of strings.

56 Quantum field theory in curved spacetime

The point-particle quantization suffers from some pathologies if we attempt to include interactions. To remedy the situation, the notion of field is introduced to replace point particles.

56.1 Defining quantum field theories

56.2 Algebraic quantum field theory

The main method by which quantum fields are defined on spacetimes will be variations on algebraic quantum field theory, or AQFT, based on Haag's axioms. We will also use perturbative QFT, which is not on as firm footing but will be necessary for heuristic arguments for interacting theories.
For now, let's define axioms for the free theory :

1. An algebraic quantum field theory is a net of abstract C^*-algebras mapping open sets $\mathcal{O}$ of the spacetime manifold to a C^*-algebra $\mathfrak{A}(\mathcal{O})$
2. Isotony axiom : For $\mathcal{O}_1 \subset \mathcal{O}_2$, there is an embedding $\alpha_{12} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2)$
3. States are defined as positive linear functionals $\omega : \mathfrak{A} \to \mathbb{C}$, such that $\omega(1) = 1$
4. A state ω must satisfy the microlocal spectrum condition
5. Covariance axiom : For every isometry g of spacetime, there exists an automorphism α_g such that $\alpha_g \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(g\mathcal{O})$
6. Timeslice axiom : if Σ is a Cauchy surface, then $\mathfrak{A}(\Sigma) = \mathfrak{A}(\mathcal{M})$
7. Microcausality axiom : Elements of $\mathfrak{A}(\mathcal{O}_1)$ commute with elements of $\mathfrak{A}(\mathcal{O}_2)$ if the two regions are spacelike separated (there are no causal curves connecting them).

Let us now define all those axioms

56.2.1 C^*-algebras and von Neumann algebras

C^*-algebras are algebras with the following properties :

- There is an automorphism $*$ such that for every $A \in \mathfrak{A}$, there exists an adjoint element $A^* \in \mathfrak{A}$, with properties
  - $(A^*)^* = A$
  - $(AB)^* = B^* A^*$
  - $(\lambda A + B)^* = \bar{\lambda} A^* + B^*$
- There's a norm $\| \cdot \|$ that satisfies $\|A^* A\| = \|A\|^2$

- The algebra is complete with respect to this norm, that is, for any Cauchy sequence (A_n)
$$\|A_n - A_m\| \rightsquigarrow_{n,m\to\infty} 0$$
then there is an element A such that (A_n) converges to A.

$\mathcal{H}$ a Hilbert space, $\mathfrak{B}(\mathcal{H})$ bounded linear operators on $\mathcal{H}$. That is, for $A \in \mathfrak{B}(\mathcal{H})$, there is an upper bound for $\langle Ax, Ax\rangle$ for all unit vectors in $\mathcal{H}$.

56.2.2 States and the GNS construction

States in AQFT are defined by linear functionals on the algebra, in a manner similar to how Hilbert space vectors interact with linear operators on a Hilbert space. This is how we will be able to construct the Hilbert space from the algebra later on via the GNS construction.
States have the property of being positive, that is, for $A \in \mathfrak{A}$, $\omega(A^*A) \geq 0$, and normalized, so that for the identity $1 \in \mathfrak{A}$, we have $\omega(1) = 1$, corresponding to the notion of normalized Hilbert vectors $\langle \psi, \psi\rangle = 1$.

Theorem 56.1. For a state ω, $\omega(A^*) = \overline{\omega(A)}$

Proof. If we write decompose A into a "real" and "imaginary" part

$$\begin{aligned} A_1 &= \frac{1}{2}(A + A^*) \\ A_2 &= -\frac{i}{2}(A - A^*) \end{aligned}$$

as $A = A_1 + iA_2$, then by linearity

$$\begin{aligned} \omega(A) &= \omega(A_1) + i\omega(A_2) \\ \omega(A^*) &= \omega(A_1) - i\omega(A_2) \end{aligned}$$

The positive norm also gives us that

$$\omega((A+I)^*(A+I)) = \omega(A^*A) + \omega(I) + \omega(A + A^*) \tag{56.1}$$

meaning that if $A = A^*$, then $\omega(A) \in \mathbb{R}$, giving us $\omega(A^*) = \overline{\omega(A)}$. □

Norm of an operator

$$\|\omega\| = \sup_{A\neq 0} \frac{|\omega(A)|}{\|A\|} \tag{56.2}$$

Theorem 56.2. A state ω defines a positive sesquilinear form on $\mathfrak{A}$.

Proof. For $A, B \in \mathfrak{A}$, we can verify that $\langle A, B\rangle = \omega(A^*B)$ defines such a form.

- $\langle B, A\rangle = \omega(B^*A) = \omega((A^*B)^*) = \overline{\omega(A^*B)} = \overline{\langle A, B\rangle}$
- $\langle A, A\rangle = \omega(A^*A) \geq 0$

□

GNS construction:

Theorem 56.3. Given a state ω on a unital C^*-algebra, we can define a Hilbert space $\mathcal{H}$ with a $*$-representation π of $\mathfrak{A}$ as linear operators on $\mathcal{H}$ such that $\pi(A^*) = \pi(A)^*$ and

$$\omega(A) = \langle \omega, \pi(A)\omega \rangle \tag{56.3}$$

Proof. If we take the sesquilinear form defined earlier

$$\langle A, B \rangle = \omega(A^* B) \tag{56.4}$$

Consider the elements $N \in \mathfrak{A}$ such that $\langle N, N \rangle = 0$. By the Schwartz inequality, we have

$$|\langle N, N' \rangle| \leq \langle N, N' \rangle^{\frac{1}{2}} \langle N, N' \rangle^{\frac{1}{2}} = 0 \tag{56.5}$$

$$|\langle aN + bN', aN + bN' \rangle| = 0 \tag{56.6}$$

The set of zero norm vectors then forms a vector space $\mathcal{N}$. To have our norm be positive definite, we can define the equivalence class

$$\hat{\mathfrak{A}} = \mathfrak{A}/\mathcal{N} \tag{56.7}$$

An element $\hat{A} \in \hat{\mathfrak{A}}$ is the set of elements $A' \in \mathfrak{A}$

$$A' - A \in \mathcal{N} \tag{56.8}$$

By the Schwartz inequality, we have that for $A', A'' \in \hat{\mathfrak{A}}$

$$* \tag{56.9}$$

[SHOW IT]
Show that Cauchy sequences converge □

56.2.3 Hadamard states and the microlocal spectrum condition

Hadamard states : The divergent structure of the state is similar to that of Minkowski space

$$\omega() \tag{56.10}$$

56.2.4 Automorphisms of isometries

For a Killing vector field K^μ generating a one-parameter group of isometries

$$\phi_t : \mathcal{M} \to \mathcal{M} \tag{56.11}$$

there corresponds a one-parameter group of automorphisms α_{ϕ_t} on the algebra $\mathfrak{A}(\mathcal{O})$ such that

$$\alpha_{\phi_t} \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\phi \mathcal{O}) \tag{56.12}$$

that leaves states invariant

$$\forall A \in \mathfrak{A},\ \omega(\alpha_{\phi_t} A) = \omega(A) \tag{56.13}$$

Identity transformation : $\alpha_{\phi_0} = \mathrm{Id}_{\mathfrak{A}}$
Representation of the automorphisms as unitary operators on $\mathcal{H}$

$$\pi(\alpha_\phi A) = U_\phi \pi(A) U_\phi^\dagger \tag{56.14}$$

56.2.5 Microcausality

56.2.6 Borcher algebra

The Borcher algebra, or Borcher-Uhlmann algebra, is a contructive example of a C^*-algebra used for constructing QFT. It is constructed as a direct sum of algebras

$$\mathcal{S} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n \tag{56.15}$$

with $\mathcal{S}_0 = \mathbb{C}$ and $\mathcal{S} = \mathscr{S}(\mathbb{R}^n)$, the Schwartz space on Minkowski space.
For a set of functions $f_n \in \mathcal{S}^n$, an element of the Borcher algebra is

$$f = (f_0, f_1, ..., f_N, 0, 0, ...) \tag{56.16}$$

with $N < \infty$. The product of the algebra is defined by

$$fg = \sum_{i=0}^{n} f_i(x_1, x_1, ..., x_i) g_{n-i}(x_{i+1}, ..., x_n) \tag{56.17}$$

The involution of an element is then defined by

$$f^* = (\bar{f}_0, \bar{f}_1, ..., \bar{f}_N, 0, 0, ...) \tag{56.18}$$

"Two formulations using differently described Hilbert spaces $\mathcal{H}^1$, $\mathcal{H}^2$ are equivalent if there exists a unitary map U from $\mathcal{H}^1$ to $\mathcal{H}^2$ such that the operators $A^{(1)}$, $A^{(2)}$ corresponding to the same observables are related by

$$A^{(2)} = U A^{(1)} U^{-1} \tag{56.19}$$

Haag's axioms :
Presheaf over open sets of spacetime to C^*-algebra.
Presheaf : For each open set U of $\mathcal{M}$, there is an object $F(U)$ in the category C
If $V \subseteq U$, there is a morphism $F(U) \to F(V)$ in C.
Net of abstract C^*-algebras $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$, $\mathcal{O}$ an open, finitely extended region of space. Self-adjoint elements are observables that can be measured in $\mathcal{O}$.
$\hat{\mathcal{O}}$ the causal completion of $\mathcal{O}$, then $\mathfrak{A}(\hat{\mathcal{O}}) = \mathfrak{A}(\mathcal{O})$
Algebraic quantum field theory : Operators belong to a C^* algebra, states are linear functionals on operators.
$\mathcal{H}$ a Hilbert space. $\mathfrak{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$

$$\forall A \in \mathfrak{B}(\mathcal{H}) \exists \|A\| \tag{56.20}$$

Category for AQFT :
Man : Category of manifolds
Obj(Man) : Oriented and time oriented globally hyperbolic Lorentzian manifolds
Mor(Man) : Oriented and time oriented preserving isometric embeddings, image is open and causally compatible
C^*Alg : Category of C^* algebras
Obj(C^*Alg) : unital C^* algebras over $\mathbb{C}$
Mor(C^*Alg) : injective unital C^* algebra homomorphisms
Locally covariant QFT : Covariant functor $\mathfrak{A}$
If $f_1 : \mathcal{M}_1 \to \mathcal{M}$ and $f_2 : \mathcal{M}_2 \to \mathcal{M}$ are causally disjoint, then $\mathfrak{A}(f_1)(\mathfrak{A}(\mathcal{M}_1))$ and $\mathfrak{A}(f_2)(\mathfrak{A}(\mathcal{M}_2))$ commute as subalgebras of $\mathfrak{A}(\mathcal{M})$ (causality axiom)
If $f : \mathcal{M} \to \mathcal{M}'$ is a Cauchy morphism ($f(\mathcal{M}) \subseteq \mathcal{M}'$ contains a Cauchy surface), then $\mathfrak{A}(f)$ is an isomorphism (time slice axiom)

56.3 Wavefunctional formalism

By a parallel with the point-particle quantization, we can also define the Hilbert space as the space of square-integrable functions, on the space of test functions with some measure defined on it.

$$\mathcal{H} = L^2(\mathcal{D}(\mathcal{M}), d\mu) \tag{56.21}$$

Measure : for $\Psi \in \mathcal{D}(\mathcal{M})$

$$\langle \Psi, \Psi \rangle = \int \Psi(p)\Psi^*(p) \tag{56.22}$$

$\Psi = \Psi[\phi(\vec{x}), t]$, $p \in \Sigma_t$
operators :

$$\Phi\Psi[\phi(\vec{x}), t] = \phi(\vec{x})\Psi[\phi(\vec{x}), t] \tag{56.23}$$

Canonical momentum :

$$\Pi\Psi[\phi(\vec{x}), t] = (\frac{\delta}{\delta\phi(\vec{x})} + V(\phi(\vec{x}))\Psi[\phi(\vec{x}), t] \tag{56.24}$$

Requirements : obeys the CCR, hermitian

56.4 Wick products of operators

Define products on distributions via the wavefront set

56.5 Renormalization

Nguyen on the renormalization of distribution products
Renormalization group
Hadamard

56.6 Perturbative Quantum Field Theory

The basic notions of perturbative quantum field theory are less rigorously defined, but they are roughly these :

1. States are represented by rays in a Hilbert space $\mathcal{H}$
2. Operator valued distributions are defined as acting on that Hilbert space. While those distributions cannot formally be multiplied (unless one uses distribution algebras), we allow it by considering some limit process on the product of their sequence representation.
3. There are rules for removing divergences (for instance of those distribution products) from the theory, called renormalization or regularization schemes.
4. Among the operator valued distributions, we define in particular field operators $\{\hat{\phi}_i, \hat{\psi}_i\}$, which represent the value of the fields themselves. the fields ϕ_i are tensorial (bosons), while the fields ψ_i are spinorial (fermions).
5. There are equations of motion satisfied by the field operators. They are linear for free fields, non-linear if they possess self-interactions or interaction betweel different fields.
6. The tensorial field operators obey commutation relations, while the spinorial field operators obey anticommutation relations.
7. There is some splitting of the Hilbert space $\mathcal{H}$ into a Fock space $\bigoplus_{n=0}^{\infty} S_\nu \mathcal{H}_1^{\otimes n}$, where $\mathcal{H}_1$ is interpreted as the Hilbert space of a single particle, and S_ν is the symmetrization operator S_+ for bosons, or the antisymmetrization operator S_- for fermions.
8. INTERACTION PICTURE There exists some unitary operator U
9. The renormalized hamiltonian operator $\hat{H}$ is bounded from below. (in a frame, there exists one or more "vacuum states" that minimize the hamiltonian)

Relation between the free theory field operator ϕ^0 and the interacting theory field operator ϕ : We assume at at a time $t = 0$ the two field operators can be considered equal :

$$\phi(0, \vec{x}) = \phi^0(0, \vec{x}) \tag{56.25}$$

Time evolution :

$$e^{-i\hat{H}t}\phi(0, \vec{x})e^{i\hat{H}t} = e^{-i\hat{H}t}\phi^0(0, \vec{x})e^{i\hat{H}t} \tag{56.26}$$

Say something about $\langle 0|\Omega\rangle \neq 0$, which implies that
LSZ formula :
Haag's theorem :

Theorem 56.4. For two irreducible sets of operator-valued distributions $\phi_{1,\alpha}[f,t]$, $\phi_{2,\beta}[f,t]$, defined on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, on which are defined continuous unitary representations of the inhomogeneous SU(2)

56.6.1 Path integral formulation

Assuming global hyperbolicity, take two Cauchy surfaces Σ_1 and Σ_2

$$\int_{\phi_1}^{\phi_2} d\mu[\phi] \tag{56.27}$$

56.6.2 BRST quantization

Quantization of gauge fields
Gauge invariance : for ϕ^i representing all fields, coordinates and momentum :

$$\delta\phi^i = R^i_\alpha \xi^\alpha \tag{56.28}$$

$$\frac{\delta S_c[\phi^i]}{\delta\phi^i} R^i_\alpha = 0 \tag{56.29}$$

Classical equation of motion

$$\frac{\delta S_c[\phi^i]}{\delta\phi^i} = 0 \tag{56.30}$$

Implies

$$\frac{\delta S_c[\phi^i]}{\delta\phi^i\delta\phi^j} R^j_\alpha = 0 \tag{56.31}$$

The second derivatives of S has zero eigenvalue eigenvectors at the stationary point ¿ no inverse, propagator is ill defined.
Orbits of equivalent fields :

$$\phi^i(\xi) = \phi^i R^i_\alpha \xi^\alpha + \mathcal{O}(\xi^2) \tag{56.32}$$

Need of a gauge fixing function fixed for a constant parameter

$$\mathcal{F}^\alpha(\phi) - f^\alpha = 0 \tag{56.33}$$

Admissibility condition :

$$\det\left(\frac{\partial \mathcal{F}^\alpha}{\partial\phi^i} R^i_\alpha\right) \neq 0 \tag{56.34}$$

Each admissible gauge function determines a surface (orbit in the principal bundle and associated bundle???)

$$\Delta_F^{-1}(\phi) = \int \mathscr{D}\Omega\,\delta(\mathcal{F}(\phi^\Omega) - f) \tag{56.35}$$

Equivalent of the identity

$$\overset{-1}{\det}\left(\frac{\partial \vec{f}}{\partial \vec{x}}\right)\Big|_{\vec{f}=\vec{\lambda}} = \int d\vec{x}\,\delta(\vec{f}(\vec{x}) - \vec{\lambda}) \tag{56.36}$$

Coleman-Mandula theorem

56.7 Quantum field theory on non-time orientable spacetimes

Lacks of advanced and retarded solutions due to the lack of time orientability

56.8 Quantum field theory on non-globally hyperbolic spacetimes

No Cauchy surfaces : no timeslice axiom

56.8.1 F-locality

F-locality : Every point in $\mathcal{M}$ should have a globally hyperbolic neighbourhood N such that the induced algebra $\mathcal{A}(M, g, N)$ (the subalgebra of $\mathcal{A}(M, g, N)$ of polynomial fields smeared by test functions supported by N) coincide with the intrinsic algebra $\mathcal{A}(M, g|_N)$ (on the spacetime $(N, g|_N)$.

Definition 56.5. A quantum field theory is called F-quantum compatible if it admits a *-algebra of local observables that obey the condition of F-locality.

Examples of spacetimes that are F-quantum compatible (Timelike cylinder)
Examples that are not

56.8.2 Loss of unitarity

In quantum field theory, there is a particular importance on the time evolution of quantum fields being unitary, that is, the time evolution operator $U(t_1, t_2)|\psi(t_1)\rangle = |\psi(t_2)\rangle$ must obey the property

$$UU^\dagger = U^\dagger U = I \tag{56.37}$$

or, in other words, $U^\dagger = U^{-1}$. Unitarity has the important consequence of preserving probabilities

$$\langle\Psi(t_2)|\Psi(t_2)\rangle = \langle\Psi(t_1)|\Psi(t_1)\rangle \tag{56.38}$$

$$|\Psi(t_2)\rangle = U(t_1, t_2)|\Psi(t_1)\rangle,\ U(t_1, t_2) = e^{-\frac{i}{\hbar}(t_2 - t_1)\hat{H}} \tag{56.39}$$

If time evolution is unitary : $\hat{H}^\dagger = \hat{H}$
S-Matrix : $\hat{S} = e^{-i\hat{H}t}$, S-matrix unitary
Canonical QFT : Problem with the time ordering operator (not defined) and the possible lack of spacelike hypersurfaces foliation, possibility to do it on partial Cauchy surfaces up to the Cauchy horizon

56.8.3 Cutkoski cutting rule

Optical theorem :
$S = 1 + i\mathcal{T}$

$$\langle f|\mathcal{T}|i\rangle = (2\pi)^4\delta^4(p_i - p_f)\mathcal{M}(i \to f) \tag{56.40}$$

Unitarity : $i(\mathcal{T}^\dagger - \mathcal{T}) = \mathcal{T}^\dagger\mathcal{T}$

$$\begin{aligned} i\langle f|\mathcal{T}^\dagger - \mathcal{T}|i\rangle &= i\langle f|\mathcal{T}|i\rangle^* - i\langle f|\mathcal{T}|i\rangle & (56.41) \\ &= i(2\pi)^4\delta^4(p_i - p_f)(\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f)) & (56.42) \end{aligned}$$

path integrals

56.8.4 Stability

One of the main issue raised concerning closed timelike curves is stability. A wide variety of non-causal spacetimes are unstable to small perturbations, such as objects trying to cross them.
Blueshifting for closed null curves
Blueshifting in Kerr black hole, Misner space

56.8.5 Path integral

57 Conformal field theory and the AdS/CFT correspondance

57.1 Conformal field theory

Field theory is conformally invariant if its action is invariant under the conformal group. It is called a conformal field theory (or CFT)

$$\delta S = \frac{1}{2}\int d^n x T^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{1}{n}\int d^n x T^\mu{}_\mu \partial_\rho \epsilon^\rho \tag{57.1}$$

If a stress energy tensor is traceless, the action is invariant under conformal transformation. (opposite isn't true)
Conserved current under dilation $x'^\mu = (1+\alpha)x^\mu$, $\mathcal{F}(\Phi) = (1-\alpha\Delta)\Phi$:

$$j^\mu_D = -\mathscr{L}x^\mu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}x^\nu \partial_\nu \Phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}\Delta\Phi \tag{57.2}$$

$$= T^\mu{}_\nu x^\nu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}\Delta\Phi \tag{57.3}$$

Example : conformal scalar field

$$\mathscr{L} = \nabla_\mu \phi \nabla^\mu \phi + m^2\phi^2 + \xi R\phi \tag{57.4}$$

57.1.1 Conformal field theory in two dimension

An important subclass of CFTs is the conformal field theory on a $(1+1)$ dimensional spacetime.
Conformal transformation in 2 dimension : $SO(3,1)$, the Lorentz group
Conformal coupling constant is zero

$$\mathscr{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi \tag{57.5}$$

Of particular use will be conformal field theory on Minkowski space, in which case we can perform the Wick rotation $t \to i\tau$ to do CFT in Euclidian space, and even moreso, by the isomorphism between $\mathbb{R}^2$ and $\mathbb{C}$, on $\mathbb{C}$.
Conformal group on $\mathbb{C}$:
A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is conformal iff holomorphic and its derivatives never vanish on U.
We'll use the notation $\partial_x + i\partial_t = \partial_z = \partial$ and $\partial_x - i\partial_t = \partial_{\bar{z}} = \bar{\partial}$, $z^\mu = (z, \bar{z})$ in which case the Klein-Gordon equation becomes

$$\partial\bar{\partial}\phi = 0 \tag{57.6}$$

with the usual plane wave solutions

$$\phi = e^{k_\mu z^\mu} \tag{57.7}$$

under the condition that $k_\mu k^\mu = 0$, or $k_z = -k_{\bar{z}}$.
Quantization

58 The AdS/CFT correspondance

$(d+1)$ dimensional metric that is Poincaré invariant in d dimensions

$$ds^2 = \Omega(z)(-dt^2 + dx_i^2 + dz^2) \tag{58.1}$$

Something something

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dx_i^2 + dz^2) \tag{58.2}$$

$$R_{\mu\nu} = -\frac{n}{L^2} g_{\mu\nu} \tag{58.3}$$

Inserted in EFE :

$$\Lambda = -\frac{n(n-1)}{2L^2} \tag{58.4}$$

$$R = -\frac{n(n+1)}{2L^2} \tag{58.5}$$

Count the degrees of freedom

Part VII

Quantum gravity

59 Generalities

Due to various arguments (that will be exposed in details in the section on semiclassical gravity), it is likely that gravity itself needs to be quantized.
This presents some problems, as the notion of time is pretty deeply ingrained in quantum theory, even in curved spacetime, but if we quantize the metric itself, time will itself become part of the equation. Problems of quantizing gravity : define time, diffeomorphism invariance
Quantization and background independence
There are a few heuristic arguments for getting the order of magnitude of the scale at which quantum gravity effects might become important. A few of them are :

- Consider a particle with a Compton wavelength equal to its own Schwarzschild radius,

$$\lambda = \frac{h}{mc} = r_S = \frac{2GM}{c^2}$$

  The

All of these explanations use quantities of around the Planck scale, which let us define the Planck mass, Planck length, Planck time and Planck energy :

$$m_p = \sqrt{\frac{\hbar c}{G}},\ l_p = \sqrt{\frac{\hbar c}{G}},\ t_p = \sqrt{\frac{\hbar G}{c^5}},\ E_p = \sqrt{\frac{\hbar c^3}{G}}$$

with the simple relations between them of $l_p/t_p = c$, $E_p = m_p c^2$, $l_p = 2Gm_p/c^2$ and $E_p = \hbar 2\pi/t_p$

59.1 Limits of quantum gravity

As our current theories of quantum field theory and general relativity are experimentally well-verified, we want our theories of quantum gravity to reproduce their predictions. Which means that in the limit $\hbar \to 0$, it should reduce to general relativity in some way, and in the limit $G \to 0$, it should reduce to quantum field theory. The full set of theories we would like in various limits is represented by the so-called Bronstein hypercube of quantum gravity, representing the limits of various theories along the axis of G, $\hbar$, c, and N, the number of degrees of freedom. With N suppressed, it gives
The limits $N \approx 1$ and $N \to \infty$ can be represented by, for instance,

- Classical mechanics $\to$ statistical mechanics
- Quantum mechanics $\to$ quantum statistical mechanics
- Special relativity $\to$ classical field theory
- Regge calculus $\to$ general relativity
- Relativistic quantum mechanics $\to$ quantum field theory

The limits for quantum gravity will of course depend on the specific theory we're dealing with.

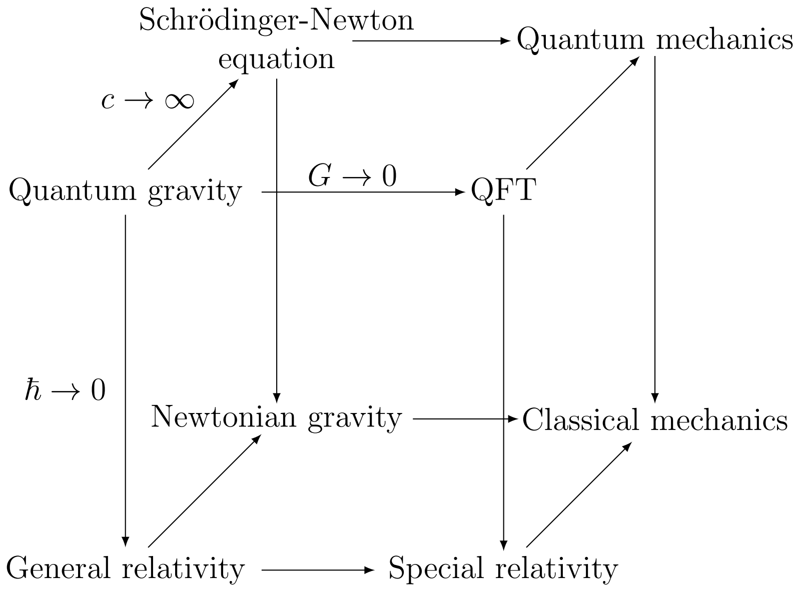


Figure 12: Limiting cases

59.2 The Schrödinger-Newton equation

It will be of some interest to consider the non-relativistic limit of quantum gravity, as it is the only one that has been well verified experimentally.
The Schrödinger-Newton equation is quite simply the Schrödinger equation with a quantization of the Newtonian classical potential, that is

$$i\hbar\frac{\partial\Psi}{\partial t}(\vec{x},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\vec{x},t) + V\Psi(\vec{x},t) + m\Phi\Psi(\vec{x},t) \tag{59.1}$$

with the Newtonian potential

$$\Delta\Phi(\vec{x},t) = 4\pi Gm|\Psi|^2 \tag{59.2}$$

60 Semiclassical gravity

The simplest form of unifying quantum field theory and general relativity is to leave them each as they are and simply connecting them by the stress-energy tensor (and the quantum field theory in curved spacetime that we have seen previously), with the form

$$T_{\mu\nu} = \langle \psi, \hat{T}_{\mu\nu}\psi \rangle \tag{60.1}$$

where we define an operator for the stress-energy tensor. The Einstein field equation then just becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \langle \psi, \hat{T}_{\mu\nu}\psi \rangle \tag{60.2}$$

This has the benefit of being the simplest possible theory of quantum gravity, and it is expected that any other theory of quantum gravity should be approximated by it within some limit.

60.1 Quantization of the stress-energy tensor

If we defined our quantum field theory with respect to a quantization procedure, the simplest way to obtain the stress energy tensor is via the quantization procedure of the classical stress-energy tensor, that is,

$$T^{\mu\nu}(\phi, x) \to \hat{T}^{\mu\nu}(\hat{\phi}, \hat{\pi}) \tag{60.3}$$

This has the problem of not being immediately well defined, as we saw previously, since the stress-energy tensor will usually be defined as the product of fields at the same point. We will instead start off by defining the stress-energy bi-tensor, which has the benefit of being well-defined at least for free fields.

$$T^{\mu\nu}(\phi, x, y) \to \hat{T}^{\mu\nu}(\hat{\phi}, \hat{\pi}, x, y) \tag{60.4}$$

As quantum fields are defined as operator-valued distributions, this will be well-defined in the case where the distribution defined by the expectation of the stress-energy tensor will itself be a bitensor (that is not too divergent). In this case, we'll define the stress-energy tensor as the coincidence limit

$$T^{\mu\nu}(x) = \lim_{y\to x} \langle \hat{T}^{\mu\nu}(x, y) \rangle_\psi \tag{60.5}$$

This will require some renormalization, even for the free field, similar to the renormalization of the Hamiltonian in QFT.

60.1.1 The scalar field

The simplest example of a semiclassical system is as usual the scalar field, with stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + m^2\varphi^2) \tag{60.6}$$

Its quantization will obviously lead to products of distributions. Instead, we will use the fact that for the propagator $D(f,g)$, we have the property

$$D(f,g) = \langle \phi[f]\phi[g]\rangle_0 \tag{60.7}$$

By linearity of the sesquilinear product, we also have that

$$\nabla_\mu D(f,g) = \langle (\nabla_\mu \phi[f])\phi[g] + \phi[f](\nabla_\mu \phi[g])\rangle_0 \tag{60.8}$$

$$\nabla_\nu\nabla_\mu D(f,g) = \langle (\nabla_\nu\partial_\mu \phi[f])\phi[g] + \phi[f](\nabla_\nu\partial_\mu \phi[g]) + (\nabla_\mu\phi[f])(\nabla_\nu\phi[g]) + (\nabla_\nu\phi[f])(\nabla_\mu\phi[g])\rangle_0 \tag{60.9}$$

$$\begin{aligned} D_{\mu\nu}(x,y,\gamma) &= \frac{1}{6}(\nabla^x_\mu \gamma_\nu{}^\alpha(x,y,\gamma)\nabla^y_\alpha + \gamma_\mu{}^\alpha(x,y,\gamma)\nabla^y_\alpha\nabla^x_\nu) \\ &- \frac{1}{12} g_{\mu\nu}(x)\gamma^{\alpha\beta}(x,y,\gamma)\nabla^x_\alpha\nabla^y_\beta \\ &- \frac{1}{12}(\nabla^x_\mu\nabla^x_\nu + \gamma_\mu{}^\alpha(x,y,\gamma)\nabla^y_\alpha\gamma_\nu{}^\beta(x,y,\gamma)\nabla^y_\beta) \\ &+ \frac{1}{48} g_{\mu\nu}(x)(g^{\alpha\beta}(x)\nabla^x_\alpha\nabla^x_\beta + g^{\alpha\beta}(y)\nabla^y_\alpha\nabla^y_\beta) \\ &- R_{\mu\nu}(x) + \frac{1}{4}g_{\mu\nu}(x)R(x) \end{aligned} \tag{60.10}$$

Classical limit :

$$T_c^{\mu\nu} = \lim_{\hbar\to 0}\langle\psi, \hat{T}_{\mu\nu}\psi\rangle \tag{60.11}$$

60.2 Domain of validity

It is generally assumed that semiclassical gravity will not be a correct description of quantum gravity. One of the common argument for this is the following :
Consider two quantum state $|\psi_1\rangle$ and $|\psi_2\rangle$, such that the support of $\langle T_{\mu\nu}\rangle_1$ and $\langle T_{\mu\nu}\rangle_2$ are disjoint. If we take the state to be

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) \tag{60.12}$$

then the initial stress-energy tensor will be

$$T_{\mu\nu} = \frac{1}{2}\langle T_{\mu\nu}\rangle_1 + \frac{1}{2}\langle T_{\mu\nu}\rangle_2 \tag{60.13}$$

After a measurement performed that would collapse the wavefunction to either ψ_1 or ψ_2, the stress energy tensor will become (for let's say ψ_1)

$$T_{\mu\nu} = \langle T_{\mu\nu}\rangle_1 \tag{60.14}$$

The stress-energy tensor has doubled in the first region and totally disappeared in the second, and this regardless of their separation. Due to this non-local effect, it is assumed

that the metric itself has to be described by a quantum field, and that semiclassical gravity can only be an appropriate description of when it nears classical behaviour.
Fluctuations of the stress energy tensor : cf "Gravitational radiation by quantum systems", should be

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(q) \rangle \approx \langle T_{\mu\nu}(p) \rangle \langle T_{\rho\sigma}(q) \rangle \tag{60.15}$$

60.3 Renormalization of the stress-energy tensor

As with most products of field operators, the stress-energy tensor operator requires some renormalization. Renormalization via the coincidence limit
Dimensional regularization

60.4 Effective action

It is also possible to express semiclassical gravity by defining an effective action W so that

$$T^{\text{Eff}}_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle_\psi \tag{60.16}$$

We then need to find an effective action W such that its functional derivative will give us the appropriate stress-energy tensor.

60.5 Energy conditions

60.6 Quantum aspects of chronology protection

One of the application of semiclassical gravity is that we can see the effect of Cauchy horizons on the stress energy tensor.
As the equations for free fields will usually be second order hyperbolic partial differential equations, it helps to express the propagator in the Hadamard form :

$$G_R(x,y) = \hbar \sum_\gamma \frac{\sqrt{\Delta_\gamma(x,y)}}{4\pi^2} [\frac{1}{\sigma_\gamma(x,y)} + v_\gamma(x,y) \ln(|\sigma_\gamma(x,y)|) + \varpi_\gamma(x,y)] \tag{60.17}$$

The Hadamard form of the Green function has the benefit of outlining the singular structure, as it can be shown that $\varpi_\gamma(x,y)$ and $v_\gamma(x,y)$ both converge to proper functions in the limit $x \to y$, so that we may focus on the remaining terms for divergences.
Stress energy tensor :

$$T_{\mu\nu}(x) = \lim_{x \to y} T_{\mu\nu}(x,y) = \lim_{x \to y} D_{\mu\nu} G_R(x,y) \tag{60.18}$$

Stress energy tensor renormalization :

$$T^{Ren}_{\mu\nu}(x) = \tag{60.19}$$

61 Stochastic gravity

Stochastic gravity is a compromise between semiclassical gravity and a full theory of quantum gravity. The metric remains classical, but a stochastic term that depends on the fluctuations of the stress-energy tensor is added.

$$\hat{t}_{\mu\nu} = \hat{T}_{\mu\nu} - \langle \hat{T}_{\mu\nu} \rangle \hat{I} \tag{61.1}$$

Noise kernel bitensor :

$$N_{\mu\nu\rho\sigma}(p,q) = \frac{1}{2}\langle \{ \hat{t}_{\mu\nu}, \hat{t}_{\rho\sigma} \} \rangle \tag{61.2}$$

$$N_{\mu\nu\rho\sigma}(p,q) = N_{\rho\sigma\mu\nu}(q,p) \tag{61.3}$$

$\hat{t}_{\mu\nu}$ is finite since we substract the UV divergences.
T is self-adjoint ¿ N is real and positive semi-definite since it's the expectation value of an anticommutator. Classical gaussian stochastic tensor $\xi_{\mu\nu}$

$$\langle \xi_{\mu\nu}(p) \rangle_s = 0,\ \langle \xi_{\mu\nu}(p) \xi_{\rho\sigma}(q) \rangle_s = N_{\mu\nu\rho\sigma}(p,q) \tag{61.4}$$

$\xi_{\mu\nu} = \xi_{\nu\mu}$ and $\nabla^m u \xi_{\mu\nu} = 0$. $\langle \cdot \rangle_s$ is a classical stochastic average. Correction to the metric tensor $g \to g + h$
Einstein-Langevin equation :

$$G^{(1)}_{\mu\nu}[g+h] = \kappa(\langle \hat{T}^{(1)}_{\mu\nu}[g+h] \rangle_{\text{ren}} + \xi_{\mu\nu}[g]) \tag{61.5}$$

g is a solution to the semiclassical equation.

62 Covariant quantum gravity

Covariant gravity is one of the first attempt at the quantization of gravity, based on the canonical quantization of the Pauli-Fierz theory of gravity. As a simple interacting QFT on flat space, it is expected to work by the exchange of gauge bosons, the graviton, represented by the perturbative metric field $h_{\mu\nu}$, or the perturbative frame field, ${h_\mu}^a$, if fermions are to be added to the theory.
As we have seen earlier, Pauli-Fierz on a Minkowski background has the following Lagrangian

$$\mathscr{L} = -\frac{1}{4} h_{\mu\nu,\sigma} h^{\mu\nu,\sigma} + \frac{1}{2} h_{\mu\nu,\sigma} h_{\sigma\nu,\mu} + \frac{1}{4}(\partial_\sigma h)(\partial^\sigma h) - \frac{1}{2}(\partial_\sigma h)(\partial_\nu h^{\nu\sigma}) + V(h)$$

With V a potential that will depend on h and its first derivatives. Its equation of motion being

$$\Box h_{\mu\nu} - .... \tag{62.1}$$

If we limit for now our attention to the linear part
Hamiltonian :
Let's define $N = -h_{00}$, $N_i = 2h_{0i}$

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4} h_{\mu\nu,\sigma} h^{\mu\nu,\sigma} + \frac{1}{2} h_{\mu\nu,\sigma} h_{\sigma\nu,\mu} + \frac{1}{4}(\partial_\sigma h)(\partial^\sigma h) - \frac{1}{2}(\partial_\sigma h)(\partial_\nu h^{\nu\sigma}) \\ &= \end{aligned} \tag{62.2}$$

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{h}_{ij}} = \tag{62.3}$$

The operator $\hat{h}$ for the free theory obeys, in the de donder gauge :

$$\Box h_{\mu\nu} = 0 \tag{62.4}$$

Free wave solutions for $k^2 = 0$:

$$h_{\mu\nu} = \varepsilon_{\mu\nu}(\vec{k}) e^{\pm i k^\alpha x_\alpha} \tag{62.5}$$

Polarization obeys [why]

$$k^\mu \varepsilon_{\mu\nu}(\vec{k}) = \varepsilon_{\mu\nu}(\vec{k}) k^\nu = 0,\ {\varepsilon_\mu}^\mu = 0 \tag{62.6}$$

Choice :

$$\varepsilon_\nu = 0,\ k^i \varepsilon_{ij}(\vec{k}) = 0, {\varepsilon_i}^i = 0 \tag{62.7}$$

Propagator :

$$\tilde{h}_{\mu\nu} = k^{-2} \tag{62.8}$$

$$h_{\mu\nu} = \int k^{-2} \tag{62.9}$$

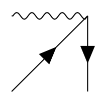


Figure 13: Tree-level gravitational process

Graviton propagator :
We expect

$$\frac{iP_{\mu\nu,\rho\sigma}}{k^2 + i\varepsilon} \tag{62.10}$$

$$[e^{\mu\nu} p_\mu p'_\nu - \frac{1}{2} e^\rho (\eta^{\mu\nu} p_\mu p'_\nu - m^2)] \tag{62.11}$$

62.1 Couplings of the graviton to matter

$$h_{\mu\nu} = \frac{\kappa}{k^2}(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T) \tag{62.12}$$

Coupling to scalar fields
Coupling to EM
Coupling to fermions

62.2 Renormalization

Renormalization
1-loop
2-loop

62.3 Classical limits

Tree level approximation

63 Canonical quantum gravity

A simple way to quantize gravity is to simply use canonical quantization on the Hamiltonian theory we already have, ADM gravity. As the GR Hamiltonian is identically zero, this will go through the usual process of BRST quantization.
First it is important to define the field variables we'll be using, by performing canonical quantization on the Lie brackets. For ADM, these were :

$$\{q\} = \tag{63.1}$$

Wheeler-DeWitt equation :

$$H\psi = 0 \tag{63.2}$$

Problem of interpretation of the equation

63.0.1 Thermal time hypothesis

Quantum theory generally uses the time

64 Loop quantum gravity

Hamilton-Jacobi equation of general relativity :

$$F^{ij}_{ab}(\vec{x})\frac{\delta S[A]}{\delta A^i_a(\vec{x})}\frac{\delta S[A]}{\delta A^j_b(\vec{x})} = 0 \tag{64.1}$$

Ashtekar variables
Spatial tetrads

$$E^a{}_\mu = e^a{}_\mu + e^b{}_\mu n^a n_b \tag{64.2}$$

Properties : $E^a{}_\mu n_a = 0$, $E^a{}_\mu n^\mu = 0$
Haar measure : measure on locally compact group ¿ gives integral for SU(2)
Properties : for $S \subset G$, $g \in G$, $\mu(gS) = \mu S$
for compact S, $\mu(S) < \infty$
Normalization condition : $\mu(G) = 1$

64.1 Spin networks

Spin networks

65 Lorentzian and Euclidian gravity

Another natural method of quantizing gravity is by using the path integral quantization method, which has the benefit of only requiring the Lagrangian, but suffers from having often poorly defined objects.
The basic idea will be to try and compute the transition amplitude between two measurements of the gravitational field and other fields on some spacelike hypersurface

$$S[h_i, \phi_i; h_f, \phi_f] = \int_{h_i,\phi_i}^{h_f,\phi_f} \mathcal{D}g[x]\mathcal{D}\phi(x) \exp[i \int_i^f dt \int d^{n-1}x(\mathscr{L}_{EH} + \mathscr{L}_M)] \tag{65.1}$$

As for other field theories, the Lorentzian path integral is hard to define properly, so we will also define the Euclidian transition function, defined by

$$S[h_i, \phi_i; h_f, \phi_f] = \int_{h_i,\phi_i}^{h_f,\phi_f} \mathcal{D}g[x]\mathcal{D}\phi(x) \exp[- \int_i^f dt \int d^{n-1}x(\mathscr{L}_{EH} + \mathscr{L}_M)] \tag{65.2}$$

with a Riemannian metric rather than a Lorentzian metric.
Lorentzian :

$$\int_{S_1}^{S_2} d\mu[g] e^{iS_{EH}[g]} \tag{65.3}$$

Euclidian action

$$\int_{S_1}^{S_2} d\mu[g] e^{-S_R[g]} \tag{65.4}$$

This present a few problems :
There is no general Wick rotation from Lorentzian metric to Riemannian metric. While this process works for the Minkowski space

$$-dt^2 + \sum_a dx^a dx^a \xrightarrow{t\to it} dt^2 + \sum_a dx^a dx^a \tag{65.5}$$

or even a static spacetime

$$-dt^2 + g_{ab}dx^a dx^b \xrightarrow{t\to it} dt^2 + g_{ab}dx^a dx^b \tag{65.6}$$

this will not be true in general, for instance, for any metric with timelike crossterms

$$-dt^2 + 2g_{at}dtdx^a + g_{ab}dx^a dx^b \xrightarrow{t\to it} -dt^2 + 2ig_{at}dtdx^a + g_{ab}dx^a dx^b \tag{65.7}$$

which is not at all a Riemannian manifold. The opposite procedure of applying the Wick procedure to a Riemannian metric will also likewise not produce a Lorentzian metric in general.
Show that the Euclidian action is unbounded from below via Weyl transformation
Einstein-Hilbert action in 2D :

$$S_H[g(x)] = -\int d^2x \sqrt{-g} \frac{1}{2\kappa}[R(x) - 2\Lambda] \tag{65.8}$$

Gauss Bonnet theorem :

$$\frac{1}{2}\int_{\mathcal{M}} RdA + \int_{\partial\mathcal{M}} k_g dS + \sum_i \theta_i = 2\pi\chi(\mathcal{M}) \tag{65.9}$$

Without boundaries,

$$\frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{-g}Rd^2x = 2\frac{\pi}{\kappa}\chi(\mathcal{M}) \tag{65.10}$$

$$S_H[g(x)] = -\int d^2x\sqrt{-g}\frac{\Lambda}{\kappa} = -\frac{\Lambda}{\kappa}V[g] \tag{65.11}$$

Path integral :

$$Z = \int \mathcal{D}g(x)e^{-\frac{i}{\hbar}S_{HE}[g]} = \int \mathcal{D}g(x)e^{\frac{i\Lambda}{\hbar\kappa}V[g]} \tag{65.12}$$

Euclidian path integral :

$$Z(\Lambda) = \int \mathcal{D}[g(x)]K_{\mathcal{M}}e^{-\frac{\Lambda}{\hbar\kappa}V[g]} \tag{65.13}$$

$$G \tag{65.14}$$

66 String theory

Instead of the usual attempts at some matter of quantization of the gravitational action, string theory takes the route of a quantum theory that reproduces the predictions of general relativity to some limit.
Originally conceived as a theory of strong interaction, it is based on the quantum theory of the relativistic string seen previously, with the Polyakov action

$$\begin{aligned} S[X,\gamma] = & -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{-\gamma}\gamma^{ab}(\sigma)g_{\mu\nu}(X)\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma) \\ & + \int_\Sigma d^2\sigma(\lambda_1 + \lambda_2{}^{(2)}R)] + \int_{\partial\Sigma} dl K \end{aligned} \tag{66.1}$$

$$T = \frac{1}{2\pi\alpha'} \tag{66.2}$$

$$\ell_s = 2\pi\sqrt{\alpha'} \tag{66.3}$$

$$M_s = \frac{1}{\sqrt{\alpha'}} \tag{66.4}$$

T : string tension α' : Regge slope ℓ_s : string length M_s : string mass scale

66.1 The relativistic string

The basic object used in string theory is the relativistic string, that we have seen in chapter [x]. The 2-dimensional timelike submanifold of the relativistic string will be referred to as the worldsheet, while the 1-dimensional submanifold is what is referred to as the string.
More generally, string theory may involve $(p+1)$-dimensional timelike submanifolds (the worldvolume) which intersect achronal spacelike surfaces with p-dimensional submanifolds, called a p-branes. Strings are then 1-branes and point particles 0-branes.
The spacetime manifold itself is referred to as the target space of the theory.
The action of the worldsheet is defined once again by the Brink-Di Vecchia-Howe-Deser-Zumino action, or Polyakov action.

$$S[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}g_{\mu\nu}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \tag{66.5}$$

Induced metric : $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X_\mu$
This is similar to a set of conformal scalar fields with internal symmetry $\mathrm{SO}(1, n-1)$
Symmetries :
If the target space is Minkowski space, Poincaré invariance in the target space :

$$X'^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \tag{66.6}$$

Diffeomorphism invariance

$$S = \frac{1}{\alpha}\int \frac{d^2\sigma}{2\pi}\sqrt{-\gamma}[(\partial_a X^\mu)P^a_\mu + \frac{1}{2}\gamma_{ab}P^a_\mu P^{b\mu}] \tag{66.7}$$

$(2\pi\alpha')^{-1}$ is the string tension and mass per unit length, P auxiliary field
Conformal gauge :

$$\gamma_{ab} = \eta_{ab} \tag{66.8}$$

η the 2D Minkowski metric
In this gauge :

$$P_a^\mu = -\partial_a X^\mu,\ \partial_a P^{a\mu} = 0 \tag{66.9}$$

$$\gamma^{ab}\partial_a\partial_b X^\mu \tag{66.10}$$

66.1.1 The light-cone gauge

A common way of solving the equation of motion in string theory is the use of the light-cone gauge. If we pick a set of null-coordinates on the worldsheet,

$$\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^1 \mp \sigma^0) \tag{66.11}$$

the worldsheet metric then simply becomes

$$ds^2 = -d\sigma^+ d\sigma^- \tag{66.12}$$

with components $\gamma_{+-} = \gamma_{-+} = -1/2$ and $\gamma^{+-} = \gamma^{-+} = -1$, giving the raising and lowering operators

$$(V_+ dx^+ + V_- dx^-)^\sharp = \tag{66.13}$$

Partial derivatives :

$$\partial_\pm = \frac{\partial}{\partial\sigma^\pm} \tag{66.14}$$

Measure :

$$d^2\sigma = \frac{1}{2} d\sigma^+ d\sigma^- \tag{66.15}$$

Polyakov action in light cone coordinates :

$$S = T\int d^2\sigma \eta_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu \tag{66.16}$$

$$\partial_+\partial_- X^\mu = 0 \tag{66.17}$$

$$X = \frac{1}{2}[\hat{X}^+(\tau+\sigma) + \hat{X}^-(\tau-\sigma)] \tag{66.18}$$

[...]
expansion :

$$\partial_+\partial_- X\text{\textasciicircum}mu = 0 \tag{66.19}$$

Left-moving and right-moving waves on the string

$$X^\mu(\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{66.20}$$

For a closed string : Fourier expansion as

$$\begin{aligned}
X_R^\mu(\sigma^-) &= \frac{1}{2}(x^\mu + c^\mu) + \frac{1}{2}\frac{2\pi\alpha'}{\ell}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z},n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{\ell}in\sigma^-} \\
X_L^\mu(\sigma^+) &= \frac{1}{2}(x^\mu - c^\mu) + \frac{1}{2}\frac{2\pi\alpha'}{\ell}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z},n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{\ell}in\sigma^+}
\end{aligned}$$

x^μ : center of mass position of the string at $\tau = 0$, p^μ : momentum of the string

$$q^\mu = \ell^{-1}\int_0^\ell d\sigma X^\mu = x^\mu + \frac{2\pi\alpha'}{\ell}p^\mu\tau \tag{66.21}$$

$$\int d\sigma \Pi^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'}\int_0^\ell d\sigma \dot{X}^\mu = p^\mu \tag{66.22}$$

66.2 String quantization

The basic

66.2.1 Canonical quantization

Canonical commutation relations

$$\begin{aligned}
[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')] &= i\eta^{\mu\nu}\delta(\sigma-\sigma') \\
[\hat{X}^\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')] &= [\hat{\Pi}^\mu(\tau,\sigma), \hat{\Pi}^\mu(\tau,\sigma')] = 0
\end{aligned}$$

This implies

$$\begin{aligned}
[x^\mu, p^\nu] &= i\eta^{\mu\nu} \\
[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= m\delta_{m+n,0}\eta^{\mu\nu} \\
[\tilde{\alpha}_m^\mu, \alpha_n^\nu] &= 0
\end{aligned}$$

X, Π are hermitian, implying $\alpha_m^\mu = (\alpha_{-m}^\mu)^\dagger$
Something something BRST quantization

66.2.2 Light-cone quantization

Since we are in Minkowski target space, $X^+ = X_+$, $X^- = X_-$ [CHECK]
Open string, $\tau \in \mathbb{R}$, $\sigma \in [0, l]$
Light cone : $\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma)$
$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$
Light cone gauge :

$$
\begin{aligned}
X^+ &= \tau \\
\partial_\sigma \gamma_{\sigma\sigma} &= 0 \\
\det \gamma_{ab} &= -1
\end{aligned}
$$

Conditions related to the Poincaré invariance of the target space, diffeomorphism invariance of the worldsheet and conformal invariance

$$S[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma[-2\gamma^{ab}\partial_a X^+ \partial_b X_- + \gamma^{ab}\partial_a X^i \partial_b X_i]$$

Under reparametrization of σ, $\gamma_{\sigma\sigma}(-\det\gamma_{ab})^{-1/2}d\sigma$ is invariant.
Define the measure

$$dl = \gamma_{\sigma\sigma}(-\det\gamma_{ab})^{-1/2}d\sigma \tag{66.23}$$

$\gamma = -1$ implies $-\gamma_{\tau\tau}\sigma_{\sigma\sigma} - \gamma_{\sigma\tau}^2 = -1$

$$\gamma_{\tau\tau} = \frac{-1+\gamma_{\sigma\tau}^2}{\sigma_{\sigma\sigma}} \tag{66.24}$$

For the inverse metric, we use the usual formula of the inverse of a 2×2 matrix :

$$
\begin{aligned}
\gamma^{\tau\tau} &= -\gamma_{\sigma\sigma} \\
\gamma^{\tau\sigma} &= -\gamma_{\tau\sigma} \\
\gamma^{\sigma\sigma} &= \frac{1-\gamma_{\sigma\tau}^2}{\sigma_{\sigma\sigma}}
\end{aligned}
$$

66.3 The bosonic string

$$\frac{1}{\alpha'}\int {P^0}_i(\sigma) = i\frac{\delta}{\delta X_i(\sigma)} \tag{66.25}$$

$$[\frac{\delta}{\delta X_i(\sigma_1)}, X_j(\sigma_2)] = \delta_{ij}2\pi\delta(\sigma_2-\sigma_1) \tag{66.26}$$

66.4 String interactions and vertex

Unlike most other quantum theories, string theory needs no interaction terms. Interaction is dealt by summing over every possible geometry

$$Z = \int \mathcal{D}X\mathcal{D}g e^{iS_p[X,g]} \tag{66.27}$$

Path integral, sum over 2-manifolds with boundary conditions (cobordism)
Scattering : initial condition of open strings and closed string, final condition same, the scattering matrix is the interpolation of all possible topologies and geometries in between

$$\sum_{\textbf{Top}}\int \mathcal{D}\gamma(\sigma)\mathcal{D}X(\sigma) \tag{66.28}$$

66.5 Superstrings

Define the worldsheet over a supermanifold instead of a manifold

66.6 Compactification

The physical universe seems to be a $(3+1)$ dimensional spacetime, while the current state of string theory requires at least 10 total dimensions to function.
Explaining the low apparent dimensionality of the universe
Decomposition of the target space $\mathcal{M}$ into

$$\mathcal{M} = \mathcal{M}_d \times \mathcal{M}_{n-d} \tag{66.29}$$

with n the dimension of the target space and d the dimension of physical space
Compactification scale : $M_c = R^{-1}$, R the typical length of $\mathcal{M}_{n-d}$
should be smaller than the string scale $M_s = 1/l_s$, $M_c \ll M_s$.
Usual experiments : $E \ll M_c \ll M_s$
Example : Kaluza-Klein as $\mathbb{R}^4 \times S$
Coordinates (x^μ, x^4), $(x^\mu, x^4) \sim (x^\mu, x^4 + 2\pi R)$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + dx^4 dx^4 \tag{66.30}$$

$$\Phi(x) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) e^{\frac{inx^4}{R}} \tag{66.31}$$

Solution of the wave equation :

$$(\nabla_\mu \nabla^\mu - \frac{n^2}{R})\phi_n = 0 \tag{66.32}$$

Equivalent to particles of mass nM_c. If $E \ll M_c$, we can ignore any mode $n > 0$.
Compactification of bosonic string on $\mathcal{M} = \mathcal{M}_9 \times S$
Calabi-Yau manifold

66.6.1 Orbifolds

66.7 String field theory

Definition with field theory

$$Q_B \Psi = 0 \tag{66.33}$$

67 Causal Dynamical Triangulation

68 Non-commutative geometry

Elements $[uv]$ defined by the Lie bracket, defines a Lie algebra of vector fields (algebra of diffeomorphisms), called Ξ.
Universal envelopping algebra $U\Xi$ defined by the tensor algebra over $\mathbb{C}$ uv generated by Ξ and $uv - vu - [uv]$.
Natural Hopf algebra structure. For the unit element 1,

$$\begin{aligned}
\Delta(u) &= u \otimes 1 + 1 \otimes u \\
\Delta(1) &= 1 \otimes 1 \\
\varepsilon(u) &= 0 \\
\varepsilon(1) &= 1 \\
S(u) &= -u \\
S(1) &= 1
\end{aligned}$$

Twist of a Hopf algebra

69 Causal sets

Causal set theory is an attempt to turn the structure of spacetime itself in a quantum theory. The basic idea stems from the abstract causal spaces defined in [x], in which the structure of spacetime itself is defined in the abstract. Such structures are a bit too general for causal set theory, as we want spacetime to be discrete as well. To guarantee discreteness, we will add some requirements of local finiteness.

Definition 69.1. A *causal set*, or causet, is a set C with a partial relation order $\prec$ with the following properties

1. Transitivity : $p \prec q \prec r \Rightarrow p \prec r$
2. Irreflexivity : $p \prec p$
3. Local finiteness : $\mathrm{Card}(\{q \in C | p \prec q \prec r\}) \in \mathbb{N}$

The ordering roughly corresponds to the strict causal relation $p < q$ seen previously, but with the added condition that there are only finitely many points in the intersection of the future and the past of two points.
No spacetime is a causal set, since the axiom of local finiteness runs afoul of its manifold structure, but it is possible to embed of causal sets in them. For instance, if we pick any covering of $\mathcal{M}$ by non-overlapping compact subsets, and pick a finite number of points in each of these subsets, this will generate a causal set.
On the other hand, some causal sets cannot be embedded in any manifold.
To generate a causal set from a given spacetime, the method used is sprinkling : we select random points in it according to a Poisson process. That is, for any compact region of volume V, we select n elements with probability

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \tag{69.1}$$

where ρ is the fundamental density, a parameter of the theory, probably related to the Planck density.
A manifold $(\mathcal{M}, g)$ is said to approximate a causet if C has a relatively high probability of coming from a springling process on it. C is then said to be faithfully embeddable in $\mathcal{M}$.
Causal set fundamental conjecture : two wildly different spacetimes on a large scale cannot be faithfully approximated by the same sprinkling.
Function $D(C)$: approximate dimension of the manifold
Estimators for timelike distances, volumes, etc
The volume is $\approx$ proportional to the cardinality of the causet, so

70 Entropic gravity

Gravity as emergent property of entropy of matter

70.1 Entropy

For a macroscopic configuration (or macrostate) X, the entropy associated is

$$S(X) = k_B \ln(\Omega(X)) \tag{70.1}$$

with $\Omega(X)$ the number of microstates associated with this macroscopic configuration.

70.2 Entropic forces

Entropic force : effective macroscopic force generated by the tendency to increase entropy.
No fundamental field associated with an entropic force.
In the canonical ensemble, entropic force $\vec{F}$ associated to a macrostate partition $\{X\}$ is

$$F(X_0) = T\nabla_X S(X)|_{X_0} \tag{70.2}$$

T the temperature, S the entropy.
Examples : colloid molecules suspended in a thermal environment of smaller particles, osmosis, polymer molecules

$$F \approx -\alpha k_B T\langle x\rangle \tag{70.3}$$

70.3 Entropic gravity

Timelike Killing vector $\approx$ temperature and entropy gradient
Geodesic motion $\approx$ entropic force
BTZ black hole in quantum gravity

Part VIII

Specific spacetimes

71 Minkowski space and variants

Minkowski space is the simplest example of a spacetime, as it is maximally symmetric, topologically trivial and almost every of its tensor quantities vanish in Cartesian coordinates. It also offers the benefit of Cartesian coordinates being defined on a single coordinate patch for the whole manifold. As such, it presents many interesting properties.

$$ds^2 = -dt^2 + \sum_{i=1}^{n} dx_i^2 \tag{71.1}$$

$$\Gamma^{\sigma}{}_{\mu\nu} = 0 \tag{71.2}$$

$$R^{\sigma}{}_{\mu\nu\rho} = 0 \tag{71.3}$$

Quotient manifolds from Minkowski space
Cylinder : identification of (x^μ) and $(x^\mu + na^\mu)$ (quotient $\mathbb{R}^n/\mathbb{Z}$)
Important : spacelike hypersurfaces Σ_t have points noted by $\vec{x}(t)$ such that $\vec{x}(t) = (t, \vec{x})$
Induced metric on Σ_t : $g|_{\Sigma_t} = \delta$, with $\delta(\partial_i, \partial_j) = \delta_{ij}$
Product $\delta(\vec{x}(t), \vec{y}(t))$ is noted $\vec{x}(t) \cdot \vec{y}(t)$, $\vec{x} \cdot \vec{x} = x^2$, $|x| = \sqrt{\vec{x} \cdot \vec{x}}$
Differential operators on Σ_t : $\partial_i = \vec{\nabla}$, $\delta^{ij}\partial_i\partial_j = \Delta$

71.1 Isometries of Minkowski space

As Minkowski space is maximally symmetric, it admits the full $(n^2+n)/2$ Killing vectors. In Cartesian coordinates, the common choice is

- n translations ∂_μ
- $(n^2 - n)/2$ rotations $x^\mu \partial_\nu - x^\nu \partial_\mu$

The Killing vectors are usually decomposed into time translations ∂_t, space translations ∂_i, spatial rotations $x^i\partial_j - x^j\partial_i$ and boosts $x^i\partial_t - x^t\partial_i$. In addition, ∂_t is a static timelike vector field, as it is orthogonal to the Cauchy surface Σ_t.
Two additional important isometries of Minkowski space are the discrete isometries of time reversal and space reversal, $T(t, x_i) = (-t, x_i)$ and $P(t, x_i) = (t, -x_i)$. Those have the effects of reversing the time and space orientation of the manifold.
The full group of isometries generated by those Killing vectors is called the Poincaré group, noted $\mathbb{R}^n \rtimes \mathrm{O}(1, n-1)$, with $\mathbb{R}^n$ representing the translation group and $\mathrm{O}(1, n-1)$ the full Lorentz group, composed of time and space reversal as well as all rotations.

71.2 Geodesics of Minkowski space

The geodesics of Minkowski space are simply the lines of Euclidian space, as can be shown easily from its geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \tag{71.4}$$

From basic calculus, we can deduce that the geodesic equation has the following solution for initial conditions (x_0, v_0)

$$x^\mu(\lambda) = v_0^\mu \lambda + x_0^\mu \quad (71.5)$$

With the constant tangent vector $v^\mu = v_0^\mu$. This is indeed the equation of a straight line. Geodesic is timelike if

$$(v^t)^2 > |\vec{v}| \quad (71.6)$$

null if

$$(v^t)^2 = |\vec{v}| \quad (71.7)$$

spacelike if

$$(v^t)^2 < |\vec{v}| \quad (71.8)$$

71.3 Causality

From the fact that it is both static and diagonal, we already know that Minkowski space is causal.
Prove that it is globally hyperbolic (from the null geodesics)
Any null geodesic will be of the form

$$x^\mu(\lambda) = v_0^\mu \lambda + x_0^\mu \quad (71.9)$$

Show that it contains no singularities

71.4 The Fourier transform

On Minkowski space, as well as any quotient manifold, we can define the Fourier transform of a function
Pontryagin duality

71.5 Solution of classical fields

Klein Gordon :

$$(-\partial_t^2 + \sum_{i=0}^{n} \partial_{x_i}^2 + m^2)\varphi = 0 \quad (71.10)$$

Green function :

$$(-\partial_t^2 + \sum_{i=0}^{n} \partial_{x_i}^2 + m^2)G(x,y) = \delta(x-y) \quad (71.11)$$

Fourier transform

$$x \quad (71.12)$$

Dirac, Maxwell equation (classical and quantum)

Maxwell in Lorentz gauge :

$$\Box A^\mu = -\partial_t^2 A^\mu + \Delta A^\mu = j^\mu \tag{71.13}$$

71.6 Important coordinates on Minkowski space

71.6.1 Spherical coordinates

$$t = t \tag{71.14}$$
$$r = \sqrt{\vec{x}\cdot\vec{x}} \tag{71.15}$$

71.6.2 Null coordinates

for a spacelike Cartesian coordinate x, take

$$u = t - x \tag{71.16}$$
$$v = t + x \tag{71.17}$$

71.6.3 Rindler coordinates

For an observer with constant acceleration along x
In the region $x > 0$

$$t = a^{-1}\operatorname{arctanh}(\frac{t'}{x'})$$
$$x = \sqrt{(x')^2 - (t')^2}$$

$$ds^2 = -g^2x^2dt^2 + dx^2 + dy^2 + dz^2 \tag{71.18}$$

71.6.4 Born coordinates

For an observer rotating at a constant angular velocity
Rigidly rotating coordinates :

$$\begin{cases} t = \sqrt{1 + R^2\Omega^2}\tau \\ r = r' \\ \phi = \phi' + \Omega\tau \end{cases} \tag{71.19}$$

71.7 Quantum Minkowski space

Minkowski space in covariant quantum gravity :
Background metric of η, vacuum state for the gravitons
Measurements of the metric :

$$\langle\Omega|h_{\mu\nu}[f]|\Omega\rangle \tag{71.20}$$

Transition function :

$$\langle \Omega, \sum_n \int d^3 p n_p \rangle \tag{71.21}$$

For LQG : trivial spin network
For string theory

72 de Sitter space

Pseudosphere submanifold of $\mathbb{R}^{n+1}$
Equivalent to a quotient of hyperbolic space $\mathbb{H}^{(n-1)}$

73 Anti de Sitter space

74 Friedmann-Lemaître-Robertson-Walker spacetime

Homogeneous spacetime :
Homogeneous, isotropic spacetime :

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{n} dx_i^2 \tag{74.1}$$

Conformal expression :
switch to conformal time

$$\tau = \int_0^t a(t')dt' \tag{74.2}$$

$$\frac{\partial t}{\partial \tau} = \frac{1}{a(\tau)} \tag{74.3}$$

We write $a(t(\tau)) = a(\tau)$, giving us

$$ds^2 = a^2(\tau)(-d\tau^2 + \sum_{i=1}^{n} dx_i^2) \tag{74.4}$$

The FLRW metric is then conformally equivalent to Minkowski space.

74.1 Isometries

Maximally symmetric on the spacelike hypersurface : $((n-1)^2 + n - 1)/2$ Killing vectors
Translations ∂_i and rotations $x^i \partial_j - x^j \partial_i$

74.2 The assumption of homogeneity

The FRLW metric is used in cosmology under the assumption that our universe is homogeneous, which is obviously wrong (you may check this experimentally by observing an object in your vicinity with a density higher than the same volume of air also in your vicinity). But it is assumed that at large enough scale, it is in some sense homogeneous, which gives that the FRLW metric is in some sense approximately true. We will make this notion clearer.
One important theorem for the observation of homogeneity is the EGS theorem :

Theorem 74.1. If every observer measures a radiation field to be exactly isotropic during a time interval I, then the spacetime is homogeneous and isotropic.

Applied to the cosmic microwave background, this gives us a good argument for homogeneity. But the CMB is not perfectly isotropic (although small we have variations of the order of one in 10^5). We then have to use weaker assumptions.

74.3 FRLW universe and the matter content

The factor a depending on the stress energy tensor

75 Schwarzschild spacetime

The Schwarzschild metric applies to spherically symmetric spacetimes that are vacuum solutions on the whole manifold or for some range $r > r_0$, by considering the submanifold where the stress-energy tensor vanishes. As per (ref), the metric of a spherically symmetric spacetime

$$ds^2 = ds^2 = -f(r,t)dt^2 + g(r,t)dr^2 + Y^2(r,t)(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{75.1}$$

75.1 Birkhoff theorem and the Schwarzschild solution

[REAL THEOREM MUCH MORE COMPLICATED CF STEPHANI]

Theorem 75.1. A spherically symmetric vacuum solution of the Einstein field equations with $\Lambda = 0$ must be static and asymptotically flat.

Proof. □

This means that our metric must be both spherically symmetric and static

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + Y^2(r)(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{75.2}$$

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{75.3}$$

75.2 Source of the Schwarzschild metric

It may seem strange that even for a totally vacuum spacetime, we end up with the relativistic equivalent of a point mass solution.
While the stress-energy tensor is everywhere vanishing, we can verify that the total mass associated with the spacetime will be indeed the parameter M.

75.2.1 Komar mass of a Schwarzschild black hole

Komar mass of the metric

75.2.2 Schwarzschild metric as a distribution

Distributional Schwarzschild :

75.3 The exterior solution

The Schwarzschild coordinate patch is only defined on the maximum range $r > r_0$, with $r_0 = 2M$, the Schwarzschild radius. The surface $r = r_0$ is the Schwarzschild horizon, at which point the metric becomes degenerate, $\det g = 0$.

As we will see later, there's no spherically symmetric static spacetimes with a mass distribution outside of its horizon, so we can safely consider the exterior solution as the spacetime outside of a spherical body.
Shell theorem
Surface gravity

Definition 75.2. The *surface gravity* κ of a static Killing horizon is the acceleration needed to keep an object at the horizon.

$$\nabla_K K = \kappa K \tag{75.4}$$

Normalization : $|K| = -1$ on the asymptotic boundary.

75.4 The interior solution

There are two types of interior solution for the Schwarzschild metric : the vacuum interior solution (corresponding to a Schwarzschild black hole), and the non-empty interior, corresponding to a spherical object. In this section we will consider the non-empty interior, where the interior starts at $r > r_S$.

75.4.1 Hydrostatic stars

Equation of state

75.5 Maximal extension

For the totally empty Schwarzdschild solution, the Schwarzschild coordinates do not give us a complete covering of the manifold by its coordinate patches, as the metric signature becomes degenerate at $r = r_S$. This is not the sign of a singularity, as can be checked by computing some scalars
To get the full spherically symmetric extension of the metric, we will have to switch to another coordinate system. It can be constructed by a few subsequence change in coordinates.
First we will switch to Tortoise coordinates

$$r^* = r + 2M \ln |\frac{r}{2M} - 1| \tag{75.5}$$

They are so called in reference to Zeno's paradox [1], because as can be checked, the coordinate r^* never reaches the horizon.

$$\lim_{r \to 2M} r^* = -\infty \tag{75.6}$$

Tortoise coordinates have the single non-trivial Jacobian element

$$\frac{\partial r^*}{\partial r} = (1 - \frac{2M}{r})^{-1} \tag{75.7}$$

making the metric

$$ds^2 = \tag{75.8}$$

Eddington-Finkelstein coordinates :
ingoing : $v = t + r^*$ outgoing : $u = t - r^*$

$$ds^2 = -(1 - \frac{2M}{r})dudv + r^2 d\Omega^2 \tag{75.9}$$

$$ds^2 = -\frac{2Me^{-\frac{r}{2M}}}{r} e^{\frac{v-u}{4M}} dudv + r^2 d\Omega^2 \tag{75.10}$$

Kruskal-Szekeres coordinates

$$\begin{aligned} U &= -e^{-\frac{u}{4M}} \\ V &= e^{\frac{v}{4M}} \end{aligned} \tag{75.11}$$

$$ds^2 = -\frac{32M^2 e^{-\frac{r}{2M}}}{r} dUdV + r^2 d\Omega^2 \tag{75.12}$$

The topology of the maximally extended Schwarzschild spacetime
If we suppress the angular coordinates, the Kruskal coordinates are conformally equivalent to Minkowski space, meaning that we can deduce its Penrose conformal diagram easily enough by embedding it into Minkowski space.

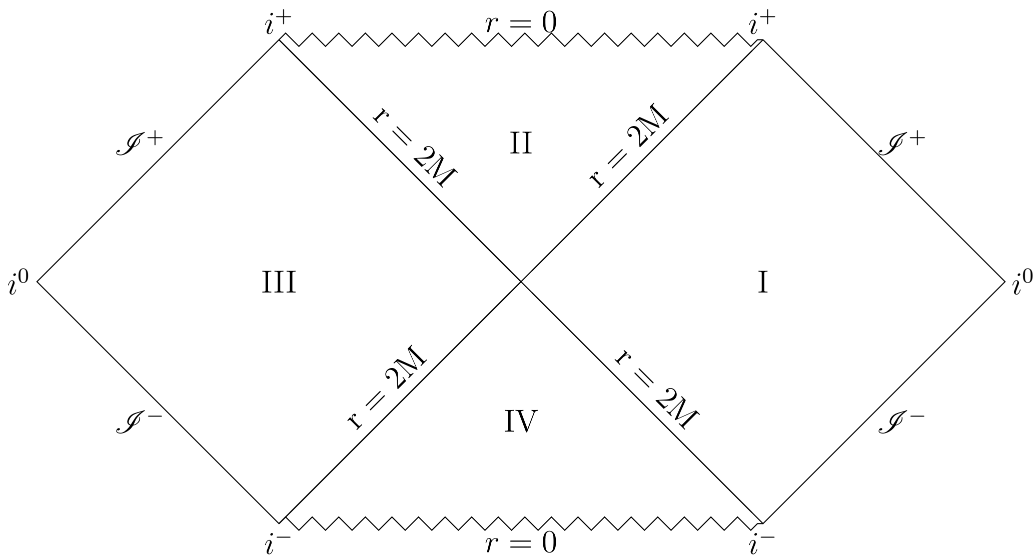


Figure 14: Conformal diagram of the Schwarzschild spacetime

Wormhole

75.6 Geodesics

$$-\kappa = g_{\mu\nu}u^\mu u^\nu = -(1 - \frac{2M}{r})\dot{t}^2 + (1 - \frac{2M}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 \tag{75.13}$$

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}(1 - \frac{2M}{r})(\frac{L^2}{r^2} + \kappa) = \frac{1}{2}E^2 \tag{75.14}$$

Timelike radial geodesics :

$$x \tag{75.15}$$

Orbits of test particles
For massive and massless particles
Photon sphere

75.7 Causality

Despite its singularities, the Schwarzschild spacetime is globally hyperbolic.
Find the Cauchy surface in Kruskal coordinates, find the null geodesics
Horizons, trapped surfaces, singularities

75.8 Black hole thermodynamics

Laws of black hole thermodynamics : analogy between surface gravity/temperature, surface area/entropy

Theorem 75.3. The horizon has constant surface gravity for a stationary black hole.

Theorem 75.4. For a perturbative change of a stationary black hole, the change in energy is related to the change of area, angular momentum and electric charge.

$$dE = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \tag{75.16}$$

Theorem 75.5. If the weak energy condition holds, the area of the event horizon is increasing.

$$\frac{dA}{dt} \geq 0 \tag{75.17}$$

Theorem 75.6. There are no black hole with vanishing surface gravity κ.

Analogy :

$$T_H = \frac{\kappa}{2\pi} \tag{75.18}$$

$$S_{BH} = \frac{A}{4} \tag{75.19}$$

Issue with black hole entropy and violation, segue into Hawking radiation

75.9 Hawking radiation

Scalar field on Schwarzschild :

$$(\Box + m^2)\varphi + \xi R = (\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + m^2)\varphi + \xi R \tag{75.20}$$

75.10 Variations on the Schwarzschild spacetime

75.10.1 The negative mass Schwarzschild metric

The negative mass Schwarzschild, like its name indicates, is just a variation on the Schwarzschild metric with a negative mass.

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{75.21}$$

75.10.2 The Vaidya metric

The Vaidya metric is a simple modification of the Schwarzschild metric to make it into a non-vacuum solution
outgoing Vaidya metric :

$$ds^2 = -(1 - \frac{2M(u)}{r})du^2 + 2dudr + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{75.22}$$

ingoing Vaidya metric :

$$ds^2 = -(1 - \frac{2M(v)}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{75.23}$$

76 Reissner–Nordström metric

Vacuum solution of the spherically symmetric Einstein-Maxwell equation

$$ds^2 = (1 - \frac{2M}{r} + \frac{\mu Q^2}{4\pi r^2})dt^2 - (1 - \frac{2M}{r} + \frac{\mu Q^2}{4\pi r^2})^{-1}dr^2 - r^2 d\Omega^2 \tag{76.1}$$

Theorem 76.1. That thing about massless neutrino fields

76.1 Causality

extremal or not
Penrose diagram

77 Kerr and associated spacetimes

Unlike for the Schwarzschild solution, which can describe entirely every vacuum spherically symmetric spacetime by a single class of solutions parametrized by the mass M, the solutions for vacuum axisymmetric spacetimes cannot be so easily classified, even assuming stationarity.

77.1 Static axisymmetric vacuum spacetimes

As we've seen previously, a general form for a static axisymmetric spacetime is

77.2 Tomimatsu metrics and Perjes metrics

77.3 Kerr-Newman metric

$$ds^2 = \tag{77.1}$$

77.4 Causality

Spacelike singularities, closed timelike curves, naked singularities, naked CTCs
Instability of the Cauchy horizon

77.5 Stellar black holes

Theorem on collapse and stable solutions

77.6 Hawking radiation

Loss of angular momentum by emission

78 Wave solutions

$$g = g_R(x) + 2dudv + H(x,u)du^2 \tag{78.1}$$

Gravitational wave solutions
EM wave solutions
pp-wave spacetimes
wave collision

79 The Gödel spacetime

Dust solution, homogeneous, axisymmetric
Topology : $\mathbb{R}^4$

$$ds^2 = \frac{1}{2\omega^2}[-(dt + e^x dz)^2 + dx^2 + dy^2 + \frac{1}{2}e^{2x}dz^2] \tag{79.1}$$

This corresponds to the frame field

$$\begin{aligned}
e_0 &= \sqrt{2}\omega\partial_t \\
e_1 &= \sqrt{2}\omega\partial_x \\
e_2 &= \sqrt{2}\omega\partial_y \\
e_3 &= 2\omega(\exp(-x)\partial_z - \partial_t)
\end{aligned}$$

Einstein tensor :

$$G^{ab} = \omega^2\eta^{ab} + 2\omega^2\partial_t \otimes \partial_t \tag{79.2}$$

79.1 Symmetries

Killing vectors : ∂_t, ∂_y, ∂_z, $\partial_x - z\partial_z$, $-2\exp(-x)\partial_t + z\partial_x + (\exp(-2x) - z^2/2)\partial_z$

79.2 Causality

Closed timelike curves through every point (no geodesics)
Lack of spacelike hypersurfaces

80 Wormholes

Wormholes are not perfectly well defined, but they can roughly be defined by the process of removing an open n-ball from two spacelike hypersurface either of the same spacetime or two different spacetimes, and identifying the boundaries together (in other words a connected sum). The region around the boundary is referred to as the throat of the wormhole, and it should be small in some sense compared to the original manifolds (otherwise Minkowski space could qualify). If the two spacetimes are different, it is sometimes referred to as an inter-universe wormhole, while if it is the same, an intra-universe wormhole.

80.1 The Morris-Thorne wormhole

Morris-Thorne wormholes are a class of simple spherically symmetric traversible wormholes. The first instance of them was in the form of the Ellis and Bronnikov drain hole. For its wormhole structure, the Morris-Thorne wormhole is simply two copies of $\mathbb{R}^{n-1}$ with the boundary of a ball identified. As such, its Cauchy surface has the topology

$$\mathbb{R}^{n-1} \# \mathbb{R}^{n-1} \approx \mathbb{R}^{n-1} \setminus \{p\} \approx \mathbb{R} \times S^{n-1} \tag{80.1}$$

with the full spacetime manifold with the topology $\mathbb{R}^2 \times S^{n-1}$, equipped with the coordinates (t, l, θ_i). Its metric is of the form

$$ds^2 = -e^{2\Phi(t,l)} dt^2 + dl^2 + r^2(t,l) d\Omega^2 \tag{80.2}$$

throat radius :

$$r_0 = \min_{l \in \mathbb{R}} (r(l)) \tag{80.3}$$

with the appropriate choice of α and r, we would get the Schwarzschild black hole, which has indeed a wormhole structure, but suffers from having an event horizon. To restrict it to the traversible case, we also require that $\phi(l)$ be everywhere finite.
In fact, we can recast the metric in a form even more suggestive of the Schwarzschild metric, which will be of some use for calculations.

$$ds^2 = -e^{2\Phi_\pm(t,r)} dt^2 + (1 - b_\pm(r)/r)^{-1} dr^2 + r^2(t,l) d\Omega^2 \tag{80.4}$$

If we restrict our analysis to the static case

$$ds^2 = -e^{2\Phi(l)} dt^2 + dl^2 + r^2(l)[d\theta^2 + \sin^2(\theta) d\varphi^2] \tag{80.5}$$

Christoffel symbols :
Riemann tensor :
Ricci scalar :
Stress energy tensor associated
Violation of the weak energy condition
Mass of the wormhole
charge

80.1.1 The Ellis-Bronnikov wormhole

The Ellis-Bronnikov wormhole is the simplest form of the Morris-Thorne wormhole, with the : $r(l) = \sqrt{l^2 + a^2}$, a the diameter of the throat

$$ds^2 = -dt^2 + dl^2 + (l^2 + a^2)[d\theta^2 + \sin^2(\theta) d\varphi^2] \tag{80.6}$$

WEC, minimization, etc
No go theorem on scalar fields

80.2 Expanding wormholes

the de Sitter wormhole

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\chi t}[(1 - \frac{b(r)}{r})^{-1} dr^2 + r^2 d\Omega] \tag{80.7}$$

80.3 Intra-universe wormholes

The intra-universe wormhole is the basic notion of a wormhole as the addition of a handle to a hypersurface. If we consider a foliated spacetime with topology $\mathbb{R} \times \Sigma$, the addition of an intra-universe wormhole will have the topology $\mathbb{R} \times (\Sigma \# S^{n-1})$.
As this process is the connected sum of a torus, and that a 1-dimensional torus will be the circle S, an identity of the connected sum, we will not have any wormholes in $(1+1)$ dimensions directly.

80.3.1 Torus universe

The simplest intrauniverse wormhole is just be the addition of a handle to $\mathbb{R} \times S^2$, which will simply be $\mathbb{R} \times T^2$. While a flat torus might be a bit of a stretch as a wormhole, it will always be possible to apply a Weyl transform to it to make the inside distance much shorter than the outside. For instance, if we define the outside of the wormhole as the region $\theta \in [\pi, 2\pi]$ and the inside as $\theta \in [0, \pi]$, all we need is to define the metric

$$g = \Omega \eta \tag{80.8}$$

with Ω rapidly diminishing between $\pm\pi$ and 0.
Method to construct a time machine from it

80.3.2 Asymptotically flat wormhole

A more "realistic" wormhole spacetime is the space with a handle, $\mathbb{R} \times (\mathbb{R}^{n-1} \# T^{n-1})$. As in general, the connected sum of a manifold with $\mathbb{R}^n$ is equivalent to that manifold with a disk removed, this
Wormhole that is $\mathbb{R}^n \# T^n$: plane with a handle, aka punctured torus $T^n \setminus D^n$
The fundamental group of the punctured n-torus is the free group of n generators

Definition 80.1. The free group F_n is the group constructed from the generators $\{a_1, ..., a_n\}$, each of which has an associated inverse a_k^{-1}, such that every sequence of the generators

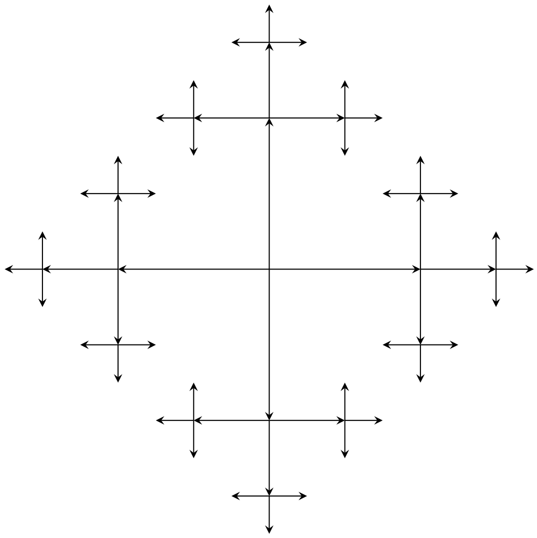


Figure 15: Cayle graph of F_2

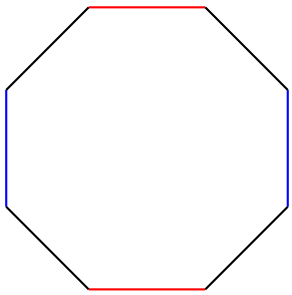


Figure 16: Fundamental polygon of the punctured torus

and their inverse give rise to a unique element of the group, unless dictated by the group structure.

For instance, the free group F_2 is defined by the generators $\{a, b\}$, with inverses a^{-1}, b^{-1}. The first few group elements are of the form

$$\begin{aligned} F_2 = \{ \quad & 1,\ a,\ ,\ b,\ a^{-1},\ b^{-1},\ ab,\ ba,\ a^{-1}b^{-1}, \\ & b^{-1}a^{-1},\ a^{-1}b,\ ba^{-1},\ ab^{-1},\ b^{-1}a,\ ...\} \end{aligned}$$

Only elements of the type $a^{-1}a$, $ab^{-1}bb$, $aa^{-1}a$, etc, are written non-uniquely, corresponding to 1, ab and a respectively.
Cayle graph of F_2 :
Universal cover : the disk, can be hyperbolic space or plane
For the covering map $\mathbb{H}^2 \to T^2 \setminus D^2$, need to find a Fuchsian group $\Gamma \approx F_2$ so that $\mathbb{H}^2/\Gamma \approx T^2 \setminus D^2$
Construction of the covering map via the Cayle complex method
Embedding of the polygon in the Poincaré disk or Poincaré half plane
Poincaré disk : $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$
Geodesics : Arcs of circles and diameters that meets ∂D orthogonally

Map from $\mathbb{H}$ to $\mathbb{D}$:

$$h(z) = \frac{z - i}{iz - 1} \tag{80.9}$$

Inverse :

$$h^{-}1(z) = \frac{-z + i}{-iz + 1} \tag{80.10}$$

$$ds^2 = 4\frac{dx_1^2 + dx_2^2}{1 - x_1^2 - x_2^2} \tag{80.11}$$

$$ds^2 = 4\frac{dz + d\bar{z}}{(Im(z))^2} \tag{80.12}$$

Symmetry group of $\mathbb{H}^2$: $SL(2, \mathbb{R})$
Möbius transformation on $\mathbb{D}$:

$$\frac{\alpha z + \beta}{\bar{\beta} z + \bar{a}},\ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 > 0 \tag{80.13}$$

Fuchsian groups : discrete subgroups of $PSL(2, \mathbb{R})$
Basic wormholes constructed from the punctured torus :
Consider the punctured torus constructed from the Clifford torus (φ, θ), $\varphi, \theta \in [0, 2\pi]$. The Clifford torus just has the flat metric

$$ds^2 = d\theta + d\varphi \tag{80.14}$$

We remove from it the closed set $A = \{(\varphi, \theta) | \varphi, \theta \in [0, \pi]\}$.
To make it into an asymptotically flat manifold, we apply a Weyl transformation to send the edges of the removed set to infinity

$$\lim_{x \to \partial A} \Omega(x) = 1 \tag{80.15}$$

$$d(x, \partial A) = \infty \tag{80.16}$$

The simplest spacetime for this hypersurface is then just the static wormhole spacetime

$$ds^2 = -dt^2 + \Omega(\omega, \varphi)(d\theta + d\varphi) \tag{80.17}$$

check for horizons

80.3.3 Cut and paste thin-shell wormholes

The simplest method to get wormholes with tractable solutions is the use of the thin-shell formalism after cutting and pasting some elements from the manifold.

Definition 80.2. A thin-shell Minkowski wormhole with mouth topology S is composed by cutting two open sets $S_1, S_2 \subset \mathcal{M}$, $S_1 \cap S_2 = \varnothing$, where S_1 and S_2 intersect every such that there exists two homeomorphisms

$$f_i : S \to S_i,\ i = 1, 2 \tag{80.18}$$

and such that the boundaries ∂S_1 and ∂S_2 are identified using the homeomorphism $f_1^{-1} \circ f_2 : S_1 \to S_2$

As the metric is homogeneous for Minkowski space, it's not hard to show that the metric on ∂S_1 is the same as the one on ∂S_2, hence the metric defined by the gluing will be at least C^0.

Definition 80.3. The acceleration of the mouth is defined by a sequence of timelike curves $\{\gamma_n\}$ such that for every spacelike hypersurface, $\gamma_n(\tau)$ converges to

CTCs : Induce a time difference between the two mouthes (identify the spheres at different moments in time)

Proposition 80.4. If a cut and paste wormhole in Minkowski space has the boundaries identified such that $\partial A_1(t)$ is identified with $\partial A_2(f(t))$, with $f(0) = 0$, the function f depends on the acceleration of ∂A_2.

Proof. If we pick a point $p \in \partial S$ □

81 Faster-than-light spacetimes

81.1 Defining faster-than-light spacetimes

A faster than light (or FTL) spacetime, also called a spacetime shortcut or hyperfast spacetime is a spacetime for which the travelling time between two points can be shorter than the travelling time of light in normal circumstances.
The exact definition of a faster than light (or FTL) metric is a bit difficult, as it is more of an engineering notion than a properly physical one. One definition might be a metric where, given two points x_1 and x_2 on a spacelike hypersurface with a distance d in between them, at a future time
The basic idea behind FTL metrics is the widening of light cones along the trip's trajectory, to have the possibility of travelling along trajectories that would be considered spacelike without those modifications.
There are several definitions available for
The notion behind a rigorous definition of a spacetime allowing FTL travel is to define it with respect to another, more "reasonable" spacetime.
Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two spacetimes, which each contain a pair of inextendible timelike curves $\mathcal{E}_i$, $\mathcal{D}_i$. Define points $S_1 \in \mathcal{E}_1$, $S_2 \in \mathcal{E}_\in$. We then define the points

$$F_i = \partial J^+(S_i) \cap \mathcal{D}_i \tag{81.1}$$

$$R_i = \partial J^+(F_i) \cap \mathcal{E}_i \tag{81.2}$$

F_i corresponds to the earliest point something leaving $\mathcal{E}_i$ could reach $\mathcal{D}_i$, whihe R_i is the earliest point something could make the trip back.

81.2 The Alcubierre warp drive metric

The Alcubierre drive was the first attempt at a FTL metric. It was inspired by the faster than light cosmological expansion, where the distance between two stars at rest might increase in such a way that they appear to move at speeds greater than the speed of light.
The metric is constructed by a widening of the light cone within a compact region, that itself moves

$$ds^2 = -dt^2 + (dx - v_s f(r_s) dt)^2 + dy^2 + dz^2 \tag{81.3}$$

$x_s(t)$ some arbitrary function of spacetime

$$v_s(t) = \frac{dx_s(t)}{dt},\ r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2} \tag{81.4}$$

$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)} \tag{81.5}$$

For a constant warp bubble velocity $v_s(t) = v_0$, consider the coordinate change

$$r = x - v_0 t \tag{81.6}$$

In the new coordinates, the metric becomes

$$ds^2 = -A(r)(dt - \frac{v_0(1-f(r))}{A(r)})^2 + dy^2 + dz^2 \tag{81.7}$$

Then proper time $\tau = t - \frac{v_0(1-f(r))}{A(r)} r$

$$ds^2 = -A(r)d\tau^2 + \frac{dr^2}{A(r)} + dy^2 + dz^2 \tag{81.8}$$

Static metric in $\mathbb{R}^4$: causal
Why the FTL part : inside the bubble, for $ds^2 = 0$ in the tx plane, $t^2 = (x - vt)^2$, $t = \pm x - vt$
Also for $v > 1$: horizon on the bubble
Modified version that violates causality :
Consider the Lorentz transform

$$\begin{aligned} t' &= \gamma(t - \beta x) \\ x' &= \gamma(x - \beta t) \\ y' &= y \\ z' &= z \end{aligned}$$

$$\begin{aligned} t &= \gamma(t' + \beta x') \\ x &= \gamma(x' + \beta t') \\ y &= y' \\ z &= z' \end{aligned}$$

$$\begin{aligned} g_{xx} &= \gamma^2(1 - \beta^2(1 - v^2 f^2(r_s)) - 4\beta v f(r_s)) \\ g_{tt} &= -\gamma^2(1 - v^2 f^2(r_s) + 4\beta v f(r_s) - \beta^2) \\ g_{xt} &= \gamma^2(\beta v^2 f^2(r_s) + (1 + \beta^2)2v f(r_s)) \end{aligned}$$

$$\begin{aligned} ds^2 &= -(1 + \gamma^2 f(r_s)(4\beta v - v^2 f(r_s)))dt^2 \\ &+ (1 + \gamma^2 \beta v f(r_s)(\beta v f(r_s) - 4))dx^2 \\ &+ v f(r_s)(\gamma^2 \beta v f(r_s) + 2)dxdt \\ &+ dy^2 + dz^2 \end{aligned}$$

Outside the bubble, the metric reduces to Minkowski space, while inside the bubble, it reduces to

$$\begin{aligned} ds^2 &= -(1 + 4\beta v - v^2)dt^2 + (1 + \gamma^2 \beta v(\beta v - 4))dx^2 \\ &+ v f(\gamma^2 \beta v + 2)dxdt + dy^2 + dz^2 \end{aligned}$$

81.3 The Krasnikov tunnel

The Alcubierre warp drive metric, beyond its physical problems with respect to horizons and energy conditions, also suffers from engineering issues due to the causal isolation of the inside of the bubble.
2D version :

$$\begin{aligned} ds^2 &= -(dt - dx)(dt + k(x,t)dx) && (81.9) \\ &= -dt^2 + (1 - k(x,t))dxdt + k(x,t)dx^2 && (81.10) \end{aligned}$$

$$k(x,t) = 1 - (2 - \delta)\theta_\varepsilon(t - x)[\theta_\varepsilon(x) - \theta_\varepsilon(x + \varepsilon - D)] \quad (81.11)$$

θ_ε is a mollified version of the Heaviside function, a smooth function satisfying

$$\theta_\varepsilon(x) = \begin{cases} 1 & x > \varepsilon \\ 0 & x < 0 \end{cases} \quad (81.12)$$

4D version :

$$ds^2 = -dt^2 + (1 - k(x,t,\rho))dxdt + k(x,t,\rho)dx^2 + d\rho^2 + \rho^2 d\phi^2 \quad (81.13)$$

$$k(x,t) = 1 - (2 - \delta)\theta_\varepsilon(\rho_{max} - \rho)\theta_\varepsilon(t - x - \rho)[\theta_\varepsilon(x) - \theta_\varepsilon(x + \varepsilon - D)] \quad (81.14)$$

CTCs : At least 3+1D, to have two non-intersecting tubes

82 Causality violating and time machine solutions

While many spacetimes violate the chronology condition, a time machine refers specifically to a spacetime that could be conceivably constructed.

Definition 82.1. A time machine is a spacetime with a chronology violating region stemming from a compactly generated Cauchy horizon.

This excludes spacetimes with no Cauchy horizons such as the Gödel spacetime or the Tipler cylinder, and non-compactly generated Cauchy horizons. This is on the assumption that our own universe doesn't include by itself closed causal curves and that we can only influence matter in a compact region of spacetime.

Part IX

The real world

83 Experimental tests of general relativity

Since this section will involve physical measurements, we will put back in every SI constant in the equations, with the measured values of

$$\begin{aligned} c &= 299\,792\,458\,\mathrm{m\,s^{-1}} \\ \kappa &= 2.076\,579 \times 10^{-43}\,\mathrm{s^2\,m^{-1}\,kg^{-1}} \\ G &= 6.674\,08(31) \times 10^{-11}\,\mathrm{s^{-2}\,m^3\,kg^{-1}} \\ \hbar &= 1.054\,571\,800(13) \times 10^{-34}\,\mathrm{J\,s} \\ M_\oplus &= 5.9724(3) \times 10^{-24}\,\mathrm{kg} \\ R_\oplus &= 6.3781 \times 10^{-6}\,\mathrm{m} \end{aligned}$$

83.1 Measurements in general relativity

To perform experiments on general relativity, we first have to define what measurable quantities are involved and how we can measure them. The two basic quantities we will be interested in are proper times and lengths, which we will measure using clocks and measuring rods.

83.1.1 Clocks

A clock in general relativity will be any process which, approximated as happening along a single timelike curve $\gamma(\tau)$ of unit speed, has recurring events such that if this event occurs at some τ_p, then it will only occur at $\tau_p + kT$, for some $T \in \mathbb{R}_{>0}$, $k \in \mathbb{Z}$. As the proper time along a geodesic of unit speed is just $\tau_2 - \tau_1$, the proper time between any two events will be kT, and the proper time between two consecutive events will be T, the *period* of the clock.
We can then define a parametrization of the curve

$$\tau \mapsto (\tau - \tau_p)/T \tag{83.1}$$

In this parametrization, every event will occur at $\tau = k$. This is the time as measured by the clock.
An important clock is the one defined by the maximal amplitude of the radiation emitted by the transition between the two hyperfine levels of the ground state of caesium 133, at rest and at a temperature of $0K$. For this, we define additionally the parametrization

$$\tau \mapsto t = (\tau - \tau_p)/(9\,192\,631\,770\ T) \tag{83.2}$$

for which we go from t to $t+1$ after 9 192 631 770 consecutive events. We then say that in this parametrization, the time elapsed between two events $\gamma(t_1)$ and $\gamma(t_2)$ is $t_2 - t_1$ seconds.
If we have to use two different clocks, we can calibrate them to measure approximately the same time. If we have two clocks of period T_1 and T_2 starting at λ_p, and we find that the last tick N_2 of the second clock occurs between the ticks N_1 and $N_1 + 1$ of the first clock, we then know that the relation between the two periods will be between $N_1 T_1 = N_2 T_2$ and $(N_1 + 1) T_1 = N_2 T_2$, or

$$T_2 \in [T_1 \frac{N_1}{N_2}, T_1 \frac{N_1+1}{N_2}] \tag{83.3}$$

which we can write as the uncertainty

$$T_2 = T_1(\frac{N_1}{N_2} + \frac{1}{2N_2}) \pm T_1 \frac{1}{2N_2} \tag{83.4}$$

The uncertainty becoming negligible for $N_2 T_2 \ll T_1$.
We can then parametrize, up to uncertainty, any timelike curve with the parametrization of that new clock. This will allow us to measure any time interval using SI units.
If the spacetime is globally hyperbolic, we can define the temporal function from a foliation such that $\nabla t = n$ has timelike curves as a flow with this parametrization for hypothetical clocks going through each points. This will be the global SI parametrization of time for this spacetime. If the spacetime isn't globally hyperbolic, we can simply pick a globally hyperbolic neighourhood of the region of interest.
Light clocks

83.1.2 Measuring rod

Measuring rod : two timelike curves such that their intersection in an achronal spacelike hypersurface is of constant separation.
Meter : a measuring rod is of 1 m if a light rays leave point A at λ_A and reaches point B at λ_B such that, in the SI parametrization,

$$(\lambda_B - \lambda_A) = \frac{1}{299\,792\,458} \text{ s} \tag{83.5}$$

or for a round trip AB, BA, the time measured along A is twice that.
Observer carries a clock and measuring rods

83.2 Special relativity

For the effects of special relativity, we will take as the manifold Minkowski space

83.2.1 Time dilation

Proposition 83.1. If we have in our coordinate system one observer A at rest and another observer B in motion at a constant speed $\vec{v}$, each carrying a clock, then on the same Cauchy surface, the proper time τ_A measured by A will be related to τ_B measured by B by

$$\tau_A = \frac{1}{\sqrt{\cdots}} \tau_B \tag{83.6}$$

Proof. □

Proposition 83.2. A measuring rod of length L_A in its own rest frame will be of length

$$L_B = \sqrt{\ldots} L_A \tag{83.7}$$

as measured by the measuring rod of an observer B in a motion of constant velocity.

The experimental measurement of those effects can be observed in the twin paradox

Proposition 83.3. For an observer A at rest on a timelike curve γ_A in the coordinate system, and an observer B following a timelike curve such that there exists two points of intersection between γ_A and γ_B, then the proper time measured by B will be shorter than the proper time measured by A.

Twin paradox on a cylinder

83.2.2 The relativistic Doppler effect

If we consider a wave source on Minkowski space with two observers, one at rest with respect to the source, another with a velocity of $\vec{v}$ with respect to it, if the first observer measures a frequency of ω

83.3 Newtonian approximation

As we saw with the Pauli-Fierz theory, it is possible to express general relativity as a field theory. In the limit of small perturbations and small derivatives around Minkowski space, the Einstein field equations become

$$\Box \bar{h}_{\mu\nu} = T_{\mu\nu} \tag{83.8}$$

Assumption of the Newtonian approximation : the spacetime is quasi-stationary :

$$h_{\mu\nu}(t, x) \approx h_{\mu\nu}(x) \tag{83.9}$$

The speeds are very low, so that the only large component of the stress energy tensor is T_{tt}.

$$\Box \bar{h}_{tt}(x) = \Delta \bar{h}_{tt} = T_{tt}(x) \tag{83.10}$$

If we put back everything in SI units, this gives us

$$\Delta \bar{h}_{tt} = \frac{8\pi G}{c^2} T_{tt}(x) \tag{83.11}$$

This is the Newtonian Gauss law, with the notation $\bar{h}_{tt} = 2\Phi + \text{const.}$, $T_{tt}/c^2 = \rho$ the mass density.

$$\Delta \Phi = 4\pi G \rho(x) \tag{83.12}$$

[CHECK WHERE THE SIGN WENT]
This means that to some approximation, every Newtonian effect will correspond to a general relativistic effect.

83.3.1 Free fall

The simplest gravitational experiment one can perform is the free fall of a body in a gravitational field. Since Earth is approximately spherical, and we can hopefully neglect its rotation, this will correspond to the motion of a radial geodesic in a Schwarzschild spacetime.

$$\dot{r}^2 + (1 - \frac{2GM}{c^2}\frac{1}{r}) = E^2 \tag{83.13}$$

If we expand around the radius of the earth, $R_\oplus$

$$\frac{1}{r} = \frac{1}{R_\oplus}\sum_{n=0}^{\infty} = \frac{1}{R_\oplus} - \frac{r - R_\oplus}{R_\odot^2} + \mathcal{O}(R_\oplus^3) \tag{83.14}$$

Cutting off the last term,

$$\dot{r}^2 + \frac{2GM}{c^2 R_\oplus^2} r = E^2 - 1 + \frac{4GM}{c^2 R_\odot} \tag{83.15}$$

Differentiate by τ :

$$2\ddot{r}\dot{r} + \frac{2GM}{c^2 R_\oplus^2}\dot{r} = 0 \tag{83.16}$$

$$\ddot{r} + \frac{GM}{c^2 R_\oplus^2} = 0 \tag{83.17}$$

This is indeed the classical equation of motion of a point mass in a uniform gravitational field. If we further define the gravitational acceleration on earth's surface as $g = GM/R_\oplus^2$, we get the familiar form

$$\ddot{r} = -g \tag{83.18}$$

with the usual solution

$$r(\tau) = -\frac{1}{2}g\tau^2 + v_0\tau + r_0 \tag{83.19}$$

which is the usual form of the Newtonian free fall.
Experiment : 60fps camera, drop of a ball, markers of distances
check local distance to Earth's barycenter, mass of the ball independently of gravity (spring pan)

83.3.2 The Cavendish experiment

Torsion balance for the attraction of two masses

83.3.3 Planetary orbits

Planetary orbits are one of the oldest method of testing gravitational laws, going all the way back to Newtonian gravity when the idea emerged.
Two body problem in Newtonian gravity

Standard gravitational parameter :

$$\mu = G(m_1 + m_2) \tag{83.20}$$

Newton's equation :

$$\begin{aligned} \vec{a}_1(t) &= Gm_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \\ \vec{a}_2(t) &= Gm_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \end{aligned} \tag{83.21}$$

Kepler's laws :

1. The orbit of a planet is an ellipse with the sun as one of its foci.
2. A line from the sun to the planet sweeps the same areas during the same time interval.
3. for the orbital period T and the semi-major axis of the orbit a, we have the relation

$$T^2 \propto a^3$$

From the Schwarzschild timelike geodesics :

83.4 The equivalence principle

One of the fundamental principle behind general relativity, as well as most metric theories of gravity, is the equivalence principle, which comes in two variety :
Weak equivalence principle : The trajectory of a test particle only subject to gravity will only depend on its initial position and velocity.
Strong equivalence principle : The outcome of any local experiment in a freely falling frame is independant of its position and velocity.

83.4.1 The Eötvös-Dicke experiment

objects of various materials dropped in a vacuum
Eötvös experiment :

83.5 Light deflection and gravitational lensing

83.5.1 Newtonian light deflection

While Newtonian gravity does not a priori affect any massless object, it is possible to make sense of light deflection in a Newtonian context by either considering it as a metric theory or by considering the massless limit of a deflection.
Deflection of a point particle of mass m :
Object coming from infinity with velocity v, hyperbolic orbit of the two body problem

83.5.2 Relativistic light deflection

Geodesic equation of a null curve around Schwarzschild metric
Eddington's measurement during the 1919 eclipse
Einstein cross

83.6 Redshifting in gravitational fields

83.6.1 The Pound-Rebka-Snider redshift experiment

Experiment to check the redshifting of some photons travelling up or down a gravitational potential.
Consider an EM wave/null particle travelling radially on a Schwarzschild metric
Blueshifting of the photon :

$$f_r = \left(\frac{1 - \frac{2GM}{(R+h)c^2}}{1 - \frac{2GM}{Rc^2}}\right)^{\frac{1}{2}} f_e \tag{83.22}$$

Emitter placed at the top of the tower, receiver at the bottom.

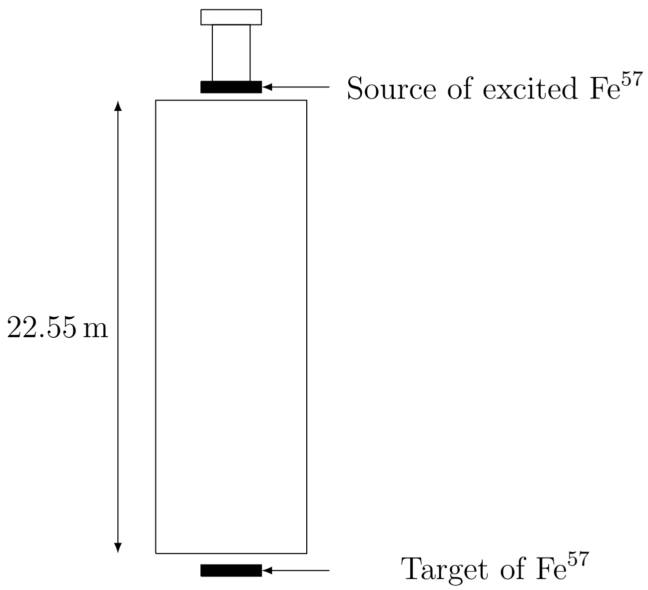


Figure 17: The Pound-Rebka experimental setup

Mossbauer effect with iron 57 : Nucleus in a lattice emits or absorbs gamma rays without significant loss of energy to nuclear recoil. The emitted gamma ray can be reabsorbed by another nucleus at rest with respect to the first one due to the very small difference.
Emitter : a source of 0.4 Cu Co^{57}, decaying to Fe^{57} in an excited state
Emission : 14.4 keV by iron 57 atoms
Distance between source and receiver : 22.55 m

83.6.2 Space MASER experiment

83.7 Time dilatation in gravitational fields

83.8 Frame dragging and the geodetic effect

Gyroscope (modelled by a spinning test particle) in orbit around a Kerr metric
Gravity Probe B

84 Astrophysics

84.1 Gravitational waves

84.1.1 Orbit decay of binary systems

Two-body problem in linearized gravity up to some order
Binary systems : loss of energy via gravitational radiation induces a decrease in the distance between the two stars

84.1.2 Direct detection by interferometry

Gravitational waves can also be detected directly by their effect on geodesics when passing through.
LIGO detection : interferometer perturbed by the passage of a gravitational wave
1064 nm neodymium-doped yttrium aluminium garnet laser beam of 20 W

84.2 Black holes

Observational evidence of black hole
Future probe for photon sphere observation

85 Cosmology

85.1 The Copernician principle

Copernician principle : the universe is at great scale homogeneous and isotropic at every point.
Matter distribution
Critical density

85.2 The cosmological expansion

Cosmological redshift : far enough objects are redshifted as if receeding at a speed that depends on their distance, including $> c$
comoving distance in the FRW metric

85.2.1 The cosmological distance ladder

To check the effect of the cosmological expansion, we first need some notion of the measure of distances for astronomical bodies.
Different methods for different objects and distances are available.

- Parallax
- Standard candles
- Cepheid variable stars

85.2.1.1 Parallax measure

Parallax measure is the oldest method available for the measuring of distances, going back to 189 BCE with Hipparcus' measurement of the distance of the moon.
Aristarchus measurement of the distance of the sun
First successful measurement : Friedrich Bessel in 1838 for the star 61 Cygni.

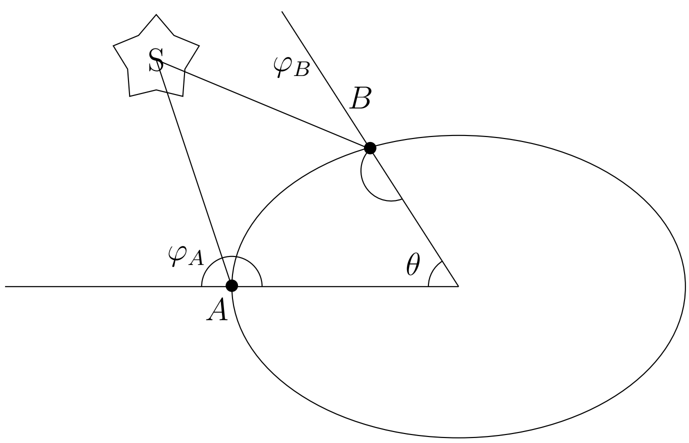


Figure 18: Measurement of distances using the parallax

If we assume the space sufficiently flat in between the points A, B and the star S, we can use to a good approximation Euclidian geometry. If we consider the spherical coordinates formed by the center of the point of origin that we consider, with the coordinates

$$\begin{aligned} A &= (\varphi_A, \theta_A, r_A) \\ B &= (\varphi_B, \theta_B, r_B) \\ S &= (\varphi_R, \theta_S, r_S) \end{aligned} \tag{85.1}$$

Distance A to S, B to S, distance A to B known
Simplest case : $r_A = r_B = R$, $\theta = \pi/2$, $\varphi_A = 0$, $\varphi_B = \pi$

$$x \tag{85.2}$$

85.2.1.2 Standard candles

As stars follow for the most part the same model, we usually have a good enough idea of their mass and luminosity simply from their composition (spectral rays).
If we can consider the curvature small enough, from the van Vleck determinant, we know that the light intensity of a source decays roughly as

$$I(r) \propto r^{-2} \tag{85.3}$$

Square decay
Distance modulus

$$m - M = 5\log_{10}(d) - 5 \tag{85.4}$$

M absolute magnitude, m apparent magnitude
[Some Hertzsprung russell diagram]
Cepheid variable stars
Stars that pulsate at a regular interval
Period of pulsation proportional to the luminosity

85.3 The cosmic microwave background

Cosmic microwave background
removal of redshifting from motions wrt the CMB
Roughly isotropic as seen from earth
PLANCK

85.4 Topology of the universe

Topology of the universe : Largest fundamental domain
Physical appearance of a wormhole, divergence of rays
Repeated patterns of the CMB outside the fundamental domain

86 Post-Newtonian parametrization

Description of most gravitational theories in terms of a handful of parameters. Compare those parameters with experiment.

87 Tests of quantum gravity

87.1 Tests of semiclassical gravity

Cavendish experiment with radioactive source

87.2 Tests of the Schrödinger-Newton equation

Due to the very small potential induced on individual particles, gravitation acts rather weakly in quantum mechanics. To palliate this, experiments usually involve particles with
Ultra cold neutron experiment

87.3 Tests for Lorentz violations

Lorentz invariance at very small scale
Experiment on quantum foam and light diffraction

88 History

88.1 Classical theories of gravitation

88.1.1 Antiquity

The study of gravity dates all the way back to the very beginning of physics, with Aristotelian physics [1]
Natural place of every elements in the world, every element attracted to its own domain, "Aristotle's Physics: a Physicist's Look" :

$$v \propto \frac{W}{\rho} \tag{88.1}$$

v the speed of the object, W the weight of the object, ρ the density of the medium
Space as absolute vs. relative

88.1.2 Classical mechanics and Newtonian gravity

Galilean relativity
Motion of falling bodies, mass on an incline
Newtonian gravity

$$\vec{F} = G\frac{m_1 m_2}{r^2} \tag{88.2}$$

Gauss equation :

$$\Delta\Phi = 4\pi G\rho(x) \tag{88.3}$$

Fluid theories of gravity
LeSage theory : particles moving throughout the universe apply uniform pressure. For two bodies : inelastic collisions change the pressure between the two bodies
Mach principle : inertial frames determined by mass distribution in the universe

88.2 Non-euclidian geometry

The assumption of Euclidian geometry
Kant epistemology
Non-Euclidian geometry : Gauss, Lobachevski, Riemann, Beltrami
That 1870's idea of having curved space for forces

88.2.1 Non-euclidian cosmologies

Cosmology has always had problems in the days of classical mechanics
Instability of Newtonian cosmology
Olbers paradox Statement : In an approximatively homogeneous distribution of stars in a euclidian, static, infinitely old universe, the sky would be infinitely bright

Theorem 88.1. If the universe has the flat spatial topology $\mathbb{R}^3$, with a star density corresponding to a Poisson distribution with density ρ of N stars in the region D

$$P(N(D) = k) = \frac{(\rho V(D))^k e^{-\rho V(D)}}{k!} \tag{88.4}$$

the total light flux at any point is infinite.

Proof. Let's consider stars of identical spectrum (this is true for any non-zero spectrum so we can just take the lowest possible one). If the star distribution is uniform, with some density of n stars per unit volume V, then for any point p, the average number of stars at a distance R will be

$$\bar{N}(B_R) = \frac{4}{3}\pi R^3 \rho \tag{88.5}$$

Each star at distance r has the intensity $I = I_0/r^2$, giving the average intensity

$$\bar{I}(B_R) = \int_0^R I_0 \frac{(\rho V(D))^k e^{-\rho V(D)}}{k! r^2} dV \tag{88.6}$$

□

Non-euclidian cosmologies : Zöllner
Solution of Olber's paradox by Zöllner : universe of constant positive curvature

88.3 Birth of general relativity

88.3.1 The necessity for relativity

A variety of anomalous effects existed by the early 20th century that would be later on explained by special and then general relativity.
One of the earliest of those effects was the anomalous precession of the perihelion of Mercury. The best computations of Urbain Le Verrier, by the analysis of the transit of Mercury from 1697 to 1848, showed that the value was of about $5600''$ per century, while the computed value

88.3.2 Special relativity

The Maxwell equations for electromagnetism
Total current (p, q, r), conduction current (p', q', r'), magnetic potential (F, G, H), magnetizing field (α, β, γ), electric field (P, Q, R), displacement field (f, g, h), electric potential Ψ

$$\begin{aligned} p' &= p + \frac{df}{dt} \\ q' &= q + \frac{dg}{dt} \\ r' &= r + \frac{dh}{dt} \end{aligned}$$

$$\mu\alpha = \frac{dH}{dy} - \frac{dG}{dz} \tag{88.7}$$
$$\mu\beta = \frac{dF}{dz} - \frac{dH}{dx}$$
$$\mu\gamma = \frac{dG}{dx} - \frac{dF}{dy}$$

Galilean invariance : measurable quantities should be invariant under the transformation $x \to x + vt$, $v \in \mathbb{R}^3$

Break of Galilean relativity in electromagnetism due to the magnetic term

This meant that either the principle of Galilean invariance was wrong or that the laws of electromagnetism were only valid in a specific frame, the later being the theory that was adopted at the time, as it was supposed that electromagnetism was the consequence of the mechanical deformation of a fluid, the aether, in a similar manner to sound waves.

If true, this would mean that the speed of light will vary in frames in motion with respect to the aether.

88.3.2.1 The Michelson-Morley experiment

To try to measure the absolute motion of earth through the aether, Albert Abraham Michelson proposd an experiment in 1881 [cf The Relative Motion of the Earth and the Luminiferous Ether]

Consider the speed of light c, speed of earth with respect to the aether v, take two points A and B separated by a distance D. During the time t_1 it takes for light to travel from A to B, Earth moves a distance d_1, while during the time t_2 it takes to travel from B to A, the earth will move a distance d_2. Were the Earth at rest in the aether, we would simply have $t_1 = t_2 = t = D/c$, but as the earth will move by d_1 during the first trip, the point B will move the same distance, hence the true distance travelled by light will be $D + d_1$, and similarly on the other way, $D - d_2$.

$$t_1 = \frac{D + d_1}{c} = \frac{d_1}{v}$$
$$t_2 = \frac{D - d_2}{c} = \frac{d_2}{v} \tag{88.8}$$

$$d_1 = D\frac{v}{c - v}$$
$$d_2 = D\frac{v}{c + v} \tag{88.9}$$

$$t_1 = \frac{D}{c - v}$$
$$t_2 = \frac{D}{c + v} \tag{88.10}$$

From this, we get

$$t_1 - t_2 = 2D\frac{c}{c^2 - v^2} = 2t\frac{1}{1 - \frac{v^2}{c^2}} \quad (88.11)$$

Michelson and Edward Morley performed this experiment (later called the Michelson-Morley experiment) [cf "On the Relative Motion of the Earth and the Luminiferous Ether", 1887] between april and july 1887.
Interferometer to measure the speed of light in two directions
Experimental setup :
To account for the constancy of the speed of light in apparently all inertial frames, the effect of Lorentz contraction was formulated by Hendryke Lorentz, in the general framework called the Lorentz ether theory.
Poincaré
Einstein's theory of special relativity ($\approx$ 1905)

88.3.3 Gravity in special relativity

The translation of classical theories to special relativistic theories is usually fairly straightforward.
Problem with the equivalence principle
attempts to have theories of gravity with it
Scalar gravity : Einstein, Nordstrom
Vector gravity
Tensor gravity : Pauli-Fierz

88.3.4 General relativity

1908 : "On the relativity principle and the conclusions drawn from it" (acceleration in SR) 1909 : Ehrenfest : "Uniform rotation of rigid bodies and the theory of relativity" 1911 : "On the influence of gravity on the propagation of light" 1913 : "Outline of a generalized theory of relativity and of a theory of gravitation" General relativity

88.4 Early years

88.4.1 Schwartzschild metric

Schwarzschild solution : 1916 "On the gravitational field of point masses in Einstein's theory"

88.4.2 Gravitational radiations

controversies with general covariance, gravitational radiations, bead argument, Eddington

88.4.3 Cosmology

Lemaître metric, Hubble observation of redshift, Einstein universe, de Sitter universe, Misner universe
Big bang cosmology, soviet union

88.4.4 Early quantum gravity

Gupta's flat space quantization

88.4.5 Causality

Weyl in space time matter
Reichenbach
Gödel

88.5 The golden years of general relativity

60's and topology, singularity theorem, Hawking radiation

88.5.1 The singularity theorem

88.6 The current era

FERMI
Gravity probe B
PLANCK
LIGO

A Topology

Basic notions and theorems of topology, for more details cf Munkres [21].
Basic definition of a topology :

Definition A.1. A topology τ on a set A is a set of subsets of A that obeys the following properties :

- Both the empty set $\varnothing$ and the set A itself are in τ.
- For any collection X_α of members of τ, $\bigcup_\alpha X_\alpha \in \tau$.
- For any two members $X, Y \in \tau$, $X \cap Y \in \tau$.

A subset $C \subset X$ is *closed* if $X \setminus C$ is open.
Corrolary : X, $\varnothing$ are both closed and open.
Cover
Basis
Compact set

A.1 Separation axioms

Definition A.2. A topological space (A, τ) is Hausdorff if for every $p, q \in A$, there exists neighbourhoods $U_p, U_q \in \tau$ such that $U_p \cap U_q = \varnothing$.

Definition A.3. A topological space X is second countable if there's a countable collection of open subsets $\{U_i\}$ such that any open set in X is the union of some subset of $\{U_i\}$.

Definition A.4. A manifold is paracompact if every open cover $\{U_\alpha\}$ has a refinement $\{V_\beta | \exists \alpha, V_\beta \subseteq U_\alpha\}$ that is locally finite, that is, such that every point of the manifold only intersects finitely many sets of that refinement.

A.2 Functions between topological spaces

Definition A.5. A function $f : X \to Y$ between two topological spaces X and Y is *continuous* if for every open set $O \subset Y$, $f^{-1}(O)$ is an open set of X.

Proposition A.6. For a continuous function $f : X \to Y$, with $O \subset Y$ and $U \subset X$,

- The pre-image of a closed set is a closed set.
- The image of the closure of a set is a subset of the closure of the image : $f(\bar{U}) \subset \overline{f(U)}$
- for every $x \in X$ and every neighbourhood O of $f(x)$, there's a neighbourhood U of x such that $f(U) \subset V$

Proof. □

Definition A.7. A *homeomorphism* is a continuous bijection with a continuous inverse.

Proposition A.8. For a homeomorphism

- The image of an open set is an open set
- The image of a closed set is a closed set
- The image of the closure of a set is the closure of the image of that set.

Proof.

- Since f^{-1} is itself a continuous function, the preimage of some open set $f(U)$ will be an open set.
- Same proof.
-

□

B Simplices

Definition B.1. A *k-simplex* is a k-dimensional polytope of $k+1$ vertices.

Examples :

- 0-simplex : point (no vertices)
- 1-simplex : line (2 points as vertices)
- 2-simplex : triangle (3 lines as vertices)
- 3-simplex : tetrahedron (4 triangles as vertices)

simplex : list of points x_i

$$C = \{\sum_{i=0}^{n} \theta_i x_i | \sum_{i=0}^{k} \theta_i = 1,\ \theta_i \geq 0\} \tag{B.1}$$

C Lie groups and Lie algebras

An important class of manifolds are Lie groups, which are manifolds equipped with some group structure.

Definition C.1. A *Lie group*

Manifold with a group structure Associated with a Lie algebra : the tangent space at $p = 1$, the identity of the group

$$T_1 G = \mathfrak{g} \tag{C.1}$$

Group structure as Lie brackets

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{C.2}$$

Properties :
Bilinearity

$$\begin{aligned} [ax + by, z] &= a[x, z] + b[y, z] \\ [z, ax + by] &= a[z, x] + b[z, y] \end{aligned}$$

Alternativity

$$[x, x] = 0 \tag{C.3}$$

Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \tag{C.4}$$

Anti-commutativity :

$$[x, y] = -[y, x] \tag{C.5}$$

Lie group for every finite dimensional algebra
Exponential map

$$\exp_1(x) = \exp(T_a x^a) \tag{C.6}$$

Examples :
Vector space V, U, O, SL, etc

Proposition C.2. The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

Proof. The general linear group is a subset of the vector space of $n \times n$ matrices, which is diffeomorphic to $\mathbb{R}^{n^2}$, defined by

$$\mathrm{GL}(n, \mathbb{R}) = \{M \in \mathrm{Mat}_{n \times n} \,|\exists M^{-1},\ M^{-1}M = MM^{-1} = \mathbb{1}\} \tag{C.7}$$

□

D Functionals and functional derivatives

An important notion for physics in general and quantum theory is the notion of functionals and their derivatives, which are functions with domain on some function space. More precisely,

Definition D.1. A *functional* is a linear function on some vector space to a field (we will only consider $\mathbb{R}$), usually a vector space of functions.

$$F : V \to \mathbb{R} \tag{D.1}$$

The most common example of a functional is some function involving the integral over a domain, for instance

$$F[\phi(x)] = \int_D \Psi(\phi(x)) d^n x \tag{D.2}$$

which is a functional from some function vector space (usually L^2) to $\mathbb{R}$. The measure $d^n x$ will just be the usual Lebesgue measure on $\mathbb{R}^n$, the volume form will be included in the function Ψ, as we will also need functional derivatives with respect to the metric components.

$$\Psi(\phi(x)) = \alpha(\phi(x))\sqrt{-g} \tag{D.3}$$

with $\alpha(\phi(x))$ some n-form.

D.1 The functional derivative

The functional derivative of a functional is a distribution defined by the Gâteaux derivative

$$\frac{\delta F[\phi]}{\delta \phi}[f(x)]dx = \lim_{\varepsilon \to 0} \frac{F[\phi + \varepsilon f] - F[\phi]}{\varepsilon} \tag{D.4}$$

with the limit defined in the topology of $\mathbb{R}$. Most of the functionals considered will depend both on ϕ and its first derivative $\mathrm{d}\phi$, so we will write it as $F[\phi, \mathrm{d}\phi]$, with its variation $F[\phi + \varepsilon f, \mathrm{d}(\phi + \varepsilon f)]$.
To solve this in general, we will take the Taylor expansion of $F[\phi + \varepsilon f, \mathrm{d}(\phi + \varepsilon f)]$ as a function of ε. If F is at least C^1, we will get

$$\begin{aligned} F[\phi + \varepsilon f, \mathrm{d}\phi + \varepsilon \mathrm{d} f] &= F[\phi, \mathrm{d}\phi] + \varepsilon(\frac{d}{d\varepsilon} F[\phi + \varepsilon f, \mathrm{d}\phi])|_{\varepsilon=0} \\ &+ \varepsilon(\frac{d}{d\varepsilon} F[\phi, \mathrm{d}\phi + \varepsilon \mathrm{d} f])|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{D.5}$$

If F is defined as an integral :

$$\begin{aligned} \frac{d}{d\varepsilon} F[\phi + \varepsilon f, \mathrm{d}\phi] &= \frac{d}{d\varepsilon} \int_D \Psi(\phi(x) + \varepsilon f(x), \mathrm{d}\phi) dx \\ &= \int_D \frac{\partial \Psi}{\partial \phi}(\phi(x) + \varepsilon f(x), \mathrm{d}\phi) f(x) dx \end{aligned}$$

$$\begin{aligned}\frac{d}{d\varepsilon}F[\phi,\partial\phi+\varepsilon\partial f] &= \frac{d}{d\varepsilon}\int_D \Psi(\phi,\partial\phi+\varepsilon\partial f)dx \\ &= \int_D \Psi'(\phi,\partial\phi+\varepsilon\partial f)\partial f(x)dx\end{aligned}$$

If it also depends on the derivatives :

$$\begin{aligned}\frac{d}{d\varepsilon}F[\phi+\varepsilon f,\partial(\phi+\varepsilon f)] &= \frac{d}{d\varepsilon}\int_D \Psi(\phi(x)+\varepsilon f(x),\partial_x(\phi(x)+\varepsilon f(x))dx \\ &= \int_D \Psi'(\phi(x)+\varepsilon f(x))f(x)dx\end{aligned}$$

Derivative :

$$\begin{aligned}\frac{\delta F[\phi]}{\delta\phi}[f(x)] &= \lim_{\varepsilon\to 0}\int \Psi'(\phi(x)+\varepsilon f(x))f(x)dx \\ &= \int \Psi'(\phi(x))f(x)dx \qquad \text{(D.6)}\end{aligned}$$

$$\begin{aligned}\frac{\delta F[\phi,\partial\phi]}{\delta\phi}[f(x)] &= \lim_{\varepsilon\to 0}\int [\Psi'(\phi(x)+\varepsilon f(x),\partial\phi)f(x)+\Psi'(\phi,\partial\phi+\varepsilon\partial f)\partial f(x)]dx \\ &= \int [\Psi'(\phi(x),\partial\phi)f(x)+\Psi'(\phi,\partial\phi)\partial f(x)]dx \qquad \text{(D.7)}\end{aligned}$$

By integration by part, we can put everything as a factor of $f(x)$ up to a surface term.

$$\int \Psi'(\phi,\partial\phi)\partial f(x)dx = [f(x)\Psi'(\phi,\partial\phi)] - \int f(x)\partial_x\Psi'(\phi,\partial\phi)dx \qquad \text{(D.8)}$$

$$\frac{\delta F[\phi,\partial\phi]}{\delta\phi}[f(x)] = \int_D [\Psi'(\phi(x),\partial\phi)-\partial_x\Psi'(\phi,\partial\phi)]f(x)dx + \int_{\partial D} f(x)\Psi'(\phi,\partial\phi) \qquad \text{(D.9)}$$

If the surface integral drops to 0, then we have that

$$\frac{\delta F[\phi,\partial\phi]}{\delta\phi}[f(x)] = \int_D [\frac{\partial\Psi}{\partial\phi}(\phi,\partial\phi)-\partial_x\frac{\partial\Psi}{\partial(\partial\phi)}(\phi,\partial\phi)]f(x)dx \qquad \text{(D.10)}$$

which means that the functional derivative is itself a function if the derivatives of Ψ are. We will note that function as

$$\frac{\delta F[\phi,\partial\phi]}{\delta\phi}(x) = \frac{\partial\Psi}{\partial\phi}(\phi,\partial\phi)-\partial_x\frac{\partial\Psi}{\partial(\partial\phi)}(\phi,\partial\phi) \qquad \text{(D.11)}$$

Properties :
Linear :

$$... \qquad \text{(D.12)}$$

Proposition D.2. The functional derivative obeys the Leibniz property

$$\frac{\delta FG[\phi]}{\delta\phi} = \frac{\delta F[\phi]}{\delta\phi}G[\phi] + F[\phi]\frac{\delta G[\phi]}{\delta\phi} \tag{D.13}$$

There are more complex ways of defining functionals and their derivatives on manifolds, involving the jet bundle and differential graded algebras.

E Sobolev spaces

For a Riemannian manifold (M, g)
A function f or tensor field T belongs to the Sobolev space $W^{k,p}$ if its derivatives up to order k admit a finite L^p norm, the L^p norm being

$$\|f\|_{k,p} = (\int_M \sum_{i=0}^{k} |\nabla^{(i)} f(x)|^p dx)^{\frac{1}{p}} \tag{E.1}$$

with the component representation

$$\nabla^{(i)} f(x) = \nabla_\mu \nabla_\nu ... \nabla_\sigma f(x) \tag{E.2}$$

and the tensor norm $|T| = T^*(T)$, with T^* the dual of T, or, in components

$$|T| = T_{\mu\nu...\sigma} T^{\mu\nu...\sigma} \tag{E.3}$$

Sobolev Banach space $W_0^{k,p} \subset W^{k,p}$: closure with respect to the norm of the space of smooth functions of compact support $\mathscr{D}$
thm : $W_0^{k,p} = L^p(M)$
if M has a non-zero injectivity radius [Injectivity radius = inf of injectivity radius at every point, which is the largest radius for which the exponential map is a diffeomorphism](ie is complete), then $W_0^{1,p} = W^{1,p}$.
corrolary : true for compact manifolds
If M has a non-zero injectivity radius, and a Riemann tensor uniformly bounded, and its derivatives up to $k-2$, then for $k \geq 2$, $W_0^{k,p} = W^{k,p}$

F Hilbert spaces and operator algebras

To define quantum theories properly, we will need to define Hilbert spaces and the linear operators that act upon them.

F.1 Hilbert spaces

F.1.1 Definition

Hilbert spaces are vector spaces with additional structures defined on them. As a reminder, here are the axioms of a vector space over a field K.
A vector space $(V, +, \cdot)$ is a set V equipped with two operations,

$$\begin{aligned} + : V \times V &\rightarrow V \\ \cdot : K \times V &\rightarrow V \end{aligned} \tag{F.1}$$

As this is usually not an issue, there will be no specific symbols to differentiate the sum and product for the vector space and the field. The vector space then obeys

- For two vectors $X, Y \in \mathcal{H}$, $X + Y \in \mathcal{H}$
- There is a zero vector $0 \in \mathcal{H}$ such that for every vector X, $X + 0 = X$.
- For a scalar value $k \in K$ and a vector $X \in \mathcal{H}$, $kX \in \mathcal{H}$
- For every vector $X \in \mathcal{H}$, there corresponds an inverse $-X$ such that $X + -X = 0$
- The addition is commutative and associative. It is also distributive with the scalar multiplication.

A Hilbert space is also equipped with a sesquilinear form $\langle \cdot, \cdot \rangle$, defined by

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}^+ \tag{F.2}$$

which is sesquilinear, that is, for $X, Y, Z \in \mathcal{H}$ and $a, b \in \mathbb{R}$,

$$\begin{aligned} \langle X, aY + bZ \rangle &= a\langle X, Y \rangle + b\langle X, Z \rangle \\ \langle aX + bY, Z \rangle &= \bar{a}\langle X, Z \rangle + \bar{b}\langle Y, Z \rangle \end{aligned}$$

$$\langle X, Y \rangle = \overline{\langle Y, X \rangle} \tag{F.3}$$

Vector space on the field K, $K = \mathbb{C}$ or $\mathbb{R}$. For quantum theory, $K = \mathbb{C}$
A Hilbert space is then defined as the quadruple $(\mathcal{H}, +, \cdot, \langle \cdot, \cdot \rangle)$.

F.1.2 Properties

The sesquilinear products naturally defines a norm on $\mathcal{H}$

$$\begin{aligned} \| \cdot \| : \mathcal{H} &\rightarrow \mathbb{R}^+ \\ X &\rightarrow \|X\| = \sqrt{\langle X, X \rangle} \end{aligned} \tag{F.4}$$

$\|X\|$ is guaranteed to be positive real by $\langle X, X \rangle = \overline{\langle X, X \rangle}$

F.1.3 Linear operators on a Hilbert space

Linear operator on a dense subset $D \subset \mathcal{H}$

$$A : D \to \mathcal{H} \tag{F.5}$$

Norm on operators :

$$|A| = \sup_{X \in D} [A(X)] \tag{F.6}$$

Adjoint of an operator A : An adjoint operator A^* is an operator such that

$$\langle AX, Y \rangle = \langle X, A^* Y \rangle \tag{F.7}$$

An operator is then defined as hermitian if it acts in the same way on vectors as its adjoints

$$\langle AX, Y \rangle = \langle X, AY \rangle \tag{F.8}$$

Self-adjoint if both hermitian and $D(A) = D(A^*)$
C^* algebra : Banach algebra over $\mathbb{C}$

F.1.4 Density matrices

Density matrix : for a quantum state $|\psi\rangle$,

$$\hat{\rho} = |\psi\rangle\langle\psi| \tag{F.9}$$

G Path integrals

A path integral is an integral defined on a function space. If we take some space of configuration $\mathcal{C}$ with the structure of a Banach space, with the σ-algebra Σ, then we define a measure

$$\mu : \Sigma_C \to \mathbb{R} \tag{G.1}$$

Unfortunately, unlike for the case of finite-dimensional Banach spaces, there is no infinite-dimensional analogue of the Lebesque measure.

Theorem G.1. The only locally finite and translation invariant Borel measure μ on an infinite-dimensional separable Banach space is the trivial measure $\mu(A) = 0$.

Proof. For X some infinite-dimensional separable Banach space, take an open ball $B(\varepsilon)$ of radius $\varepsilon > 0$, with a ε such that, by local finiteness, $\mu(B(\varepsilon)) < \infty$. As X is infinite-dimensional, it is possible to find an infinite sequence of pairwise disjoint open balls $\{B_i(\varepsilon/4)\}_{i\in\mathbb{N}}$ such that for all i, $B_i(\varepsilon/4) \subset B(\varepsilon)$.
As the measure is translation invariant, for all $i, j \in \mathbb{N}$, $\mu(B_i(\varepsilon/4)) = \mu(B_j(\varepsilon/4))$, and since they are subsets of $B(\varepsilon)$, $\mu(B_i(\varepsilon/4)) \leq \mu(B(\varepsilon)) < \infty$.
Then, by property of the measure, we have

$$\mu(B(\varepsilon)) \leq \sum_{i=0}^{\infty} B_i(\varepsilon/4) \tag{G.2}$$

which will only be finite if $\mu(B_i(\varepsilon/4)) = 0$, and then $\mu(B(\varepsilon)) = 0$. Since X is separable, there is an open cover by open balls of radius $\varepsilon/4$. As all of them have measure 0, $\mu(X) = 0$, and so μ is the trivial measure. □

Instead of using the Lebesgue measure, the most common measure used (for free theories, anyway) is the standard Gaussian (or Wiener) measure.

Definition G.2. A *standard Gaussian measure* γ on an infinite-dimensional separable Banach space X is a Borel measure with the following properties :

-

For some completion of the Borel σ-algebra $B_0(\mathbb{R}^n)$
Standard Gaussian measure $\gamma^n : B_0(\mathbb{R}^n) \to [0, 1]$

$$\gamma^n(A) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int A \exp(-\frac{1}{2}\|x\|^2) d\lambda^n(x) \tag{G.3}$$

Infinite dimensional gaussian measure is well defined
Classical Wiener space :

$$H = L_0^{2,1}([0, T], \mathbb{R}^n) \tag{G.4}$$

This corresponds to the space of paths $\gamma : [0, T] \to \mathbb{R}^n$ with L^2 first derivatives.

Proposition G.3. *H is an infinite-dimensional separable Banach space.*

Proof. Inner product :

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^T \langle \dot{\gamma}_1(t), \dot{\gamma}_2(t) \rangle dt \tag{G.5}$$

□

G.1 The Feynman path integral

Less rigorously, we can define the path integral by a limit process commonly used in physics, the Feynman path integral. For a

$$\begin{aligned} \langle \phi_1, t_1 | F | \phi_2, t_2 \rangle &= \int_{\phi_1}^{\phi_2} \mathscr{D}\phi(t) F e^{iS[\phi]} \\ &= \lim_{n \to \infty} \int \prod_{i=0}^{n} \frac{d}{A} \end{aligned} \tag{G.6}$$

The fact that the Lebesgue measure is used here, despite being shown to be 0 in the infinite-dimensional case, is due to a conflict of the various terms : as $n \to \infty$, the measure goes to 0 while the exponent diverges.

Theorem G.4. The Feynman path integral is not defined by a measure.

Proof. □

H Probabilities and information theory

H.1 Kolmogorov probabilities

Probability space : (Ω, F, P)
Ω the set of probability events, F a σ-algebra of Ω, $P : \Omega \to \mathbb{R}$ the probability function
Probability axioms :
The probability of any event is non-negative

$$P(X) \geq 0 \tag{H.1}$$

The probability of any event occuring is 1

$$P(\Omega) = 1 \tag{H.2}$$

σ-additivity :

H.2 Quantum probabilities

H.3 Information theory

I Full spacetime structure

$$
\begin{array}{ccccccc}
P\mathcal{M} & & F\mathcal{M} & & T\mathcal{M} & \underset{\sharp}{\overset{\flat}{\rightleftarrows}} & T^*\mathcal{M} \\
 & & \downarrow \pi_F & & \downarrow \pi_T & & \downarrow \pi_{T^*} \\
\mathcal{M} & & \mathcal{M} & & \mathcal{M} & & \mathcal{M} \\
 & & & & \downarrow \phi & & \\
 & & & & \mathbb{R}^n & &
\end{array}
$$

Figure 19: Structure of the spacetime manifold

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