Commentary on nLab's Science of Logic and other matters

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Couples are things whole and things not whole, what is drawn together and what is drawn asunder, the harmonious and the discordant. The one is made up of all things, and all things issue from the one.

Heraclitus

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Introduction

While not a new phenomenon by any mean, there is a certain recent trend of trying to mathematize certain philosophical theories, in particular ideas relating to dialectics. Dialectics can refer to quite a lot of somewhat related topics, such as a general method to arrive at the truth by trying to reconcile contradictions between opposing ideas, but in this case we are talking more specifically about the focus of dialectics on oppositions between concepts and their contradictions. In the case of metaphysics, those oppositions can be for instance

- One / Many
- Sameness / Difference
- Being / Nothing
- Space / Quantity
- General / Particular
- Objective / Subjective
- Qualitative / Quantitative
- Finite / Infinite

The oldest of such thoughts goes back to ancient greek traditions, to such philosophers as Parmenides[1] and Heraclitus. Heraclitus in particular was very much involved in the notion that oppositions such as these do not form the division that people would typically expect, and in fact are very much related to each other.

”What opposes unites, and the finest attunement stems from things bearing in opposite directions, and all things come about by strife.” [Aristotle’s Nicomachaen Ethics, book VIII]

The contradictions involved would be for instance that a collection of objects is both one and many at the same time,

PARMENIDES: If one is, the one cannot be many?
SOCRATES: Impossible.
PARMENIDES: Then the one cannot have parts, and cannot be a whole?
SOCRATES: Why not?
PARMENIDES: Because every part is part of a whole; is it not?
SOCRATES: Yes.
PARMENIDES: And what is a whole? would not that of which no part is wanting be a whole?
SOCRATES: Certainly.
PARMENIDES: Then, in either case, the one would be made up of parts; both as being a whole, and also as having parts?
SOCRATES: To be sure.
PARMENIDES: And in either case, the one would be many, and not one?
SOCRATES: True.
PARMENIDES: But, surely, it ought to be one and not many?
SOCRATES: It ought.
PARMENIDES: Then, if the one is to remain one, it will not be a whole, and will not have parts?
SOCRATES: No.
PARMENIDES: But if it has no parts, it will have neither beginning, middle, nor end; for these would of course be parts of it.
SOCRATES: Right.
PARMENIDES: But then, again, a beginning and an end are the limits of everything?
SOCRATES: Certainly.
PARMENIDES: Then the one, having neither beginning nor end, is unlimited?
SOCRATES: Yes, unlimited.
PARMENIDES: And therefore formless; for it cannot partake either of round or straight.

[Plato’s Parmenides]

with a similar notion in Heraclitus

Couples are things whole and things not whole, what is drawn together and what is drawn asunder, the harmonious and the discordant. The one is made up of all things, and all things issue from the one. [Pseudo-Aristotle, On The Cosmos]

But while Parmenides' solution was to deny plurality, Heraclitus was to welcome the opposition.

Sameness and difference?

Other such examples can be found in the medieval era with the work of Nicholas of Cusa, De Docta Ignorantia (on learned ignorance) [2].

"Now, I give the name "Maximum" to that than which there cannot be anything greater. But fullness befits what is one. Thus, oneness—which is also being—coincides with Maximality. But if such oneness is altogether free from all relation and contraction, obviously nothing is opposed to it, since it is Absolute Maximality. Thus, the Maximum is the Absolute One which is all things. And all things are in the Maximum (for it is the Maximum); and since nothing is opposed to it, the Minimum likewise coincides with it, and hence the Maximum is also in all things. And because it is absolute, it is, actually, every possible being; it contracts nothing from things, all of which [derive] from it."

The first major author for the exact field that we will broach here is Kant and his transcendental logic [3]

The main author for these recent trends target is Hegel and his Science of Logic [4], where he describes his *objective logic*. The "classical" logic originally described by Aristotle, the logic of propositions and such, is described under the term of *subjective logic*, the logic of merely assembling

Heidegger, being and time?

The original attempt at the formalization of those ideas (or at least ideas similar to it) was by Grassmann[5], giving his theory of extensive quantities [vector spaces], which while it was commented on and inspired some things, mostly did not go much further.

In modern time, this programme was originally started by Lawvere[6]

"It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy."

This is a somewhat recurring theme in nlab[7]

[8, 9, 10, 11, 12, 13]

As is traditional for such types of philosophy, the writings are typically fairly abstract and lacking example. For a more pedagogical exposition, I have tried here to include more examples and demonstrations to such ideas. I am not an expert in algebraic topology by any mean and have tried my best to explain those notions without using notions from this field. These are mostly written from the perspective of a physicist, so while I may not expect prior knowledge in schemes and type theory, I will likely commonly use notions of quantum theory or mechanics.

There are a few caveats to bring up here. First, Hegel's system is primarily about *thought*. Although it tries its best to explain the modern science of its era (the 19^{th} century) through this framework, the focus is more on the way that the mind can conceptualize those notions. This may not on the other hand be the primary goal of the categorical approach here, or at least a very difficult goal to attain, although that is certainly true of Hegel himself as he has not been known to be the most obvious to read.

Furthermore, while some of these ideas could be argued to be fairly faithful translations of the philosophical ideas, others seem more to be generally inspired by them, the original notion of unity of opposites being more of a collection of different ideas on that theme than a rigorous construction. While such notions as quantity from the abstraction to pure being seem to have some parallels, I do not believe that Hegel had particularly in mind the concept of a graded algebra when he spoke of the opposition between das Licht and die Körper des Gegensatzes (in particular, this opposition is true regardless of dynamics, and therefore should not be particularly relevant to the solidity of a body), but if the opposition can somehow mirror the adjunction of bosonic and fermionic modalities, why not look into it. The notion of being-for-itself and being-for-one are [according to X] more related to consciousness than etc.

And in the other direction, as is common in the context of formal systems, this is merely one semantics that we can apply to it.

The notions described here are furthermore not firmly rooted in the formalism but merely described by it, as many of these notions are already needed to *define* the formalism. In particular, it is difficult to define any theory without the concept of discrete objects (in our case, by rooting the formalism of categories in the notion of sets, and in fact in general in any thought process requiring to have different ideas as discrete entities). The rooting of the notion of oneness in terminal objects are somewhat superficial, as the very notion of a terminal object already requires the notion of oneness, ie in the unique morphism demanded by universal properties.

Actual applications of dialectical logic [14, 15] also seem fairly disconnected from the formalism developed here, so that we will have to look into it separately.

It is therefore best to keep in mind that this is not a faithful formalization of Hegel's system but merely reflects some of its ideas.

[12, 16, 17, 18, 19, 20, 21, 22, 23]

Transitions Into, With, and From Hegel's Science of Logic

Before going into those various formalizations, we will first have a rather in depth look at the formalisms on which these are based, which are type theory and category theory.

2
Types

The first element of the theory discussed is that of *types*[24], which will relate to the notion of categories and logic later on, through the notion of *computational trinitarianism*.

Relation with whatever Kant idk

A *type* is, as the name implies, a sort that some mathematical object can be. We denote that an object c is of type C by

$$c : C \tag{2.1}$$

and say that c is an *instance* of type C.

From the computational trinitarianism interpretation, there are roughly three main interpretations of a type. In terms of logic, a type represents a proposition. In terms of category theory, a type is an object, and in our focus in particular, a space. And in terms of type theory, a type is a construction.

The typical simple example, as used in mathematics and computation theory, is that of integers. An integer n is an instance of the integer type, $n : \mathbb{N}$.

As a space :

Types being themselves a mathematical object, we also have some type for types, denoted Type, although to avoid some easily foreseeable Russell style paradox (called the Girard paradox[25]), we will instead use some hierarchy of such types, called type universes :

$$C : \mathrm{Type}_0, \mathrm{Type}_0 : \mathrm{Type}_1, \mathrm{Type}_1 : \mathrm{Type}_2, \dots \tag{2.2}$$

Although as we will not really require much foray into the hierarchy of type universes, we will simply refer Type_0 as Type from now on.

"A proposition is interpreted as a set whose elements represent the proofs of the proposition"

The notation of instances belonging to types that we've seen is one example of such a judgment, called a *typing judgment*. A typing judgment is any

2.1 Formulas and judgements

The basic element of type theory is the *formula*, similarly to the case of logic, which is some statement about the type theory.

Definition 2.1.1 *A* formula

The basic formulas that we will deal with are the declaration of type $c : C$, and the equivalence $c \equiv d$. The main structure we will use on formulas is that of a *judgement*. Given a type theory TT, a judgment is given by two lists of formulas as

$$A_1, A_2, \dots, A_n \vdash_{\mathrm{TT}} B_1, B_2, \dots, B_m \tag{2.3}$$

with the semantics of "within the type theory TT, and assuming all the formulas (A_i), then at least one formula of (B_j) is true". We will not often work in different type theories at the same time, so that we can omit the subscript on $\vdash_{\mathrm{TT}}$.

It should be noted that those lists are indeed lists and not sets. While many type theories do not place any importance on the ordering of those formulas, some (such as quantum logic) do, so that the commutativity of formulas is a specific axiom of the system.

To shorten notation, as we may be typically dealing with rather long lists of arbitrary formulas, we will use the notion of *context*, which is defined as a (possibly empty) list of formulas.

$$\Gamma = A_1, A_2, \dots, A_n \tag{2.4}$$

where the concatenation of contexts and formulas is understood to mean the obvious concatenation of all formulas within :

$$\Gamma, A = A_1, A_2, \ldots, A_n, A \tag{2.5}$$

$$\Gamma_1, \Gamma_2 = A_{1,1}, A_{1,2}, \ldots, A_{1,n}, A_{2,1}, A_{2,2}, \ldots, A_{2,n} \tag{2.6}$$

As with formulas in general, we can perform a substitution operation. If a formula A contains a variable x of type X, we denote its substitution by a term t of type X by

$$A[t/x] \tag{2.7}$$

Example 2.1.1 *If we have a formula with free variable $f(x)$*

2.2 Sequent calculus

The transformation rules of statements about types are given by the sequent calculus. In a type theory, the basic entities that we manipulate are the *formulas* and the *judgements*. A formula is simply some statement we have on our type theory, the basic one being the typing of a term. For instance, the statement "a is of type A" is a formula :

$$a : A \tag{2.8}$$

A judgement of a formula is the notion that the formula that we have can be proven in our system. For instance, if we consider the formula that 0 is an integer, $0 : \mathbb{N}$, our judgement is that this is indeed true in our system. If we write down our specific system by S, this is denoted by the turnstyle $\vdash_S$:

$$\vdash_S 0 : \mathbb{N} \tag{2.9}$$

As we will here not typically work with many different systems however, we will keep S implicit unless necessary, so that we will just use $\vdash$.

Judgements in general are done with specific assumptions, that is, a formula is valid in the system only assuming another formula. We write this assumption on the left as

$$A \vdash B \tag{2.10}$$

The left side of the judgement is called the *antecedent*, while the right side is called the *consequent*.

In general, in Gentzen style sequent calculus, we can have multiple formulas on both sides.

$$A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_m \tag{2.11}$$

The semantics of which are meant to be read as if *all* the formulas A_i are valid, then at least one of the formula B_j is valid. This specific semantics is meant to emulate the notion of implication in propositional logic, as

$$A_1 \wedge A_2 \wedge \ldots \wedge A_n \to B_1 \vee B_2 \vee \ldots \vee B_m \tag{2.12}$$

As we can have quite a lot of formulas on either side, it is common to write large numbers of formulas in a variable called a *context*.

$$\Gamma = \{A_i\}_{i \in I} \tag{2.13}$$

where I is a possibly empty finite set.

Above : premises, below : conclusion

$$\frac{\Gamma \vdash P}{\Gamma \vdash P}$$

A simple universal inference rule in sequent calculus for instance is that a formula entails itself :

$$A \vdash A$$

Meaning that even assuming no previous judgments, we can deduce that the hypothesis of assuming A entails A.

Example of judgement : type judgment C : Type, type equality judgment $A \equiv B$Type, element $c : C$, equal element judgment $c = c' : C$

Context as a list of type instance declarations? $a : A,\ b : B,\ c : C, \ldots$

Definition 2.2.1 *An* equality type*, denoted by $a = b$, is a formula indicating the equality between two terms.*

Example 2.2.1 *A useful example of a type theory in our context is that of intuitionist logic, where the basic type is that of propositions,* Prop : Type*, with two terms* $\top, \bot$: Prop

Substitution rule : If we replace a variable in a formula

$$\frac{\Gamma, x : X \vdash A : \text{Type} \qquad \Gamma \vdash t : X}{\Gamma \vdash A[t/x] : \text{Type}}$$

[...]

For the various types and constructions involved in type theory, we can generally split the rules as follow.

Formation rules allow the existence of a given type from other types (possibly none if they are fundamental). If we have some list of types $(A, B, C, \ldots)$, then a formation rule is the formation of a new type from those :

$$\frac{\vdash A : \text{Type} \qquad \vdash B : \text{Type} \qquad \vdash C : \text{Type} \qquad \ldots}{F(A, B, C, \ldots) : \text{Type}}$$

Likewise, *introduction rules* give us a way to construct terms from existing terms.

Elimination rules

Computation rules

To give an example of a formalism we will not detail later on, let's see how this definition applies to functions in typed lambda calculus. As we will use dependent typing to define our functions, this is not quite how we will define our functions later on.

A lambda term in type theory is given by some formula f and variable x

$$\lambda x.f(x) \tag{2.14}$$

eta conversion, beta conversion

$$(\lambda x.f(x))y = f(x)[x/y] \tag{2.15}$$

$$(\lambda x.y)z = y \tag{2.16}$$

2.3 Equality type

Separate from the notion of judgmental equality, where we judge two terms or types to be equal by definition, somewhat externally to the theory, is the notion of equality type, also called identity type. An equality type is a type associated to the equality between two terms of the same type, ie for $a : A$ and $a' : A$, we have the existence of a type $a =_A a'$. In terms of formation rule, any type induces an equality type :

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, a : A, a' : A \vdash (a =_A a') : \mathrm{Type}}$$

Unlike the judgmental equality, the equality type is meant to represent an actual equality of the theory and not merely a definition of terms.

Two terms are equal if their equality type is inhabited, ie if we have

$$a : A, b : A \vdash c : a =_A b \tag{2.17}$$

Like any definition of equality, an equality type has some notion of reflexivity, symmetry and transitivity. The reflexivity is given by its introduction rule :

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, a : A \vdash \mathrm{refl}_A(a) : (a =_A a)}$$

In the interpretation of the unit type as truth, this means that a term is equal to itself.

2.4 Dependent types

We speak of *dependent types* for a type that depends on a value, ie a "type" that is actually a function from one type to the universe of types. If we pick for instance the function type $B : A \to \mathrm{Type}$, its evaluation for each different instance of A may lead to a different type :

$$a : A \vdash B(a) : \mathrm{Type} \tag{2.18}$$

B as a whole is the dependent type of A, with each instance $B(a)$

An example of this is the dependent type of vector spaces, where for some integer type $n : \mathbb{N}$, we associate a type of n-dimensional vector space, $\mathrm{Vect}(n)$, where we have the series of types

$$\text{Vect}(0) : \text{Type}, \text{Vect}(1) : \text{Type}, \text{Vect}(2) : \text{Type}, \dots \tag{2.19}$$

Example 2.4.1 *Indexed set*

Example 2.4.2 *The type of $n \times n$ matrices is a dependent type indexed by an integer type $n : \mathbb{N}$:*

$$n : \mathbb{N} \vdash \text{Mat}(n) : \text{Type} \tag{2.20}$$

2.5 Function types

Given any two types $A, B \in \text{Type}$, we can define another type called the *function type* of A to B :

$$f : A \to B \tag{2.21}$$

which are meant to model functions, ie given a term $a : A$, we will have a corresponding term in B given by something of the form $f(b) : B$.

As we can form a function type for any two types, the formation rule is given by

$$\frac{\Gamma \vdash A : \text{Type} \qquad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A \to B : \text{Type}}$$

$$\frac{\Gamma, x : X \vdash f(x) : Y}{\Gamma \vdash (x \mapsto f(x)) : X \to Y}$$

$$\frac{\Gamma \vdash f : X \to Y \qquad \Gamma \vdash x : X}{\Gamma \vdash f(x) : Y}$$

2.6 Sums and products

The sum type constructor takes two types $A, B : \text{Type}$ and combines it in a single type $A + B : \text{Type}$, which can be understood as a type containing the terms of both A and B. A term of $A + B$ will therefore correspond to either a term of A or a term of B.

This means that there is some map from each of those type to the sum type, denoted

$$\iota_A : A \rightarrow A + B \tag{2.22}$$
$$\iota_B : B \rightarrow A + B \tag{2.23}$$

We therefore have the formation rule that given two types, there exists a sum type for those types

$$\frac{\Gamma \vdash A : \text{Type} \qquad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A + B : \text{Type}}$$

and given this, we also have the existence of our injection maps

Example 2.6.1 *If we consider the type of even number* Even *and odd numbers* Odd*, then integers are the sum type of both :*

$$\mathbb{N} \equiv \text{Even} + \text{Odd} \tag{2.24}$$

The sum of two types gives us a pair of the individual types, ie

$$a : A, b : B \vdash (a, b) : A \times B \tag{2.25}$$

Dependent sum : the type of the second element might depend on the value of the first.

$$\sum_{n:\mathbb{N}} \text{Vect}(\mathbb{R}, n) \tag{2.26}$$

Dually to the sum type is the product type, where given two types A, B, we have the product type $A \times B$.

2.7 Inductive type

One of the primary construction to create a type is that of *induction*, by which we define a type by declaring the existence of a term in that type, and by declaring functions mapping terms of that type to other terms.

There are two different ways to deal with inductive types

[26]

[...]

The classic example of an inductive type is given by the integers $\mathbb{N}$, which is a type constructed inductively by a single object $0 \in \mathbb{N}$ and a function type

$$S : \mathbb{N} \to \mathbb{N} \tag{2.27}$$

2.8 Martin-Löf type theory

1

The basis for our type theory will usually be some Martin-Löf type theory[27], which corresponds to intuitionistic logic in the trinitarianism view, and is generally a rather universal sort of approach to logic. This will be the basic form of logic for any topos later on.

A few rules exist which are entirely independent from the specific types we will define later on. These are the *structural rules*, which tell us how the judgements we do depend on the context. While obvious enough in a context of classical logic, these are not in fact universal in type theory, and will in fact be broken in the case of quantum logic later on.

First is the identity rule, that a formula entails itself :

$$A \vdash A \text{ (I)}$$

The weakening rules are given by the property that additional context preserves judgement. For the *left weakening rule*, we add additional context to the antecedent :

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (WL)}$$

In other words, an additional hypothesis does not change the validity of the deduction. Conversely we have the *right weakening rule*

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ (WR)}$$

The contraction rules allow us to remove duplicated context without changing judgement. For the *left contraction rule*,

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (CL)}$$

right contraction rule,

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ (CR)}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}$$

There are three basic types for it, called the *finite types* :

- The zero type $\mathbf{0}$, or empty type $\varnothing$ or $\perp$, which contains no terms.
- The one type $\mathbf{1}$, or unit type $*$, which contains one canonical term.
- The two type $\mathbf{2}$, which contains two canonical term.

Formation rule :

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbf{1} : \text{Type}}$$

Introduction rule :

$$\frac{\Gamma \vdash}{\Gamma \vdash * : \mathbf{1}}$$

Empty type for nothingness, something that doesn't exist

Unit type for existence

Two type for a choice between two values, such as boolean values.

As with any type theory, those types also give us function types

$\nvdash \to \nvdash$: the empty function (no term or 1 term?) $\nVdash \to \nVdash : \nVdash \to \nvDash$: Two functions (can be interpreted as some boolean?)

$\nvDash \to \nvDash$: unary boolean functions

In addition to these types, we have a variety of *type constructors*, which allow us to construct additional types from those basic types.

First are the constructions which just combine different types together. These are given by the sum type and the product type.

Indexed sets

Equality type

Inductive type

2.9 Classical logic as a type theory

A useful example of a type theory is that of classical logic.

A common model of classical logic as a type theory is done via the "proposition as types". In terms of Martin-Löf type theory,

The basic type in classical logic is the *boolean type*, which is the two type we saw in Martin-Löf type theory. This is the type

2.10 Homotopy types

[28]

A further refinement of type theory is the notion of *homotopy type*, where in addition to identity types, we also include the more general notion of equivalence types.

Definition 2.10.1

Definition 2.10.2 *Two types* $A, B : \text{Type}$ *are said to be* equivalent*, denoted* $A \cong B$*, if there exists an equivalence between them.*

$$(A = B) \to (A \cong B) \tag{2.28}$$

Univalence axiom :

$$(A = B) \cong (A \cong B) \tag{2.29}$$

Correspondence between type theory and category theory :

- A universe of types is a category
- Types are objects in the category $T \in \text{Obj}(C)$
- A term $a : A$ of A is a generalized element of A
- The unit type $*$ if present is the terminal object
- The empty type $\varnothing$ if present is the initial object
- A dependent type $x : A \dashv B(x) : \text{Type}$ is a display morphism $p : B \to A$, the fibers $p^{-1}(a)$ being the dependent type at $a : A$.

2.11 Modalities

Definition 2.11.1 *A* modality $\square$ *on a type theory is a unary operator between two types*

$$\square : \text{Type} \to \text{Type} \tag{2.30}$$

along with a modal unit Induction principle Computation rule Equivalence

The classic example of a modal theory is that of the necessity monad, or S4 modal logic.

Introduction rule :

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

Elimination rule :

$$\frac{\Gamma \vdash \Diamond A \qquad \Gamma, A \vdash \Diamond B}{\Gamma \vdash \Diamond B}$$

2.12 Interpretation

Notions and ideas?

3
Categories

As is often the case in foundational issues in mathematics, the foundations used to define mathematics can easily become circular. In our case, although category theory can be used to define set theory and classical logic, as well as your other typical foundational field like model theory, type theory, computational theory, etc, we still need those concepts to define category theory itself. In our case we will simply use implicitly classical logic (what we will call the *external logic*, as opposed to the internal logic of a category we will see later on) and some appropriate set theory like ETCS[29] (as ZFC set theory will typically be too small to talk of important categories). We will not get too deeply into this, but it can be an important issue.

Definition 3.0.1 *A* category $\mathbf{C}$ *is a structure composed of a class of objects* $\mathrm{Obj}(\mathbf{C})$ *and a class of morphisms* $\mathrm{Mor}(\mathbf{C})$ *such that*

- *There exists two functions* $s, t : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{C})$*, the* source *and* target *of a morphism. If* $s(f) = X$ *and* $t(f) = Y$*, we write the morphism as* $f : X \to Y$*.*
- *For every object* $X \in \mathrm{Obj}(\mathbf{C})$*, there exists a morphism* $\mathrm{Id}_X : X \to X$*, such that for every morphism* g_1 *with* $s(g_1) = X$ *and every morphism* g_2 *with* $t(g_2) = X$*, we have* $g_1 \circ \mathrm{Id}_X = g_1$ *and* $\mathrm{Id}_X \circ g_2 = g_2$*.*
- *For any two morphisms* $f, g \in \mathrm{Mor}(\mathbf{C})$ *with* s

To simplify notation, if there is no confusion possible, we will write the set of objects and the set of morphisms as the category itself, ie :

$$X \in \mathrm{Obj}(\mathbf{C}) \quad := \quad X \in \mathbf{C} \tag{3.1}$$
$$f \in \mathrm{Mor}(\mathbf{C}) \quad := \quad f \in \mathbf{C} \tag{3.2}$$

Categories are often represented, in totality or in part, by *diagrams*, a directed graph in which objects form the nodes and morphism the edges, such that the direction of the edge goes from source to target. For instance, if we consider some category of two objects A, B with some morphism f with $s(f) = A$ and $t(f) = B$, we can write it as

$$A \xrightarrow{f} B$$

We will also use a lot the function notation, where this morphism is denoted by $f : A \to B$.

Throughout this section we will use a variety of common categories for examples. Some of them will be seen in more details later on6, using all the tools we have accumulated. For now, we will just mostly make our intuition on those categories by either considering categories with sets and functions for objects and morphisms, or elements and partial order relations.

Before we go on detailing examples of categories, first a quick note on skeletonized categories. It is common in category theory to more or less assume the identity of objects that are isomorphic (we will see the exact definition of isomorphism later on but we can assume the usual definition here). This is not necessarily the case formally speaking (the category of sets can be seen as having multiple isomorphic sets in it, like $\{0, 1\}$, $\{1, 2\}$, etc), but for some purposes (such as trying to get a broad view of that category) it will be useful to consider the category where the set of objects is given as the equivalence class up to isomorphism.

Definition 3.0.2 *A category* $\mathbf{C}$ *is* skeletal *if all isomorphisms are identities, and the* skeleton *of a category* $\mathbf{C}$*, written* $\mathrm{sk}(\mathbf{C})$*, is given by the equivalence class of objects up to their isomorphisms, ie*

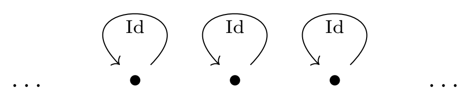
$$\mathrm{Obj}(\mathrm{sk}(\mathbf{C})) = \{[X] | \forall X' \in [X],\ \exists f \in \mathrm{iso}(\mathbf{C}),\ f : X \to X'\} \tag{3.3}$$

It will be pretty typical as we go on to implicitely consider the skeletal version of whatever category we talk about, as we will talk about *the* set of one element, *the* vector space of n dimensions, etc. If not specified, just assume that it is implicitly "up to isomorphism".

[30]

3.1 Examples

A few categories can easily be defined in categorical terms alone, such as the *empty category* $\varnothing$, which is the category with no objects and no morphisms (with the empty diagram as its diagram). We also have the *discrete categories* $\mathbf{n}$ for $n \in \mathbb{N}$, which consist of all the categories of exactly n objects and n morphisms (the identities of each object)

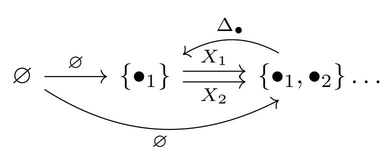


The empty category is in fact itself the discrete category $\mathbf{0}$.

Many categories can also be defined using typical mathematical structures built on set theory, using sets as objects and functions as morphisms.

Example 3.1.1 *The category of sets* **Set** *has as its objects all sets (*Obj(**Set**) *is the* class *of all sets), and as its morphisms the functions between those sets.*

If we consider the skeletonized version of **Set**, where we only consider unique sets of a given cardinality, its diagram will look something like this for the first three elements classified by cardinality :

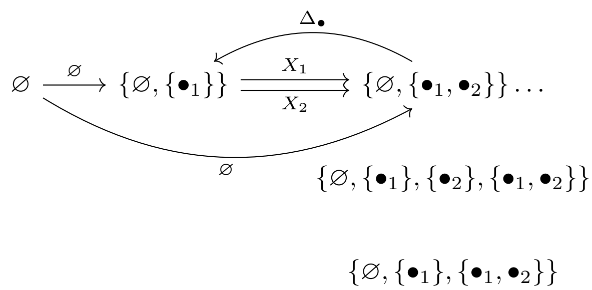


where $\varnothing$ is the empty function, X_1, X_2 are the functions that map the only element of $\{\bullet_1\}$ to either $\bullet_1$ or $\bullet_2$, and $\Delta_\bullet$ maps all elements to $\bullet_1$. The rest of the elements of the category follow similar patterns.

Example 3.1.2 *The category of topological spaces* **Top** *has as its objects the topological spaces* (X, τ_X)*, and as morphisms* $f : (X, \tau_X) \to (Y, \tau_Y)$ *the continuous functions between two such spaces.*

If we denote topologies on finite sets by their open sets,

As isomorphic sets can have different topologies, we can see more branching in **Top** :



[empty topology, singleton topology, for two elements : trivial, discrete and Sierpinski topology]

Example 3.1.3 *The category of vector spaces* $\mathbf{Vect}_k$ *over a field* k *has as its objects the vector spaces over* k*, and as its morphisms the linear maps between them. The hom-set between* V_1 *and* V_2 *is therefore the set of linear maps* $L(V_1, V_2)$ *(see later for enriched category)*

Diagram : finite dimensional case classified by dimension

$$k^0 \xrightarrow{x \in k} k \xrightarrow{L} k^2 \dots$$

Example 3.1.4 *The category of rings* **Ring** *has as its objects rings, and as morphisms ring homomorphisms.*

Example 3.1.5 *The category of groups* **Grp** *has as its objects groups* G*, and as its morphisms group homomorphisms.*

An important category for geometry is the one given by Cartesian spaces. There's a few different ways this can be interpreted. The objects can be either the real spaces $\mathbb{R}^n$ themselves, the open subsets of $\mathbb{R}^n$, or open subsets obeying certain properties, such as connectedness, simple connectedness, etc. The morphisms can be continuous maps, smooth maps, etc.

Choice :

Example 3.1.6 *The category of smooth Cartesian spaces* **CartSp** *has as its objects the real spaces* $\mathbb{R}^n$*, and as morphisms smooth maps between them.*

Besides concrete categories, another common type of category is *partial orders*, which are defined as usual in terms of sets, ie a partial order $(X, \leq)$ is a set X with a relation $\leq$ on $X \times X$, obeying

- Reflexivity :
$$\forall x \in X,\ x \leq x$$

- Antisymmetry :

$$\forall x, y \in X,\ x \leq y \wedge y \leq x \rightarrow x = y$$

- Transitivity :

$$\forall x, y, z \in X,\ x \leq y \wedge y \leq z \rightarrow x \leq z$$

As a category, a partial order is simply defined by $\mathrm{Obj}(\mathcal{C}) = X$. Its morphisms are defined by the relation $\leq$: if $x \leq y$, there is a unique morphism between x and y, which we will call $\leq_{x,y}$ formally, or simply $\leq$ if there is no risk of confusion. Otherwise, there is no morphisms between the two. This obeys the categorical axioms for morphisms as the identity morphism Id_x is simply given by reflexivity, $\leq_{x,x}$, and obeys the triangular identities by transitivity :

$$\leq_{x,x} \circ \leq_{x,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq x \wedge x \leq y \rightarrow x \leq y \tag{3.4}$$

$$\leq_{x,y} \circ \leq_{y,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq y \wedge y \leq y \rightarrow x \leq y \tag{3.5}$$

Morphisms : $f : X \rightarrow Y$ means $X \leq Y$. Between any two objects, there are exactly zero or one morphisms. The identity is $X \leq X$, composition is transitivity

Example 3.1.7 *Given a topological space (X, τ), its* category of opens $\mathbf{Op}(X)$ *is given by its set of open sets τ with the partial order of inclusion $\subseteq$.*

Example 3.1.8 *The integers $\mathbb{Z}$ as a totally ordered set $(\mathbb{Z}, \leq)$ forms a category.*

$$\ldots \xrightarrow{\leq} -2 \xrightarrow{\leq} -1 \xrightarrow{\leq} 0 \xrightarrow{\leq} 1 \xrightarrow{\leq} 2 \xrightarrow{\leq} \ldots$$

Other total orders of interest are the rational numbers $(\mathbb{Q}, \leq)$ and real numbers $(\mathbb{R}, \leq)$.

As category theory is meant to describe mathematical structures, there is also a category of categories. To avoid

The category of (locally small) categories **Cat**

Another type of such categories is the *simplicial category* Δ. There are a few different interpretation for it, one of them being that of totally ordered finite sets with monotone functions between them, ie

$$\forall f : \vec{m} \rightarrow \vec{n},\ f(m_i \rightarrow m_j) = f(m_i) \rightarrow f(m_j) \tag{3.6}$$

or alternatively, as its equivalent poset category, where each object of Δ is the finite total order $\vec{\mathbf{n}}$.

Example 3.1.9 *The* interval category I *is composed of two elements* $\{0,1\}$ *and a morphism* $0 \to 1$.

$$I = \{0 \to 1\} \tag{3.7}$$

Since categories are constructed to be mathematical structures, there is in fact a category of categories, **Cat**. To avoid any Russell-type paradox, we will only consider specific kinds of categories here, typically the category of small categories, where $\mathrm{Obj}(\mathbf{C})$ is small enough to be a set, and larger categories fit into some hierarchy. As the category of categories require a bit more machinery from category theory, we will look at its definition in more details later on.

As our categories are simply built from sets, one thing we can do is also construct a product of categories.

Definition 3.1.1 *A product of two categories* $\mathbf{C} \times \mathbf{D}$ *is the category given by the objects*

$$\mathrm{Obj}(\mathbf{C} \times \mathbf{D}) = \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{D}) \tag{3.8}$$

with the morphisms

$$\mathrm{Mor}(\mathbf{C} \times \mathbf{D}) = \mathrm{Mor}(\mathbf{C}) \times \mathrm{Mor}(\mathbf{D}) \tag{3.9}$$

with every object a pair (X, Y) *and every morphism* (f, g)*, such that the source and target obey*

$$s((f,g)) = (s(\pi_1((f,g))), s(\pi_2((f,g)))) \tag{3.10}$$
$$t((f,g)) = (t(\pi_1((f,g))), t(\pi_2((f,g)))) \tag{3.11}$$
$$\tag{3.12}$$

or more succinctly,

$$s((f,g)) = (s(f), s(g)) \tag{3.13}$$
$$t((f,g)) = (t(f), t(g)) \tag{3.14}$$
$$\tag{3.15}$$

While we can do this comfortably outside the categorical framework here simply with set theory, it is also something that we will be able to do internally later on as in fact a product category is given by a product in the category of categories, **Cat**.

3.2 Morphisms

As our categories are fundamentally built from sets and classes, we can look at specific subsets of our set of morphisms, $\mathrm{Mor}(\mathbf{C})$, with specific properties.

A simple example is simply the set of all morphisms between two objects, called hom-set :

Definition 3.2.1 *The* hom-set *between two objects* $X, Y \in \mathbf{C}$ *is the set of all morphisms between* X *and* Y *:*

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathrm{Mor}(\mathbf{C}) | s(f) = X,\ t(f) = Y\} \tag{3.16}$$

This notion is more rigorously only valid for what are called *locally small categories*, where the cardinality of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is small enough to be a set. If the cardinality is too large, and could only fit in say a class, it is more properly described as a *hom-object*. This will not affect our discussion much as most categories of interest are locally small.

Example 3.2.1 *The hom-set on the category of sets* $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ *is the set of all functions between* X *and* Y

Example 3.2.2 *The hom-set on the category of vector spaces* $\mathrm{Hom}_{\mathbf{Vec}}(X, Y)$ *is the set of all linear functions between* X *and* Y*, also known as* $L(X, Y)$*.*

Example 3.2.3 *The hom-set on the category of topological spaces* $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ *is the set of all continuous functions between* X *and* Y*, also known as* $C(X, Y)$*.*

A specific type of hom-set is the set of endomorphisms of an object.

Definition 3.2.2 *The endomorphisms of an object* X *are the hom-set*

$$\mathrm{End}(X) = \mathrm{Hom}_{\mathbf{C}}(X, X) \tag{3.17}$$

We can also define an induced function given an object between hom-sets thusly. For a morphism $f : X \to Y$, the induced function f_* on the hom-sets

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \to \mathrm{Hom}_{\mathbf{C}}(Z, Y) \tag{3.18}$$

is defined by composition. In other words, if we have a function $g : Z \to X$, we can map it to a function $g' : Z \to Y$ by post-composition with f :

$$g' = f_*(g) = f \circ g \tag{3.19}$$

Likewise, we have a similar induced function by pre-composition, defined by

$$f^* : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \to \mathrm{Hom}_{\mathbf{C}}(X, Z) \tag{3.20}$$

$$g \mapsto f^*(g) = g \circ f \tag{3.21}$$

(These will be transformations induced by the hom-functors later on)

3.2.1 Monomorphisms

Definition 3.2.3 *A* monomorphism $f : X \to Y$ *is a morphism such that, for every object* Z *and every pair of morphisms* $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \to g_1 = g_2 \tag{3.22}$$

We say that a monomorphism is left-cancellable.

Example 3.2.4 *on* **Set**, *a monomorphism is an injective function.*

Proof 3.2.1 *If* $f : X \to Y$ *is a monomorphism, then given two elements* $x_1, x_2 \in X$, *represented by morphisms* $x_1, x_2 : \{\bullet\} \to X$, *then we have*

$$f \circ x_1 = f \circ x_2 \to x_1 = x_2 \tag{3.23}$$

So that we can only have the same value if the arguments are the same, making it injective. Conversely, if f *is injective, take two functions* $g_1, g_2 : Z \to X$. *As the functions in sets are defined by their values, we need to show that*

$$\forall z \in Z,\ f(g_1(z)) = f(g_2(z)) \to g_1(z) = g_2(z) \tag{3.24}$$

By injectivity, the only way to have $f(g_1(z)) = f(g_2(z))$ *is that* $g_1(z) = g_2(z)$. *As this is true for every value of* z, $g_1 = g_2$.

Theorem 3.2.1 *A monomorphism is equivalent to the fact that the induced function* f_* *is injective on hom-sets :*

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \hookrightarrow \mathrm{Hom}_{\mathbf{C}}(Z, Y) \tag{3.25}$$

Proof 3.2.2 *If* $f : X \to Y$ *is a monomorphism, then for some hom-set* $\mathrm{Hom}_{\mathbf{C}}(Z, X)$, *we need to show that given two functions* g_1, g_2 *in there, the elements of* $\mathrm{Hom}_{\mathbf{C}}(Z, Y)$ *obtained by the induced map* f_* *obey*

$$f_*(g_1) = f_*(g_2) \to g_1 = g_2 \tag{3.26}$$

This is true by the definition of f_* *and left cancellability. Conversely, if* f_* *is an injective function on* $\mathrm{Hom}_{\mathbf{C}}(Z, X)$, *we have that*

Theorem 3.2.2 *The composition of two monomorphisms is a monomorphism.*

Proof 3.2.3 *Given two composable monomorphisms* $f : X \to Y$ *and* $g : Y \to Z$, *we must show that the composition* $g \circ f : X \to Z$ *is a monomorphism, ie for any two morphisms* $h_1, h_2 : W \to X$, *we have*

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \to h_1 = h_2 \tag{3.27}$$

By associativity, we have that this is equivalent to

$$g \circ (f \circ h_1) = g \circ (f \circ h_2) \tag{3.28}$$

and by the mono status of g, *this means that* $f \circ h_1 = f \circ h_2$. *And likewise, as* f *is a monomorphism,* $h_1 = h_2$.

It is tempting to try to view monomorphisms as generalizing injections, but categories may lack elements on which to define such notions and even if it does possess elements, monomorphisms may fail to be injective.

Counterexample 3.2.1 *Take the category of divisible abelian groups, whose objects are Abelian groups G for which for any positive integer $n \in \mathbb{N}$ and any group element $g \in G$, we have existence of some other element h such that*

$$h^n = g \tag{3.29}$$

so that any element is the sum of some element arbitrarily many times. Its morphisms are there is a monomorphism from $\mathbb{Q}$ to $\mathbb{Q}/\mathbb{Z}$,

$$\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \tag{3.30}$$

given by

3.2.2 Epimorphisms

Definition 3.2.4 *An* epimorphism *$f : X \to Y$ is a morphism such that, for every object Z and every pair of morphisms g_1, g_2, we have*

$$g_1 \circ f = g_2 \circ f \to g_1 = g_2 \tag{3.31}$$

we say that an epimorphism is right-cancellable*.*

Example 3.2.5 *On* **Set***, an epimorphism is a surjective function.*

Proof 3.2.4

Theorem 3.2.3 *The composition of two epimorphisms is an epimorphism.*

Proof 3.2.5 *Given two composable epimorphisms $f : X \to Y$ and $g : Y \to Z$, we must show that the composition $g \circ f : X \to Z$ is an epimorphism, ie for any two morphisms $h_1, h_2 : Z \to W$, we have*

$$h_1 \circ (g \circ f) = h_2 \circ (g \circ f) \to h_1 = h_2 \tag{3.32}$$

By associativity, we have that this is equivalent to

$$(h_1 \circ g) \circ f = (h_2 \circ g) \circ f \tag{3.33}$$

and by the fact that f is an epimorphism, this means that $g \circ h_1 = g \circ h_2$. And likewise, as g is an epimorphism, $h_1 = h_2$.

3.2.3 Isomorphisms

Definition 3.2.5 *An* isomorphism $f : X \to Y$ *is a morphism with a two-sided inverse, ie there exists a morphism* $f^{-1} : Y \to X$ *such that*

$$f \circ f^{-1} = \mathrm{Id}_Y \tag{3.34}$$

$$f^{-1} \circ f = \mathrm{Id}_X \tag{3.35}$$

Example 3.2.6 *In the category of sets, isomorphisms are bijections.*

Proof 3.2.6 *This is the usual proof of bijective functions having an inverse.*

For $f : X \to Y$ *an injective and surjective function, define the function* $f^{-1} : Y \to X$ *to be such that, for any* $y \in Y$*, we associate the value* x *such that* $f(x) = y$*, which exists as* f *is surjective. And since* f *is injective, this* x *is unique, so that* f^{-1} *is well-defined.*

Conversely, if $f : X \to Y$ *has an inverse* $f^{-1} : Y \to X$*, for any element of* y *there is a corresponding element of* x *mapped onto* x*,* $f^{-1}(y)$*, as* $f(f^{-1}(y)) = y$*, making it surjective, and for any two elements* x_1, x_2 *of* X*, if we have* $f(x_1) = f(x_2)$*,* f^{-1} *fails to be defined at that point since it will fail to have a unique image, meaning that by contradiction,* f *is injective.*

Example 3.2.7 *In the category of topological spaces, isomorphisms are homeomorphisms.*

Example 3.2.8 *In the category of smooth manifolds, isomorphisms are diffeomorphisms.*

Despite the most typical examples, it is not in general true that a morphism that is both a monomorphism and an epimorphism is an isomorphism.

Counterexample 3.2.2 *Take the Sierpinski category* $0 \to 1$*. Its unique non-trivial morphism* $f : 0 \to 1$ *is mono (the only other morphism* $0 \to 0$ *is the identity), and epi (same for* $1 \to 1$*), but there does not even exist a morphism* $1 \to 0$.

Counterexample 3.2.3 *Continuous bijections aren't homeomorphisms.*

A specific type of isomorphisms are the isomorphic endomorphisms, called the *automorphisms*. Together, they are called the core of an object.

Definition 3.2.6 *For a given object* X*, the subset of all its endomorphisms which are isomorphisms are called its* core *:*

$$\mathrm{core}(X) = \{f \in \mathrm{Hom}_{\mathbf{C}}(X, X) \mid f \textit{ isomorphism}\} \tag{3.36}$$

Corrolary 1 *In a poset category, the core of an object is always the identity morphism.*

Theorem 3.2.4 *The core of an object has a group structure.*

Proof 3.2.7 *If we pick as the set of the group* Core(X)*, as its group product the composition of morphisms* $\circ$*, as unit the identity morphism and as inverse the function mapping a morphism to its inverse, we have that this fulfills all the group axioms.*

Example 3.2.9 *The core of a set is its symmetric group*

$$\mathrm{core}(X) = S_X \tag{3.37}$$

Example 3.2.10 *The core of a smooth manifold is its diffeomorphism group*

$$\mathrm{core}(M) = \mathrm{Diff}(M) \tag{3.38}$$

Definition 3.2.7 *A category* **C** *for which any morphism that is both mono and epi is an isomorphism is called* balanced*.*

Example 3.2.11 **Set** *is balanced.*

Proof 3.2.8 *This simply stems from the equivalence of monos with injections and epis with surjections.*

3.2.4 Properties

Theorem 3.2.5 *For* f, g *two morphisms that can be composed as* $g \circ f$*, if* $g \circ f$ *and* g *are monomorphisms, then so is* f*.*

Proof 3.2.9 *For* f *to be a monomorphism, we need*

$$f \circ h_1 = f \circ h_2 \to h_1 = h_2 \tag{3.39}$$

we can compose this with g *to obtain*

$$g \circ (f \circ h_1) = g \circ (f \circ h_2) \tag{3.40}$$

and by associativity and given that $g \circ f$ *is a monomorphism, this leads to*

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \to h_1 = h_2 \tag{3.41}$$

As monomorphisms represent embeddings and inclusions, this means in particular that if we have the inclusion $S \hookrightarrow X$ and

3.2.5 Split morphisms

From 3.2.2, we've seen that the notions of epis and monos do not generally directly correspond to injective and surjective functions, even in the case where objects are actual sets, but we do have a more accurate categorical notion for these, which are split epis and monos.

Definition 3.2.8 *A* split monomorphism $f : X \to Y$ *is a monomorphism possessing a left inverse* $r : Y \to X$*, called its* retraction*, so that*

$$r \circ f = \mathrm{Id}_X \tag{3.42}$$

Conversely, we say also that r is a *section* of f.

Theorem 3.2.6 *The backward composition of the split monomorphism* (f, r) *is idempotent, so that for the morphism* $f \circ r : Y \to Y$*, we have*

$$(f \circ r)^{\circ n} = (f \circ r) \tag{3.43}$$

Proof 3.2.10 *By recursion, this is simply true for* $n = 1$*, and for* $n + 1$*, we have*

$$\begin{aligned}
(f \circ r)^{\circ(n+1)} &= (f \circ r)^{\circ(n-1)} \circ (f \circ (r \circ f) \circ r) && (3.44)\\
&= (f \circ r)^{\circ(n-1)} \circ (f \circ \mathrm{Id}_X \circ r) && (3.45)\\
&= (f \circ r)^{\circ n} && (3.46)\\
&= f \circ r && (3.47)
\end{aligned}$$

Example 3.2.12 *In the category of vector spaces* **Vec***, any monomorphism is split. The monomorphism is the inclusion map of a subspace* $\iota : W \hookrightarrow V$*, and its retract can be constructed by some projection* P *onto that subspace. More strictly, if we consider the projection* $P : V \to V$ *of any point in* V *onto that subspace, and define our vector space via the direct sum* $V = W \oplus U$ *for a complementary subspace* U *with projection* $1 - P$*, the retraction is given by the projection onto the first element of this product.*

$$r = p_1 \tag{3.48}$$

The idempotency of the morphism $f \circ r$ *stems from that of the projection*

Example 3.2.13 *In the category of sets* **Set***, any monomorphism is split. The monomorphism is a subset relation* $\iota : S \hookrightarrow X$*, and its retract* r *is a given choice function, ie some function defined as*

$$\forall x \in X,\ \exists s \in S,\ r(x) = s \tag{3.49}$$

which is guaranteed by the axiom of choice. [diagram]

Definition 3.2.9 *A* split epimorphism $f : X \to Y$ *is an epimorphism possessing a right inverse* $s : Y \to X$*, called a* section*, so that*

$$f \circ s = \mathrm{Id}_Y \tag{3.50}$$

Example 3.2.14 *Given a bundle between two topological spaces* $\pi : E \to B$*, a section is a function* s *mapping points* $x \in B$ *to points of* $E_x = \pi^{-1}(X)$*. This defines a subspace* $s(B) \subseteq E$*.*

Theorem 3.2.7 *A morphism that is both split epi and split mono is an isomorphism.*

Proof 3.2.11 *If* $f : X \to Y$ *is split epi and split mono, there is a section* $g : Y \to X$ *for which* $f \circ g = \mathrm{Id}_Y$*. Applying* f *on the left, we have*

$$f \circ g \circ f = f \tag{3.51}$$

and furthermore, $f = f \circ \mathrm{Id}_X$*. Since* f *is a monomorphism, it is left cancellable, meaning that*

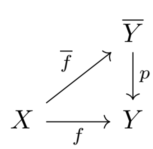
$$f \circ g \circ f = f \circ \mathrm{Id}_X \to g \circ f = \mathrm{Id}_X \tag{3.52}$$

meaning that g is a left inverse in addition to a right inverse, making it a double sided inverse of f*.*

Left-unique, left-total?

3.2.6 Lifts and extensions

Definition 3.2.10 *A* lift *of a morphism* $f : X \to Y$ *through some morphism* $p : \overline{Y} \to Y$ *is a morphism* $\overline{f} : X \to \overline{Y}$ *such that* $f = p \circ \overline{f}$*, ie it is the commutative triangle*



Definition 3.2.11 *An* extension *of a morphism*

Right and left lift

Lifting property

Example 3.2.15 *In* **Set***, any morphism that has the right lifting property with respect to the inclusion*

$$\iota : \varnothing \hookrightarrow \{\bullet\} \tag{3.53}$$

is a surjective function.

Proof 3.2.12

Example 3.2.16 *A classic example of an extension is that of a* group extension*, which is an extension in* **Grp**

3.2.7 Subobjects

As monomorphisms are similar to injective functions, it is tempting to see a way to define subobjects with them. To formalize this notion, we define subobjects that way

Definition 3.2.12 *A subobject is an equivalence class of monomorphisms to the same object up to isomorphism,*

$$[S] \in \mathrm{Sub}_{\mathbf{C}}(X) \leftrightarrow [S] = \{S' \in \mathbf{C} \wedge \exists\} \tag{3.54}$$

Equivalently, monomorphism in the skeletal category?

Example 3.2.17 *A subobject in* **Set** *is a subset up to isomorphism, ie the equivalence class of sets of some lower cardinality, where each specific subset is related to each other by the symmetric group.*

Example 3.2.18 *In* **Top***, two examples of subobjects are the subspaces,*

$$\iota : S \to X \tag{3.55}$$

and the continuous inclusion of a topological space in a space of coarser topology

$$\forall \tau_1 \subseteq \tau_2,\ (X, \tau_2) \hookrightarrow (X, \tau_1) \tag{3.56}$$

Example 3.2.19 *A subobject in* **Vec** *is a subspace up to isomorphism, $\iota : W \hookrightarrow V$.*

Example 3.2.20 *A subobject in* SmoothMan *is a submanifold, up to diffeomorphism. This is in particular the definition of a path (as opposed to a curve),*

$$[\gamma] = \gamma / \mathrm{Diff}(I) \tag{3.57}$$

Definition 3.2.13 *The* image *of a morphism $f : X \to Y$, denoted* $\mathrm{im}(f)$*, if it exists, is the smallest subobject of Y which factors through f, ie*

$$X \overset{f|^{\mathrm{im}(f)}}{\longrightarrow} \mathrm{im}(f) \overset{\iota_{\mathrm{im}(f)}}{\longrightarrow} Y \tag{3.58}$$

Example 3.2.21 *In* **Set***, this is the ordinary notion of image, where for a function $f : X \to Y$, we can consider the corestriction $f|^{\mathrm{im}(f)}$*

For instance, the function $(-)^2 : \mathbb{R} \to \mathbb{R}$ has image $\mathrm{im}((-)^2) = \mathbb{R}_{\geq 0}$*, so that we can decompose it as*

$$\mathbb{R} \overset{(-)^2|^{\mathbb{R}_{\geq 0}}}{\longrightarrow} \mathbb{R}_{\geq 0} \overset{\iota_{\mathbb{R}_{\geq 0}}}{\longrightarrow} \mathbb{R} \tag{3.59}$$

3.2.8 Factorization system

In many categories, there are important pairs of classes of morphisms called *factorization systems*, which are composed of two classes of morphisms, (L, R), where

$$L, R \subseteq \mathrm{Mor}(\mathbf{C}) \tag{3.60}$$

such that every morphism is a composition of those two, ie

$$\forall f \in \mathrm{Mor}(\mathbf{C}),\ \exists l, r \in (L, R),\ f = l \circ r \tag{3.61}$$

[31]

Definition 3.2.14 *A* weak factorization system *on a category* **C** *is a pair of classes of morphisms (L, R) such that*

- *For any morphism $f \in \mathbf{C}$, it can be factored by a left and right morphism $l \in L,\ r \in R :\ f = r \circ l$*
- *Left and right morphisms have the opposite lifting properties to each other : for any $r \in R$,*

Definition 3.2.15 *An* orthogonal factorization system *is a weak factorization system for which every left and right lifting is unique.*

Definition 3.2.16 *An* (epi, mono) factorization system *is an orthogonal factorization system for which the left class is that of epimorphisms and the right class is that of monomorphisms.*

Theorem 3.2.8 *A category with an (epi, mono) factorization system is balanced.*

Proof 3.2.13

Example 3.2.22 *The category of sets has an (epi, mono) factorization system.*

Proof 3.2.14

3.3 Opposite categories

Definition 3.3.1 *The* opposite *of a category* $\mathbf{C}$*, denoted* $\mathbf{C}^{\text{op}}$*, is a category for which the two categories share the same objects and morphisms*

$$\text{Obj}(\mathbf{C}) = \text{Obj}(\mathbf{C}^{\text{op}}) \tag{3.62}$$

$$\text{Mor}(\mathbf{C}) = \text{Mor}(\mathbf{C}^{\text{op}}) \tag{3.63}$$

but for which the source and target map are reversed. That is,

$$s_{\mathbf{C}} = t_{\mathbf{C}^{\text{op}}} \tag{3.64}$$

$$t_{\mathbf{C}} = s_{\mathbf{C}^{\text{op}}} \tag{3.65}$$

Simply speaking, the opposite category is the category for which all arrows are reversed.

Theorem 3.3.1 *The opposite of a category is an involution,*

$$(\mathbf{C}^{\text{op}})^{\text{op}} = \mathbf{C} \tag{3.66}$$

Proof 3.3.1

$$s_{(\mathbf{C}^{\text{op}})^{\text{op}}} = t_{\mathbf{C}^{\text{op}}} = s_{\mathbf{C}} \tag{3.67}$$

$$t_{(\mathbf{C}^{\text{op}})^{\text{op}}} = s_{\mathbf{C}^{\text{op}}} = t_{\mathbf{C}} \tag{3.68}$$

Despite its rather simple definition, the semantics of opposite categories can be rather obscure, which will lead to some difficulties whenever definitions will involve them. This is because for many categories of interest, the morphisms are functions in some sense, and it is not generally clear what the inverse object for a function will be in the general case.

A good example of this is given by the opposite of **Set**, which is one of the simplest category. The more tractable case of this is to consider the category of finite sets, **FinSet**

Example 3.3.1 *The opposite category of* **FinSet** *is equivalent to the category of finite boolean algebras.*

Proof 3.3.2

Example 3.3.2 *Any group interpreted as a category is the opposite group, for which any element g becomes its inverse g^{-1}.*

Example 3.3.3 *The opposite category of a poset $(X, \leq)$ is the poset of the reverse ordering relation, $(X, \geq)$.*

3.4 Functors

Functors are roughly speaking functions on categories that preserve their structures. In more details,

Definition 3.4.1 *A* functor $F : \mathbf{C} \to \mathbf{D}$ *is a function between two categories $\mathbf{C}, \mathbf{D}$, mapping every object of $\mathbf{C}$ to objects of $\mathbf{D}$ and every morphism of $\mathbf{D}$ to morphisms of $\mathbf{D}$, such that categorical properties are conserved :*

$$\begin{aligned} s(F(f)) &= F(s(f)) && (3.69) \\ t(F(f)) &= F(t(f)) && (3.70) \\ F(\mathrm{Id}_X) &= \mathrm{Id}_{F(X)} && (3.71) \\ F(g \circ f) &= F(g) \circ F(f) && (3.72) \end{aligned}$$

Example 3.4.1 *The* identity functor $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ *maps every object and morphism to themselves.*

Example 3.4.2 *The* constant functor $\Delta_X : \mathbf{C} \to \mathbf{D}$ *for some object $X \in \mathbf{D}$ is the functor mapping every object in $\mathbf{C}$ to X and every morphism to Id_X.*

Example 3.4.3 *A functor between two partial order categories $(X, \leq)$ and $(Y, \leq)$ is simply an order-preserving function (or monotone function).*

Example 3.4.4 *For a category where the objects are sets and the morphisms are functions (such as* **Top** *or* $\mathbf{Vect}_k$*), the* forgetful functor $U_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set}$ *is the functor sending every object to their underlying set, and every morphism to their underlying function on sets.*

A common functor is the *forgetful functor*, which maps a category that is composed of a set with extra structure its underlying set. For instance, the category **Top** has a forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$, which maps every topological space to its set, and every continuous function to the corresponding function on sets. If we have a set X and on it are all the different topologies (X, τ_i), then the forgetful functor maps

$$U((X, \tau_i)) = X \tag{3.73}$$

The forgetful functor on **Top** is obviously not injective, as two topological spaces with the same underlying set (such as any set of cardinality ≥ 1 with the discrete or trivial topology) will map to the same set.

Example : negation

Example 3.4.5 *The skeletonization functor is a functor* $\mathrm{Sk} : \mathbf{Cat} \to \mathbf{Cat}$ *which maps categories to their equivalent skeleton category.*

Definition 3.4.2 *A* contravariant functor *is a functor from the opposite category, so that $F : C \to D$ is a contravariant functor equivalently to a functor $F : C^{\mathrm{op}} \to D$ is a functor. In particular, this changes the rules as*

$$s(F(f)) = F(t(f)) \tag{3.74}$$
$$t(F(f)) = F(s(f)) \tag{3.75}$$
$$F(g \circ f) = F(f) \circ F(g) \tag{3.76}$$

Theorem 3.4.1 *The composition of contravariant and covariant functors works as follow :*

$$C \tag{3.77}$$

A generalization of functors is the notion of multifunctors (we mean here specifically the *jointly functorial* multifunctor)

Definition 3.4.3 *A multifunctor $F : \prod_i \mathbf{C}_i \to \mathbf{D}$ is a function from a product of category to another category.*

Theorem 3.4.2 *Any multifunctor $F : \prod_i \mathbf{C}_i \to \mathbf{D}$ can be constructed by a tuple of functors $F : \mathbf{C}_i \to \mathbf{D}$*

Proof 3.4.1

"Functor categories serve as the hom-categories in the strict 2-category Cat."

3.4.1 The hom-functor

The hom-bifunctor $\mathrm{Hom}_{\mathbb{C}}(-,-)$ is the map

$$\mathrm{Hom}_{\mathbb{C}}(-,-) : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{Set} \tag{3.78}$$
$$(X,Y) \mapsto \mathrm{Hom}_{\mathbf{C}}(X,Y) \tag{3.79}$$

mapping objects of $\mathbf{C}$ to their hom-sets. Given a specific object X, we can furthermore define two types of hom functors : the covariant functor $\mathrm{Hom}(X,-)$, also denoted by h^X, and the contravariant functor $\mathrm{Hom}(-,X)$, denoted by h_X.

Example :

Subobject functor :

$$\mathrm{Sub}_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{Set} \tag{3.80}$$

3.4.2 Full and faithful functor

As functors can be interpreted as functions on categories, we will need some notion similar to injectivity and surjectivity. This is given by the notions of a full functor and faithful functor.

Definition 3.4.4 *A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ induces the function*

$$F_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X,Y) \rightarrow \mathrm{Hom}_{\mathbf{D}}(F(X),F(Y)) \tag{3.81}$$

F is said to be

- faithful *if $F_{X,Y}$ is injective*
- full *if $F_{X,Y}$ is surjective*
- fully faithful *if $F_{X,Y}$ is bijective*

Despite the analogy of full and faithful functors with surjective and injective functions, functors being full or faithful does not imply that they are surjective or injective on either objects or morphisms.

Counterexample 3.4.1 *Given the terminal category* $\mathbf{1}$ *and a category of two objects and one isomorphism between them, $X \overset{\cong}{\rightleftarrows} Y$, then we have a*

However, there is some sense in which it is true that such functors are equivalent to injective and surjective functions.

Theorem 3.4.3 *On a skeletal category, a conservative functor (preserves isomorphisms) is*

Theorem 3.4.4 *A fully faithful functor is conservative.*

Proof 3.4.2

[essentially injective/surjective, pseudomonic]

[image, essential image?]

Monomorphisms are reflected by faithful functors.

3.4.3 Subcategory inclusion

An important type of functor is the inclusion of a subcategory. If we take a category $\mathbf{C}$, and then create a new category $\mathbf{S}$ for which $\mathrm{Obj}(\mathbf{S}) \subseteq \mathrm{Obj}(\mathbf{C})$

Definition 3.4.5 *An inclusion of a subcategory* $\mathbf{S}$ *in a category* $\mathbf{C}$ *is a functor* $\iota : \mathbf{S} \hookrightarrow \mathbf{C}$*, such that* $\iota(\mathrm{Obj}(\mathbf{S})) \subseteq \mathrm{Obj}(\mathbf{C})$*,* $\iota(\mathrm{Mor}(\mathbf{S})) \subseteq \mathrm{Mor}(\mathbf{C})$*, and*

- *If* $X \in \mathbf{S}$*, then* $\mathrm{Id}_X \in \mathbf{S}$
- *For any morphism* $f : X \to Y$ *in* $\mathbf{S}$*, then* $X, Y \in \mathbf{S}$*.*
-

For discrete categories, $\mathbf{n}$ in $\mathbf{m}$ if $n \leq m$

Full subcategories

Example 3.4.6 *The linear order of the integers* $\mathbb{Z}$ *has inclusion functors to* $\mathbb{R}$

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{R} \tag{3.82}$$

If treated as Canonical inclusion :

$$\iota_h : \mathbb{Z} \to \mathbb{R} \tag{3.83}$$

$$k \mapsto h + k \tag{3.84}$$

Example 3.4.7 *The category of finite sets* **FSet** *is a subcategory of* **Set***, via the identity functor restricted to finite groups.*

Example 3.4.8 *The category of Abelian group* **Ab** *in* **Grp**

Full subcategory : ie every morphism of $\mathbf{C}$ is a morphism of $\mathbf{D}$, and such that every object $d \in \mathrm{Obj}(\mathbf{D})$ and morphism $(f : d \to d') \in \mathrm{Mor}(\mathbf{D})$ have a reflection in $\mathbf{C}$.

3.4.4 Pseudofunctors

3.5 Natural transformations

Natural transformations are a type of transformations on functors

Definition 3.5.1 *For two functors $F, G : \mathbf{C} \to \mathbf{D}$, a* natural transformation *η between them is a map $\eta : F \to G$ which induces for any object $X \in \mathbf{C}$ a morphism on $\mathbf{D}$*

$$\eta_X : F(X) \to G(X) \tag{3.85}$$

and for every morphism $f : X \to Y$ the identity

$$\eta_Y \circ F(f) = G(f) \circ \eta_X \tag{3.86}$$

[Commutative diagram]

Example 3.5.1 *The* identity transformation $\mathrm{Id}_F : F \to F$ *on the functor $F : \mathbf{C} \to \mathbf{D}$ is the natural tranformation for which every component $\mathrm{Id}_{F,X} : F(X) \to F(X)$ for $X \in \mathbf{D}$ is the identity map. This obeys the identity as*

$$\begin{aligned} \eta_Y \circ F(f) &= \mathrm{Id}_Y \circ F(f) && (3.87)\\ &= F(f) && (3.88)\\ &= F(f) \circ \mathrm{Id}_X && (3.89) \end{aligned}$$

Example 3.5.2 *The category of groups* **Grp** *has a functor to the category of Abelian groups* **AbGrp***, the Abelianization functor*

$$\begin{aligned} \mathrm{Ab} : \mathbf{Grp} &\to \mathbf{AbGrp} && (3.90)\\ G &\mapsto G/[G, G] && (3.91) \end{aligned}$$

[show functoriality] There is a natural transformation from the identity functor on groups to the abelianization endofunctor

$$\eta : \mathrm{Id}_{\mathbf{Grp}} \to \mathrm{Ab} \tag{3.92}$$

Example 3.5.3 *Given the category $\mathbf{Vect}_k$, for any vector space V we have the dual space V^* [see later in the internal hom section for why] of linear maps $V \to k$, and its double dual V^** of linear maps $V^* \to k$. We would like to show that there is an equivalence between V and V^{**}.*

Example 3.5.4 *The opposite group functor is simply given by the opposite category functor on* Grp*. Groups to opposite group*

For constant F and G : cone and cocone

A special case of a natural transformation is if the underlying functors are bifunctors $F, G : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$

Dinatural transformations

Definition 3.5.2 *Given two bifunctors* $F, G : \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{D}$*, a dinatural transformation* $\alpha : F \to G$

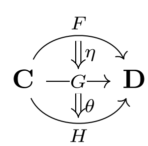
3.5.1 Composition

There are three ways to compose natural transformations.

Definition 3.5.3 *Given two natural transformations* $\eta : F \to G$ *and* $\theta : G \to H$ *between three functors* $F, G, H : \mathbf{C} \to \mathbf{D}$*, the* vertical composition *of those natural transformations is the natural transformation* $\theta \circ \eta : F \to H$*, defined component-wise by*

$$(\theta \circ \eta)_X = \theta_X \circ \eta_X \tag{3.93}$$

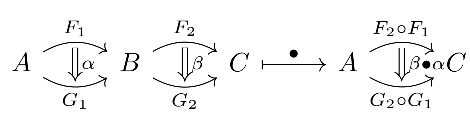
Giving the diagram



Definition 3.5.4 *For two natural transformations* $\eta : F \to G$ *and* $\theta : J \to K$*, with* $F, G : \mathbf{C} \to \mathbf{D}$ *and* $J, K : \mathbf{D} \to \mathbf{E}$*, their* horizontal composition *is given by the composition of their functors* $\theta \bullet \eta : J \circ F \to K \circ G$*, which is given components-wise as*

$$(\theta \bullet \eta)_X = \theta_{G(X)} \circ J(\eta_X) = K(\eta_X) \circ \epsilon_{F(X)} \tag{3.94}$$

Giving the diagram



Finally, a natural transformation can be composed with a functor by pre or post-composition in an operation called *whiskering* :

Definition 3.5.5 *For a natural transformation $\eta : F \to G$ between two functors $F, G : \mathbf{C} \to \mathbf{D}$, we talk of* left whiskering *for a functor $H : \mathbf{D} \to \mathbf{E}$ post-composed with it, $H \triangleleft \eta : H \circ F \to H \circ G$, with components*

$$(H \triangleleft \eta)_X = H(\eta_X) \tag{3.95}$$

and of right whiskering *for a functor $K : \mathbf{B} \to \mathbf{C}$ which is pre-composed with it, $\eta \triangleright K : F \circ K \to G \circ K$, with components*

$$(\eta \triangleright K)_X = \eta_{K(X)} \tag{3.96}$$

Theorem 3.5.1 *Whiskering with respect to a functor F is equivalent to horizontal composition with the identity natural transformation on F,*

$$H \triangleleft \eta = \mathrm{Id}_H \bullet \eta \tag{3.97}$$

$$\eta \triangleright K = \eta \bullet \mathrm{Id}_K \tag{3.98}$$

Proof 3.5.1 *Given the components of the horizontal composition, we have*

$$(\mathrm{Id}_F \bullet \eta)_X = \theta_{G(X)} \circ J(\eta_X) = K(\eta_X) \circ \mathrm{Id}_{F(X)} \tag{3.99}$$

In terms of components, left whiskering can be understood as a transformation applied to a diagram

[diagram]

While a right whiskering can be understood as [...]

3.6 Functor categories

Functors between two categories themselves form a category.

Definition 3.6.1 *A* functor category *between two categories $\mathbf{C}$ and $\mathbf{D}$ is a category, denoted by $\mathbf{D}^{\mathbf{C}}$ or $[\mathbf{C}, \mathbf{D}]$, for which the objects are all the functors $F : \mathbf{C} \to \mathbf{D}$ and all morphisms are natural transformations, with identity the identity natural transformation and the composition is the vertical composition of natural transformations.*

This indeed forms a category as the natural transformations obey all the appropriate requirements of morphisms in a category.

Example 3.6.1 *Given the terminal category $\mathbf{1}$, the functor category $[\mathbf{1}, \mathbf{C}]$ is isomorphic to $\mathbf{C}$.*

Proof 3.6.1 *Any functor of* $[\mathbf{1}, \mathbf{C}]$ *is constrained to be a function from* $* \in \mathbf{1}$ *to some object* $X \in \mathbf{C}$*, mapping* Id_* *to* Id_X*. These functions are exactly in bijection with the objects of* $\mathbf{C}$*, as there is always exactly one such function per object of* $\mathbf{C}$*. The natural transformations of those functors are*

$$\eta : \Delta_X \to \Delta_Y \tag{3.100}$$

To verify the naturality of the morphisms in this category, for the morphism $\mathrm{Id}_* : * \to *$*, mapped to* $\Delta_X(*) \cong X$*,* $\Delta_Y(*) \cong Y$ *The naturality square is just the unique component*

$$\eta_X \cong \eta_Y : X \to Y \tag{3.101}$$

This naturality is obeyed for any morphism $f \in \mathbf{C}$*, so that every morphism in* $\mathbf{C}$ *gives rise to a natural transformation in* $[\mathbf{1}, \mathbf{C}]$*.*

Example 3.6.2 *For the discrete category* $\mathbf{n}$*, the functor category* $[\mathbf{n}, \mathbf{C}]$ *is the category of families of objects indexed by* $\mathbf{n}$*.*

Proof 3.6.2

Whiskering in a functor category?

3.7 Yoneda lemma

One of the common philosophical idea underlying category theory is that of the specific objects involved in a category are not as meaningful as the equivalence of all such objects under isomorphisms. That is, if we have two objects X, X' in a category, such that those two objects have identical behaviour in the category (all morphisms to other objects are mirrored on the other), then they are in essence the same object. In philosophical terms, this

"every individual substance expresses the whole universe in its own manner and that in its full concept is included all its experiences together with all the attendent circumstances and the whole sequence of exterior events. There follow from these considerations several noticeable paradoxes; among others that it is not true that two substances may be exactly alike and differ only numerically, solo numero."

[Leibniz discourse on metaphysics]

This is true in some sense in category theory as expressed by the Yoneda lemma. If we consider all the relationship of an object X to all other objects, this is given by the hom-sets of X to every other objects, that is,

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{3.102}$$

What we mean by two objects having the same relationships to every other objects is that if we consider their respective hom-sets to every other objects, they are isomorphic :

$$\forall Y \in \mathbf{C},\ \mathrm{Hom}_{\mathbf{C}}(X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{3.103}$$

Meaning that for any object Y and any morphism $f : X \to Y$, we will have some corresponding function $f' : X' \to Y'$, and the relationships between all those morphisms reflect each other. If we have $f : X \to Y$ and $g : Y \to Z$, then we have

$$\exists g',\ f' \circ g' : X \tag{3.104}$$

Equivalently (converse)

This equivalence is in fact an equivalence of the hom functors,

$$h_X \cong h_{X'} \tag{3.105}$$
$$h^X \cong h^{X'} \tag{3.106}$$

As an equivalence of functors, this means that we have a pair of natural transformations between them which are two-sided inverses of each other.

For a functor $F : \mathbf{C} \to \mathbf{Set}$, for any object $X \in \mathbf{C}$

$$よ : \mathbf{C} \to [\mathbf{C}, \mathbf{Set}] \tag{3.107}$$

Functor lives in the functor space $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

$$\mathrm{Nat}(h_A, F) \cong F(A) \tag{3.108}$$

Example 3.7.1 *Consider the category of a single group ($G = \mathrm{Aut}(*)$). A functor $F : \mathrm{G} \to \mathbf{Set}$ is a set X and a group homomorphism to its permutation group $G \to \mathrm{Perm}(X)$ (A G-set). Natural transformation is an equivariant map Cayley's theorem*

Example 3.7.2 *Applied to a poset, the hom-functor $\mathrm{Hom}_P(x, -)$ gives us the set containing all the elements superior*

3.8 Enriched categories

By default, we consider the hom-sets of a category $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ to be sets, but many categories may have additional structure on the hom set. For instance, if we consider the category $\mathbf{Vect}_k$ of vector spaces over the field k, its morphisms are k-linear maps, and its hom-sets are

$$\mathrm{Hom}_{\mathbf{Vect}_k}(V, W) = L_k(V, W) \tag{3.109}$$

However, in addition to being a set, $L_k(V, W)$, the k-linear maps, also form themselves a vector space, as we can define the sum $f + g$ of two linear maps, and the scaling αf, $\alpha \in k$, of a linear map.

To generalize this notion, we define enriched categories

Definition 3.8.1 *An* enriched category $\mathbf{C}$ *over* $\mathbf{V}$ *a monoidal category* $(\mathbf{V}, \otimes, I)$ *is a category such that each hom-set* $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ *is associated to a* hom-object $C(X, Y) \in \mathbf{V}$*, such that every hom-object in* $\mathbf{V}$ *obeys the same rules regarding composition and identity, which are*

$$\circ_{X,Y,Z} : C(Y, Z) \otimes C(X, Y) \to C(X, Z) \tag{3.110}$$

$$j_X : I \to C(X, X) \tag{3.111}$$

with the following commutation diagrams :

[composition is associative]

[Composition is unital]

Example 3.8.1 *A category enriched in* **Set** *is a locally small category.*

Example 3.8.2 *A* k-linear category *is enriched over* Vect_k*.*

3.9 Comma categories

The notion of a comma category can be used to describe categories whose objects are the morphisms of another category.

Definition 3.9.1 *The* comma category $(f \downarrow g)$ *of two functors* $f : C \to E$ *and* $g : D \to E$ *is the category composed of triples* (c, d, α) *such that* $\alpha : f(c) \to g(d)$ *is a morphism in* E*, and whose morphisms are pairs* (β, γ)

$$\beta : c_1 \to c_2 \tag{3.112}$$
$$\gamma : d_1 \to d_2 \tag{3.113}$$

that are morphisms in C and D, such that $\alpha_2 \circ f(\beta) = g(\gamma) \circ \alpha_1$ *[Commutative diagram]*

Composition

Def via pullback

Comma categories are rarely used directly, but are more typically used to define more specific operations. The three important one we will see are arrow categories, slice categories and coslice categories.

3.9.1 Arrow categories

The simplest kind of comma category is the arrow category, where we just consider the arrows of a category as a category.

Definition 3.9.2 *An arrow category is the comma category for the case where the two functors are the identity functors,* $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$

$$\mathrm{Arr}(\mathbf{C}) = (\mathrm{Id}_{\mathbf{C}} \downarrow \mathrm{Id}_{\mathbf{C}}) \tag{3.114}$$

Theorem 3.9.1 *The arrow category* $\mathrm{Arr}(\mathbf{C})$ *is equivalent to the category whose objects are the morphisms of* $\mathbf{C}$ *:*

$$\mathrm{Obj}(\mathrm{Arr}(\mathbf{C})) = \mathrm{Mor}(\mathbf{C}) \tag{3.115}$$

and its morphisms are given by pairs of morphisms (f, g) *in* $\mathbf{C}$ *obeying the commutative square*

3.9.2 Slice categories

Given an object $X \in \mathbf{C}$, we can define the *over category* (or *slice category*) $\mathbf{C}_{/X}$ by taking all morphisms emanating from X as objects :

$$\mathrm{Obj}(\mathbf{C}_{/X}) = \{f | s(f) = X\} \tag{3.116}$$

As a comma category, this is the comma category of the two functors $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ and $\Delta_X : \mathbf{1} \to \mathbf{C}$, of the identity functor on $\mathbf{C}$ and the inclusion of the object X, in which case $\mathbf{C}_{/X} = (\mathrm{Id}_{\mathbf{C}} \downarrow \Delta_X)$ is defined by the triples $(c, *, \alpha)$ of objects $c \in \mathbf{C}$, the unique object $* \in \mathbf{1}$, and morphisms in $\mathbf{C}$ $\alpha : c \to X$. As there is only one object in the terminal category, we can drop it as it is isomorphic to simply (c, α), and furthermore, c is implied by α as simply being the source term. Our slice category is therefore indeed just defined by the set of morphisms from objects of the category to our selected object.

Slice categories are useful to consider objects in a category as a category in themselves, where the objects are simply all the relations they have with all other objects in the category.

Example 3.9.1 *In* **Set***, given a set* X*, the slice category* $\mathbf{Set}_{/X}$ *has as its objects all functions with codomain* X*,* $f : Y \to X$*, and as morphisms all functions between sets* $g : Y \to Y'$ *for which*

$$f'(g(y)) = f(y) \tag{3.117}$$

Category of X-indexed collections of sets, object $f : Y \to X$ is the X-indexed collection of fibers $\{Y_x = f^{-1}(\{x\})| \in X\}$, morphisms are maps $Y_x \to Y'_x$

[fiber product $Y \times_X Y'$ is the product in the slice category]

If we look for instance at $\mathbb{N}$ as a set (the natural number object of sets), its slice category $\mathbf{Set}_{/\mathbb{N}}$ is the category of all functions to numbers

Example 3.9.2 *For a poset* P*, the slice category* $\mathbf{P}_{/p}$ *is isomorphic to the down set of* p*, ie the subcategory of every element* $\{q|q \leq p\}$*.*

Objects : Id_p*, and every map* $\leq_{q,p}$ *(corresponding to* p *and object inferior to* p*), morphisms are*

Example 3.9.3 $\mathbf{Top}_{/X}$ *is the category of covering spaces over* X*.*

Example 3.9.4 *The slice category of smooth manifolds* $\mathbf{SmoothMan}_{/X}$ *is a subcategory of smooth bundles over* X*, consisting only of its epimorphisms.[?]*

Theorem 3.9.2 *Given a category* **C** *with pullbacks and a morphism* $f : X \to Y$ *in that category, there is an induced functor between the slice categories*

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{3.118}$$

where for morphisms $p : K \to Y$ *in* $\mathbf{C}_{/Y}$*, we define the equivalent morphism to* X *in* $\mathbf{C}_{/X}$ *by pullback. An object of* $\mathbf{C}_{/Y}$*, some morphism* $p : K \to Y$*, is mapped to the pullback defined by*

$$\begin{array}{ccc} X \times_Y K & \xrightarrow{p_A} & K \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and the bundle morphisms $(p : K \to Y) \to (p' : K' \to Y)$ *are given by*

Definition 3.9.3 *Given a functor* $F : \mathbf{C} \to \mathbf{D}$*, we can defined a sliced functor for* $X \in \mathbf{C}$ *via :*

$$F_{/X} : \mathbf{C}_{/X} \to \mathbf{D}_{/F(X)} \tag{3.119}$$

Theorem 3.9.3 *If a category* **C** *has a limit for a given functor* $F : I \to$

Theorem 3.9.4 *If* $\mathbb{C}$ *has an initial object* 0*,* $\mathbf{C}_{/X}$ *has the initial object* $\varnothing_X : 0 \to X$*.*

Proof 3.9.1

3.9.3 Coslice categories

Definition 3.9.4 *Given a category* $\mathbf{C}$ *and an object* $X \in \mathbf{C}$*,a coslice category, or under category,* $\mathbf{C}^{X/}$*, is the comma category of the identity functor* $\mathrm{Id}_{\mathbf{C}}$ *and the constant functor* Δ_X*,*

$$\mathbf{C}^{X/} = \Delta_X \downarrow \mathrm{Id}_{\mathbf{C}} \tag{3.120}$$

Theorem 3.9.5 *A coslice category* $\mathbf{C}^{X/}$ *is the category whose objects are morphisms in* $\mathbf{C}$ *with source* X*, and whose morphisms are morphisms in* $\mathbf{C}$ *which obey*

3.9.4 Base change

Base change functor

Definition 3.9.5 *Given a category* $\mathbf{C}$ *with pullbacks and a morphism* $f : X \to Y$*, and its induced functor on slice categories*

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{3.121}$$

Two useful case to keep in mind from slice categories is that if $\mathbf{C}$ has a terminal object, $\mathbf{C}_{/1} \cong \mathbf{C}$

Theorem 3.9.6 *The left adjoint of the base change functor* f^* *is equivalent to the dependent sum on the morphism* f *:*

$$\sum_f : \mathbf{C}_{/X} \to \mathbf{C}_{/Y} \tag{3.122}$$

Proof 3.9.2 *The base change functor transforms our morphism* $p : K \to Y$ *to the morphism* $p^* : X \times_Y K \to X$

Adjoint :

$$\mathrm{Hom}_{\mathbf{C}_{/X}}(A \to X, f^*(B \to Y)) \cong \mathrm{Hom}_{\mathbf{C}_{/Y}}(\sum_f (A \to X), B \to Y) \tag{3.123}$$

Theorem 3.9.7 *The right adjoint of the base change functor* f^* *is equivalent to the dependent product of the morphism* f

Example 3.9.5 *In the category of manifolds* **SmoothMan***, the slice categories* $\mathbf{SmoothMan}_{/X}$ *are the bundles over* X*,* $p : E \to X$*.*

Example 3.9.6 *In a poset category* $\mathbf{P}$*, the slice category* $\mathbf{P}_x$ *is the down set* $\downarrow (p)$*, the subposet of elements* $x' \leq x$*.*

3.10 Limits and colimits

In category theory, a limit or a colimit are roughly speaking a construction on a category. For some given set of objects $A, B, \dots$ in our category C, and some morphisms between them, a limit or colimit of those objects will be some construction performed using those. Those constructions can be quite different, but overall, a limit will often be like a ”subset”, while a colimit is more of an ”assemblage” of those.

A (co)limit is done using an indexing category, which is roughly the ”shape” that our construction will take. An indexing category is a small category, ie it has a countable number of objects and morphisms small enough that you could fit them into sets. Typically they are fairly simple ones. As we are only interested in their shape, it's common to denote the objects by simple dots. Examples include the discrete categories of n elements $\mathbf{n}$,

$$\bullet \qquad \bullet \qquad \bullet \qquad \dots$$

The span category :

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

and the cospan :

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

and the parallel pair :

$$\bullet \rightrightarrows \bullet$$

Directed and codirected set, sequential and cosequential limit

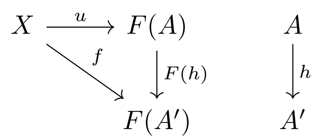
To do our constructions, we need to send this diagram's shape into our category C, which we do by using some functor $F : I \to C$, producing a diagram :

Definition 3.10.1 *A* diagram *of shape I into C is a functor $F : I \to C$.*

The image of this functor in C will then be some subset of C that "looks like" I, although we are not guaranteed that the objects and morphisms of I will be mapped injectively into C (this simply corresponds to cases where our construction will use the same object or morphisms several times).

with this diagram, to find a (co)limit, we would like to define a universal construction on it.

Definition 3.10.2 *For a functor $F : \mathbf{C} \to \mathbf{D}$, a* universal morphism *from an object $X \in \mathbf{D}$ to F is a unique pair of an object $A \in \mathbf{C}$ and a morphism $u : X \to F(A)$ which obey the universal property : for any object $A' \in \mathbf{C}$ and morphism $f : X \to F(A')$ in $\mathbf{D}$, there is a unique morphism $h : A \to A'$ such that $f = F(h) \circ u$.*



Let's now consider the constant functor $\Delta_X : I \to C$, which for $X \in C$ sends every object of I to X. If we can find a natural transformation between Δ_X and our diagram $F : I \to C$, we will have either a cone over F $\eta : \Delta_X \Rightarrow F$ or a cone under F $\eta : F \Rightarrow \Delta_X$.

Definition 3.10.3 *The* limit *of a diagram $F : I \to M$ is an object* $\lim F \in$ Obj(C) *and a natural transformation $\eta : \Delta_{\lim F} \to F$, such that for any $X \in$* Obj(C) *and any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f : X \to \lim F$ such that $\alpha = \eta \circ F$. The cone of $\Delta_{\lim F}$ over F is the* universal cone *over F.*

In other words, if we pick any object X in our category C and define some collection of morphisms from X to other objects

Theorem 3.10.1 *The presheaf limit of a functor $F : D^{\text{op}} \to C$ is the presheaf defined by*

$$(\lim F)(X) = \mathrm{Hom}_{\mathbf{Set}^{\mathbf{D}^{\text{op}}}}(\mathrm{pt}, \mathrm{Hom}_{\mathbf{C}}(X, F(-))) \tag{3.124}$$

and if this presheaf is representable, the object associated is the limit.

Let's consider for instance the case of the trivial category $\mathbf{1}$. Any functor $F : \mathbf{1} \to C$ is simply a choice of an object in C, mapping $\bullet$ to $F(\bullet) = A$, ie it is just the constant functor Δ_A for some A. A natural transformation $\eta : \Delta_X \to F$ is them simply $\eta : \Delta_X \to \Delta_A$, and conversely, $\eta : F \to \Delta_X$ is $\eta : \Delta_A \to \Delta_X$. The components of this natural transformations are simply a morphism from X to A (and a morphism from A to X).

Limit and colimit

Types of indexing category I and their limit and colimit :

3.10.1 Initial and terminal objects

The simplest kind of limit is the one done on the empty indexing category, **0**, as this leads to a unique diagram in the category **C**, the empty functor.

Definition 3.10.4 *Given the empty category* **0***, the limit* Lim$F$ *of a diagram* $F : \mathbf{0} \to C$ *is its* terminal object*, and the colimit is its* initial object*.*

Generally we will denote the terminal object as 1 and the initial object as 0, since they often correspond to objects of one and zero elements in concrete categories, except for a few cases where we will use a more specific notation for the given category, such as $\{\bullet\}$ and $\varnothing$ for **Set** (the singleton and empty set), or **1** and **0** for the category of categories **Cat**.

As there exists only one functor from the empty category to any other category (the empty functor $F_\varnothing$), the initial and terminal objects do not depend on specific objects and are simply special objects of the category. Every "constant functor" Δ_X sending objects of I to $X \in \text{Obj}(C)$ is also the empty functor, sending them trivially to X by simply not having any objects to send. This is therefore also true of the constant functor to the limit $\lim F_\varnothing$, meaning that the natural transformation $\eta : \varnothing \to \varnothing$ is simply the identity transformation $\text{Id}_{\varnothing^\varnothing}$. This means that the limit of the empty diagram in a category C is the object (defined by no other objects in the category) $\lim \mathbf{0}$ such that for any natural transformation $\alpha : \Delta_X \to F$ (as we've seen, only possibly the identity transformation), there exists a unique morphism $f : X \to \lim \mathbf{0}$.

This means that the terminal object $1 = \lim \mathbf{0}$ of a category, if it exists, is therefore an object for which there exists only one morphism from any object $X \in$ C to 1.

Dually, the *initial object* $i = \text{colim}\varnothing$ of a category, if it exists, is an object for which there exists only one morphism from i to any object $X \in \text{Obj}(C)$.

Theorem 3.10.2 *Initial and terminal objects are unique in a category up to isomorphisms.*

Proof 3.10.1 *If we have two different terminal objects* $1, 1'$*, by their universal property, there is exactly one morphism to themselves (the identity), and between them, meaning that there is a morphism* $1 \to 1'$ *and* $1' \to 1$*, and their composition in both direction leads to the unique morphisms* Id_1 *and* $\text{Id}_{1'}$*, meaning those maps are isomorphisms. A similar reasoning shows the same property for the terminal object.*

Initial and terminal objects occur in quite a lot of important categories, and tend to be somewhat similar objects. In **Set**, the initial and terminal objects are the empty set and the singleton set.

Definition 3.10.5 *An object that is both an initial and terminal object is called a* zero object*, and denoted by* 0*.*

This type of object is often found in the case of categories for which the objects are understood to have a distinguished element that morphisms are meant to preserve.

Example 3.10.1 *The zero dimensional vector space is a zero object of* $\mathbf{Vect}_k$*, with its unique map in being the projection to* 0 *and its unique map out the map pointing to* 0 *in its image.*

Definition 3.10.6 *The trivial group* $\{e\}$ *in* **Grp** *is a zero object, with its unique map in being the projection to the neutral element and its unique map out being the map pointing to the neutral element in its image.*

Zero object in rings

Theorem 3.10.3 *Any map* $f : 1 \to X$ *is a split monomorphism.*

Proof 3.10.2 *For any two morphisms* $g_1, g_2 : Y \to 1$*, by the universal property of the terminal object, we have that* $g_1 = g_2$*, therefore any such morphism is a monomorphism. Additionally, given the unique map* $!_X : X \to 1$*, we have* $!_X \circ f = \mathrm{Id}_1$*, as there is only one endomorphism for* 1*.*

Theorem 3.10.4 *An epimorphism* $f : 1 \to X$ *is an isomorphism.*

Proof 3.10.3 *Being a split monomorphism and an epimorphism, it is an isomorphism.*

Theorem 3.10.5 *Any morphism*

It is quite common in categories that there does not exist any morphism from any object to the initial object outside of its own identity map, Id_0, since in many cases the terminal object is ”empty” in some sense.

Definition 3.10.7 *An initial object is a* strict initial object

3.10.2 Products and coproducts

The product and coproduct are the limits where the diagrams are the discrete categories of n elements $\mathbf{n}$. This means obviously that the trivial case $\mathbf{0}$ of the diagram of zero object is the initial and terminal object, and this will correspond to the trivial product and coproduct as we will see later :

$$\sum_{\varnothing} = 0,\ \prod_{\varnothing} = 1 \tag{3.125}$$

Any diagram of shape $\mathbf{n}$ simply selects n objects in the category (and their identity functions),

$$\forall F : \mathbf{n} \to \mathbf{C},\ \exists X_1, \dots X_n \in \mathbf{C},\ \mathrm{Im}(F) = (X_1, \dots, X_n) \tag{3.126}$$

The constant functor Δ_X is the functor sending each of those points to X, $\bullet_i \to X$, and the identity of those points to the identity on X. We will denote the limit of a diagram F on the discrete category, selecting the objects $\{X_i\}$, by $\prod_i X_i$. The product is therefore some object $\prod_i X_i \in \mathbf{C}$ along with morphisms $\pi_i : \prod_j X_j \to X_i$

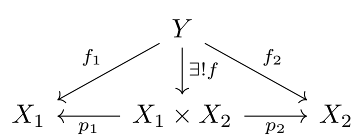
such that for any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f = \Delta_X \to \prod X_i$ such that $\alpha = \eta \circ F$.

For any object $Y \in \mathbf{C}$ with morphisms $f_i : Y \to X_i$ for each object X_i, we therefore have a unique morphism to $\prod_i$ which make the maps commute :

$$\exists! f : Y \to \prod_i X_i,\ \forall i \in I,\ f_i = \pi_i \circ f \tag{3.127}$$

What this property means for the product is that given any object X picked in $\mathbf{C}$,

Universal property



The semantics of the product is typically that we are considering elements of those objects *together*, as pairs. This can easily be seen in the case of some concrete category where objects are sets

[...]

You should however beware of overextending this interpretation to all categories, as it is typically only really valid in the case of objects being seen as some collection or space. As we've seen, categories such as preorders can have radically different interpretations.

Example 3.10.2 *For a preorder category* $(X, \leq)$*, the product of two objects is their (meet, join?). If we take the product of two elements, the universal property tells us that* $\prod_i X_i$ *has morphisms to all* X_i*, so that* $\prod_i X_i \leq X_i$ *(*$\prod_i X_i$ *is a lower bound of all elements). Furthermore, for any other object* Y *with morphisms to all* X_i*, so that* Y *is also a lower bound, we have a morphism* $Y \to \prod_i X_i$*, so that* $Y \leq \prod_i X_i$*. Therefore any other lower bound is inferior or equal to it.* $\prod_i X_i$ *is the* greatest lower bound*,*

$$\prod_i X_i = \bigwedge_i X_i \tag{3.128}$$

A special case of the product is the case of the product of an object with itself. This leads to a special morphism.

Definition 3.10.8 *For any object* X *in a category with products, the diagonal morphism* $\Delta : X \to X \times X$ *is the one given by the universal property of the product for the case where the two objects are* X *: [diagram]*

The name of diagonal morphism can be easily seen in the case of $\mathbb{R}$, where it corresponds to

$$\delta : \mathbb{R} \quad \to \quad \mathbb{R}^2 \tag{3.129}$$

$$x \quad \mapsto \quad (x, x) \tag{3.130}$$

which corresponds to the graph of the identity function, a diagonal in $\mathbb{R}^2$.

Likewise for the coproduct,

Definition 3.10.9 *For any object* X *in a category with products, the codiagonal morphism* $\nabla : X + X \to X$ *is the one given by the universal property of the product for the case where the two objects are* X *:*

Theorem 3.10.6 *The product of an object* X *by the terminal product is isomorphic to the object itself :*

$$X \times 1 \cong X \cong 1 \times X \tag{3.131}$$

Proof 3.10.4 *As for any product, there is a morphism*

$$p_1 : X \times 1 \to X \tag{3.132}$$

From the properties of the product, for the two morphisms $\mathrm{Id}_X : X \to X$ *and* $!_X : X \to 1$*, we have some unique morphism* $f : X \to X \times 1$ *such that* f

$$X$$

Theorem 3.10.7 *The product of two monomorphisms is a monomorphisms.*

Proof 3.10.5 *Given two monomorphisms* $f_1 : Y \hookrightarrow X_1$*,* $f_2 : Y \hookrightarrow X_2$*, their product* $(f_1, f_2) : Y \to X_1 \times X_2$

3.10.3 Equalizer and coequalizer

The equalizer and coequalizer are the limits and colimits corresponding to the diagram of a pair of parallel morphisms, ie :

$$\bullet \rightrightarrows \bullet \tag{3.133}$$

This diagram maps to some pair of morphisms in our category between two objects,

$$X \underset{f}{\overset{g}{\rightrightarrows}} Y \tag{3.134}$$

$$\mathrm{Hom}_{\mathbf{C}}(X, \lim F) \cong \mathrm{Nat}_{\mathbf{C}}(X, F) \tag{3.135}$$

Constant functor : $\Delta_X : I \to \mathbf{C}$ maps A, B to X, f, g to Id_X

Natural transformation $\eta : \lim F \to F$

Equalizer corresponds roughly to a solution of an equation, coequalizer to a quotient?

Example 3.10.3 *The equalizer of two morphisms* $f, g : X \to Y$ *in* **Set** *is the subset of* X *on which those morphisms agree, ie*

$$\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\} \tag{3.136}$$

Example 3.10.4 *In the category of group* **Grp***, given a group homomorphism $f : G \to H$ and the trivial homomorphism $\epsilon : G \to H$ (the homomorphism that factors through the zero group $\epsilon : G \to 0 \to H$), the equalizer of those morphisms is the kernel of f :*

$$\mathrm{eq}(f, \epsilon) = \ker(f) \tag{3.137}$$

with the morphism to G its inclusion map

$$\iota : \ker(f) \to G \tag{3.138}$$

Proof 3.10.6 *As the trivial homomorphism is the one mapping every element to the neutral element,*

$$\forall g \in G,\ \epsilon(g) = e_H \tag{3.139}$$

The universal property of the equalizer gives us that

$$f(\iota(\ker(f))) = \epsilon(\iota(\ker(f))) \tag{3.140}$$

so that every element of G in the kernel is mapped to the neutral element/

Coequalizer

Example 3.10.5 *The coequalizer of two morphisms $f, g : X \to Y$ in* **Set** *is the quotient set for the equivalence defined by those morphisms :*

$$\mathrm{coeq}(f, g) = \{f(x) = g(x)\} \tag{3.141}$$

Example 3.10.6 *The coequalizer of a group homomorphism with the trivial group homomorphism ϵ is the cokernel*

$$\mathrm{coeq}(f, \epsilon) = \mathrm{coker}(f) \tag{3.142}$$

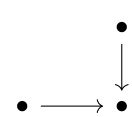
Theorem 3.10.8 *The morphism of an equalizer is a monomorphism.*

Equalizer is mono, coequalizer is epi

Equalizer for terminal object

3.10.4 Pullbacks and pushouts

The pullback and pushout are the limits of dual diagrams, the span and cospan, where the span is given by two morphisms to the same object,



or $\bullet \to \bullet \leftarrow \bullet$ for short, and two morphisms from the same object for the cospan :

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

or $\bullet \leftarrow \bullet \rightarrow \bullet$. Those are opposite diagrams, in the sense that $\mathbf{Span} = \mathbf{Cospan}^{\mathrm{op}}$

The pushout is the limit of the span, while the pullback is the limit of the cospan. From their opposition, we can also say that the pushout is the colimit of the cospan and the pullback the colimit of the span.

The span diagram can be mapped to any three objects connected thusly. For $A, B, C \in \mathbf{C}$, and $f : A \to C$, $g : B \to C$, the diagram of shape I will be

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

If we now look at the constant functor $\Delta_X : I \to \mathbf{C}$, this will map our three objects A, B, C to X, and f and g to Id_X. To find the limit of F, we therefore need to find the natural transformation $\eta : \Delta_{\lim F} \to F$

Component-wise :

$$\bullet_1 \xrightarrow{\alpha_1} \bullet_3 \xleftarrow{\alpha_2} \bullet_2 \tag{3.143}$$

$$\eta_{\bullet_i} : \lim F \to A, B, C \tag{3.144}$$

For any morphism, ie either the first or second morphism in I :

$$\eta_C = f \circ \eta_A \tag{3.145}$$

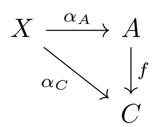
$$\eta_C = g \circ \eta_B \tag{3.146}$$

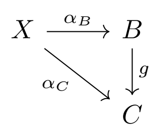
For any $X \in \mathbf{C}$, and any $\alpha : \Delta_X \to F$, there is a unique morphism $h : X \to \lim F$ such that $\alpha = \eta \circ F$.

Components of α : for every $Y \in I$, a morphism $\alpha_Y : \Delta_X(Y) \to F(Y)$, so

$$\alpha_Y : X \to F(Y) \tag{3.147}$$

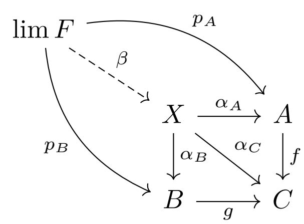
$F(Y)$ can only be A, B, C, so we have three components for objects, and for $f : A \to C$ and $g : B \to C$, then using what we know of Δ_X we have the following commuting diagrams :



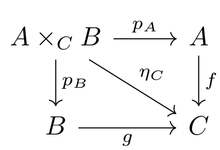


$$F(f) \circ \eta \tag{3.148}$$

The resulting cone is



The limit is the pullback, denoted as $A \times_C B$, along with the two and the universal cone that we have constructed gives us the commutative square



This means that our pullback diagram is given by this object and the two projectors, obeying

$$f \circ p_A = g \circ p_B \tag{3.149}$$

The interpretation of this is the *dependent sum* of the equality

$$\sum_{a:A} \sum_{b:B} (f(a) = g(b)) \tag{3.150}$$

Example 3.10.7 *In* **Set***, the pullback by* $f : A \to C$*,* $g : B \to C$ *is the set*

$$A \times_C B = \{(a, b) \in A \times B | f(a) = g(b)\} \tag{3.151}$$

$$\bigcup_{c \in f(A) \cap g(B)} f^{-1} \tag{3.152}$$

Example 3.10.8 *A particular type of pullback in* **Set** *is the pullback of* $f : A \to B$ *by the identity function on* B, $\mathrm{Id}_B : B \to B$

Definition 3.10.10 *For two subobjects* $\iota_1 : U_1 \hookrightarrow X$*,* $\iota_2 : U_2 \hookrightarrow X$*, the pullback of* $U_1 \to X \leftarrow U_2$ *is called the intersection of* U_1 *and* U_2

$$U_1 \cap U_2 = U_1 \times_X U_2 \tag{3.153}$$

This is easy enough to see in terms of **Set**, where this pullback gives us

$$U_1 \times_X U_2 = \{(x_1, x_2) \in U_1 \times U_2 | \iota_1(x_1) = \iota_2(x_2)\} \tag{3.154}$$

which is the set of all points that are in both U_1 and U_2 in X [diagonal morphism?]

Likewise, we can define the intersection of two subobjects thusly

Definition 3.10.11 *The intersection of two subobjects* $\iota_1 : U_1 \hookrightarrow X$*,* $\iota_2 : U_2 \hookrightarrow X$ *is the pushout of the inclusion maps of their intersection, that is*

$$\iota_{1, U_1 \cap U_2} : U_1 \cap U_2 \to U_1 \tag{3.155}$$

$$\iota_{2, U_1 \cap U_2} : U_1 \cap U_2 \to U_2 \tag{3.156}$$

and so

$$U_1 \cup U_2 = U_1 \times_{U_1 \cap U_2} U_2 \tag{3.157}$$

Example 3.10.9 *A typical example of the pullback is given in fiber bundles, for instance in the category of manifolds, where given a bundle morphism* $\pi : E \to B$ *and some morphism* $f : B' \to B$*, the pullback* $B' \times_B E$ *is the* pullback bundle*, which is a fiber bundle with the same typical fiber as* E *but done over* B'*, given by the equation*

$$f^* E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \tag{3.158}$$

Semantics of an equation

Properties :

Theorem 3.10.9 *The pullback of any morphism $f : X \to Y$ by the identity $\mathrm{Id}_X : X \to X$, $X \times_X Y$, is an isomorphism.*

Theorem 3.10.10 *The pullback to the terminal object, $X \to 1 \leftarrow Y$, is the product $X \times Y$.*

$$\begin{array}{ccc} A \times B & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle !_A} \\ B & \xrightarrow[!_B]{} & 1 \end{array}$$

Proof 3.10.7 *As the morphisms from any object to the terminal object are unique, every span $A \to 1 \leftarrow B$ is in unique correspondence with the pair of objects (A, B). As the universal cone over the pullback will induce a universal cone over (A, B) similar to that of the product, it is isomorphic to the product.*

Example 3.10.10 *In the context of differential geometry, it is common to call "pullback" simply the act of composition of two maps. That is, for two smooth maps between manifolds $f : M \to N$ and $g : N \to P$, the pullback of g by f is*

$$f^* g = g \circ f \tag{3.159}$$

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle f}\downarrow & \searrow^{\alpha_C} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[g]{} & P \end{array}$$

As spans and cospans are dual diagrams, we therefore have a duality between pullback and pushout, ie for a pullback $X \times_Z Y$ in $\mathbf{C}$, there is a dual pushout which is given by $X \sqcup_{X \times_Z Y} Y$, and is isomorphic to Z

Example 3.10.11

Theorem 3.10.11 *Isomorphisms always have a pullback.*

Proof 3.10.8 *Given some isomorphism $f : X \to Y$ and any other morphism $g : Z \to Y$, we can form a pullback square as*

$$\begin{array}{ccc} Z & \xrightarrow{f^{-1} \circ g} & X \\ {\scriptstyle \mathrm{Id}_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

As pullback squares are always unique up to isomorphism, this means that the pullback of any isomorphism along a function will itself be an isomorphism.

Dually, we define the pushout as the limit of a cospan diagram,

$$\bullet_1 \xleftarrow{\alpha_1} \bullet_3 \xrightarrow{\alpha_2} \bullet_2 \tag{3.160}$$

which corresponds to a functor mapping this diagram to a diagram of the form

$$X \xleftarrow{f} Z \xrightarrow{g} Y \tag{3.161}$$

so that a pushout is an operation on a pair of morphisms with the same target.

[...]

Example 3.10.12 *Given the inclusion maps of the intersection of two sets, $\iota_1 : U_1 \cap U_2 \hookrightarrow U_1$ and $\iota_2 : U_1 \cap U_2 \hookrightarrow U_2$, their pushout is the union of the two :*

$$U_1 \cap U_2 = U_1 +_{U_1 \cap U_2} U_2 \tag{3.162}$$

In terms of coproduct and coequalizer, this is the disjoint sum of the two sets with their overlaps identified.

Example 3.10.13 *Given a subspace $\iota : X \to Y$ in* **Top**,

Theorem 3.10.12 *Given a pushout $X +_Z Y$ for the functions $f : X \to Z$ and $g : Y \to Z$, if g is an epimorphism, then $f_* g$ is an epimorphism.*

Proof 3.10.9 *Given two morphisms $h_1, h_2 : X +_Z Y \to W$ such that $h_1 f_* g = h_2 f_* g$, if we precompose it with g, we obtain*

$$\begin{aligned} h_1(f_* g)g &= h_2(f_* g)g & (3.163)\\ &= h_1(g_* f)f & (3.164)\\ &= h_2(g_* f)f & (3.165) \end{aligned}$$

Since f is an epimorphism, we have that $h_1(g_* f) = h_2(g_* f)$. By the universal property of the pushout, for the two morphisms $h_1(g_* f) : h_2(g_* f)$

Fibers and cofibers

A specific type of pullback that we will commonly use is the fiber of a morphism.

Definition 3.10.12 *The* fiber *of a morphism $f : X \to Y$ by a given base point in Y, $p : 1 \to Y$, is the pullback with the terminal object :*

$$X \xrightarrow{f} Y \xleftarrow{p} 1 \tag{3.166}$$

which we denote by

$$\mathrm{Fib}_p(f) = X \times_Y 1 \tag{3.167}$$

As monomorphisms are stable under pullback, and $x : 1 \to Y$ is always a monomorphism, we have that $\mathrm{Fib}(f) \to X$ is always a monomorphism, and therefore corresponds to a subobject of X.

From the definition of pullbacks as equalizers of products, we can rewrite this notion as the equalizer of f and p as factored through the projectors of the product

$$X \times 1 \rightrightarrows Y \tag{3.168}$$

As $X \times 1$ is isomorphic to X itself, this will simply be some subobject of X, given by

$$X \times_Y 1 = \mathrm{eq}(f \circ \mathrm{pr}_1, p \circ \mathrm{pr}_2) \tag{3.169}$$

In other words, the subobject of X which, mapped through f, is the point p of Y.

Example 3.10.14 *As the name implies, the classic example of a fiber of a morphism is given by the case of the fiber of a bundle. If f is an epimorphism[?], such as in the case of a bundle space $\pi : E \to B$, the fiber $\mathrm{Fib}(\pi, x) = E \times_B 1$ is the subobject $E_x \hookrightarrow E$ which projects down to the point x :*

$$\pi(\iota(E_x)) = x(1) \tag{3.170}$$

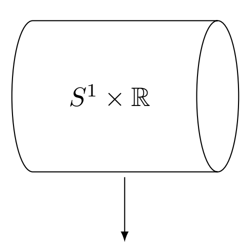
which is the property of the fiber in a bundle.

For instance, if we take the trivial circle bundle on the line, it is given by

$$\pi : S^1 \times \mathbb{R} \to \mathbb{R} \tag{3.171}$$

its fiber at any given point $x \in \mathbb{R}$ is the circle

$$S^1_x = \left\{(\theta, y) \in S^1 \times \mathbb{R} \mid \mathrm{pr}_2(\theta, y) = x\right\} \tag{3.172}$$



Example 3.10.15 *More generally, for any function on* **Set***,*

Example 3.10.16 *Another example of a fiber is given by additive categories, such as* **Vec** *or* **Grp***, where the fibers over the zero object are the kernels of the maps.*

$$\ker(f) = \mathrm{Fib}(f, 0) = f \times_0 1 \tag{3.173}$$

which are also given by the set of points mapped to the zero subspace by f.

Theorem 3.10.13 *The fiber of an isomorphism f is the terminal object*

$$\mathrm{Fib}_p(f) \cong 1 \tag{3.174}$$

Proof 3.10.10 *Pullback of isomorphisms*

Dually to the fiber is also the cofiber, the equivalent for the pushout.

Definition 3.10.13 *A* cofiber *of a morphism $f : X \to Y$ is the pushout of the span with the terminal object*

$$1 \longleftarrow X \xrightarrow{f} Y \tag{3.175}$$

$$\mathrm{Cofib}(f : X \to Y) = \tag{3.176}$$

Unlike the fiber, the cofiber does not depend on any base point as there is a unique morphism $X \to 1$.

Cokernels in additive categories

Example 3.10.17 *In an additive category such as* **Vec** *or* **Grp***, the cofiber of a map $f : X \to Y$ is the* cokernel*, it is the subobject of Y which is*

Theorem 3.10.14 *The fiber of the identity is the object :*

$$\mathrm{Fib}(\mathrm{Id}_X) = X \tag{3.177}$$

Theorem 3.10.15 *The cofiber of an epimorphism is the terminal object.*

Proof 3.10.11 *As pushouts preserve epimorphisms, the pointed object $1 \to$* Cofib(f) *is also an epimorphism. However, the only object for which a morphism out of the terminal object is an epimorphism in a topos is the terminal object itself[?].*

Theorem 3.10.16 *The fiber of the terminal morphism $!_X$ is the identity.*

Proof 3.10.12

Theorem 3.10.17 *The double fibration of an object is the terminal object.*

Proof 3.10.13 *Given a morphism $f : X \to Y$,*

$$\begin{array}{ccc} \mathrm{Fib}_p(f) & \xrightarrow{!} & 1 \\ {\scriptstyle p^*f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

If we try to further find the fiber of p^*f,

$$\begin{array}{ccccc} \mathrm{Fib}_q(\mathrm{Fib}_p(f)) & \longrightarrow & \mathrm{Fib}_p(f) & \xrightarrow{!} & 1 \\ \downarrow{\scriptstyle !} & & {\scriptstyle p^*f}\downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{q} & X & \xrightarrow[f]{} & Y \end{array}$$

As both squares are pullbacks, we have that the outer square is a pullback as well, so that the double fiber of f is the fiber product

$$\mathrm{Fib}_p(\mathrm{Fib}_p(f)) \cong 1 \times_Y 1 \cong \mathrm{eq}(p, f \circ q) \tag{3.178}$$

For any object Z and morphism $g : Z \to 1$, we therefore have that there must be a unique morphism $u : Z \to \mathrm{Fib}_p(\mathrm{Fib}_p(f))$ such that $\mathrm{eq} \circ u = m$. As there can be only one such morphism from Z to 1, this means $!_{\mathrm{eq}} \circ u = !_Z$. As this is true for any morphism, for it to be unique, we must simply have

$$\mathrm{Fib}_p(\mathrm{Fib}_p(f)) \cong 1 \tag{3.179}$$

[true?]

Dependent products and sums

The notion of dependent sum and product from type theory is in fact a variant of that of pullback and pushout[32].

Definition 3.10.14 *The dependent product $\prod_f$ for a morphism $f : X \to 1$ is defined as the pullback of*

Dependent product/sum, indexed objects

3.10.5 Directed limits

Given some directed set J forming our diagram, a directed limit is a limit of a functor $F : J \to \mathbf{C}$

Definition 3.10.15 *If the directed set J is furthermore a well-ordered set,*

$$\ldots \to 2 \to 1 \to 0 \tag{3.180}$$

the limit of this diagram is a sequential limit*.*

Example 3.10.18 *Given a sequence of inclusions*

$$\ldots \to X_2 \to X_1 \to X_0 \tag{3.181}$$

its sequential limit is the union over all indexes

$$\lim_J F = \bigcup_{i \in I} X_i \tag{3.182}$$

Proof 3.10.14

Example 3.10.19 *Given some directed set in* **Set** *ordered by inclusion, its directed limit is the union of all sets involved*

Projective limit

3.10.6 Properties of limits and colimits

Theorem 3.10.18 *For two diagrams I, J, and a functor $F : I \times J \to \mathbf{C}$,*

$$\mathrm{colim}_I \lim_J F \to \lim_J \mathrm{colim}_I F \tag{3.183}$$

Definition 3.10.16 *If we furthermore have the inverse morphism*

$$\lim_J \mathrm{colim}_I F \to \mathrm{colim}_I \lim_J F \tag{3.184}$$

we say that the limit and colimit commute.

The limits and colimits can also be used to define specific morphisms,

Definition 3.10.17 *An epimorphism $f : X \to Y$ is* effective *if it is the morphism of the kernel pair*

$$X \times_Y X \rightrightarrows X \xrightarrow{f} Y \tag{3.185}$$

"behaves in the way that a covering is expected to behave, in the sense that "Y is the union of the parts of X", identified with each other in some specified way"."

$$\coprod_i U_i \to U \tag{3.186}$$

Limits and colimits in functor categories

Theorem 3.10.19 *Given any limit or colimit of* $\mathbf{D}$*, then* $[\mathbf{C}, \mathbf{D}]$ *has the same limit or colimit, computed pointwise.*

Proof 3.10.15

3.10.7 Presentation of a category

While categories can, in set theoretical terms, be quite large, even larger than a set can accomodate, it is common for them to at least be able to be generated from smaller sets of objects. This is the *presentation* of a category.

Definition 3.10.18 *A localy small category is said to be* κ*-accessible if*

Example 3.10.20 *We can define a variety of subcategories of* **Set** *as the* κ*-accessible.* **FinSet** *is the*

Proof 3.10.16 *proof? For any set A, we have a directed diagram* $\mathrm{Sub}(A)$ *of its subset, and the finite subsets given by*

$$\mathrm{Sub}^{\aleph_0}(A) \tag{3.187}$$

3.10.8 Limits and functors

A common tool in category theory to use is the behavior of limits under the action of a functor.

Definition 3.10.19 *For a functor* $F : \mathbf{C} \to \mathbf{D}$ *and a diagram* $J : \mathbf{I} \to \mathbf{C}$*, a functor is said to preserve the limit* $\lim_J$ *if*

$$F \circ \lim_J \cong \lim_{F \circ J} \tag{3.188}$$

Preservation of limits and colimits

Left and right exact functors :

Definition 3.10.20 *A functor is* left exact *(resp.* right exact*)*

Maps inital objects to initial objects, products to products, and equalizers to equalizers

Even if we do not know the explicit limits and colimits of a category, we can verify that a functor preserves them, using the notion of a flat functor.

Definition 3.10.21 *A functor is* flat

[Flat functors preserve any limit and colimit]

Example 3.10.21 *The covariant hom-functor preserve limits :*

$$h^X(\lim F) = \lim(h^X F) \tag{3.189}$$

and the contravariant hom-functor preserves limits in the category $\mathbf{C}^{\mathrm{op}}$*, ie colimits in* $\mathbf{C}$ *:*

$$h_X() \tag{3.190}$$

Proof 3.10.17

Example 3.10.22 *In the category of vector spaces* **Vec***, the covariant hom functor* h^V *gives us the set of linear transformations* $L(V, -)$*. If we take a look at various cases, we have for the terminal object* k^0 *:*

$$h^V(k^0) = L(V, k^0) \tag{3.191}$$
$$= \{0\} \tag{3.192}$$
$$\cong \{\bullet\} \tag{3.193}$$

The product of two vector spaces is the direct sum

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \tag{3.194}$$

The kernel of a linear map f *can be described as the equalizer of this map with the* 0 *map, Equalizer : for* $f, 0 : X \to Y$*, the equalizer is* $\ker(f)$*. The two functions* f, g *map to*

$$h^V(f) = \{a \in L(X, Y)\} \tag{3.195}$$

$$h^V(0) = \{0\} \tag{3.196}$$

$$h^V(\ker(f)) = L(V, \ker(f)) \tag{3.197}$$
$$= \mathrm{eq}(h^V(f), h^V(0)) \tag{3.198}$$
$$= \mathrm{eq}(h^V(f), \{0\}) \tag{3.199}$$
$$= \ker() \tag{3.200}$$

$$h^V(f) = L(V, \ker(f)) \tag{3.201}$$
$$= \tag{3.202}$$

The contravariant one : initial object is also k^0, therefore terminal object in op

$$h_V(k^0) = L(k^0, V) \tag{3.203}$$
$$= \{0\} \tag{3.204}$$
$$\cong \{\bullet\} \tag{3.205}$$

Same deal with the coproduct in op, which is also $\oplus$

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \tag{3.206}$$

but $\lim(h^V \circ *) = \lim \varnothing = *$

Example 3.10.23 *In the category of sets, terminal object :*

$$h^X(\{\bullet\}) = \{!_X\} \tag{3.207}$$

Product :

$$h^X(Y \times Z) = h^X(Y) \times h^X(Z) \tag{3.208}$$

Definition 3.10.22 *Given a functor $F : \mathbf{C} \to \mathbf{D}$, and a diagram $J : \mathbf{I} \to \mathbf{C}$, we say that F reflects the limits of J if for a cone $\eta : \Delta_X \to \mathbf{J}$ over J in $\mathbf{C}$ for which $F(\eta)$ is a limit of $F \circ J$ in $\mathbf{D}$, then η was already a limit of J in $\mathbf{C}$.*

Created limits

3.11 Monoidal categories

It is common in categories to have some need of defining a binary operation, some function of the type

$$A \cdot B = C \tag{3.209}$$

Given two objects $A, B \in \mathbf{C}$, we want to find a third object C, such that there exists a bifunctor $(-) \cdot (-) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$

$$A \cdot B = C \tag{3.210}$$

While the notion of bifunctor covers this well enough, we often need to have additional conditions. A common case is that of a *monoid*, usually denoted as $\otimes$, where we ask that the operation be associative and unital, so that for any triple of objects A, B, C, we have the isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C) \tag{3.211}$$
$$\exists I \in \mathbf{C},\ I \otimes A \cong A \otimes I \cong A \tag{3.212}$$

To categorify this notion, we define monoidal categories

Definition 3.11.1 *A* monoidal category $(\mathbf{C}, \otimes, I)$ *is a category* $\mathbf{C}$*, a bifunctor* $\otimes$*, and a specific object* $I \in \mathbf{C}$*, along with three natural transformations :*

$$a : ((-) \otimes (-)) \otimes (-) \overset{\cong}{\to} (-) \otimes ((-) \otimes (-)) \tag{3.213}$$
$$(X \otimes Y) \otimes Z \mapsto X \otimes (Y \otimes Z) \tag{3.214}$$
$$\lambda : (I \otimes (-)) \overset{\cong}{\to} (-) \tag{3.215}$$
$$I \otimes X \mapsto X \tag{3.216}$$
$$\rho : ((-) \otimes I) \overset{\cong}{\to} (-) \tag{3.217}$$
$$(X \otimes I) \mapsto X \tag{3.218}$$

called the associator*, the* left unitor *and the* right unitor*, which obey the following rules, the triangle identity :*

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{Id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{Id}_X \otimes \lambda_Y} & \\ & X \otimes Y & \end{array}$$

and the pentagonal identity :

$$\begin{array}{ccccc} W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{k} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{k} & ((W \otimes X) \otimes Y) \otimes Z)) \\ \downarrow {\scriptstyle I_W \otimes a_{X,Y,Z}} & & & & \downarrow {\scriptstyle f_i} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\qquad a_{W,X \otimes Y,Z} \qquad} & & W \otimes (X \otimes Y)) \otimes Z \end{array}$$

We do not ask the equality for the associator and unitors, as equivalence is typically what we ask in general for a category. If in addition those equivalences are equality, we say that this is a *strict monoidal category.*

Example 3.11.1 *The tensor product of two k-vector spaces is a monoid in* $\mathbf{Vect}_k$*, as* $(\mathbf{Vec}_k, \otimes, k)$

Proof 3.11.1 *First we need to show that the tensor product is bifunctorial. Given two morphisms $f : X \to X'$, $g : Y \to Y'$, the product $f \otimes g : X \otimes Y \to X' \otimes Y'$*

Proof that it is functorial, unit k, associator, unitor, obeys the identities

Given the tensor product with k, consider the map

$$\lambda : k \otimes V \quad \to \quad V \tag{3.219}$$

$$(a, v) \quad \mapsto \quad av \tag{3.220}$$

A basic example of monoidal structures on a category is the product and coproduct. If a category $\mathbf{C}$ has all finite products, it is automatically a monoidal category, called a *Cartesian monoidal category*, given by $(\mathbf{C}, \times, 1)$.

Likewise, a category with all finite coproducts is called a *co-Cartesian monoidal category*, given by $(\mathbf{C}, +, 0)$

Example 3.11.2 *The product and coproduct are both monoidal in* **Set***, leading to the monoidal category* $(\mathbf{Set}, \times, \{\bullet\})$.

A weaker notion to this is that of a *semi-Cartesian* monoidal category, which is when the unit of the tensor product is the terminal object. This is a useful notion to keep in mind due to this example :

Example 3.11.3 *The category of Poisson spaces,* **Poiss***, is a semi-Cartesian category.*

This will have consequences on the differences between classical and quantum mechanics later on.

Definition 3.11.2 *A* bimonoidal category

3.11.1 Braided monoidal category

Definition 3.11.3 *A* braided monoidal category $(\mathbf{C}, \otimes, I, B)$ *is a monoidal category equipped with a natural isomorphism B between the functors*

$$B : X \otimes (-) \to (-) \otimes X \tag{3.221}$$

Definition 3.11.4 *A monoidal category is* symmetric *if it is a braided monoidal category obeying the commutation law*

$$B_{Y,X} \circ B_{X,Y} \cong \mathrm{Id}_{X \otimes Y} \tag{3.222}$$

Example 3.11.4 *The category of vector spaces over a field k $\mathbf{Vec}_k$ is a symmetric monoidal category.*

3.11.2 Monoids in a monoidal category

3.11.3 Category of monoids

Definition 3.11.5 *Given a monoidal category $(\mathbf{C}, \otimes, I)$, the category of monoids* $\mathrm{Mon}(\mathbf{C})$ *is the category with objects*

3.12 The interplay of mathematics and categories

As a field meant to emulate the behavior of mathematical structures, there are many interactions between any given mathematical structure and category theory.

3.12.1 Decategorification

Definition 3.12.1 *For a small category $\mathbf{C}$, its decategorification $K(\mathbf{C})$ is its set of objects quotiented by isomorphisms, or equivalently, the object set of its skeleton.*

In terms of internal operation, we can say that decategorification is a functor from the category of categories to that of sets, given by

$$K : \mathbf{Cat} \to \mathbf{Set} \tag{3.223}$$

$$\mathbf{C} \mapsto \mathrm{Obj}(\mathrm{Sk}(\mathbf{C})) \tag{3.224}$$

Furthermore, structures on the category itself may lead to structures on the set. For instance, given any multifunctor on the category

$$F : \prod_i \mathbf{C}_i \to \mathbf{D} \tag{3.225}$$

we have an equivalent function between the decategorified sets,

$$f(c_1, c_2, \ldots, c_n) = d \tag{3.226}$$

Example 3.12.1 *The categorification of a monoidal category $(\mathbf{C}, \otimes, I)$ is a monoid $(M, \cdot, e)$ with $M = K(\mathbb{C})$,*

$$K(X \otimes Y) = K(X) \cdot K(Y) \tag{3.227}$$

and

$$K(I) = e \tag{3.228}$$

Example 3.12.2 *The decategorification of the category of finite sets is the set of integers,*

$$K(\mathbf{FinSet}) = \mathbb{N} \tag{3.229}$$

Its monoidal category given by the coproduct is the addition, while the monoidal category of the product is the multiplication, with $K(\{\bullet\}) = 1$ and $K(\varnothing) = 0$, and more generally for any set

$$K(X) = |X| \tag{3.230}$$

The rules of arithmetic are given by the distributivity of the product and coproduct in sets

3.12.2 Categorification

The reverse process of decategorifcation is that of *categorification.* If we view decategorigication as a functor

$$K : \mathbf{Cat} \quad \to \quad \mathbf{Set} \tag{3.231}$$
$$\mathbf{C} \quad \mapsto \quad \mathrm{Obj}(\mathrm{Sk}(\mathbf{C})) \tag{3.232}$$

Then a simple way to view categorification is merely as some functor in the opposite direction,

$$F : \mathbf{Set} \quad \to \quad \mathbf{Cat} \tag{3.233}$$

Such that F is the right inverse of K : $K \circ F = \mathrm{Id}$, if we decategorify our categorification, we should end up on our original set. This notion of categorification is known as *vertical categorification.*

With our earlier example of **FinSet**,

Horizontal categorification

3.12.3 Internalization

If a category admits set-like properties, typically properties such as finite limits, monoidal structures or Cartesian closedness, it is possible to recreate many types of mathematical structures inside the category itself. This is called an *internalization* of the structure.

3.12.4 Algebras

One particular type of categorification is that of the categorification of universal algebras. In this context, "algebra" is to be understood as a theory of operations of arbitrary arities and not in the sense of a binary operation.

Definition 3.12.2 *Given an endofunctor $F \in \mathrm{End}(\mathbf{C})$, an* algebra (X, α) *of F is composed of an object $X \in \mathbf{C}$ and a morphism $\alpha : F(X) \to X$, the* carrier *of the algebra.*

Definition 3.12.3 *An algebra homomorphism on $\mathbf{C}$ between two algebras (X, α) and (Y, β) is defined by a morphism $m : X \to Y$ such that the following square commute.*

Theorem 3.12.1 *The algebras of a category along with their homomorphisms form a category*

Example 3.12.3 *If we take the functor associating to each object its successor in* **Set***,*

$$S(X) = X + \{\bullet\} \tag{3.234}$$

corresponding to X with a new element, and which maps morphisms to morphisms identical on all elements of X but preserving the new element,

$$S(f) : S(X) \to S(Y) \tag{3.235}$$

with

$$S(f)(x) = \begin{cases} \{\bullet_Y\} & x = \{\bullet_X\} \\ f(x) & x \neq \{\bullet_X\} \end{cases} \tag{3.236}$$

S-algebras are then given by some set X and function

$$\alpha : X + \{\bullet\} \to X \tag{3.237}$$

By the universal property of the coproduct, α can be written as some pair of function (f, g) such that f is a function $X \to X$ while g is a function $1 \to X$. We will write them as (z, s). This means that any S-algebra is defined by some element in X and some endomorphism.

Given two such S-algebras,

This category of S-algebras has an initial object N, ie some object for which there is only one S-algebra homomorphism from N to any other, meaning that for any other algebra $(X, (z', s'))$, we need

$$\begin{array}{ccc} S(N) & \xrightarrow{(z,s)} & N \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle f} \\ S(X) & \xrightarrow{(z',s')} & X \end{array}$$

So that we need for an arbitrary map $(z', s') : X + \{\bullet\} \to X$ that, composed with the morphism $S(f)$ which acts identical to f on X but preserves the extra element, that

$$\forall n \in N,\ (z', s')(S(f)(n)) = f(\text{succ}(n)) \tag{3.238}$$

now if we pick $n \in N$, the copairing (z, s) will simply be s, and the successor functor $S(f)$ will have the same behaviour as f, so that we get

$$f(s(n)) = f(s'(n)) \tag{3.239}$$

but if we pick $\bullet \in S(N)$, we get $z(\bullet)$ and the successor functor $S(f)(\bullet_N) = \bullet_X$

$$f(z'(\bullet)) = f(\bullet_X) \tag{3.240}$$

For f to be unique,

The initial object in the category of S-algebras is the set of natural numbers. [Because $S(N) = N$? and it's the smallest set to do so or something?]

3.12.5 Monoid in a monoidal category

The most generic form of internalization is that of an internal monoid (an even more general case would be that of a magma but we will not look at it here). Given some object M in the category, we would like to define what it means for M to be a monoid.

Definition 3.12.4 *In a monoidal category $(\mathbf{C}, \otimes, I)$, a monoid in $\mathbf{C}$ is given by a triplet (M, m, e), of an object $M \in \mathbf{C}$, a morphism $m : M \otimes M$, and a morphism $e : I \to M$, such that they make the following diagrams commute :*

All internal objects we will see as we go on will be some variation of a monoid in a monoidal category, most of them being more specifically Cartesian monoids, where the monoid will be that of the product, $(\mathbf{C}, \times, 1)$.

A monoid M can always be defined as a category itself, via the construction of a category $\mathbf{M}$ with a single object $\bullet$, and such that the endomorphisms of this object are isomorphic to the monoid as a set,

$$\mathrm{End}(\bullet) \cong |M| \tag{3.241}$$

in which the monoid operation is implemented by the composition rules of those morphisms.

3.12.6 Internal group

Definition 3.12.5 *In a category $\mathbf{C}$ with finite products, a group object G is an object $G \in \mathbf{C}$ equipped with the morphisms*

- *The unique map to the terminal object $p : G \to 1$*
- *A neutral element morphism from the terminal object : $e : 1 \to G$*
- *An inverse endomorphism : $(-)^1 : G \to G$*
- *A binary morphism on the product : $m : G \times G \to G$*

such that all the following diagrams commute

If we want to write it out explicitly, the group object is not merely the object G itself but the quintuple $(G, p, e, (-)^{-1}, m)$.

Example 3.12.4 *Every category with finite product has the trivial group object $\{e\}$ which is the terminal object and the unique map to itself, so that*

$$\{e\} \cong (1, \mathrm{Id}_1, \mathrm{Id}_1, \mathrm{Id}_1, \mathrm{pr}_1) \tag{3.242}$$

Example 3.12.5 *As groups can be defined using sets, the category of sets contains every group as group objects using the traditional definition of groups.*

We can work out an explicit example of this. The set of two elements 2 has two functions $f : 1 \to 2$, which are the functions mapping to the first and second element, and has four endomorphisms, corresponding to all unary boolean functions (the constant functions, the identity and the involution $\bullet_0 \to \bullet_1$, $\bullet_1 \to \bullet_0$, equivalent to the negation). The product object $2 \times 2 \cong 4$ has 16 functions, corresponding to all the binary boolean functions.

A possible model of the internal group $\mathbb{Z}_2$ can be defined as an assignment of $\bullet_0$ to the identity 1 and $\bullet_1$ to the element -1, the inverse map as the involutive function $2 \to 2$, and the multiplication map being the equivalent of the XNOR function

$$f(\bullet_0, \bullet_0) = \bullet_1 \tag{3.243}$$
$$f(\bullet_1, \bullet_0) = \bullet_0 \tag{3.244}$$
$$f(\bullet_0, \bullet_1) = \bullet_0 \tag{3.245}$$
$$f(\bullet_1, \bullet_1) = \bullet_1 \tag{3.246}$$

$$\mathbb{Z}_2 \cong (2, e, \neg, \mathrm{XNOR}) \tag{3.247}$$

Example 3.12.6 *The group objects in the category* Top *are the topological groups, where all group operations are continuous functions. This can be done for instance by considering the set of all such quintuples that are mapped to a group object by the forgetful functor to* **Set***.*

Example 3.12.7 *The group objects in the category of smooth manifolds are the Lie groups, where the group operations are smooth maps.*

As the notation $(G, p, e, (-)^{-1}, m)$ is fairly cumbersome, from now on internal groups will be simply defined by their object G, with all morphisms left implicit.

Theorem 3.12.2 *For a category* $\mathbf{C}$ *with an internal group* G*, the hom-set* $\mathrm{Hom}_{\mathbf{C}}(X, G)$ *for any object* X *has a group structure. [Abelian only?]*

Proof 3.12.1 *If we consider two morphisms* $f, g \in \mathrm{Hom}_{\mathbf{C}}(X, G)$ *in* **Set** *(in other words an element in the product of two hom-sets), there*

Definition 3.12.6 *In a category with internal groups, the left action of an internal group* G *on an object* X *is defined by a morphism*

$$\rho : G \times X \to X \tag{3.248}$$

such that the following diagrams commute :

As a generalization of internal groups, we also have the notion of an internal groupoid.

Definition 3.12.7 *An internal groupoid* $\mathcal{G}$ *in a category with products is given by three objects in a category* (E, G, X)*, with* X *the underlying space of the groupoid,* G *the and the following morphisms :*

- *The morphism $e : X \to G$ associating the unit element to the underlying object X*
- *The inverse morphism $(-)^{-1} : G \to G$*
- *The*

3.12.7 Internal ring

Definition 3.12.8 *In a category* **C** *with finite products, a* ring object *R is an object $R \in$* **C** *along with the following morphisms :*

- *The addition morphism $a : R \times R$*
- *The multiplication morphism $m : R \times R \to R$*
- *The addition identity morphism $0 : 1 \to R$*
- *The multiplication identity morphism $1 : 1 \to R$*
- *The additive inverse morphism : $-(-) : R \to R$*

such that all those morphisms make the following diagrams commute :

Example 3.12.8 *The real line object $\mathbb{R}$ in many categories is an internal ring, with the elements $0, 1 : 1 \to \mathbb{R}$, and the expected morphisms associated to it. Those morphisms are for instance smooth functions and continuous functions, so that both* **SmoothMan** *and* **Top** *have the internal ring of the real line.*

It is common for many categories to have some variant of the real line, as we can find such an object in **Set**, **Top**, **CartSp**, **SmoothMan**, etc.

3.13 Groupoids

In many contexts, the objects of a category can themselves be categories, such as the category of categories **Cat**, the category of groupoids **Grpd** (where the monoid structure on the elements of the groupoids is built from morphisms) and so on.

If a category of that sort is equipped with a terminal object 1 which represents a category of a single object **1**, such as is the case for groupoids for instance, and we have that the object set of some other object category **C** is given by the hom-set

$$\mathrm{Obj}(\mathbf{C}) \cong \mathrm{Hom}_{?}(\mathbf{1}, \mathbf{C}) \tag{3.249}$$

In those circumstances, we can consider the morphisms of those objects by functors preserving 1 and C, as a functor will map those objects $X, Y : 1 \to C$ to each other (associativity?)

From this, we also have that the natural transformations η between the functors (or here morphisms) $f, g \in \mathrm{Mor}(\mathbf{C})$

Definition 3.13.1 *A* groupoid $\mathcal{G}$ *is a pair of two sets* G_1, G_0*, where* G_0

3.14 Subobjects

Given an object X in a category $\mathbf{C}$, a *subobject* S of X is an isomorphism class of monomorphisms $\{\iota_i\}$

$$\iota_i : S_i \hookrightarrow X \tag{3.250}$$

so that the equivalence classes of $\{\iota_i\}$ is given by any two such monomorphisms if there exists an isomorphism between the two subobjects S_i;

$$S = [S_i]/(S_i \cong S_j \leftrightarrow \exists f : S_i \to S_j,\ \exists f^{-1} S_j \to S_j,\ f \circ f^{-1} = \mathrm{Id}) \tag{3.251}$$

Theorem 3.14.1 *In a skeletal category, the subobjects are equivalent to the monomorphisms.*

This is meant to define the common mathematical notion of an object being *part* of another object in some sense.

Example 3.14.1 *On* **Set***, subobjects are subsets (defined by injections up to the symmetric group), where*

Example 3.14.2 *On* $\mathbf{Vect}_k$*, subobjects are subspaces (defined by injections up to the general linear group?)*

So that a subobject is a *line*, $[k] \hookrightarrow V$, and not a specific line given by a specific linear map $k \hookrightarrow V$.

Example 3.14.3 *On* **Top***, subobjects are subspaces with the subspace topology (defined up to homeomorphisms)*

On **Ring**,

On **Grp**, subobjects are subgroups

Example 3.14.4 *On the category of smooth manifolds* **SmoothMan***, subobjects are* submanifolds, $\iota : S \hookrightarrow M$*, where the set of all submanifolds with the same image up to diffeomorphism of the base* $\mathrm{Diff}(S)$ *are equivalent.*

"Let C_c be the full subcategory of the over category C/c on monomorphisms. Then C_c is the poset of subobjects of c and the set of isomorphism classes of C_c is the set of subobjects of c. "

3.15 Simplicial categories

The simplicial category Δ is made of simplicial objects, which can be defined in a variety of equivalent ways, either as being finite total orders

[...]

or as being categories themselves for total orders

$$\left[\vec{\mathbf{0}}\right] = \{\bullet\} \tag{3.252}$$

$$\left[\vec{\mathbf{1}}\right] = \{\bullet \to \bullet\} \tag{3.253}$$

$$\left[\vec{\mathbf{2}}\right] = \{\bullet \to \bullet \to \bullet\} \tag{3.254}$$

$$\left[\vec{\mathbf{3}}\right] = \{\bullet \to \bullet \to \bullet \to \bullet\} \tag{3.255}$$

The simplicial morphisms are the order-preserving functions $[\vec{\mathbf{m}}] \to [\vec{\mathbf{n}}]$.

Only defined for $m \leq n$

Examples :

$$\mathrm{Hom}_\Delta([\vec{0}], [\vec{n}]) = n \tag{3.256}$$

This morphism maps the single point to the various objects of $[\vec{n}]$

$$\mathrm{Hom}_\Delta([\vec{1}], [\vec{n}]) = n \tag{3.257}$$

interpretation as embedding of simplices

Despite the name, the simplicial categories (and its overarching simplicial 2-category) is not made of simplices, but they will be of use later on to *construct* simplices in the category of simplicial sheaves.

3.16 Equivalences and adjunctions

Like many objects in mathematics, it is possible to try to define some kind of equivalence between two categories. Like most such things, we try to consider two mappings between our categories. Let's consider two categories $\mathbf{C}, \mathbf{D}$, and two functors F, G between them,

$$\mathbf{C} \underset{\leftarrow G -}{\overset{- F \rightarrow}{}} \mathbf{D}$$

The usual process of finding equivalent objects in such cases is to have those two maps be inverses of each other, ie $FG = \mathrm{Id}_{\mathbf{D}}$ and $GF = \mathrm{Id}_{\mathbf{C}}$. The composition functor FG maps every object and morphism of D to itself and likewise for GF on C, so that in some sense, the objects X and $F(X)$ are the same objects, and likewise, $f : X \to Y$ and $F(f) : F(X) \to F(Y)$ represent the same morphism.

If two categories admit such a pair of functors, we say that they are *isomorphic*. While this is the most obvious definition of equivalence, it is in practice not commonly used, as very few categories of interest are actually isomorphic, and this is generally considered too strict a definition in the philosophy of category theory, as we are generally more interested in the relationships between objects rather than the objects themselves. A good example of this overly strict definition is that the product of two objects done in a different order will not be isomorphic, or shifting a total order to the left and back right.

If we weaken the notion of equivalence, we can look at the case where our two functors are merely isomorphic to the identity, $FG \cong \mathrm{Id}_{\mathbf{D}}$ and $GF \cong \mathrm{Id}_{\mathbf{C}}$, where there exists a natural transformation η taking FG to $\mathrm{Id}_{\mathbf{D}}$ and ϵ taking $\mathrm{Id}_{\mathbf{C}}$ to GF (this ordering for the two will make more sense later on)

$$\eta : \mathrm{Id}_{\mathbf{C}} \to GF \tag{3.258}$$

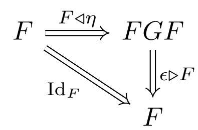
$$\epsilon : FG \to \mathrm{Id}_{\mathbf{D}} \tag{3.259}$$

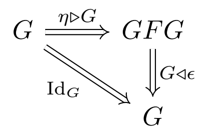
with η and ϵ both being two-sided inverses, that is, for all objects of either category, we have that η_X is an isomorphism, and likewise, ϵ_X is an isomorphism.

$$\mathbf{C} \qquad \mathbf{D}$$

Furthermore, we say that this equivalence is an *adjoint equivalence* if those natural transformations satisfy the *triangle identities* :

Definition 3.16.1 *The triangle identities for two functors F, G is given by the two commutative diagrams*





In other words, adding the "almost" identity functors FG or GF on one side by transforming the identity into them, before removing them on the other by transforming it back into the identity is an isomorphism.

$$(\epsilon \triangleright F) \circ (F \triangleleft \eta) = \mathrm{Id}_F \tag{3.260}$$
$$(G \triangleright \epsilon) \circ (\eta \triangleright G) = \mathrm{Id}_G \tag{3.261}$$

In terms of components, this means that for every objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, the components of the relevant natural transformations are

$$\epsilon_{F(Y)} \circ F(\eta_Y) = \mathrm{Id}_{F(Y)} \tag{3.262}$$
$$G(\epsilon_X) \circ \eta_{G(X)} = \mathrm{Id}_{G(X)} \tag{3.263}$$

Counterexample 3.16.1 *An equivalence that is not an adjoint equivalence can be seen in the case of two monoid categories* $\mathbf{M}_1$, $\mathbf{M}_2$ *with a single object each*

Now if we weaken the condition on η and ϵ, merely requiring them to be natural transformations rather than natural isomorphisms, although still obeying the triangle identity, this is what is called an *adjunction* of functors. The adjunction of F and G is denoted by $(F \dashv G)$, where F is called the *left adjoint*, while G is the *right adjoint*, with η and ϵ called the *unit* and *counit* respectively. The diagram for this will be given as

$$(F \dashv G) : \mathbf{C} \; \underset{-\,G\,\rightarrow}{\overset{\leftarrow\,F\,-}{\rightleftarrows}} \; \mathbf{D}$$

where the left functor F will always be on top of its right adjoint G.

Adjoint categories are not (adjoint) equivalent, they do not share the same objects and morphisms, even up to equivalence

Applied to a morphism :

For some morphism $f \in \mathbf{C}$,

$$L(f) : L(X) \to L(Y) \tag{3.264}$$

For some morphism $g \in \mathbf{D}$,

$$R(g) : R(V) \to R(W) \tag{3.265}$$

$$\epsilon_{L()} \tag{3.266}$$

Definition 3.16.2 *Given two objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, with an adjunction $(L \dashv R)$, we say that a morphism $f \in \mathbf{D}$ and $g \in \mathbf{C}$ are* adjunct *if f is of the form*

$$f : L(X) \to Y \tag{3.267}$$

and g

$$g : X \to R(Y) \tag{3.268}$$

such that

$$g = R(f) \circ \eta_X \tag{3.269}$$

Theorem 3.16.1 *Conversely, we have*

$$f = \epsilon_Y \circ L(g) \tag{3.270}$$

Proof 3.16.1 *Given the equality*

$$g = R(f) \circ \eta_X \tag{3.271}$$

We have

$$\begin{aligned} \epsilon_Y \circ L(g) &= \epsilon_Y \circ L(R(f) \circ \eta_X) & (3.272) \\ &= \epsilon_Y \circ (L \circ R)(f) \circ L(\eta_X) & (3.273) \\ &= \epsilon_Y \circ \epsilon(\mathrm{Id}_{\mathbf{D}})(f) \circ L(\eta_X) & (3.274) \\ &= f \circ \epsilon_{LRL(X)} \circ L(\eta_X) & (3.275) \end{aligned}$$

Theorem 3.16.2 *For two adjoint functors $(L \dashv R) : \mathbf{C} \rightleftarrows \mathbf{D}$, the triangle identities are equivalent to the identity*

$$\mathrm{Hom}_{\mathbf{D}}(L(-), -) \cong \mathrm{Hom}_{\mathbf{C}}(-, R(-)) \tag{3.276}$$

so that for any object $X \in \mathbf{C}$ and $Y \in \mathbf{D}$,

$$\mathrm{Hom}_{\mathbf{D}}(L(X), Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, R(Y)) \tag{3.277}$$

Proof 3.16.2

Example 3.16.1 *Many categories have what are called* free-forgetful adjunctions*, where we consider the forgetful functor U to some other category (the prototypical example being* Set*, sending a structure to its underlying set), and the left adjoint to that functor F being its left adjoint, called the* free functor*, which constructs an object of this category that is in some sense the "best approximation" of the set from the left, that is, for every other object X in the category, morphisms of the form $FS \to X$ are exactly the image of the functions $S \to UX$, but not the other way around.*

These are for instance given by the free topological space on a set (the discrete space on that set), where every function from a discrete space as a space is also a continuous function, the free group FS, where every function from the generators to another group is also a group homomorphism, or the free vector space, where all functions from its generators to another vector space are linear functions.

Example 3.16.2 *A basic non-trivial example of adjoint functors is the even and odd functors. If we consider $\mathbb{Z}$ as a linear order category, with $\leq$ as its morphisms, functors are its order-preserving functions. Two specific functors that we have are the even and odd functors, give by*

$$\forall k \in \mathbb{Z},\ \mathrm{Even}(k) = 2k,\ \mathrm{Odd}(k) = 2k+1 \tag{3.278}$$

These do indeed preserve the order so that for the unique morphism $k_1 \leq k_2$, it is mapped to the unique morphism $2k_1 \leq 2k_2$ and similarly for the odd functor. The corresponding "inverse" functor is the functor mapping any integer to the floor of its division by 2 :

$$\lfloor -/2 \rfloor : \mathbb{Z} \to \mathbb{Z} \tag{3.279}$$

Even *is then the left adjoint and* Odd *the right adjoint of $\lfloor -/2 \rfloor$. The unit and counit of* Even *are :*

$$\varepsilon_l = f \circ \lfloor -/2 \rfloor \tag{3.280}$$

[...]

This is however not an equivalence, as the floor function is not strictly an inverse of the even and odd functor (and not being a faithful functor to begin with), as $\lfloor (2n)/2 \rfloor = \lfloor (2n+1)/2 \rfloor$, and we have

Example 3.16.3 *Take the two linear order categories of $\mathbb{Z}$ and $\mathbb{R}$, with their elements being the objects and their order relations are the morphisms. The inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$, mapping $n \in \mathbb{Z}$ to its equivalent real number, is a functor*

We can try to define left and right adjoints for it, by finding two functions $f, g : \mathbb{R} \to \mathbb{Z}$ for which there exists

- *A left and right counit $\epsilon_l : f\iota \to \mathrm{Id}_{\mathbb{Z}}$ and $\epsilon_r : \iota g \to \mathrm{Id}_{\mathbb{R}}$*

- *A left and right unit* $\eta_l : \mathrm{Id}_{\mathbb{R}} \to \iota f$ *and* $\eta_r : \mathrm{Id}_{\mathbb{Z}} \to g\iota$

And all these must obey the triangle identities

If we take for instance the left adjoint, we need that our function f *be such that there exists a natural transformation between the identity on* $\mathbb{R}$ *(*$\mathrm{Id}_{\mathbb{R}}(x) = x$*) and our function* f *reinjected into* $\mathbb{R}$ *:* $\iota(f(x))$*. For every* $x, y \in \mathbb{R}$*, there's a morphism*

$$\begin{array}{c} x \\ \Big\downarrow{\scriptstyle \leq} \\ y \end{array} \overset{\eta}{\Longrightarrow} \begin{array}{ccc} x & \xrightarrow{\eta_{l,x}} & \iota(f(x)) \\ \Big\downarrow{\scriptstyle \leq} & & \Big\downarrow{\scriptstyle \iota(f(\leq))} \\ y & \xrightarrow{\eta_{l,y}} & \iota(f(y)) \end{array}$$

In the context of our linear order, this means that for any two numbers x, y *such that* $x \leq y$*, we have* $x \leq \iota(f(x))$*,* $y \leq \iota(f(y))$ *and* $\iota(f(x)) \leq \iota(f(y))$

$$x \leq y \leq \iota(f(y)) \tag{3.281}$$

and for the counit, we need a natural transformation between the mapping of an integer into $\mathbb{R}$ *and then back into* $\mathbb{Z}$ *via* f *with* $f(\iota(n))$*, and the identity on* $\mathbb{Z}$*,* $\mathrm{Id}_{\mathbb{Z}}(n) = n$*. For every* n, m*,* $n \leq m$*,*

$$\begin{array}{c} n \\ \Big\downarrow{\scriptstyle \leq} \\ m \end{array} \overset{\epsilon}{\Longrightarrow} \begin{array}{ccc} f(\iota(n)) & \xrightarrow{\epsilon_{r,n}} & n \\ \Big\downarrow{\scriptstyle f(\iota(\leq))} & & \Big\downarrow{\scriptstyle \leq} \\ f(\iota(m)) & \xrightarrow{\epsilon_{r,m}} & m \end{array}$$

We have the condition that if we inject n *into* $\mathbb{R}$*, its left adjoint will be such that* $f(\iota(n)) \leq n$*,* $f(\iota(m)) \leq m$ *and* $f(\iota(n)) \leq f(\iota(m))$*. If we take the case* $m = n+1$ *and ignoring the injection* ι *for now, this means that* $f(n) \leq f(n+1) \leq n+1$ *and* $f(n) \leq n$*. As* $f(n) \leq n+1$ *cannot be equal to* $n+1$*,* $f(n)$ *can only be equal to* n *or smaller. If we pick the case* $n-1 \leq n$ *instead, the natural transformation implies* $f(n-1) \leq f(n) \leq n$ *and* $f(n-1) \leq n-1 \leq n$*, so that* $f(n-1) < n$ *and*

Triangle identities : for any $n \in \mathbb{Z}$*,*

$$\mathrm{Id}_{\iota(n)} = \iota(\epsilon_{l,n}) \circ \eta_{l,\iota(n)} \tag{3.282}$$

The components of this natural transformations give us that, if we transform our integer n *to a real and back,*

$$\begin{array}{c} x \\ \Big\downarrow{\scriptstyle \leq} \\ y \end{array} \overset{\eta}{\Longrightarrow} \begin{array}{ccc} x & \xrightarrow{\eta_{l,x}} & \iota(f(x)) \\ \Big\downarrow{\scriptstyle \leq} & & \Big\downarrow{\scriptstyle \iota(f(\leq))} \\ y & \xrightarrow{\eta_{l,y}} & \iota(f(y)) \end{array} \overset{\epsilon}{\Longrightarrow}$$

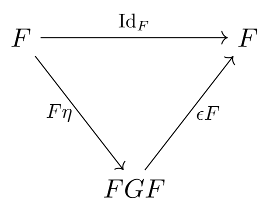
Different definitions of adjunction

Adjoint functors : For two functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$, the functors form an *adjoint pair* $F \dashv G$, F the left adjoint of G and G the right adjoint of F, if there exists two natural transformations, η and ϵ

$$\eta : \mathrm{Id}_{\mathbf{C}} \to G \circ F \tag{3.283}$$

$$\epsilon : F \circ G \to \mathrm{Id}_{\mathbf{D}} \tag{3.284}$$

obeying the triangle equalities



Adjunct : for an adjunction of functors $(L \dashv R) : \mathbf{C} \leftrightarrow \mathbf{D}$, there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(LX, Y) \cong \mathrm{Hom}_{\mathbf{D}}(X, RY) \tag{3.285}$$

Two morphisms $f : LX \to Y$ and $g : X \to RY$ identified in this isomorphism are *adjunct*. g is the right adjunct of f, f is the left adjunct of g.

$$g = f^\sharp,\ f = g^\flat \tag{3.286}$$

Adjunction for vector spaces

Galois connection

Example 3.16.4 *A common type of adjunction (although typically not a formal one) is the free-forgetful adjunction, where given some forgetful functor $U : \mathbf{C} \to \mathbf{D}$, we have the left adjoint of the free functor $F : \mathbf{D} \to \mathbf{C}$, with the adjunction $(F \dashv U)$. In particular, for the case of a concrete category where the forgetful functor maps to* **Set***,*

Theorem 3.16.3 *Given an adjunction*

$$(L \dashv R) \tag{3.287}$$

L preserves any colimits, while R preserves any limits.

Proof 3.16.3 *As the hom-functor preserves limits, we have that*

$$\mathrm{Hom}_{\mathbf{C}}() \cong \mathrm{Hom}_{\mathbf{C}}() \tag{3.288}$$

Example 3.16.5 *Given some free-forgetful adjunction, such as some concrete category with $U : \mathbf{C} \to \mathbf{Set}$, the free functor preserves colimits while the forgetful functor preserves limits. For* **Top** *: $(\mathrm{Disc} \dashv \Gamma)$, The terminal topological space (point) has a single element :*

$$\Gamma(1_{\mathbf{Top}}) = \{\bullet\} \tag{3.289}$$

The free space from the initial set is the empty space

$$\mathrm{Disc}(\varnothing) = 0_{\mathbf{Top}} \tag{3.290}$$

The product of two spaces has the point set of the product of two sets The free space on the sum of two sets is the coproduct of the two free spaces

Overall, we typically expect the forgetful functor to have a left adjoint free functor, as those free spaces behave like the underlying category with respect to the preservation of colimits. Given some forgetful functor to sets, we expect the free object $F(X)$ to preserve coproducts. The terminal set $\{\bullet\}$ leads to the the coproduct of two sets, a set of cardinality $|A| + |B|$, leads to a free group

Example 3.16.6 *For a functor between two poset categories, the adjoints of those functors are Galois connection.*

Theorem 3.16.4 *A right adjoint functor R preserves monomorphisms, so that*

$$f \in \mathrm{Mono} \to R(f) \in \mathrm{Mono} \tag{3.291}$$

Proof 3.16.4

3.16.1 Dual adjunctions

A dual adjunction between two contravariant functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ is given by a pair of natural transformations

$$\eta : \mathrm{Id}_{\mathbf{C}} \to GF \tag{3.292}$$

$$\theta : \mathrm{Id}_{\mathbf{D}} \to FG \tag{3.293}$$

which obey the triangle equalities $F\eta \circ \theta F = \mathrm{Id}_F$ and $G\theta \circ \eta G = \mathrm{Id}_G$:

$$\begin{array}{ccc} F & \xrightarrow{\theta F} & FGF \\ & \searrow^{\mathrm{Id}_F} & \downarrow{\scriptstyle F\eta} \\ & & F \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{\mathrm{Id}_G} & \downarrow{\scriptstyle G\theta} \\ & & G \end{array}$$

this is a notion that is simply the adjunction of the two covariant functors

$$F : \mathbf{C} \to \mathbf{D}^{\mathrm{op}} \tag{3.294}$$
$$D : \mathbf{D}^{\mathrm{op}} \to \mathbf{C} \tag{3.295}$$

This notion will be mostly useful for the notion of duality of two categories.

3.16.2 Kan extension

A lot of concepts in category theory are fundamentally about finding a commutative triangle of functors.

Definition 3.16.3 *Given two functor $F : \mathbf{C} \to \mathbf{D}$ and $p : \mathbf{C} \to \mathbf{C}'$, let's consider the induced functor*

$$p^* : [\mathbf{C}', \mathbf{D}] \to [\mathbf{C}, \mathbf{D}] \tag{3.296}$$

given by composition,

$$p^*(h : \mathbf{C}' \to \mathbf{D}) = h \circ p \tag{3.297}$$

its left Kan extension *is a functor* $\mathrm{Lan}_p F : \mathbf{C}' \to \mathbf{D}$*, given by the left Kan extension operation*

Example 3.16.7 *The definition of limits and colimits we have seen is the right and left Kan extension of the functor $F : I \to \mathbf{C}$ where $\mathbf{C}'$ is the terminal category $\mathbf{1}$, so that $p : I \to \mathbf{1}$ is simply the constant functor. Its induced functor is then the functor $p^* : [I, \mathbf{C}] \to [\mathbf{1}, \mathbf{C}]$ which maps diagrams on $\mathbf{C}$ to some functor selecting a unique element of $\mathbf{C}$. If we apply it with some object $\bullet$ of the diagram, we find*

$$p^*(F)(\bullet) = F(\Delta_*(\bullet)) \tag{3.298}$$

3.17 Concrete categories

We have seen quite a few times the notion of a category being composed of sets with functions between them, so let's now formalize this notion internally to category theory.

Definition 3.17.1 *A category* **C** *is* concretizable *if there exists a faithful functor*

$$U : \mathbf{C} \to \mathbf{Set} \tag{3.299}$$

A concrete category is then a pair of the category **C** with a specific such functor. Every object X in a concrete category then has an *underlying set* $U(X)$ which we identify in some sense with the object itself. The faithfulness condition lets us associate to every morphism $f : X \to Y$ a unique function on their underlying sets, $U(f) : U(X) \to U(Y)$.

Theorem 3.17.1 *The opposite of a concrete category is concrete.*

Proof 3.17.1

Stuff, structure, and properties [20]

3.18 Pointed objects

In a category **C** with a terminal object, a *pointed object* is an object X of a category along with a specific morphism $1 \to X$, called the *basepoint*. This can be described by an object of the coslice category $\mathbf{C}^{1/}$

Example 3.18.1 *The category of pointed sets is the category of pointed objects of* **Set** *:*

$$\mathbf{Set}_* = \mathbf{Set}^{1/} \tag{3.300}$$

where objects are the pointed sets $x : 1 \to X$ of sets X with basepoint x, and the morphisms

As with any coslice category, there is a forgetful functor

$$U : \mathbf{C}_* \to \mathbf{C} \tag{3.301}$$

which simply gives back the original object,

$$U(p : 1 \to X) \cong X \tag{3.302}$$

3.19 Ends and coends

An end is meant to be a categorification of a

First we need to define the notion of an extranatural transformation. It is a generalization of a natural transformation where we deform the naturality condition.

Example : Given the hom-bifunctor $\hom_{\mathbf{Set}} : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \to \mathbf{Set}$, take the identity transformation

$$\mathrm{Id}_{\hom} : \hom_{\mathbf{Set}} \to \hom_{\mathbf{Set}} \tag{3.303}$$

Its components are of the form

$$\mathrm{Id}_{X,Y} : X^Y \to X^Y \tag{3.304}$$

Naturality in Y : for any $g : Y \to Y'$

$$\mathrm{Id}_{X,Y} X^g = X^g \mathrm{Id}_{X,Y'} : X^{Y'} \to X^Y \tag{3.305}$$

[diagram]

Definition 3.19.1 *For two functors of the form*

$$F : \mathbf{A} \times \mathbf{B} \times \mathbf{B}^{\mathrm{op}} \to \mathbf{D} \tag{3.306}$$

and

$$G : \mathbf{A} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{D} \tag{3.307}$$

an extranatural transformation *is*

Definition 3.19.2 *Given a functor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{D}$, a* wedge $e : X \to F$

in $\mathbf{D}$ is an object $X \in \mathbf{D}$ and a universal extranatural transformation

$$\theta : \mathbf{D} \to F \tag{3.308}$$

Example 3.19.1 *If we consider some total orders such as $\mathbb{R} \times \mathbb{R}^{\mathrm{op}} \to \mathbb{R}$, a functor will here be an order-preserving function in the first variable and order-reversing in the second. The end of such a function will therefore be some value $x \in \mathbb{R}$. A natural transformation between two such functions $\eta : f \to g$ on this will have components obeying*

$$\begin{array}{ccc} (x_1, x_2) & f(x_1,x_2) \xrightarrow{\eta_x} g(x_1,x_2) \\ \downarrow{\scriptstyle (\leq,\geq)} & \downarrow{\scriptstyle f(\leq)} \qquad\qquad \downarrow{\scriptstyle g(\leq)} \\ (y_1,y_2) & f(y_1,y_2) \xrightarrow[\eta_y]{} g(y_1,y_2) \end{array}$$

Example 3.19.2 *Given the category of finite dimensional spaces over a field k, $\mathbf{FVect}_k$, consider the functor*

$$F : \mathbf{FVect}_k \times \mathbf{FVect}_k^{\text{op}} \rightarrow \mathbf{FVect}_k \tag{3.309}$$

$$(V, W) \mapsto V \otimes W* \tag{3.310}$$

$$\int_{V \in \mathbf{FVect}_k} F(V, V) \cong k \tag{3.311}$$

Structure map :

$$\epsilon_V : V \otimes V^* \rightarrow k \tag{3.312}$$

$$(v, \omega) \mapsto \omega(v) \tag{3.313}$$

Coend of $\text{Hom}_{\mathbf{FVect}_k}(W, V) \cong V^* \otimes W$

Example 3.19.3 *Given a metric space (X, d) considered as an $\mathbb{R}_+$-enriched category, where points $x \in X$ are objects and morphisms correspond to the distance between two points. Take some*

3.20 Grothendieck construction

Grothendieck fibration

Definition 3.20.1 *Given a functor $F : \mathbf{C} \rightarrow \mathbf{Cat}$, the Grothendieck construction of F, denoted by $\int_{\mathbf{C}} F$, is the category with objects the pairs*

$$(c, x) \in \text{Obj}(\mathbf{C}) \times \text{Obj}(F(c)) \tag{3.314}$$

and whose morphisms are given by pairs of morphisms (f, ϕ) acting on objects as

$$f : c \rightarrow c' \tag{3.315}$$

$$\phi : F(f)(a) \rightarrow a' \tag{3.316}$$

Example 3.20.1 *For $X \in \mathbf{C}$, given a representable functor*

$$\mathrm{Hom}_{\mathbf{C}}(-, X) : \mathbf{C} \to \mathbf{Set} \tag{3.317}$$

its Grothendieck construction is the slice category $\mathbf{C}_X$:

$$\mathbf{C}_X = \int_{Y \in \mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(Y, X) \tag{3.318}$$

Proof 3.20.1 *The Grothendieck construction's objects are given by pairs of objects Y of $\mathbf{C}$ and elements of $\mathrm{Hom}_{\mathbf{C}}(Y, X)$, ie morphisms from Y to X :*

$$(Y, f : Y \to X) \tag{3.319}$$

Which are isomorphic to the appropriate slice objects via

$$(Y, f : Y \to X) \rightleftarrows f : Y \to X \tag{3.320}$$

3.21 Reflective subcategories

Definition 3.21.1 *Given two categories $\mathbf{C}$, $\mathbf{D}$, we say that $\mathbf{C}$ is a reflective subcategory of $\mathbf{D}$ if it is a full subcategory, $\iota : \mathbf{C} \hookrightarrow \mathbf{D}$, and the inclusion functor has a left adjoint called the* reflector

$$(T \dashv \iota) : \mathbf{C} \underset{\hookrightarrow \iota \to}{\overset{\leftarrow T -}{}} \mathbf{D}$$

In other words, for any object $X \in \mathbf{D}$, we can project it to the subcategory $\mathbf{C}$ so that its interactions with other objects $Y \in \mathbf{C}$ is given by the adjunction

$$\mathrm{Hom}_{\mathbf{C}}(T(X), Y) = \mathrm{Hom}_{\mathbf{C}}(X, \iota(Y)) \tag{3.321}$$

ie functions with X as source interacts the same way that it would in the larger category.

Likewise, we have its dual, the coreflective subcategory

Definition 3.21.2 *Given two categories $\mathbf{C}$, $\mathbf{D}$, we say that $\mathbf{C}$ is a coreflective subcategory of $\mathbf{D}$ if it is a full subcategory, $\iota : \mathbf{C} \hookrightarrow \mathbf{D}$, and the inclusion functor has a right adjoint called the* coreflector

$$(\iota \dashv T) : \mathbf{C} \underset{\leftarrow T -}{\overset{\hookrightarrow \iota \to}{}} \mathbf{D}$$

$$\mathrm{Hom}_{\mathbf{C}}(X, T(Y)) = \mathrm{Hom}_{\mathbf{C}}(\iota(X), Y) \tag{3.322}$$

"reflective v. coreflective : the subcategory is made of objects such that every object has a maximal "nice" quotient v. subobject, given by the reflector v. coreflector?"

[...]

Example 3.21.1 *The category of Abelian groups* **Ab** *is a full subcategory of the category of groups,*

$$\iota_{\mathbf{Ab}} : \mathbf{Ab} \hookrightarrow \mathbf{Grp} \tag{3.323}$$

ie there are no group homomorphisms between two Abelian groups which is not in **Ab***. This inclusion admits a left adjoint* Ab*, called the* Abelianization functor*.*

Proof 3.21.1 *In terms of hom-sets, we have that, if* $\iota_{\mathbf{Ab}}$ *admits a left-adjoint* Ab*,*

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Ab}(G), H) = \mathrm{Hom}_{\mathbf{Grp}}(G, \iota_{\mathbf{Ab}}(H)) \tag{3.324}$$

Then for any group homomorphism $f : G \to \iota_{\mathbf{Ab}}(H)$*, we have a corresponding Abelian group homomorphism* $r : \mathrm{Ab}(G) \to H$

[...]

Look at group operations using the free group of one element **Z***?*

Example 3.21.2 *The category of metric spaces* **Met** *with isometries as morphisms has as a full subcategory the category of complete metric spaces* **CompMet***.*

$$\iota_{\mathbf{CompMet}} : \mathbf{CompMet} \hookrightarrow \mathbf{Met} \tag{3.325}$$

The reflector associates the completion of the metric space to any metric space.

3.22 Monads

There are multiple definitions of what a monad is, depending on the context, from its use in computer science where it is understood as adding extra information to a function's return type, to a categorification of monoids.

The famed definition of it is that a monad is a monoid in the category of endofunctors.

Definition 3.22.1 *A* monad (T, η, μ) *in a category* **C** *is composed of*

- *An endofunctor* $T : \mathbf{C} \to \mathbf{C}$
- *A natural transformation* $\eta : \mathrm{Id}_{\mathbf{C}} \to T$

- *A natural transformation* $\mu : T \circ T \to T$

such that the multiplication is associative, $\mu \circ T\mu = \mu \circ \mu T$ *:*

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

and there exists an identity element : $\mu \circ T\eta = \mu \circ \eta T = \mathrm{Id}_T$

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & TT & \xleftarrow{T\eta} & T \\ & \searrow\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\swarrow & \\ & & T & & \end{array}$$

If we consider the category of endofunctors, in other words the functor category $\mathrm{End}(\mathbf{C}) = [\mathbf{C}, \mathbf{C}]$, then a monad as defined here is indeed a monoid, in the sense that it is an algebra on an endofunctor 3.12.4. The category is $\mathrm{End}(\mathbf{C})$, the object is T, and the functor is the component-wise map of composition with T :

$$\forall X \in \mathrm{End}(\mathbf{C}),\ F_T(X) = X \circ T \tag{3.326}$$

From this the carrier of the algebra is the natural transformation $\mu : F_T(T) \cong T \circ T \to T$

Example 3.22.1 *The simplest example of a monad is the identity functor which simply maps the category to itself, where the unit is an equivalence and the multiplication as well.*

Example 3.22.2 *The* maybe monad Maybe *on* **Set** *adds a single new element to a set,*

$$\mathrm{Maybe}(X) = X + \{\bullet\} \tag{3.327}$$

and transforms functions to

$$\mathrm{Maybe}(f : X \to Y) = f' : (X + \{\bullet\}) \to (Y + \{\bullet\}) \tag{3.328}$$

with the property that $f'(x \in X) = f(x)$ *but* $f'(\bullet) = \bullet$.

The unit of the monad is given by, in component form,

$$\eta_X : X \quad \to \quad X + \{\bullet\} \tag{3.329}$$

$$x \quad \mapsto \quad x \tag{3.330}$$

so that the original set is simply embedded into the new one via the obvious map, and the multiplication is given by

$$\mu_X : X + \{\bullet_1, \bullet_2\} \to X + \{\bullet\} \tag{3.331}$$

which is that upon another application of the maybe monad, the extra new element added is identified with the old.

The name of "maybe monad" stems from its use in programming to describe the formalization of error handling, where a function sends back a Maybe(Y)-typed value instead of M with the term $\bullet$ representing an error of the function, such as

$$a : \mathbb{R}, b : \mathbb{R} \dashv div(a, b) = \begin{cases} \bullet & b = 0 \\ a/b & \end{cases} \tag{3.332}$$

in which case any function can be made to handle this new type Maybe($\mathbb{R}$). The unit [return] is simply that the usual type is embedded in the new one, while the multiplication (binding) is so that only one of error element is allowed to exist. A function of type $\mathbb{R}$ can therefore be extended to a function of type Maybe($\mathbb{R}$) via

$$\eta(f) : (f : \mathbb{R} \to Y) \to (f' : \text{Maybe}(\mathbb{R}) \to \text{Maybe}(Y)) \tag{3.333}$$

where a value of an error in the input sends back an error value in the output, which is usually the behaviour of NaN.

Furthermore we also have the avoidance of the multiplication of error values from the multiplication map, so that if we attempt divisions in a row, where the input might itself be an error

$$\eta(\text{div}) : (f : \mathbb{R} \to \mathbb{R} + \{\bullet\}) \to (f' : \mathbb{R} + \{\bullet\} \to \mathbb{R} + \{\bullet_1, \bullet_2\}) \tag{3.334}$$

The multiplication map will get rid of this extra error value :

Example 3.22.3 *The* state monad

Theorem 3.22.1 x

Monads from adjunctions

Theorem 3.22.2 *Any adjunction of two functors*

$$(L \dashv R) : \mathbf{C} \underset{\leftarrow R -}{\overset{- L \rightarrow}{}} \mathbf{D}$$

defines a monad

$$T = R \circ L : \mathbf{C} \to \mathbf{C} \tag{3.335}$$

and a comonad

$$\overline{T} = L \circ R : \mathbf{D} \to \mathbf{D} \tag{3.336}$$

Proof 3.22.1 *The composition obviously defines an endofunctor on the appropriate category, so that we only need to prove the existence of a unit and multiplication map. As the two functors are adjoint, we have the existence of a unit and counit,*

$$\eta \ : \quad \mathrm{Id}_{\mathbf{C}} \to R \circ L \tag{3.337}$$

$$\epsilon \ : L \circ R \to \mathrm{Id}_D \tag{3.338}$$

Left whiskering by R the triangle identity give us

$$\begin{array}{ccc} RL & \xrightarrow{R\circ L\circ\eta} & RLRL \\ & \searrow^{\mathrm{Id}_{RL}} & \downarrow{\scriptstyle R\circ\eta\circ L} \\ & & RL \end{array}$$

Right whistering by L :

$$\begin{array}{ccc} RL & \xrightarrow{R\circ L\circ\eta} & RLRL \\ & \searrow^{\mathrm{Id}_{RL}} & \downarrow{\scriptstyle R\circ\eta\circ L} \\ & & RL \end{array}$$

Definition 3.22.2 *Given a monad (T, η, μ) on $\mathbf{C}$, we say that an object X is* T-modal *if the unit on it is an isomorphism*

$$\eta_X : X \xrightarrow{\cong} TX \tag{3.339}$$

If it is merely a monomorphism, it is submodal.

Definition 3.22.3 *Given a comonad (G, ϵ, δ) on $\mathbf{C}$, we say that an object X is* T-comodal *if the counit on it is an isomorphism*

$$\epsilon_X : GX \xrightarrow{\cong} X \tag{3.340}$$

If it is merely a epimorphism, it is supcomodal.

Modal object, comodal object, submodal object, supcomodal object, anti-modal type

[33]

3.22.1 Adjoint monads

Like any functor, monads can have adjoint functors, but in the case of monads, the term of adjoint typically has a more specific meaning.

Definition 3.22.4 *Given a monad (T, μ, η) and a comonad (G, δ, ϵ), we say that they are left (resp. right) adjoint if T is left (resp. right) adjoint to G*

Example 3.22.4 *The basic adjoint modality example is the* even/odd modality *pair,*

$$\mathrm{Even} \dashv \mathrm{Odd} \tag{3.341}$$

This is done on the category of integers as an ordered set, $(\mathbb{Z}, \leq)$, for which the morphisms are the order relations, and endofunctors are order-preserving functions.

The functor we consider here is the largest integer which is smaller to $n/2$:

$$\lfloor -/2 \rfloor : (\mathbb{Z}, \leq) \rightarrow (\mathbb{Z}, \leq) \tag{3.342}$$

$$n \mapsto \lfloor n/2 \rfloor \tag{3.343}$$

This functor has a left and right adjoint functor,

$$\mathrm{even} : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{Z}, \leq) \tag{3.344}$$

$$n \mapsto 2n \tag{3.345}$$

$$\mathrm{odd} : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{Z}, \leq) \tag{3.346}$$

$$n \mapsto 2n+1 \tag{3.347}$$

$$\tag{3.348}$$

Proof :

$\lfloor -/2 \rfloor$ has as a domain the whole category

For a total order, the hom-set $\mathrm{Hom}(X, Y)$ is simply empty if $X > Y$ and has a single element otherwise. For $\lfloor -/2 \rfloor$, the hom-set is then gonna be that $\mathrm{Hom}_{\mathbb{Z}}(X, \lfloor Y/2 \rfloor)$ is empty if $2X > Y$ if Y is even, and $2X + 1 > Y$ if Y is odd.

The left adjoint of $\lfloor -/2 \rfloor$ is a functor such that

$$\mathrm{Hom}_{\mathbb{Z}}(L(-), -) \cong \mathrm{Hom}_{\mathbb{Z}}(-, \lfloor -/2 \rfloor) \tag{3.349}$$

In the case of a total order, the isomorphism simply means that both sets have the same cardinality, ie they either have no elements (the two objects are not ordered) or one (the two objects are ordered). So

$$\mathrm{Hom}_{\mathbb{Z}}(L(n), m) \cong \mathrm{Hom}_{\mathbb{Z}}(n, \lfloor m/2 \rfloor) b \Leftrightarrow L(n) \leq m \leftrightarrow n \leq \lfloor m/2 \rfloor \tag{3.350}$$

If we have $L(n) = 2n$, we need to show this equivalence both ways.

If $2n \leq m$, then dividing by 2, we have $n \leq m/2$, which we can then apply the floor to both sides (it is monotonous), so $\lfloor n \rfloor \leq \lfloor m/2 \rfloor$. As n is an integer, $n \leq \lfloor m/2 \rfloor$

Converse : If $n \leq \lfloor m/2 \rfloor$: From properties of floor :

$$n \leq \lfloor \frac{m}{2} \rfloor \leftrightarrow 2 \leq \frac{\lceil m \rceil}{n} \tag{3.351}$$

As m is an integer, $2n \leq m$.

So the even function is indeed left adjoint.

Odd function is right adjoint :

From these three functions, we can define adjoint monads :

$$(\mathrm{Even} \vdash \mathrm{Odd}) \tag{3.352}$$

which send numbers to their half floor and then to their corresponding even and odd number :

$$\mathrm{Even}(n) = 2\lfloor n/2 \rfloor \tag{3.353}$$

$$\mathrm{Odd}(n) = 2\lfloor n/2 \rfloor + 1 \tag{3.354}$$

n	Even(n)	Odd(n)
-2	-2	-1
-1	-2	-1
0	0	1
1	0	1
2	2	3
3	2	3

Table 3.1: Caption

Monad and comonad, unit and counit, multiplication

Example 3.22.5 *Integrality modality : Given the two total order categories* $(\mathbb{Z}, \leq)$ *and* $(\mathbb{R}, \leq)$*, the inclusion functor*

$$\iota : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{R}, \leq) \tag{3.355}$$

$$n \mapsto n \text{ (as a real number)} \tag{3.356}$$

Left and right adjoints : $(L \dashv \iota \dashv R)$

$$\mathrm{Hom}_{\mathbb{Z}}(k, R(x)) \cong \mathrm{Hom}_{\mathbb{R}}(\iota(k), x) \tag{3.357}$$

$$R : \mathbb{R} \to \mathbb{Z} \tag{3.358}$$

meaning that the right adjoint is an integer which is superior to any integer k *if*

3.22.2 Monoidal monad

3.22.3 Algebra of a monad

Monads naturally form an algebra over each of the objects that they act upon.

Definition 3.22.5 *For a monad* (T, η, μ) *on a category* $\mathbf{C}$*, a* T*-algebra is a pair* (X, f) *of an object* $X \in \mathbf{C}$ *and a morphism* $\alpha : TX \to X$ *making the following diagrams commute :*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & T(X) \\ & \searrow^{\mathrm{Id}_X} & \downarrow \alpha \\ & & X \end{array}$$

$$\begin{array}{ccc} T^2X & \xrightarrow{T(\alpha)} & TX \\ \downarrow \mu & & \downarrow \alpha \\ TX & \xrightarrow{\alpha} & X \end{array}$$

The precise category in which this algebra is defined

Definition 3.22.6 *The* Eilenberg-Moore category *of a monad is a category*

Example 3.22.6 *The free monoid monad, or list monad, is the composition of the free monoid functor* $F : \mathbf{Set} \to \mathbf{Mon}$ *with the forgetful functor* $U : \mathbf{Mon} \to \mathbf{Set}$*. For some set* S*, we send it to the free monoid* $F(S)$*, which has as elements the* n*-tuples of elements of* S *of arbitrary size, with free monoidal operation their concatenation. For* $S = (a, b, c, \ldots)$*,*

$$(a, b, c, \ldots) \cdot (\alpha, \beta, \gamma, \ldots) = (a, b, c, \ldots, \alpha, \beta, \gamma, \ldots) \tag{3.359}$$

with underlying set all the tuples of finite length of S :

$$U(F(S)) = \coprod_{k\in\mathbb{N}} S^{\times k} \tag{3.360}$$

so that the list monad is given by

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} && (3.361)\\ S &\mapsto \coprod_{k\in\mathbb{N}} S^{\times k} && (3.362)\end{aligned}$$

Its unit is the morphism that associates to every element its singleton list,

$$\begin{aligned} \eta_S : S &\to \mathrm{List}(S) && (3.363)\\ x &\mapsto (x) && (3.364)\end{aligned}$$

and its multiplication takes a list of list and bind them by concatenation,

$$\begin{aligned} \mu : \mathrm{List}\circ\mathrm{List} &\to \mathrm{List} && (3.365)\\ \begin{pmatrix}(x_{11}, x_{12}, \dots x_{1n})\\ \vdots \\ (x_{k1}, x_{k2}, \dots x_{kn})\end{pmatrix} &\mapsto (x_{11}, x_{12}, \dots x_{1n}, \dots, x_{k1}, x_{k2}, \dots x_{kn}) && (3.366)\end{aligned}$$

Example 3.22.7 *The maybe monad defines the algebra of the smash product over pointed sets. The category of objects with at least one element and morphisms preserving a specific element is the category of pointed sets, so that*

$$\mathrm{EM}(\mathrm{Maybe}) = \mathbf{Set}_{\bullet} \tag{3.367}$$

If we pick some set X and a T-action $\alpha : \mathrm{Maybe}(X) \to X$, this commutes if

$$\alpha(\mathrm{Maybe}(x)) = x \tag{3.368}$$

So that a must map any element from the original set back to itself, leaving as the only freedom the element to which $\bullet$ is mapped. Given some chosen element $\overline{x}$, the T-action will be

$$\alpha_{\overline{x}}(\bullet) = \overline{x} \tag{3.369}$$

In other words, this defines a pointed set with a specific element $\overline{x}$ rather than our abstract pointed set.

[No algebra for $\varnothing$?]

Also :

$$\alpha() \tag{3.370}$$

The algebra defined on pointed sets is the one given by the smash product.

[...]

Free algebra : The free T-algebra of the maybe monad is given for some set X by the algebra over $X + \{\bullet\}$ with the multiplication map component $\mu : T^2 X \to TX$ as its T-action. This means that we have some pointed set $X_\bullet \in \mathbf{Set}_\bullet$

Maybe monad is monoidal : the monad unit is given by

$$\epsilon : \{\bullet\} \tag{3.371}$$

composition law :

$$\mu_{X,Y} : \mathrm{Maybe}(X) \times \mathrm{Maybe}(Y) \to \mathrm{Maybe}(X \times Y) \tag{3.372}$$

An important class of monads are the ones which are associated (in the internal logic of the category, cf. [X]) to the classical *modalities*, ie necessity and possibility.

Kripke semantics

Example 3.22.8 *Necessity/Possibility modalities*

3.22.4 Idempotent monads

One class of monads we will use extensively are the idempotent monads (and comonads),

Definition 3.22.7 *A monad (T, η, μ) is said to be an* idempotent monad *if*

$$\mu : T^2 \to T \tag{3.373}$$

is an isomorphism.

In components, this means that the action of a monad is such that every component of the multiplication map is an isomorphism.

$$\eta_X : T^2 X \xrightarrow{\cong} TX \tag{3.374}$$

and has an inverse,

$$\eta_X^{-1} : TX \to T^2 X \tag{3.375}$$

So that the two are, up to isomorphism, the same object,

$$T^2X \cong X \tag{3.376}$$

For simplicity, as this will be a very common operation, we will throughout this book mostly not write down the isomorphism μ_X for any operation involving T^2X on an idempotent monad, so that we will write morphisms $f : T^2X \to Y$ equivalently as $f : TX \to Y$ rather than the more proper $f \circ \mu_X$, but it is implicit.

Example 3.22.9 *The free-forgetful adjunction $(F \dashv U)$ between groups and Abelian groups is an idempotent monad.*

Theorem 3.22.3 *The Eilenberg-Moore category of an idempotent monad* $\mathrm{EM}(T)$ *is a reflective subcategory of* $\mathbf{C}$*.*

Proof 3.22.2

With this we can consider a variety of monads easily simply by looking at various reflective subcategories we have seen in 3.21.

Example 3.22.10 *From the reflector T of the inclusion functor ι :* $\mathrm{CompMet} \hookrightarrow \mathrm{Met}$*, we can construct the monad of completion,*

$$\mathrm{Comp} : \mathbf{Met} \to \mathbf{Met} \tag{3.377}$$

which sends any metric space to its completion. [Moore closure?]

On functions :

$$\mathrm{Comp}(f : X \to Y) = \overline{f} : \mathrm{Comp}(X) \to \mathrm{Comp}(f) \tag{3.378}$$

Sends functions of the underlying space to the one that's a reflection idk

Algebra?

We will see quite a lot more examples of idempotent monads and comonads in the chapter on objective logic 10, as this is the main tool by which this operates.

Theorem 3.22.4 *Adjoint idempotent monads commute with limits and colimits*

Proof 3.22.3 *As we can decompose any idempotent adjoint monad into an adjoint triple of functors $(L \dashv C \dashv R)$ on the Eilenberg-Moore category, we have that, as C is both left and right adjoint, L is left adjoint and R is right adjoint,*

$$(C \circ L)\mathrm{colim}F = \mathrm{colim}(C \circ L)F \tag{3.379}$$

$$(C \circ R)\lim F = \lim(C \circ R)F \tag{3.380}$$

[check]

3.23 Linear and distributive categories

Definition 3.23.1 *A category is* distributive

4

Spaces

One of the main type of category we will use for objective logic are categories which are *spaces* or relate to spaces, in a broad sense, such as frames, sheaves and topoi.

[34, 35]

4.1 General notions of a space

Before looking into how spaces work in category theory, let's first look at how spaces are treated both intuitively, in philosophical analysis, and the most common ways to treat spaces in mathematics.

4.1.1 Mereology

The most basic aspect of a space in philosophical terms is that of *mereology*. The mereology of a space is the study of its parts, where we can decompose a space into regions with some specific properties. A space X is composed of a collection of regions $\{U_i\}$, which are ordered by a relation of inclusion $(\{U_i\}, \subseteq)$, called *parthood*, which obeys the usual partial order relations :

- Reflection :

$$U \subseteq U$$

- Symmetry :
$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_1 \to U_1 = U_2$$
- Transitivity :
$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_3 \to U_1 \subseteq U_3$$

Those are the typical notion of a partial order : reflexivity (a region is part of itself), antisymmetry (if a region is part of another, and the other region is part of the first, they are the same region) and transitivity (if a region is part of another region, itself part of a third region, the first is part of the third). The antisymmetry allows us to define equality in terms of parthood, simply as $U_1 = U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_2 \subseteq U_1$.

As we will deal with posets extensively in this chapter, we will use Hasse diagrams to illustrate our mereologies here.

Definition 4.1.1 *A* Hasse diagram *of a poset* $(X, \leq)$ *is a directed graph for which the nodes are elements of the set* X*, and the edges represent at least the minimal amount of order relations to generate the full order by transitivity.*

For simplicity the direction of edges is usually implicit, simply being that the source of the edge is the one higher and the target is the one lower.

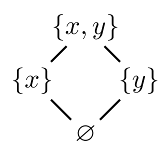


Figure 4.1: Hasse diagram of the power set of the set of two elements.

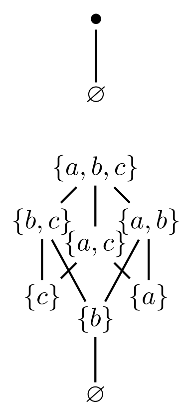
Let's look at a few basic examples of mereologies with Hasse diagrams. The simplest one is the *trivial mereology*, which is just a single node with no edges.

$$X$$

The interpretation of this diagram will depend slightly on what we impose as semantics, as it could either be an empty region (the mereology of a non-existing space) or a totally unified space with no subregions, including the empty region, à la Parmenides [Duality of nothingness and being?].

Another simple one is that of the *abstract point*. This is a single region whose only subobject is the empty region.

A general kind of mereology is the one given by discrete sets, where we consider the various subsets as subregions. For instance, in the case of a "space" of 3 regions, we have (assuming we also consider the empty region as a region) the following structure



As a space this could be interpreted as a discrete space with a finite number of points.

From the definition of a mereology, we can define some further operations. The most basic derived relation is that of *proper parthood* $\subset$, which is defined simply as parthood excluding equality :

$$U_1 \subset U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_1 \neq U_2 \tag{4.1}$$

The converse of proper parthood is *proper extension* : we say that U is a proper extension of U' if $U' \subset U$ and $U \neq U'$. In other words, U is a region that contains U' but is larger than it.

Those are roughly all the notions that can be expressed without quantifiers.

A first basic notion of mereology involving quantification is that of *overlap* : Two regions U_1, U_2 overlap if there exists a third region U_{12} which is part of both :

$$O(U_1, U_2) \leftrightarrow \exists U_{12},\ (U_{12} \subseteq U_1 \wedge U_{12} \subseteq U_2) \tag{4.2}$$

If we were to interpret this in set theoretical terms, this is equivalent to $U_1 \cap U_2 \neq \varnothing$.

The converse of overlap is *underlap*, where there exists a third region containing the first two : two regions U_1, U_2 underlap if there exists a third region U_{12} which contain them both :

$$U(U_1, U_2) \leftrightarrow \exists U_{12},\ (U_{12} \subseteq U_1 \wedge U_{12} \subseteq U_2) \tag{4.3}$$

If we were to interpret this in set theoretical terms, this is equivalent to $\exists U,\ U_1 \cup U_2 \subseteq U$.

We say that two regions are *disjoint* if they are not overlapping :

$$D(U_1, U_2) = \neg O(U_1, U_2) \tag{4.4}$$

Definition 4.1.2 *The* overlap *of two regions is the existence of a third region which is a part of both :*

$$U_1 \circ U_2 \leftrightarrow \exists U_3,\ [U_3 \subseteq U_1 \wedge U_3 \subseteq U_1] \tag{4.5}$$

Unless a mereological nihilist, we also typically define an operation to turn several regions into one, the *fusion* :

Definition 4.1.3 *Given a set of regions $\{U_i\}_{i \in I}$, we say that U is the* fusion *of those region, $\sum(U, \{U_i\})$,*

Definition 4.1.4 *If a region does not have any proper part, we say that it is* atomic

$$\mathrm{Atom}(U) \leftrightarrow \nexists U',\ U' \subset U \tag{4.6}$$

Atomic regions will, depending on the exact model, be either assimilable to points, or be the empty region. If there is an empty region, a point will then be a region whose only part is the empty region. We will generally denote points by the usual symbols $x, y, \ldots$.

Mereologies can vary quite a lot depending on what you wish to model or your own philosophical bent. *Mereological nihilism* will assume for instance that there are no objects with proper parts (so $U_1 \subseteq U_2$ implies $U_1 = U_2$), and we can only consider a collection of atomic points with no greater structure (in particular, there is no *space* itself which is the collection of all its regions), while on the other end of the spectrum is *monism* (such as espoused by Parmenides[1]), where the only region is the whole space itself, with no subregion.

Typically however, we tend to consider some specific base axioms for a mereology. Beyond the partial ordering axioms (which are referred to as M1 to M3), we also have

Axiom M4 ***Weak supplementation*** *: if U_1 is a proper part of U_2, there's a third region U_3 which is part of U_2 but does not overlap with U_1 :*

$$U_1 \subset U_2 \to \exists U_3,\ [U_3 \subseteq U_2 \wedge \neg U_3 \circ U_1] \tag{4.7}$$

[diagram]

Counterexample 4.1.1 *A linear mereology $U_0 \subset U_1 \subset U_2 \subset \ldots \subset U_N$ is not weakly supplemented : for $U_{k-1} \subset U_k$, any other region $U \subseteq U_k$ will overlap with U_{k-1}*

Axiom M5 ***Strong supplementation*** *: If U' is not part of U, there exists a third region U'' which is part of U' but does not overlap with U.*

$$\neg(U' \subseteq U) \to \exists U'',\ U'' \subseteq U' \wedge \neg(O(U', U)) \tag{4.8}$$

[diagram]

Axiom M5' ***Atomistic supplementation*** *: If U' is not a part of U, then there exists an atom x that is part of U' but does not overlap with*

Axiom TOP ***Top*** *: There is a universal object W of which every region is a part of*

$$\exists W,\ \forall U,\ U \subseteq W \tag{4.9}$$

Axiom BOT ***Bottom*** *: There is an atomic null object N which every region contains*

$$\exists N,\ \forall U,\ N \subseteq U \tag{4.10}$$

[...]

Mereologies typically do not include all possible such axioms, but we have common systems that will include a variety of them[36, 37]. The smallest of these is just bare mereology, **M**, which is just given by M1, M2 and M3, and is therefore simply the theory of partial orders.

Minimal mereology **MM** is simply **M** with weak supplementation M4

extensional mereology **EM** is **M** with M5

classical extensional mereology **CEM** is **EM** with M6 and M7

general mereology **GM** is **M** with M8

general extensional mereology **GEM** is **EM** with M8

atomic general extensional mereology **AGEM** is **M** with M5' and M8

Set theory may be considered as some kind of mereology, as we can take the class of all sets and the subset relation as its parthood system. As such as system, it is [...]

A caveat however is that the mereology of set theory does not define it uniquely[38]

4.1.2 Topology

A common approach for space in mathematics is the notion of *topology*. We have already briefly defined the category **Top** of topological spaces, but as they form the basis for most of the common understanding of spaces in math, we should look into them more deeply : what their motivations are, what they are for, and how they may relate to other objects.

If we look at one of the common structuration of mathematics, popularized by Bourbaki[ref on structuralism], spaces are built first as sets, then as topological spaces, and they may afterward get further specified into other structures, such as metric spaces, etc.

This is only a convention, as there are many other structures one may choose, that can be easier, more general, more specific to a given property, etc. The point of topological spaces is that they are a good compromise between those constraints, being fairly easy to define and allowing to talk about quite a lot of properties.

First, let's look at the basic structure of sets. From the perspective of mereology, sets are a rather specific choice of structure, corresponding to an atomic unbounded relatively complemented distributive lattice 1, [see definition of points in philosophy too etc]

Historically, the notion that spaces are made of point is quite ancient [cf. Sextus Empiricus], but it has not had the modern popularity it now has until the works of [Riemann?] Cantor, Hausdorff, Poincaré, etc, and the notion was put into the modern mathematical canon with such works as Bourbaki, etc.

If we consider spaces only as sets however, the informations we can derive from them is rather limited. This is an observation from antiquity [Sextus again]

In modern terms, we can talk about subsets, cardinalities, overlap and unions, but we would be missing on quite a lot of intuitively important properties of a space. If we consider physical space as our example, as we've seen from mereology, some points seem to "belong together" more than other points, some may be "next to" a give subset even if they do not belong to it, two subsets may "touch" without any overlap, and so on.

To illustrate those notions, we can consider some subsets of the plane. If we compare let's say some kind of continuous shape, a disk and an uncountable set of points sprinkled in an area (for instance the two-dimensional Cantor dust),

We would expect the first two objects to have more in common than with the third, having what we will later call the same *shape*, but as sets, they are all isomorphic, simply by the virtue of containing the same amount of points. The same goes for comparing one connected shape and one composed of two disconnected shapes

[...]

If we consider the set D of all points that are at a distance strictly inferior to a given value r from a central point o, we would like to say that a point at exactly a distance of r is somehow closer to D than other points, but as a set, this point is simply equivalent to every point outside of D, as $D \cap \{p\} = \varnothing$.

Convergence

quasitopological spaces, approach spaces, convergence spaces, uniformity spaces, nearness spaces, filter spaces, epitopological spaces, Kelley spaces, compact

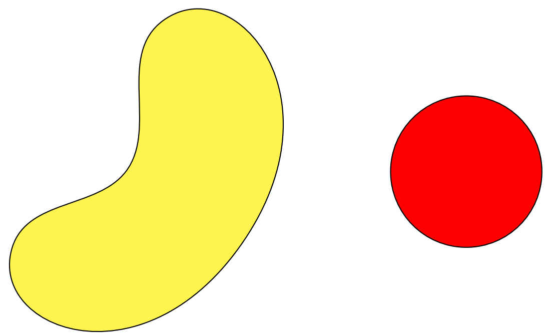


Figure 4.2: Three subsets of the plane of identical cardinality

Hausdorff spaces, δ-generated spaces Cohesion, pretopology, proximity spaces, convergence spaces, cauchy spaces, frames, locales

Definition 4.1.5 *A topological space (X, τ) is a set X and a set of subsets $\tau \subset \mathcal{P}(X)$, called* open sets*, such that for any collection of open sets $\{U_i\} \subseteq \tau$, we have*

- *The entire space X and the empty set $\varnothing$ are both open sets : $X, \varnothing \in \tau$*
-

Approaches via open/closed sets, closure operators, interior operators, exterior operators, boundary operators, derived sets

Definition 4.1.6 *Given a set X, a* net *on X is a directed set $(A, \leq)$ and a map $\nu : A \to X$*

In most of basic topology, nets are rarely used in their generality and we use instead the narrower notion of a sequence,

Example 4.1.1 *A* sequence *is a net with the directed set* $(\mathbb{N}, \leq)$.

A sequence would be in our case something like a family of nested open sets

$$\{U_i \mid i \in \mathbb{N},\ \forall n \in \mathbb{N},\ U_{n+1} \subseteq U_n\} \tag{4.11}$$

Example 4.1.2 *Net of neighbourhoods*

Definition 4.1.7 *A net $x_\bullet$ on a directed set A is said to be eventually in $S \subseteq X$ if*

Definition 4.1.8 *A net is said to* converge *to an element $x \in X$ if*

Definition 4.1.9 *A* filter

Theorem 4.1.1 *A topological space (X, τ) defines a filter for every point x called the* neighbourhood filter*, which is generated by the set of open neighbourhood of x.*

Proof 4.1.1 *Given the point x, take the set of its open neighbourhood*

$$N_o(x) = \{U \mid x \in U,\ U \in \tau\} \tag{4.12}$$

for which no element is non-empty by definition. If we look at its closure under supersets,

$$N(x) = \{U \mid \exists U' \in N_o(x),\ U' \subseteq U\} \tag{4.13}$$

this set will be closed under finite intersection, as for any two $U_1, U_2 \in N(x)$, we have that their intersection necessarily contains x as U_1 and U_2 both do, and therefore also contain an open neighbourhood of x.

Definition 4.1.10

4.1.3 Complexes

An additional notion of space common in mathematics, opposed to the "geometric" notion of space, is the so-called "combinatorial" notion of space, where instead of a continuous medium, we take some discrete set of spatial elements and connect them together.

There are many possible ways to define them, depending with what generality we want them or within which framework they would work best

Definition 4.1.11 *An* abstract simplicial complex

Example 4.1.3 *The basic examples of an abstract simplicial complex is the* simplex*, which is the case where, given some finite set X_0, the associated simplex of dimension $n = |X_0| - 1$ is*

$$X = \mathcal{P}(X_0) \tag{4.14}$$

Example :

- The -1-simplex based on the empty set is a singleton $\{\varnothing\}$, and corresponds to an empty space.

- The 0-simplex $\{\{0\}\varnothing\}$ composed of a single point and an empty set, corresponds to a single point.
- The 1-simplex $\{\{0,1\},\{0\},\{1\},\varnothing\}$

[simplex diagrams]

Examples of more complex complexes

Definition 4.1.12 *The* geometric realization *of a complex is a mapping of an abstract simplicial complex to a topological space*

Example : geometric realization in $\mathbb{R}^n$

Definition 4.1.13 *A simplicial set X*

4.2 Frames and locales

4.2.1 Order theory

All those notions of mereology and topology can be formalized within the context of category theory using the notion of frames and locales.

As we've seen, any formalization of a space can be at least formalized as a poset ordered by inclusion, already a category. All further notions relating to spaces will therefore be extra structures on posets, typically relating to their limits.

First we need to define the notion of semilattices for joins and meets.

Definition 4.2.1 *A meet-semilattice is a poset $(S,\leq)$ with a meet operation $\wedge$ corresponding to the greatest lower bound of two elements (which is assumed to always exist in a meet-semilattice) :*

$$m = a \wedge b \leftrightarrow m \leq a \wedge m \leq b \wedge (\forall w \in S,\ w \leq a \wedge w \leq b \to w \leq m) \tag{4.15}$$

Example 4.2.1 *In $\mathbb{Z}$ and $\mathbb{R}$ (in fact for any total order), the meet of two numbers is the min function :*

$$k_1 \wedge k_2 = \min(k_1, k_2) \tag{4.16}$$

Theorem 4.2.1 *In a poset category, the meet is the coproduct.*

Proof 4.2.1 *By the semantics of morphisms in a poset category, if we look at the universal property of the coproduct, for any two objects X, Y with morphisms to a third object Z (so $Z \leq X$ and $Z \leq Y$), then there exists an object $X + Y$ with morphisms from X and Y (so $X + Y \leq X$, $X + Y \leq Y$) such that there is a unique morphism $Z \to X + Y$ (so that $Z \leq X + Y$). in other words, $X + Y$ is a lower bound, and if Z is also a lower bound, it is inferior to it, making $X + Y$ the greatest lower bound, ie the meet.*

Example 4.2.2 *In the partial order defined by the power set of a set, the meet is the intersection of two sets.*

Proof 4.2.2 *The meet of two sets A, B is the largest set C that is a subset of both A and B. The intersection $A \cap B$ is by definition such a set, $A \cap B \subseteq A, B$. If $A \cap B$ is a strict subset of another lower bound C,*

$$A \cap B \subset C \subseteq A, B \tag{4.17}$$

This means that

Definition 4.2.2 *A join-semilattice is a poset $(S, \leq)$ with a join operation $\vee$ corresponding to the least upper bound of two elements (which is assumed to always exist in a join-semilattice) :*

$$m = a \vee b \leftrightarrow a \leq m \wedge b \leq m \wedge (\forall w \in S,\ a \leq w \wedge b \leq w \to m \leq w) \tag{4.18}$$

Theorem 4.2.2 *In a poset category, the join is the product.*

Proof 4.2.3 *As with the proof for the meet, the universal property tells us that for any two morphisms $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$ (so that $Z \leq X_1, X_2$), we have a unique morphism $(f_1, f_2) : Z \to X_1 \times X_2$*

$$x \tag{4.19}$$

Natural transformation (by components) :

$$\eta_X : \prod_i a_i \to a_i \tag{4.20}$$

There is one morphism from the join to each element, therefore $\prod_i a_i \leq a_i$

Upper bound is least : for any other b such that $b \leq a_i$ (ie the natural transformation $\alpha_b : b \to a_i$ for some a_i), then the unique morphism $f : b \to \prod a_i$ (b smaller than a_i)

Properties :

Proposition 4.2.1 *The meet is commutative :* $a \wedge b = b \wedge a$*.*

Proof 4.2.4 *As the roles of* a *and* b *in the definition of the meet are entirely symmetrical, due to the commutativity of the logical conjunction, this is true. Alternatively, this is simply the commutativity of the coproduct.*

Proposition 4.2.2 *The meet is associative :* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof 4.2.5 *If* $m = b \wedge c$*, then* $a \wedge (b \wedge c) = a \wedge m$*, meaning that the meet can be defined by some element* m' *such that*

$$(m' \leq a) \wedge (m' \leq m) \wedge (m \leq b) \wedge (m \leq c) \tag{4.21}$$
$$\wedge (\forall w \in S,\ w \leq b \wedge w \leq c \to w \leq m) \tag{4.22}$$
$$\wedge (\forall w' \in S,\ w' \leq a \wedge w \leq m \to w \leq m') \tag{4.23}$$

as $(m' \leq m) \wedge (m \leq b)$

Definition 4.2.3 *A* lattice *is a poset that is both a meet and join semilattice, such that* $\wedge$ *and* $\vee$ *obey the* absorption law

$$a \vee (a \wedge b) = a \tag{4.24}$$
$$a \wedge (a \vee b) = a \tag{4.25}$$

Theorem 4.2.3 *In a lattice, the join and meet are idempotent :*

$$a \vee a = a \tag{4.26}$$
$$a \wedge a = a \tag{4.27}$$

Proof 4.2.6 *By the second absorption law,*

$$a \wedge (a \vee a) = a \tag{4.28}$$

then we can write by the first absorption law

$$a \vee a = a \vee (a \wedge (a \vee a)) = a \tag{4.29}$$

Likewise for $a \wedge a$*,*

$$a \vee (a \wedge a) = a \tag{4.30}$$

and

$$a \wedge a = a \wedge (a \vee (a \wedge a)) = a \tag{4.31}$$

Example 4.2.3 *For sets, we have* $A \cap A = A$ *and* $A \cup A = A$

Definition 4.2.4 *A lattice is* distributive *if it obeys the distributy laws*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{4.32}$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{4.33}$$

Theorem 4.2.4 *If a lattice obeys any of the two distributivity laws, it obeys both.*

Proof 4.2.7

Example 4.2.4 *For sets, the distributivity law is the distributivity of intersection and union :*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{4.34}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{4.35}$$

Definition 4.2.5 *A* Heyting algebra *is a bounded lattice, such that there exists both a bottom element* 0 *such that* $0 \leq x$ *for all* x*, and a top element* 1 *such that* $x \leq 1$ *for all* x*.*

This top and bottom element will correspond to the equivalent top and bottom element of a mereology, generally interpreted as the ”total space” and the empty space.

Theorem 4.2.5 *Heyting algebras are always distributive.*

Proof 4.2.8

Example 4.2.5 *The algebra of open sets on a topological space is a Heyting algebra.*

Proof 4.2.9

A common poset structure that we will use is the algebra generated by a family of subsets. If we have a set X, and a family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, $\mathcal{B}$ forms a poset by the inclusion relation ordering, $(\mathcal{B}, \subseteq)$. Some common families of subsets of interest are the power set $\mathcal{P}(X)$, and more generally the set of opens $\mathrm{Op}(X)$ for a given topology.

It is a directed set with X the top element

In this context, the meet is the intersection

Theorem 4.2.6 *The intersection of two sets is their meet.*

Proof 4.2.10 *As the intersection of A and B is defined via*

$$A \cap B = \{x | x \in A \wedge x \in B\} \tag{4.36}$$

We can

We already know that $A \cap B \subseteq A, B$*. If we assume a set* $C \neq A \cap B$ *such that* $A \cap B \subseteq C$ *and* $C \subseteq A, B$*, this means that C contains all the same elements as* $A \cap B$ *with some additional elements (since* $A \cap B$ *is a subset, we have* $C = (A \cap B) \cup (A \cap B)^C$*, and as they are different,* $(A \cap B)^C \neq \varnothing$*). However, as C is a subset of A and B, that complement can only contains elements of A and B*

Theorem 4.2.7 *The union of two sets is their join.*

Proof 4.2.11 *As the union of A and B is defined via*

$$A \cup B = \{x | x \in A \vee x \in B\} \tag{4.37}$$

A family of sets is therefore a meet semilattice if it is closed under intersection, and a join semilattice if it is closed under union. If it is both, it is automatically a lattice as the absorption laws are obeyed by union and intersection.

[proof]

If the empty set is furthermore included, it is a bounded lattice, with $1 = X$, $0 = \varnothing$

Semi-lattice, lattice, Heyting algebra, frame (complete Heyting algebra)

Definition 4.2.6 *A* Heyting algebra *H is a bounded lattice for which any pair of elements* $a, b \in H$ *has a greatest element x, denoted* $a \to b$*, such that*

$$a \wedge x \leq b \tag{4.38}$$

Definition 4.2.7 *The* pseudo-complement *of an element a of a Heyting algebra is*

$$\neg a = (a \to 0) \tag{4.39}$$

Example 4.2.6 *A bounded total order* $0 \to 1 \to \ldots \to n$ *is a Heyting algebra given by*

$$a \to b = \begin{cases} n & a \leq b \\ b & a > b \end{cases} \tag{4.40}$$

The pseudo-complement is therefore just $\neg a = 0$*.*

Example 4.2.7 *For the power set $\mathcal{P}(X)$ poset, the relative pseudo-complement of two sets A, B is*

$$C = (X \setminus A) \cup B \tag{4.41}$$

This follows the property as $A \cap C = A \cap (B \setminus A)^C$

(eq. to the discrete topology)

Definition 4.2.8 *A Heyting algebra is* complete *if it*

Definition 4.2.9 *A* boolean algebra *is a Heyting algebra satisfying the law of excluded middle,*

$$a \wedge \neg a = 0 \tag{4.42}$$

Definition 4.2.10 *A* frame *$\mathcal{O}$ is a poset that has all small coproducts (called joints $\vee$) and all finite limits (called meets $\wedge$), and satisfied the distribution law*

$$x \wedge (\bigvee_i y_i) \leq \bigvee (x \wedge y_i) \tag{4.43}$$

Definition 4.2.11 *The category of frames* **Frm** *is the category whose objects are frames and morphisms are frame homomorphisms*

In terms of objects, frames are the same as complete Heyting algebras, but categorically this is not true, as frame homomorphisms are not the same as complete Heyting algebra homomorphisms.

Frames define a mereology by considering its objects as regions, its poset structure by the parthood relation, and joins and meets by

Mereological axiom for distribution law?

Definition 4.2.12 *A* locale *is an object in the dual category of frames, the category of locales* **Loc** *:*

$$\mathbf{Frm}^{\mathrm{op}} = \mathbf{Loc} \tag{4.44}$$

Locales are therefore formally frames, but locale homomorphisms are not

Example 4.2.8 *A power set is a boolean algebra*

Proof 4.2.12 *The power set $\mathcal{P}(X)$ is as we've seen a Heyting algebra, and furthermore, we have*

$$A \cap A^C = A \cap (X \setminus A) = \tag{4.45}$$

The basic example of a frame in math is that of the frame of opens for a topological space (X, τ).

Example 4.2.9 *The category of open sets of a topological space X, $\mathrm{Op}(X)$, is a frame.*

Proof 4.2.13 *If we consider the poset of opens, as a union and intersection of open sets is itself an open set, we have a lattice, which is bounded by X itself and the empty set $\varnothing$.*

The frame of open is not boolean typically, as the negation $\neg$ can be defined as $\neg a \to 0$, and the implication

$$U \to V = \bigcup \{W \in \mathrm{Op}(X) | U \cap W \subseteq V\} \tag{4.46}$$
$$= (U^c \cup V)^\circ \tag{4.47}$$

$$\neg U = (U^c \cup \varnothing)^\circ \tag{4.48}$$
$$= (U^c)^\circ \tag{4.49}$$
$$= X \setminus \mathrm{cl}(U \cap X) \tag{4.50}$$
$$= X \setminus \mathrm{cl}(U) \tag{4.51}$$

The interior of the complement

$$U \cup (X \setminus U)^\circ = (X \cup U) \setminus (\mathrm{cl}(U) \setminus U) \tag{4.52}$$
$$= X \setminus \partial U \tag{4.53}$$
$$\tag{4.54}$$

Therefore a frame of open is boolean if open sets never have a boundary, which is that every open set is a clopen set.

Stone theorem [39]

Theorem 4.2.8 *The category* **Sob** *of sober topological spaces with continuous functions and the category* **SFrm** *of spatial frames are dual to each other.*

Examples :

Example 4.2.10 *For a given set X, the partial order defined by inclusion of the power set $\mathcal{P}(X)$, is a complete atomic Boolean algebra.*

Definition 4.2.13 *A sober topological space*

Theorem 4.2.9 *Stone duality : The category of sober topological spaces* **Sob** *is dual to the category of spatial frames* **SFrm**

In terms of categories, the various formalizations of mereology are expressed by different types of algebraic structures on posets. **M** is simply a poset with no extra structure.

The TOP axiom corresponds to the existence of a greatest element in this partial order (if we consider this applying to spaces, this is the object X of the space itself), BOTTOM to a least element (the empty set).

Most axiomatizations of mereology do not include the bottom element, but we will keep it for a better analogy with spaces in terms of a category, as they typically include one.

4.2.2 Subobjects of lattices

Definition 4.2.14 *A* sublocale *of a locale* $L \in$ **Loc** *is a regular subobject of* L.

Example 4.2.11 *For any object* U *in a locale* L*, the down set (the slice category* L_U*) is a sublocale*

Proof 4.2.14

Moore closure

Theorem 4.2.10 *The*

double negation sublocale

Consider the map

$$\begin{aligned} \neg\neg : L &\to L && (4.55)\\ U &\mapsto \neg\neg U && (4.56) \end{aligned}$$

A nucleus on L (a frame) is a function $j : L \to L$ which is monotone ($j(a \wedge b) = j(a) \wedge j(b)$), inflationary ($a \leq j(a)$) and $j(j(a)) \leq j(a)$

A meet-preserving monad.

Properties :

- $j(\top) = \top$
- $j(a) \leq j(b)$ if $a \leq b$

- $j(j(a)) = j(a)$

Quotient frames : L/j is the subset of L of j-closed elements of L (such that $j(a) = a$).

Double negation sublocale :

4.2.3 Lattice of subobjects

If we are to consider some category or object of a category as representing a space in some sense, a useful method to model it is to consider the structure given by its subobjects, as this is the best analogue that we have to a subregion of a space.

Given a category $\mathbf{C}$, we can look at the *poset of subobjects* $\mathrm{Sub}(X)$ for a given object X, with the following definition :

Definition 4.2.15 *The poset of subobjects of X is the skeletal subcategory of the slice category $\mathbf{C}_{/X}$ from which we take every object of $\mathbf{C}_{/X}$ which is a monomorphism in $\mathbf{C}$, and identify every isomorphism to the identity.*

Theorem 4.2.11 *Every morphism in $\mathrm{Sub}(X)$ is a monomorphism in $\mathbf{C}$*

Proof 4.2.15 *As our objects are only monomorphisms, and any morhpism in $\mathrm{Sub}(X)$ will be a slice morphism g between two monomorphisms f, f', so that $f' \circ g = f$, we can just use the property that of $f \circ g$ is a monomorphism, then so is g. Therefore all morphisms in $\mathrm{Sub}(X)$ are monomorphisms in $\mathbf{C}$.*

They are in fact monomorphisms between the subobjects of X themselves.

Theorem 4.2.12 $\mathrm{Sub}(X)$ *is a poset category*

Proof 4.2.16 *As the category of subobjects includes the identity morphism, it is reflexive. As we are assuming the equivalence class for everything here, if we have two morphisms f, g between two objects $\alpha, \beta \in \mathrm{Sub}(X)$, this means that we have two subobjects A, B of X with inclusion $\alpha : A \hookrightarrow X$ and $\beta : B \hookrightarrow X$*

It is best to try to keep in mind when a statement about a category regards the category itself versus when it is about the category of subobjects, as those are typically categories of spaces versus poset categories, in which the different terms have vastly different meanings, and it is common for textbooks to not be terribly clear on this point.

As a poset category, we can say a few things about the limits of $\mathrm{Sub}(X)$, if they exist.

As we've seen in the chapter on limits, the initial and terminal object in a poset category correspond to the bottom and top element of the poset. We always have a top element of a poset of subobjects, the object itself, and if the category has an strict initial object, it will always be the bottom, as there is no monomorphism to 0 in this case. Notationally, we have

$$(0_X : 0 \hookrightarrow X) \cong 0 \in \mathrm{Sub}(X) \tag{4.57}$$
$$(\mathrm{Id}_X : X \hookrightarrow X) \cong 1 \in \mathrm{Sub}(X) \tag{4.58}$$

The product and coproduct in the poset of subobjects correspond to the join and meet, ie the greatest lower bound and least upper bound. In terms of limits in $\mathbf{C}$, we have that the product in $\mathrm{Sub}(X)$ corresponds to a

If our subobject poset is a join semilattice, this means that it is equipped with a product and a terminal object (X itself), meaning that we have in fact a monoidal category.

In this case, we can define the functor $(-) \times S$, which is simply a function on the poset mapping every object to their meet with S, which is a map from the subobject poset $\mathrm{Sub}(X)$ to the subobject poset $\mathrm{Sub}(S)$ by intersection.

If the class of functors $(-) \times S$ admits a right-adjoint for every S, we will also have an internal hom. The adjunction gives us

$$\mathrm{Hom}_{\mathrm{Sub}(X)}(S, [X, Y]) \cong \mathrm{Hom}_{\mathrm{Sub}(X)}(S \times X, Y) \tag{4.59}$$

Meaning that

$$S \leq [X, Y] \leftrightarrow S \wedge X \leq Y \tag{4.60}$$

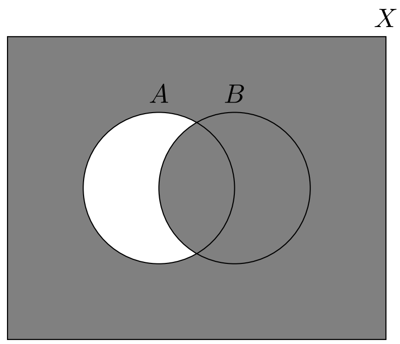
The internal hom $[X, Y]$ is therefore some subobject for which every subobjects is such that their meet with X is a subobject of Y. As we will see, this is what is defined as an implication in a Heyting algebra, denoted by

$$Y^X = X \to Y \tag{4.61}$$

Example 4.2.12 *In* **Set***, the implication between subsets $A \to B$ is the union of B and the complement $X \setminus A$,*

$$A \to B = A^c \cup B \tag{4.62}$$

with the Venn diagram



Properties :

Proposition 4.2.3 *X belongs to* $\mathrm{Sub}(X)$ *and is the top element.*

Proposition 4.2.4 *If the category has an initial object* 0*, it is in* $\mathrm{Sub}(X)$ *and is the bottom element.*

Theorem 4.2.13 *If* $\mathbf{C}$ *admits all finite limits,* $\mathrm{Sub}(X)$ *is a meet-semilattice.*

Proof 4.2.17 *As we've seen before, the intersection of two subobjects is defined by the pullback of their inclusion map. If a category has all finite limits, this is in particular true of the pullback, so that any two subobjects of X also admit an intersection, itself a subobject, such that $A \cap B$ is indeed a lower bound of both A and B. This lower bound is indeed the greatest by the universal property of the pullback : if we have any other lower bound C of A and B (monomorphisms to A and B, then there is a unique morphism $\iota_{C,A\cap B} : C \to A \cap B$ which is such that*

$$\iota_{A\cap B,A} \circ \iota_{C,A\cap B} = \iota_{C,A} \tag{4.63}$$

which must be a monomorhpism itself since its composition here is also a monomorphism, so that we indeed have $C \subseteq A \cap B$ for any other lower bound.

By the same reasoning, admitting all colimits will also make it a join-semilattice, with the join the union [?]

Definition 4.2.16 *If the poset of subobjects* $\mathrm{Sub}(X)$ *admits*

Theorem 4.2.14 *The implication $A \to B$ in a poset of subobject is given by the exponential object*

Proof 4.2.18 *As an implication, we have that the subobject $A \to B$ is a subobject of X for which all subobjects C of $A \to B$ are such that*

$$(C \wedge A) \leq B \tag{4.64}$$

Definition 4.2.17 *If the poset of subobjects has a bottom element* 0 *(the inclusion* $0 \hookrightarrow X$ *in* $\mathbf{C}$*) and an implication, the negation* $\neg_X A$ *of a subobject* A *is the operation*

$$\neg_X A = A \to 0 \tag{4.65}$$

This negation (what is called the pseudo-complement) means that for any subobject of the pseudocomplement,

$$S \subseteq \neg_X A \tag{4.66}$$

we have that

$$S \wedge A \subseteq 0 \tag{4.67}$$

As the bottom element, this simply means that any subobject of $\neg_X A$ is disjoint from A, hence its name of pseudocomplement. The pseudo being here due to the fact that we are however not guaranteed that this pseudocomplement contains *all* subobjects not in A, which is only true for the boolean case

Definition 4.2.18 *If the poset of subobjects is a Heyting algebra and has the boolean property,*

$$a \to b = \neg a \vee b \tag{4.68}$$

then it is a boolean category idk

In a boolean category we have indeed that $\neg_X A = A \to 0 = \not A \vee 0$

[...]

One type of property that we might want to impose on the poset of subobjects is to have some notion of connectedness. What we would like generally is that, given a coproduct $A + B$, it is in some sense "disconnected".

The basic notion for this is that of a disjoint coproduct,

Definition 4.2.19 *A coproduct is* disjoint *if the intersection of its components is the initial object,*

$$X \times_{X+Y} Y = 0 \tag{4.69}$$

Definition 4.2.20 *A category* $\mathbf{C}$ *is* finitely extensive *if its slice categories behave as*

$$\mathbf{C}_{/X} \times \mathbf{C}_{/Y} \cong \mathbf{C}_{/(X+Y)} \tag{4.70}$$

4.3 Coverage and sieves

To define a space in categorical terms, we need to have some formalization of an equivalent notion to mereology, open sets, frames or such that we saw earlier. The notion of *coverage* that we will see will be more general than that (in particular not necessarily be about subregions) but contain those as a special case.

Definition 4.3.1 *A* cover *of an obhect X is given by a morphism $\pi : U \to X$. For a collection of covers, we speak of a* covering family*,*

$$\{\pi_i : U_i \to X\}_{i \in I} \tag{4.71}$$

where I is some indexing set.

The raw definition does not give much properties to a covering family, but it is common to consider them to be

[define cover/covering family first?]

Definition 4.3.2 *Given an object X in a category $\mathbf{C}$, a* coverage *J of X is a covering family for X*

$$J = \{U_i \to X\}_{i \in I} \tag{4.72}$$

such that morphisms between two objects of $\mathbf{C}$ induce a coverage. For $g : Y \to X$, there exists a covering family $\{h_j : V_j \to Y\}_{j \in J}$ such that gh_j factors through f_i for some i :

$$\begin{array}{ccc} V_j & \xrightarrow{k} & U_i \\ \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow{g} & X \end{array}$$

If we take the case of topology that we've seen as an example, we define the standard coverage of a space X to be the collection of all families of open subsets that cover it, ie

$$J(X) = \{\{U_i \hookrightarrow X\} \mid U_i \subseteq X, \bigcup_i U_i = X\} \tag{4.73}$$

Its stability under pullback corresponds to the fact that for any continuous function $f : Y \to X$, as the pre-image of any open set is itself an open set, we can define a family

$$\{f^{-1}(U_i) \to Y\} \tag{4.74}$$

and as any point in X is covered by some U_i, any point in Y will similarly be covered by $f^{-1}(U_i)$, obeying the properties of a coverage.

"Another perspective on a coverage is that the covering families are "postulated well-behaved quotients." That is, saying that $\{f_i : U_i \to U\}_{i\in I}$ is a covering family means that we want to think of U as a well-behaved quotient (i.e. colimit) of the U_i. Here "well-behaved" means primarily "stable under pullback." In general, U may or may not actually be a colimit of the U_i; if it always is we call the site subcanonical. "

To define spaces in the mathematical sense of the word, we need to have some sort of equivalent definition of a *topology*.

If $\mathbf{C}$ has pullback : the family of pullbacks $\{g^*(f_i) : g^*U_i \to V\}$ is a covering family of V.

Grothendieck topology :

An important class of coverage is the *Grothendieck topology*

Čech nerve

Sieve

Definition 4.3.3 *For a covering family* $\{f_i : U_i \to U\}$ *in a coverage* J*, its* sieve *is the coequalizer*

$$\coprod_{j,k} よ(U_j) \times_{よ(U)} よ(U_k) \rightrightarrows \coprod_i よ(U_i) \to S(\{U_i\}) \tag{4.75}$$

with よ *the Yoneda embedding* $よ : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C})$

Example 4.3.1 *For an open cover of a topological space,*

Other definition : A sieve $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ on $X \in \mathbf{C}$ is a subfunctor of $\mathrm{Hom}_{\mathbf{C}}(-, X)$

Objects $S(Y)$ are a collection of morphisms $Y \to X$, and for any morphism $f : Y \to Z$, $S(f)$

Pullback by a sieve :

Ordering : $S \subseteq S'$ if $\forall X$, $S(X) \subseteq S'(X)$

Category of sieves is a partial order, with intersection and union, it is a complete lattice

Grothendieck topology : covering sieves

4.3.1 Čech nerves

Definition 4.3.4 *the* Čech nerve *of a morphism $f : U \to X$ is the simplicial object where the k-simplices are given by the k-fold pushout of U with itself over X,*

$$C(U) = \left(\ \ldots \longrightarrow U \times_X U \times_X U \,\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\, U \times_X U \rightrightarrows U \ \right) \tag{4.76}$$

Given a covering sieve $\{U_i \to X\}$ with respect to a coverage,

4.4 Subobject classifier

In a category **C** with finite limits, a subobject classifier is given by an object Ω (the object of truth values) and a monomorphism

$$\top : 1 \to \Omega \tag{4.77}$$

from the terminal object 1, such that for every monomorphism [inclusion map] $\iota : U \hookrightarrow X$, there is a unique morphism $\chi_U : X \to \Omega$ such that U is the pullback of $* \to \Omega \leftarrow X$

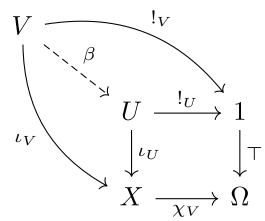
$$\begin{array}{ccc} U & \xrightarrow{!_U} & 1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow{\chi_U} & \Omega \end{array}$$

so that $U \cong X \times_\Omega 1$, or, if we look at it via the equalizer of a product,

$$U \cong X \times_\Omega 1 \to X \times 1 \underset{\chi_U \circ \mathrm{pr}_1}{\overset{\top \circ \mathrm{pr}_2}{\rightrightarrows}} \Omega$$

From our rough intuition of the pullback, we can see imagine that this means something like the object for which χ_U is equal to true, which is the usual sense of what a characteristic function is.

This diagram is furthermore universal, in the sense that for any other subobject V of X, with $\iota_V : V \to X$, the following diagram only commutes if V is itself a subobject of U :



ie that V has the same type of valuation in Ω as U through the characteristic function χ_U.

Theorem 4.4.1 *In a locally finite category* **C** *with a terminal object,* **C** *has a subobject classifier if and only if there is some object* Ω *for which the hom functor is naturally isomorphic to the subobject functor.*

A particularly important case of the pullback is the subobject defined by the monomorphism $0 \hookrightarrow 1$ (if the category admits an initial object), in which case we get the following diagram

$$\begin{array}{ccc} 0 & \xrightarrow{!_0} & 1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \top} \\ 1 & \xrightarrow{\chi_0} & \Omega \end{array}$$

The morphism $\chi_0 : 1 \to \Omega$ is another truth value of Ω, which is the *false* truth value, denoted as

$$\bot : 1 \to \Omega \tag{4.78}$$

As this defines a subobject for Ω itself, we also have the pullback

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\scriptstyle \bot} & & \downarrow{\scriptstyle \top} \\ \Omega & \xrightarrow{\chi_\bot} & \Omega \end{array}$$

$\chi_\bot$ is then some endomorphism on Ω, which we call the *negation*, $\neg$, as it can be understood to map the false value $\bot$ in Ω to the true value $\top$, being the characteristic function of the false value. From this diagram, we have the first equality on Ω,

$$\neg \circ \bot = \top \tag{4.79}$$

the negation of falsity is truth. Furthermore, if we apply the negation to the truth, we get

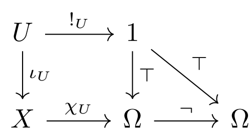
Theorem 4.4.2 *The negation of truth* $\top$ *is falsity* $\bot$*.*

Proof 4.4.1 *If we take*

Theorem 4.4.3 *If the category has no zero object* $1 \cong 0$*, the composition of the negation with any characteristic morphism* $\chi_U : X \to \Omega$ *causes the failure of* U *or any of its subobject to be subobjects of* X*.*

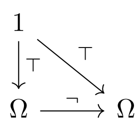
Proof 4.4.2 *If we have the span*

$$X \xrightarrow{\neg\chi_U} \Omega \longleftarrow 1 \tag{4.80}$$



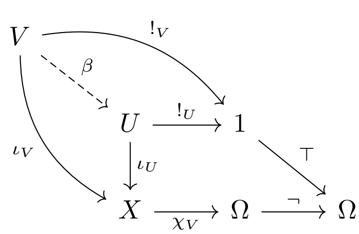
For U *to be a subobject of* X*, we need the lower composition to form a pullback with*

However, the triangle here does not commute



as from the definition, we have $\neg \circ \bot = \top$ *[proof?]*

Furthermore, if we call the pullback of this span $\neg U$*, and we try to look at any subobject of* U *itself, none of those subobjects are subobjects of* $\neg U$*.*



So that the negation of a characteristic function contains none of the original subobjects. We will call this the *pseudo-complement* of a subobject.

Definition 4.4.1 *For a characteristic function $\chi_U : X \to \Omega$, its pseudocomplement is the negation*

$$\neg_X \chi_U = \neg \circ \chi_U \tag{4.81}$$

and its pullback defines the pseudocomplemented subobject

$$\neg_X U = X \times_{\Omega, \neg \chi_U} 1 \tag{4.82}$$

While the pseudocomplement depends on the containing object X, it is typically obvious with respect to which object we are taking it, so that we will usually write it as $\neg$ unless there is some ambiguity.

The exact nature of this pseudo-complement will depend on the exact category that we are working on. As we will see, it is not necessarily true that the pseudo-complement is to be understood as "everything in X not in U" (that is, $U \cup \neg U = X$), but it does have a few of the characteristics we would expect from the complement.

Theorem 4.4.4 *A subobject and its pseudo-complement are disjoint :*

$$U \cap \neg U = 0 \tag{4.83}$$

or in other words, the pullback $U \times_X \neg U$ is the initial object.

but one thing that is true is that it does split the points of X into either. To show this, we need to consider a few things.

Proof that set is boolean

Proof that the hom set commutes with negation?

Theorem 4.4.5 *In a category with a strict initial object and a two-valued subobject classifier, given the hom-functor h^X, the negation morphism on $\mathbf{C}$ is mapped to a negation morphism on* **Set** *(???)*

Proof 4.4.3 *First we have to show that the falsity morphism is preserved. By the strictness of the initial object, we have that* $\mathrm{Hom}_{\mathbf{C}}(1, 0) = 0$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(X, 0) & \xrightarrow{!_0} & 1 \\ \downarrow{\iota} & & \downarrow{\top} \\ 1 & \xrightarrow{\chi_0} & \Omega \end{array}$$

By preservation of limits, we simply map the pullback

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\bot} & & \downarrow{\top} \\ \Omega & \xrightarrow{\neg} & \Omega \end{array}$$

to

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\scriptstyle \bot} & & \downarrow{\scriptstyle \top} \\ \Omega & \xrightarrow{\chi_\bot} & \Omega \end{array}$$

(using $\mathrm{Hom}_{\mathbf{C}}(X,1) \cong 1$*)*

Theorem 4.4.6 *Any point of* X*,* $x : 1 \to X$ *in* $\mathrm{Hom}_{\mathbf{C}}(1,X)$*, is either in* U *or* $\neg U$*.*

Proof 4.4.4 *As the pullback preserves limits, if we take the hom functor* h^1*, this leads to the following pullback in* $^{\mathbf{Set}}$ *:*

$$\begin{array}{ccc} \mathrm{Hom}(1,U) & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \top} \\ \mathrm{Hom}(1,X) & \xrightarrow{\chi} & \Omega \end{array}$$

As **Set** *is a boolean category*

Theorem 4.4.7 *Conjunction*

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ {\scriptstyle (\top,\top)}\downarrow & & \downarrow{\scriptstyle \top} \\ \Omega \times \Omega & \xrightarrow{\cap} & \Omega \end{array}$$

test

Theorem 4.4.8 *For any object* X*, the initial object* 0 *is always a subobject.*

Proof 4.4.5

This is best exemplified by the simple case for sets :

Example 4.4.1 *In* **Set***,* Ω *is the set containing the initial object,* $\Omega = \{\varnothing, \{\bullet\}\}$*, also noted as* $2 = \{0,1\}$*.*

For a subset $S \subseteq X$ *with an inclusion map* $\iota : S \hookrightarrow X$*, the characteristic function* $\chi_S : X \to 2$ *is the function defined by* $\chi_S(x) = 1$ *for* $x \in S$ *and* $\chi_S(x) = 0$ *otherwise. The truth function simply maps* $*$ *to* 1 *in* Ω*.*

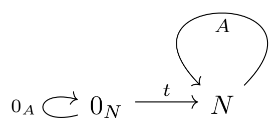
$$\forall x \in U,\ \chi_U(\iota_U(x)) = 1 \tag{4.84}$$

And conversely, if we look at another subobject $V \subseteq X$, the pullback works out

$$\forall x \in V,\ \chi_U(\iota_U(x)) = 1 \tag{4.85}$$

only if $V \subseteq U$, ie there exists a monomorphism from $V \to U$

Subobject classifiers can be more complex than the simple boolean domain true/false. A good illustration of this is the subobject classifier in the category of graphs[40]. A graph is composed by two sets, those of nodes N and of arrows A, with two functions s, t for the source and target of each arrow.



For a subgraph $\iota : S \hookrightarrow G$, the classifying map χ_S has the following behaviour :

- If a node in G is not in S, it is mapped to 0_N.
- If a node in G is in S, it is mapped to N.

Subobject classifier for a topological space

Negation complement :

Definition 4.4.2 *Given the classifying arrow $\bot : 1 \to \Omega$ of the initial object $!_0 : 0 \hookrightarrow 1$, with the associated negation morphism $\neg : \Omega \to \Omega$ the classifying arrow of $\bot$, the pseudocomplement $\neg_X U$ of a subobject $U \hookrightarrow X$ is the pullback of $\neg\chi_U$ by $\top$:*

$$\neg_X U = \mathrm{Fib}_\top(\neg\chi_U) \tag{4.86}$$

Theorem 4.4.9 *A subobject and its pseudocomplement have no overlap $U \cap \neg_X U \cong 0$*

Proof 4.4.6

To preserve negation : preserve the pullback and terminal object and initial object?

4.4.1 Algebra on the subobject classifier

Just as we have an algebra on the poset of subobjects by its use of limits and colimits, we will likewise have an equivalent algebra with the subobject classifier, being an internalization of the category of subobjects. As there is the isomorphism $\mathrm{Hom}_{\mathbf{C}}(X, \Omega) \cong \mathrm{Sub}(X)$

4.4.2 In a sheaf topos

Given the sheaf topos $\mathbf{H} = \mathrm{Sh}(\mathbf{C}, J)$, there is a natural subobject classifier

Example 4.4.2 *A trivial case of a subobject classifier in a sheaf topos is the initial topos* $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$*. As the only sheaf is the empty sheaf, it is also trivially the sheaf corresponding to the subobject classifier of a Grothendieck topos. This means the only truth value, as given by the map* $1 \to \Omega$*, is the truth, or falsity, as we have that the terminal object, initial object and subobject classifier are the same object.*

Example 4.4.3 *The subobject classifier of* **Set** *is, as we've seen,* 2*. In terms of a presheaf category* $\mathbf{Set} = \mathrm{Sh}(\mathbf{1})$*, this is given by the presheaf to sieves of the probe. As there is only one such object, this is the set of sieves on the terminal category, ie the empty sieve and the maximal sieve.*

Example 4.4.4 *Given a spatial topos on a topological space,*

4.5 Elements

One of the important difference between set theory and category theory is that while sets are composed of elements, as defined by the $\in$ relation, categories (for which the objects are often somewhat similar to sets themselves) do not seem to have a naturally equivalent notion. While we often have the notion of a concrete category, where objects of a categories are sets, this cannot be generalized to all categories. Poset categories in particular are particularly resistant to this interpretation.

If we wish to define elements of a set in terms of the morphisms of sets (functions), this is best done via the use of functions from the singleton set $\{\bullet\}$, as those functions are in bijection with the elements of a set

$$\mathrm{Fun}(\{\bullet\}, X) \cong X \tag{4.87}$$

As functions from the singleton are all of the form $\{(\bullet, x)\}$ for every $x \in X$ (From the properties of the Cartesian product)

Generalized elements : Given the yoneda embedding $Y : C \hookrightarrow [C^{\text{op}}, \mathbf{Set}]$, the representable functor

$$\text{GenEl}(X) : C^{\text{op}} \to \mathbf{Set} \tag{4.88}$$

Sends each object U of C to the set of generalized elements of X at stage U.

For an object $X \in \text{Obj}(\mathbf{C})$, its *global elements* are morphisms $x : 1 \to X$. It's a generalized element at stage of definition 1.

Definition 4.5.1 *An object $S \in \mathbf{C}$ is a* separator *if for every pair of morphisms $f : X \to Y$, and every morphism $e : S \to X$, then $f \circ e = g \circ e$ implies $f = g$.*

In other words, the global elements generated by the separator are enough to entirely define the morphisms. For instance, in the case of **Set**, $\{\bullet\}$ is a separator, essentially saying that the elements of a set entirely define its functions : a function $f : X \to Y$ is defined by its value $f(x)$ for every $x \in X$.

If we have a topos E such that its terminal object 1 is a separator, and $1 \neq 0$, we say that the topos is *well-pointed*, meaning that

Other definitions : global section functor is faithful

Prop : well-pointed topos are boolean, its subobject classifier is two-valued,

Definition 4.5.2 *A* concrete category $\mathbf{C}$ *is such that there exists a faithful functor F*

$$F : \mathbf{C} \to \mathbf{Set} \tag{4.89}$$

What the faithful functor implies

$$\text{Hom}_{\mathbf{C}}(X, Y) \overset{F_{X,Y}}{\hookrightarrow} \text{Hom}_{\text{Set}}(F(X), F(Y)) \tag{4.90}$$

ie all the morphisms between two elements are injective. Given any two morphisms, if they are mapped to the same function, then they are the same morphism. Roughly speaking this is to say that we can consider functions on our category to just be a subset of all functions on their underlying set. If we have some function $f : X \to Y$, then there is a corresponding function $|f| : |X| \to |Y|$.

In the general case of a concrete category, we can define its set of points internally to the category by considering its left adjoint to the forgetful functor, the free functor

Theorem 4.5.1 *For a concrete category with a forgetful functor $U : \mathbf{C} \to \mathbf{Set}$, and a left adjoint free functor $F : \mathbf{Set} \to \mathbf{C}$, the set of all points of a given object X taken as a set $U(X)$ is equivalent to the hom-set of its free object on a single element $F_1 = F(\{\bullet\})$.*

Proof 4.5.1

Example 4.5.1 *For the category of topological spaces, the free functor maps sets to their equivalent discrete topologies, and the free object of a single element* F_1 *is the terminal object in the category. Therefore*

$$U(X) \cong \mathrm{Hom}_{\mathbf{Top}}(F_1, X) \tag{4.91}$$

Example 4.5.2 *For the category of vector spaces, the free functor maps sets to the vector space generated by a basis of those elements (a set of* n *elements generates an* n*-dimensional vector space). The free object of a single element is the vector space isomorphic to the underlying field,* $F_1 = k$*, so that*

$$U(V) \cong \mathrm{Hom}_{\mathbf{Vec}}(k, V) \tag{4.92}$$

which is just the statement that a vector space is isomorphic to the vector space of linear maps from the field to itself.

Example 4.5.3 *For the category of groups, the free functor maps sets to their free group (the group of all strings generated by those elements by concatenation). The free object of one element is the group of integers* $\mathbb{Z}$

$$U(G) \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \tag{4.93}$$

This is commonly the case in monoidal categories, such as **Hilb**

[Concrete categories and well-pointed ones do not imply each other in any direction, depends on if elements are global elements?]

Given a set-valued functor $F : \mathbf{C} \to \mathbf{Set}$, its *category of elements*

Category of elements

[...]

Global elements also interact with the subobject classifier.

What is the relation of global elements wrt the subobject classifier?

4.5.1 Points

As we have seen, we can represent a general notion of space as that of a frame, but a more contentious issue is how to define *points* in a space. This is an issue that goes back all the way to the foundation of geometry [41], and to this day is not an uncontentious one, as the assumption of point-like structures in space is still a thorny issue.

Categories for which we have fairly simple notions of discrete elements such as finite sets do not have too much trouble defining what a point could be, corresponding in some sense to the notion of discrete objects that were used uncontroversially in antiquity, but given a frame like that of continuous physical space, this becomes a more complex notion to define, as there is no internal notion of what a point is in the context of the frame of opens $\mathcal{O}(X)$.

In the abstract, we can define a point just as we would define an element for another category, simply as morphisms from some terminal object to the regions of space, but we are neither guaranteed the existence of such an object nor that the space is in some sense *composed* by those points rather than just those merely inhabiting it.

The intuitive notion, going back at least as far as [41], would be to consider a point as the limit of a shrinking family of open sets, but we could have for instance a family of regions $\{U_i\}$ which converge to another (non-point like) region, such as a family of disks of radius $r_n = 1 + 2^{-n}$. Furthermore, two different such sequences can converge to the same point so that we also need to be able to define the equivalence of such sequences.

To represent the notion of several sequences of regions converging to the same result, we need to use the notion of *filter*

A subset F of a poset L is called a filter if it is upward-closed and downward-directed; that is:

If $A \leq B$ in L and $A \in F$, then $B \in F$; for some A in L, $A \in F$; if $A \in F$ and $B \in F$, then for some $C \in F$, $C \leq A$ and $C \leq B$.

Points given a locale?

Given a locale X, a concrete point of X a completely prime filter on $O(X)$. [Show equivalence with a continuous map $f : 1 \to X$: treat $f^* : O(X) \to O(1)$ as a characteristic function]

Completely prime filter :

A filter F is prime if $\perp \notin F$ and if $x \vee y \in F$, then $x \in F$ and $y \in F$. For every finite index set I, $x_k \in F$ for some k whenever $\bigvee_{i \in I} x_i \in F$.

[Some descent of open sets for a topological space?]

Example 4.5.4 *Take the frame defined by the poset* $0 \leq A \leq 1$*, the simplest frame which is not boolean. Its joins and meets can all be deduced from the properties of the join and meet with respect to the top and bottom element and idempotency. Let's attempt to find a frame homomorphism* ϕ *to the terminal(?) frame* $\{0 \leq 1\}$*. As it must preserve joins and meets and bounds, we should have* $\phi(0) = 0$*,* $\phi(1) = 1$*, and therefore* $\phi(A)$ *must be mapped to either* 1 *or* 0*. We should therefore have that*

$$\phi(A \to 0) = \phi(0) \tag{4.94}$$
$$\phi(A) \to 0 = 0 \tag{4.95}$$

and

$$\phi(A \to 1) = \phi(1) \tag{4.96}$$
$$\phi(A) \to 1 = 1 \tag{4.97}$$

The first implies that $\phi(A) = 0$, while the second implies that $\phi(A) = 1$. There is therefore no such map from this frame to the terminal frame, and therefore no points.

Example 4.5.5 *The classic example of a pointless locale is the locale of surjections from the discrete space $\mathbb{N}$ to the continuous space $\mathbb{R}$ with its standard topology[42]. This locale is defined by the*

As $|\mathbb{N}| < |\mathbb{R}|$, there is no such surjection.

Stone theorem

4.6 Internal hom

One component of the definition of a topos regards the behaviour of its *internal homs*, a way to internalize the hom-set of the category in its objects. In other words, every space of morphisms between two objects of the topos is itself an object of the topos, allowing us to talk about such function spaces internally to the topos itself.

Definition 4.6.1 *In a symmetric monoidal category $(\mathbf{C}, \otimes, I)$, an* internal hom *is a bifunctor*

$$[-,-] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C} \tag{4.98}$$

such that for any object $X \in \mathbf{C}$, the functor $[X,-]$ is right adjoint to the functor $(-) \otimes X$:

$$((-) \otimes X \dashv [X,-]) : \mathbf{C} \to \mathbf{C} \tag{4.99}$$

The simplest way to see the content of internal homs from their definition is via the adjunction of hom sets :

$$\mathrm{Hom}_{\mathbf{C}}(Z, [X,Y]) \cong \mathrm{Hom}_{\mathbf{C}}(Z \otimes X, Y) \tag{4.100}$$

If we consider the case where $Z = I$, we have the equivalence

$$\mathrm{Hom}_{\mathbf{C}}(I, [X,Y]) \cong \mathrm{Hom}_{\mathbf{C}}(I \otimes X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X,Y) \tag{4.101}$$

In other words, if we look at $\mathrm{Hom}_{\mathbf{C}}(I, [X,Y])$, the set of generalized elements of the monoidal object, then it is isomorphic to the actual hom-set of the category.

This internal hom is isomorphic to the hom-set as a set, but also carry the structure of **C** objects. If we pick the case of the Cartesian monoidal category $(\mathbf{Set}, \times, \{\bullet\})$ in particular, we recover an exact definition of the hom-set.

As an adjunction of two functors, we have two natural transformations

$$\eta : \mathrm{Id}_{\mathbf{C}} \to [X, ((-) \otimes X)] \tag{4.102}$$
$$\epsilon : ([X, -] \otimes X) \to \mathrm{Id}_{\mathbf{C}} \tag{4.103}$$

The natural transformation ϵ is the notion of an *evaluation map*. If we look at its components at Y, we can define the map on objects

$$\mathrm{eval}_{X,Y} = \epsilon_X = ([X, Y] \otimes X) \to Y \tag{4.104}$$

The meaning here is that given a function from X to Y and a value in X, we can obtain a value in Y, hence its name of evaluation map.

If we have a morphism $Y \otimes X \to Z$, there is equivalently some morphism $Z \to [X, Y]$

Example : take $Z = [X, Y]$, take the morphism $\mathrm{Id}_{[X,Y]} : [X, Y] \to [X, Y]$. Its adjunct is

$$Y \otimes X \to [X, Y] \tag{4.105}$$

Evaluation map :

The counit of the adjunction for $[X, -]$ is called (evaluated at a component Y) the *evaluation map*

$$\mathrm{eval}_{X,Y} : [X, Y] \otimes X \to Y \tag{4.106}$$

Internal hom bifunctor

Example 4.6.1 *In the category of vector spaces* **Vec***, where the monoidal structure is the tensor product, the internal hom $[V, W]$ is the space of linear maps from V to W, which is indeed itself a vector space. The space of dual vectors is for instance given by $[V, I]$, leading to the equivalence A map between two vector spaces can be considered as the tensor product of the target space and the dual space of the domain.* $[X \otimes Y, Z] \cong [X, [Y, Z]]$

Example 4.6.2 *There is a category of smooth spaces which is the generalization of the category of smooth manifolds, as we will see later [x]. Partly this a way to include the internal hom for smooth manifolds, so that maps between manifolds are themselves a type of generalized manifold, which can be thought of as diffeological spaces, infinite dimensional manifolds like Hilbert manifoldd, inverse limit manifolds, pro-manifolds, etc.*

4.6.1 Cartesian closed categories

As the product is an example of a monoidal structure, it is quite common to define the internal hom for it.

Definition 4.6.2 *A category is* Cartesian closed *if it is a closed monoidal category with respect to the product,* $(\mathbf{C}, \times, 1)$.

The internal hom of the Cartesian product is generally called more specifically the *exponential object* and denoted by

$$[X, Y] = Y^X \tag{4.107}$$

By analogy with the notation for the set of functions between two sets, which is the prototypical example of a Cartesian closed category.

Adjunction

$$(- \times A \dashv (-)^A) \tag{4.108}$$

Theorem 4.6.1 *As an adjunction, the monoidal product functor preserves colimits and the internal hom functor preserves limits.*

Example : the set of all function from X to some object $[X, -]$ preserves the terminal object :

$$[X, 1] \cong 1 \tag{4.109}$$

which is just the basic property of the hom-set for terminal object. Similarly, for two morphisms $f : Y \to X_1$ and $g : Y \to X_2$, we have

$$[Y, X_1 \times X_2] \cong [Y, X_1] \times [Y, X_2] \tag{4.110}$$

where the functions to a product are the product of those functions to each, ie morphism of the form (f, g)

Pullback : $[X, Y_1 \times_Z Y_2] \cong [X, Y_1] \times_{[X,Z]} [X, Y_2]$

Equalizer : $[X, \mathrm{eq}(f, g)] = \mathrm{eq}([X, g], [X, g])$.

Similarly for $(-) \otimes X$, $0 \otimes X = 0$. This is true for the Cartesian product with the empty set in **Set**

Theorem 4.6.2 *Isomorphism*

$$[X \otimes Y, Z] \cong [X, [Y, Z]] \tag{4.111}$$

4.7 Presheaves

Definition 4.7.1 *A* presheaf *on a small category* **C** *is a functor* F

$$F : \mathbf{C}^{\text{op}} \to \mathbf{Set} \tag{4.112}$$

This definition also generalizes to any category. If we replace **Set** with any category **S**, we speak of an S-valued presheaf, defined as

$$F : \mathbf{C}^{\text{op}} \to \mathbf{S} \tag{4.113}$$

In a similar manner, we have the dual of presheaves, called *copresheaves*, and defined as sheaves on the opposite category :

$$F : \mathbf{C} \to \mathbf{Set} \tag{4.114}$$

And similarly, for an S-valued copresheaf,

$$F : \mathbf{C} \to \mathbf{S} \tag{4.115}$$

Fundamentally, any functor can be described as a (co)presheaf, as any functor from a category **C** (or its opposite) fits the definition, but a presheaf is typically gonna be studied with more specific goals in mind, usually to turn them into sheaves or topos.

4.7.1 Presheaf on a topological space

An example for the motivation of (co)presheaves is to consider a topological space (X, τ). The category of interest here is the frame of opens $\text{Op}(X)$. A sheaf on the frame of open is some functor associating a set to every open set :

$$\forall U \in \text{Op}(X),\ F(U) = A \in \mathbf{Set} \tag{4.116}$$

or, in the case of an **S**-presheaf, some other object, typically something like a ring or Abelian group. This association is done in a way that preserves the functions contravariantly. In particular, if we have an inclusion $\iota : U \hookrightarrow U'$, its opposite is $\iota^{\text{op}} : U' \to U$, and the functor maps it to

$$F(\iota^{\text{op}}) : F(U') \to F(U) \tag{4.117}$$

Therefore for any inclusion, there exists some morphism sending the object of the larger open set to the smaller open set.

A common example of presheaves for the topological case is that of *structure presheaves*, which map those open sets to some function set (or ring or Abelian group). Common examples of this would be the set of continuous functions to some specific codomain, like $\mathbb{R}$

$$F(U) = C(U, \mathbb{R}) \tag{4.118}$$

or more restricted functions like smooth or analytic functions, $C^\infty(U, \mathbb{R})$ or $C^\omega(U, \mathbb{R})$. In those cases, the functor map $F(\iota^{\mathrm{op}})$ corresponds to the *restriction function*.

$$\forall f \in F(U'),\ \mathrm{res}_{U',U}(f) = f\big|_U \tag{4.119}$$

where the function f in the set of functions $F(U')$ is mapped to a function $f\big|_U$ in $F(U)$.

Properties : the restriction of an open set to itself is the identity :

$$\mathrm{res}_{U,U} = F(\mathrm{Id}_U) = \mathrm{Id}_{F(U)} \tag{4.120}$$

$$\mathrm{res}_{U'',U} \circ \mathrm{res}_{U',U} = \mathrm{res}_{U'',U} \tag{4.121}$$

We can show that this function

We will see in the section on sheaves the meaning of this construction.

Example 4.7.1 *An S-valued presheaf on* $\mathbf{C}$ *is a* constant presheaf *if it is a constant functor, ie for some element $X \in \mathbf{S}$, the presheaf is just*

$$\Delta_X : \mathbf{C}^{\mathrm{op}} \to \mathbf{S} \tag{4.122}$$

As presheaves are merely functors, there is a category of presheaves simply defined by the appropriate functor category, so that an **S**-presheaf category on **C** is

$$\mathrm{PSh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{S}] \tag{4.123}$$

and likewise for copresheaves. This means in particular that morphisms of presheaves in this context are given by natural transformations between two presheaves.

injectivity, surjectivity, etc.

Definition 4.7.2 *A* subpresheaf $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *of a sheaf C :* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *is a presheaf for which we have for any element of the site $U \in \mathbf{C}$*

$$S(U) \subseteq X(U) \tag{4.124}$$

This notion generalizes to presheaves valued in other categories using monomorphisms.

Theorem 4.7.1 *Subpresheaves define monomorphisms in the category of presheaves.*

Proof 4.7.1 *Check component-wise?*

4.7.2 The Yoneda embedding

As sheaves are fundamentally functors, this means that we can treat them within the context of the Yoneda lemma.

interpretation of presheaves $X(U)$ as a function U to X via Yoneda

Definition 4.7.3 *A* representable presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$ *is a presheaf that has a natural isomorphism to the hom-functor* h_X *fr some object* $X \in \mathbf{C}$ *:*

$$\eta : F \xrightarrow{\cong} h_X = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{4.125}$$

In other words a representable presheaf sends every object $Y \in \mathbf{C}$ to the set $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ of all morphisms from Y to X, and every morphism $f : Y \to Z$ to the function sending any morphism $g : Y \to X$ to its composite $f \circ g$.

As we will see later on, representable presheaves are often use to carry the notion that the objects of a category $\mathbf{C}$ correspond in some sense to some of the objects of a presheaf category. This is used for instance in the notion of constructing spaces from simpler spaces as presheaves, in which case the basic spaces are also included. For instance, we can consider manifolds as constructed by atlases to be presheaves on the category of open sets of $\mathbb{R}^n$, in which case the representable presheaves in this category will correspond to the manifolds which are simply raw open sets of $\mathbb{R}^n$.

Theorem 4.7.2 *A representable presheaf is determined by a unique object* $X \in \mathbf{C}$.

Proof 4.7.2 *By the Yoneda lemma,*

Theorem 4.7.3 *Representable functors preserve all limits.*

Proof 4.7.3

This embedding in fact allows us to interpret presheaves as some sort of exten sion of a category, which has the benefit of being better behaved than the original category. This is the *Yoneda embedding.*

Definition 4.7.4 *The* Yoneda embedding $よ : \mathbf{C} \hookrightarrow [\mathbf{C}^{\text{op}}, \mathbf{Set}]$ *is the embedding of a category* $\mathbf{C}$ *into its category of presheaves via its representable presheaves, so that*

$$よ(X) = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{4.126}$$

Theorem 4.7.4 *The Yoneda embedding is fully faithful.*

Definition 4.7.5 *The* Yoneda extension *of a functor* $F : \mathbf{C} \to \mathbf{D}$ *is the left Kan extension by the Yoneda embedding,*

$$\tilde{F} = \mathrm{Lan}_{よ}(F) : [\mathbf{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{D} \tag{4.127}$$

Theorem 4.7.5 *Given a functor* $F : \mathbf{C} \to \mathbf{D}$*, if the induced functor* Lan_F : $[\mathbf{C}^{\text{op}}, \mathbf{Set}] \to [\mathbf{D}^{\text{op}}, \mathbf{Set}]$ *preserves limits and all those limits are representable, then* F

Does the Yoneda embedding preserve monomorphisms,

Theorem 4.7.6 *The Yoneda embedding preserves monomorphisms and epimorphisms.*

Proof 4.7.4 *Given a morphism* $f : X \to Y$*, the Yoneda embedding applied to it gives us*

$$よ(f) : よ(X) \to よ(Y) \tag{4.128}$$

4.7.3 Simplices

[43]

A basic example of presheaves is given by simplicial sets, which are presheaves over the simplex category $\mathbf{\Delta}$:

$$X : \mathbf{\Delta}^{\text{op}} \to \mathbf{Set} \tag{4.129}$$

By the Yoneda embedding [representable presheaves etc], any object in the simplex category is a simplicial set

Example 4.7.2 *To start with we can look at a few representable presheaves. If we look at the zero case, of a single element*

$$I_0 = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{0}) \tag{4.130}$$

As a terminal object in the simplex category, there is only a single morphism for every object $X \in \mathbf{\Delta}$,

$$\mathrm{Hom}_{\mathbf{\Delta}}(X, \mathbf{0}) \cong \{\bullet\} \tag{4.131}$$

and every k-simplex morphism gets mapped to the identity

For the 1*-simplex,*

$$I_1 = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{1}) \tag{4.132}$$

$I_1(0)$ *: two maps injecting the point to either object* $I_1(1)$ *: A single identity functor*

For the 2*-simplex :*

$$I_2 = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{2}) \tag{4.133}$$

$I_2(0)$ *: three maps* $I_2(1)$ *: Two maps*

Furthermore, we can consider simplexes which are constructed from the combination of different simplexes

Example 4.7.3 *Take the basic simplex of the triangle. Let's call this sheaf* Δ_3*. It is composed of a* 2*-simplex, three* 1*-simplices, and three* 0*-simplices. We should therefore expect those elements (at least for the non-degenerate ones). Take the three simplexes* $\vec{\mathbf{2}}$, $\vec{\mathbf{1}}$ *and* $\vec{\mathbf{0}}$*. We want to identify the edges of each to form a triangle.*

$$S_3(\vec{\mathbf{0}}) = \varnothing \tag{4.134}$$
$$S_3(\vec{\mathbf{1}}) = \{\bullet\} \tag{4.135}$$
$$S_3(\vec{\mathbf{2}}) = \{\bullet\} \tag{4.136}$$

and this maps the morphism (functor) which maps each extremity of 1 *and* 2 *together :*

$$S_3(f : \vec{\mathbf{1}} \to \vec{\mathbf{2}}) = f'(S_3(\vec{\mathbf{1}}) \to S_3(\vec{\mathbf{2}})) \tag{4.137}$$

[...]

As an equalizer? $1 + 2/f$

4.8 Site

A site is roughly speaking the elements from which a (Grothendieck) topos is stitched together. This is done by considering some category of those elements, as well as some spatial structure on it, so as to be able to define those constructions properly using descent.

Definition 4.8.1 *A site* (C, J) *is a category* C *equipped with a coverage* J*.*

Sieve

Example 4.8.1 *The terminal category* 1 *is a site. The covering family is simply the only function,* $\{\mathrm{Id}_* : * \to *\}$*. As there are no other objects in the category, we only need to check the induced coverage on itself. Given the morphism* $\mathrm{Id}_* : * \to *$*, the diagram commutes trivially by using the identity function everywhere.*

$$\begin{array}{ccc} * & \xrightarrow{\mathrm{Id}_*} & * \\ {\scriptstyle \mathrm{Id}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_*} \\ * & \xrightarrow{\mathrm{Id}_*} & * \end{array}$$

Example 4.8.2 *The category of opens of a topological space is a site*

Example 4.8.3 *The category of Cartesian spaces* $\mathbf{CartSp}_{\mathrm{smooth}}$ *[with coverage?] is a site.*

"Every frame is canonically a site, where U is covered by $\{U_i\}$ precisely if it is their join."

Is there some kind of relationship between the sheaves of a Grothendieck topos, and the elements of the site taken as (representable) sheaves + coproduct and equalizer

4.8.1 Site morphisms

Definition 4.8.2 *Given two sites* $(\mathbf{C}, J)$ *and* $(\mathbf{D}, K)$*, a functor* $F : \mathbf{C} \to \mathbf{D}$ *is a morphism of sites if it is covering-flat and preserves covering families : for every covering* $\{p_i : U_i \to U\}$ *of* $U \in C$*,* $\{f(p_i) : f(U_i) \to f(U)\}$ *is a covering of* $f(U) \in D$*.*

Covering-flat :

For a set-valued functor $F : C \to \mathbf{Set}$,

Filtered category : A filtered category is a category in which every diagram has a cocone.

For any finite category D and functor $F : D \to C$, there exists an object $X \in C$ and a nat. trans. $F \to \Delta_X$.

Simpler version :

- There exists an object of C (non-empty category)
- For any two objects $X, Y \in C$, there is an object Z and morphisms $X \to Z$, $Y \to Z$

- For any two parallel morphism, $f, g : X \to Y$, there exists a morphism $h : Y \to Z$ such that $hf = hg$.

Every category with a terminal object is filtered.

Every category which has finite colimits is filtered.

Interpretation : for any limit that the site has, they are preserved.

4.9 Descent

To deal with the notion of structures on spaces, we need to take into account the notion that spaces can be in some sense composed by its regions. If we wish to define a structure on a space, we need to figure out

- Is this structure also reflected on its constitutive regions
- Can we construct this structure by patching together the ones from its regions

The former is the type of notion that we saw with the presheaf's restriction map, and the latter will be that given by *descent*. This is the process by which we will consider our various spaces as a collection of subspaces and how the various structures on them are meant to work together.

First let's see how a space can come together as the union of its regions. For some topological space (X, τ), we can reconstruct X by the quotient of the coproduct of its open sets, ie

$$X \cong \coprod_i U_i / \cong \tag{4.138}$$

where we define the equivalence relation to be that if two open sets have a non-empty intersection, then the points in the intersection are identified. If we have for instance two open sets U_1, U_2 and their intersection $U_{12} = U_1 \cap U_2$,

$$\forall x, y \in U_1 \sqcup U_2,\ x \cong y \leftrightarrow x = y \vee x \in \iota(U_1) \wedge y \in \iota(U_2) \wedge \exists U_{12} \tag{4.139}$$

This notion can be used to derive a variety of global structures from local ones

Before this we need to talk about the notion of fibered categories, and first of Cartesian morphisms.

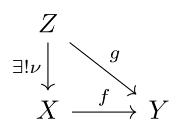
Definition 4.9.1 *Given a functor $P : \mathbf{C} \to \mathbf{D}$, we say that a morphism $f : X \to Y$ in $\mathbf{C}$ is P-*Cartesian *if for every $Z \in \mathbf{C}$ and every morphism $g : Z \to Y \in \mathbf{C}$, and for every morphism $\alpha : P(Z) \to P(X)$ in $\mathbf{D}$ such that $P(g) = P(f) \circ \alpha$, there is a unique morphism $\nu : Z \to X$ in $\mathbf{C}$ such that*

$$g = f \circ \nu \tag{4.140}$$

and

$$\alpha = P(\nu) \tag{4.141}$$

ie we have the following commuting diagram



Definition 4.9.2 *For a functor $P : \mathbf{C} \to \mathbf{D}$, this is a* fibration *if for all $X \in \mathbf{C}$ and $f_0 : Y_0 \to P(X) \in \mathbf{D}$, there exists a Cartesian morphism $f : Y \to X \in \mathbf{C}$ such that $P(f) = f_0$.*

Example 4.9.1 *Category of arrows*

4.10 Sheaves

The more important construction based on (co)presheaves is that of (co)sheaves, which are (co)presheaves with some additional conditions, meant to signify the spatial nature of the construction : the category corresponds in some sense to the piecing together of regions.

4.10.1 Sheaves on topological spaces

As for presheaves, the pedagogical model of sheaves is the one for structure sheaves on topological spaces, where we consider a functor from the frame of opens $\mathrm{Op}(X)$ to some ring, Abelian group or algebra of functions

$$F : \mathrm{Op}(X) \to \mathbf{CRing} \tag{4.142}$$

$$U \to F(U) = C(U) \tag{4.143}$$

Where typical cases are the rings of continuous, smooth or analytic real functions $U \to \mathbb{R}$.

In addition to the usual properties of a presheaf, simply stemming from its functoriality, we also have the additional properties

Definition 4.10.1 *A sheaf on a topological space is a presheaf obeying the following properties :*

- Locality *: Given some open set $U \in \mathrm{Op}(X)$ and some open cover $\{U_i \hookrightarrow U\}_{i\in I}$, and two sections $s, t \in F(U)$. Then if we have $\mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t)$ for every i of its cover, $s = t$.*
- *Given some open set $U \in \mathrm{Op}(X)$ and some open cover $\{U_i \hookrightarrow U\}_i$, and a family of sections $\{s_i \in F(U_i)\}_{i\in I}$. If those sections agree pairwise on their overlaps, ie*

$$\mathrm{res}_{U_i,U_i\cap U_j}(s_i) = \mathrm{res}_{U_j,U_i\cap U_j}(s_j) \tag{4.144}$$

  then there exists a section $s \in F(U)$ which restricts to s_i on each U_i.

Counterexample 4.10.1 *The presheaf of bounded functions does not form a sheaf.*

Proof 4.10.1 *While the presheaf of bounded function obeys locality [proof?], it is possible to construct a family of sections that does not have a global section. For instance if we pick $\{(x-1, x+1) \mid x \in \mathbb{R}\}$, which forms a cover of $\mathbb{R}$, we can form sections of bounded functions for each open set, for instance $f(x) = x$, with bound $(x-1, x+1)$, but its global section on $\mathbb{R}$ would be the identity function on $\mathbb{R}$, which is not bounded.*

Counterexample 4.10.2 *The constant presheaf does not generally form a sheaf.*

Proof 4.10.2 *Given the topological space of two points with the discrete topology,* Disc(2)*, with open sets*

$$\varnothing, \{\bullet_1\}, \{\bullet_2\}, \{\bullet_1, \bullet_2\} \tag{4.145}$$

4.10.2 General sheaves

The more general notion of a sheaf is given by sieves

In the topological case,

[...]

Consider the Yoneda embedding of **C** :

$$j : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C}) \tag{4.146}$$

Definition 4.10.2 *Given a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, and given a coverage J of $\mathbf{C}$, F is a sheaf with respect to J if*

- *for every covering family* $\{p_i : U_i \to U\}_{i \in I}$ *in* J
- *for every compatible family of elements* $(s_i \in F(U_i))_{i \in I}$*,*

there is a unique element $s \in F(U)$ *such that* $F(p_i)(s) = s_i$ *for all* $i \in I$*.*

If we consider the case where a covering family is composed of monomorphisms with subobjects (assuming the equivalence something something), then $p_i : U_i \hookrightarrow U$ can be considered [something], and the morphism generated by the sheaf is understood to be a *restriction* : $F(p_i)(s) = s_i$, we are restricting the section s on U to the subobject on U_i.

[The section of a sheaf is defined by its local elements]

The easiest example of this is to pick once again a presheaf on the frame of open $F \in \mathrm{Psh}(\mathrm{Op}(X))$, and as the coverage, pick the subcanonical coverage. For any open set $U \subseteq X$, a subcanonical coverage is a family of open sets $\{U_i\}$ such that

$$\bigcup_i U_i = U \tag{4.147}$$

[...]

A nice class of such sheaves are the sheaves given by function spaces over the appropriate sets, *sheaves of functions*. The prototypical example would be the sheaf of real valued continuous functions on a topological space, $C(-)$. For a given topological space X, take its frame of opens $\mathrm{Op}(X)$, and the sheaf associates to any open set $U \in \mathrm{Op}(X)$ the set of those functions $C(U)$.

As a presheaf, we have the notion that this assignment is a contravariant functor. In $\mathrm{Op}(X)$, any morphism $U \to X$ is an inclusion relation, $U \to X$ means that there is some inclusion map $\iota : U \hookrightarrow X$. The contravariant functorial character is then that, for the inclusion $U \hookrightarrow X$, with opposite category morphism $X \to U$, we have

$$C(X \to U) = C(X) \to C(U) \tag{4.148}$$

The corresponding operation is that of the restriction of a function. Given a function $f \in C(X)$, there is a corresponding function $C(\iota_U)(f)$, which is the restriction :

$$C(\iota_U)(f) = f\big|_U : U \to \mathbb{R} \tag{4.149}$$

Where the functorial rules furthermore say that $f\big|_X = f$ (since $F(\mathrm{Id}) = \mathrm{Id}$) and for $V \subset U \subset X$, $f\big|_U\big|_V = f\big|_V$.

The sheaf properties are usually expressed as a notion of *locality*, is then that given a covering family $\{\iota_i : U_i \hookrightarrow X\}$

that for some function on X, $f \in C(U)$,

Example 4.10.1 *For an S-valued sheaf on (C, J), the* constant sheaf

Examples with sheaves on frames of opens

For $\mathrm{Op}(X)$ with the canonical coverage, a presheaf F is a sheaf if for every complete subcategory $\mathcal{U} \hookrightarrow \mathrm{Op}(X)$,

$$F(\varinjlim \mathcal{U}) \cong \varprojlim F(\mathcal{U}) \tag{4.150}$$

Proof 4.10.3 *Complete full subcategory is a collection $\{\iota_i : U_i \hookrightarrow X\}$ closed under intersection. The colimit*

$$\varinjlim(\mathcal{U} \hookrightarrow \mathrm{Op}(X)) \cong \bigcup_i U_i \tag{4.151}$$

is the union of these open subsets. By construction,

Example 4.10.2 *The presheaf mapping all objects to the empty set $\Delta_\varnothing$ is a sheaf, called the* empty sheaf *(or* initial sheaf *as it is the initial object in the category of sheaves). As every covering family is mapped to the compatible family of elements of the empty set,*

Example 4.10.3 *The presheaf mapping all objects to the singleton $\{\bullet\}$ is a sheaf, called the* terminal sheaf*.*

Definition 4.10.3 *A* sheaf morphism *$\phi : F \to G$ for two sheaves on a site* $\mathbf{C}$ *is a natural transformation on the functors F, G, in particular preserving restriction maps via the commutative diagram*

$$\begin{array}{ccc} F(V) & \xrightarrow{\rho_{V,U}} & F(U) \\ \downarrow{\scriptstyle \phi_V} & & \downarrow{\scriptstyle \phi_U} \\ G(V) & \xrightarrow{\rho'_{V,U}} & G(U) \end{array}$$

As **Set** has a partial order relation on it by the subset relation, we similarly have

Definition 4.10.4 *A sheaf $S \in [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is a* subsheaf *of F if for all objects $U \in \mathbf{C}$, we have*

$$S(U) \subseteq F(U) \tag{4.152}$$

4.10.3 Grothendieck topology

Definition 4.10.5 *J-sheaves*

4.10.4 Subsheaves

Similarly to subpresheaves, we have that a sheaf S will be the subsheaf of another sheaf X

Theorem 4.10.1 *Subsheaves are subobjects in the category of sheaves on that site.*

Proof 4.10.4

Subobject category for sheaves

Theorem 4.10.2 *The empty sheaf is the bottom element of the poset of subsheaves.*

Proof 4.10.5 *As the empty sheaf simply maps every site object to the empty set, this simply stems from $\varnothing \subseteq A$ for any set A.*

4.10.5 Concrete sheaves

While we have seen that it is possible to define generalized spaces as sheaves, the notion of space involved may be too general in some cases, in that the "topology" of the space may not be spatial : the poset of subobjects may not be such that it could be define by relations on sets of points as we do in topological spaces.

Example 4.10.4 *From the non-spatial frame that we've defined4.5.4, we could define a sheaf over it. Take our frame*

$$\mathbf{F} = \{1 \to A \to 0\} \tag{4.153}$$

with the natural frame site structure, ie for any $U \in \mathbf{F}$, we consider the families of morphism $\{U_i \to U_i\}$ given by joins :

$$\bigvee_i U_i = U \tag{4.154}$$

For this frame, there are no such coverages outside of the trivial one, $\{U \to U\}$.

define the sheaf $\mathrm{Sh}(\mathbf{F})$

A simple enough presheaf to consider is just the representable presheaf of one of those element,

$$X_A = \mathrm{Hom}_{\mathbf{F}}(-, A) \tag{4.155}$$

which has the following probes :

$$X_A(0) = \{\leq_{0,A}\} \tag{4.156}$$
$$X_A(A) = \{\leq_{A,A}\} \tag{4.157}$$
$$X_A(1) = \{\} \tag{4.158}$$

To try to recover our more usual notion of spaces as collections of points, we need to look into the notion of concrete sheaves.

The intuition behind a concrete sheaf is that its spatial behavior is entirely determined by set of points, similarly to a topological space. That is, to a concrete sheaf X representing a space, we associate some set of points $|X|$, and to all subsheaves $S \subseteq X$ their own set of points, in a way that the interaction of those subsheaves behaves in a way consistent with them being sets of points.

First, we need to define the notion of a concrete site, so that the elements from which the sheaf is constructed itself has points :

Definition 4.10.6 *A* concrete site *is a site with a terminal object* 1 *for which*

- *The hom-functor* $h^1 = \mathrm{Hom}_{\mathbf{C}}(1, -)$ *is faithful*
- *For a covering family* $\{f_i : U_i \to U\}$,

  $$\coprod_i \mathrm{Hom}_{\mathbf{C}}(1, f_i) : \coprod_i \mathrm{Hom}_{\mathbf{C}}(1, U_i) \to \mathrm{Hom}_{\mathbf{C}}(1, U) \tag{4.159}$$

  is surjective.

For a concrete site $\mathbf{C}$, there is an *underlying set* to any object, which is given by the h^1 functor. For brevity, it will be denoted by

$$\forall U \in \mathbf{C},\ |U| \cong \mathrm{Hom}_{\mathbf{C}}(1, U) \tag{4.160}$$

This definition means that if we look at U's points, they can be entirely covered by the points of its covering family, ie for any point $x \in |U|$, there exists corresponding points in $\sqcup |U_i|$ which will map onto it. In this sense, the covering family actually "covers" U pointwise, as opposed to say the case of an object with some number of point whose covering family contains no points.

[example of the pointless frame?]

To have a concrete presheaf, in addition to being constructed from concrete sites, we will also need those concrete patches to mesh together in a way that behaves properly for a set.

Definition 4.10.7 *A presheaf* $X \in \mathrm{PSh}(\mathbf{C}, J)$ *is a* concrete presheaf *if for any* $U \in \mathbf{C}$*, the function*

$$\tilde{X}_U : X(U) \to \mathrm{Hom}_{\mathbf{Set}}(|U|, X(1)) \tag{4.161}$$

is injective.

Just as for the site, a concrete sheaf also has an underlying set, given by its probes from the terminal object, $X(1)$. We can also write

$$|X| \cong X(1) \tag{4.162}$$

This map is meant to parallel the case of the Yoneda lemma, where for this same presheaf, we also have

$$X(U) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathbf{C})}(よ(U), X) \tag{4.163}$$

So that we can in fact rewrite the condition as

$$\mathrm{Hom}_{\mathrm{PSh}(\mathbf{C})}(よ(U), X) \to \mathrm{Hom}_{\mathbf{Set}}(|U|, X(1)) \tag{4.164}$$

In the interpretation of the Yoneda lemma as the plots of a space, this means that every plot of the sheaf gives rise to a different function on its underlying sets, ie there is no case where a plot acts the same on points but differ in a purely ”spatial” manner.

”A concrete presheaf is a subobject of the presheaf

$$U \mapsto \mathrm{Hom}_{\mathbf{Set}}(|U|, |X|)$$

”

”First, for $U \in C$, write $|U|$ for the underlying set $\mathrm{Hom}_C(*, U)$, and note that we can regard $\mathrm{Hom}_C(U, V) \subseteq \mathrm{Hom}_{Set}(|U|, |V|)$ since $\mathrm{Hom}_C(*, -)$ is faithful.

Then a concrete presheaf X is given by a set $|X|$ together with, for each $U \in C$ a $|U|$-ary relation $X(U) \subseteq |X|^{|U|}$, such that for any $f : U \to V$ in C, $g \in X(V)$ implies $gf \in X(U)$, and such that X(*)=—X—.

This data defines a concrete presheaf $X : C^{op} \to Set$, and every concrete presheaf is isomorphic to one of this form.

To give a natural transformation between concrete presheaves $X \to Y$ is to give a function $|X| \to |Y|$ that preserves the relations.”

The hom-set $\mathrm{Hom}_{\mathbf{Set}}(|U|, X(1))$ is meant to represent the probes of the sheaf in terms of points.

Example : take a sheaf of two points, $\{\bullet_1, \ \bullet_2\}$, with the concrete site $\mathbf{1}$. There is exactly two such maps, Δ_1, Δ_2.

Is there a map from concrete sheaves to topological spaces?

Theorem 4.10.3 *There is a functor from a concrete site* $\mathbf{S}$ *to the category of topological spaces* **Top**.

$$\mathrm{TopReal} : \mathbf{S} \to \mathbf{Top} \tag{4.165}$$

Proof 4.10.6 *Given some object* $U \in \mathbf{S}$*, we take its set of underlying points as the points of the topological space*

$$|U| = \mathrm{Hom}_{\mathbf{S}}(1, U) \tag{4.166}$$

and we define on it the final topology making

Theorem 4.10.4 *Given the subcategory of concrete (pre?)sheaves in a (pre?)sheaf category, there exists a functor to the category of topological spaces. (Topological realization?)*

Proof 4.10.7 *Given a concrete sheaf* X*, we can consider its set of points to be* $|X|$*. The topology on this underlying set is then*

4.11 Topos

[44, 45, 46]

One important type of categories that will be the main focus of study here is that of a *topos*. There are many possible definitions and intuitions of what a topos is, many of them listed in [42], but for our purpose, a topos will mostly be about

- A universe of types in which to do mathematics
- A category of spaces
- A categorification of some types of logics

There are a few different nuances to what a topos can be, but the most general case we will look at for now (disregarding things such as higher topoi) is that of an elementary topos.

Definition 4.11.1 *An* elementary topos $\mathbf{H}$ *is a category which has all finite limits, is Cartesian closed, and has a subobject classifier.*

This definition of an elementary topos fits best in the first sense of the definitions, in that it is a universe in which to do mathematics. These properties are overall modeled over **Set**, and in some sense it is the generalization of a set. As we will see in details in 6.2, **Set** itself is a topos.

Its use as a mathematical universe is given by its closure under limits (and as we will show, colimits) and exponentiation. We can easily talk about any (finite) construction of objects in a topos, as well as any function between two elements of a topos, without leaving the topos itself, and the subobject classifier will give us some notion of subobjects in the category.

Theorem 4.11.1 *An elementary topos has all finite colimits.*

Proof 4.11.1 *Contravariant power set functor :*

$$\Omega^{(-)} : \mathbf{H}^{op} \to \mathbf{H} \tag{4.167}$$

Properties : locally Cartesian closed, finitely cocomplete, Heyting category

Giraud-Rezk-Lurie axioms

Theorem 4.11.2 *There are no finite topoi except for the initial topos.*

Proof 4.11.2 *If a topos is not the initial topos* $\mathrm{Sh}(\mathbf{0})$*,*

4.11.1 Lattice of subobjects

From the existence of all limits and colimits, we can deduce that the poset of subobjects has itself all the limits and colimits deriving from those of the underlying category, since any (co)limit in $\mathrm{Sub}(X)$ is the (co)limit in $\mathbf{C}$ with a diagram of two extra arrows to X.

Theorem 4.11.3 *In a topos, the subobjects* $\mathrm{Hom}_{\mathbf{H}}(X, \Omega)$ *of an object* X *form a Heyting algebra.*

Proof 4.11.3 *From its completeness and cocompleteness, the poset of subobject is equipped with meet and join, making it a lattice, a strict initial object, making it a bounded lattice,*

internal hom?

Is Sub(X) also a closed cartesian category?

Heyting algebra of subobject : given two subobjects $A, B \in \mathbf{H}$

$$A \wedge \tag{4.168}$$

In addition to this, as the topos is guaranteed a subobject classifier, the subobject presheaf of X is isomorphic to the hom-set

$$\mathrm{Sub}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \Omega) \tag{4.169}$$

We have therefore that every (co)limit in $\mathbf{C}$ corresponds to [some algebra on Ω].

Theorem 4.11.4 *Every object X of an elementary topos has a power object PX, given by*

$$PX = \Omega^X \tag{4.170}$$

Proof 4.11.4

Inclusion map : $X \hookrightarrow PX$

Theorem 4.11.5 *Pullback of a function to a power set :*

$$\begin{array}{ccc} U & \xrightarrow{g} & Y \\ f \big\downarrow & & \big\downarrow \{\} \\ X & \xhookrightarrow{h} & PY \end{array}$$

[42]

Theorem 4.11.6

Theorem 4.11.7 *An elementary topos is finitary extensive.*

Proof 4.11.5 *To show this, let's show that coproducts are disjoint. Consider the pushout square given by the inclusion maps of the initial object, $X +_0 Y \cong X + Y$*

$$\begin{array}{ccc} 0 & \xrightarrow{0_Y} & Y \\ 0_X \big\downarrow & & \big\downarrow \iota_Y \\ X & \xhookrightarrow{\iota_X} & X + Y \end{array}$$

By the previous theorem, this is also a pushout square of $X \hookrightarrow X + Y \hookleftarrow Y$, which is the intersection of X and Y, so that the coproduct is disjoint. As all colimits are stable under pullback in a topos, this means that the category is finitary extensive.

Theorem 4.11.8 *Any elementary topos is a distributive category, ie*

$$X \times Y + X \times Z \cong X \times (Y + Z) \tag{4.171}$$

topos is always a distributive category

As a Heyting algebra, any algebra of subobjects of a topos admits the double negation property

$$U \to \neg_X \neg_X U \tag{4.172}$$

$$\neg\neg = [[-, \varnothing], \varnothing] \tag{4.173}$$

4.11.2 Grothendieck topos

The most common type of topos, and the one we will typically use, is the Grothendieck topos, based around the category of sheaves on a given site.

As most of the proofs related to this will be easier going first through presheaves as a topos, let's first look at the presheaf topos.

Definition 4.11.2 *A* presheaf topos *on a category* $\mathbf{C}$ *is the functor category of all presheaves on* $\mathbf{C}$*,*

$$\mathrm{PSh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \tag{4.174}$$

Theorem 4.11.9 *A presheaf topos is an elementary topos.*

Proof 4.11.6 *By the property of limits on presheaves, and as* **Set** *has all limits, this means that* $\mathrm{Psh}(\mathbf{C})$ *has all limits. The*

Definition 4.11.3 *A* Grothendieck topos E *on a site* $\mathbf{C}$ *with coverage* J *is the category of sheaves over the site* $(\mathbf{C}, \mathcal{J})$

$$\mathcal{E} \cong \mathrm{Sh}(\mathbf{C}, \mathcal{J}) \tag{4.175}$$

Proposition 4.11.1 *A Grothendieck topos is an elementary topos*

Proof 4.11.7

Example 4.11.1 *A trivial example of a Grothendieck topos is the initial topos, which is the sheaf topos over the empty category with the empty topology (which is the maximal topology on this category),* $\mathrm{Sh}(\mathbf{0})$*. The only element of this topos is the empty functor, with the identity natural transformation on it (as there is no possible components to differentiate them on the empty category, this is the only one). We therefore have*

$$\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1} \tag{4.176}$$

Its only object $*$ *is both the initial and terminal object (therefore a zero object), its product and coproduct are simply* $* + * \cong *$ *and* $* \times * \cong *$

One important nuance in topos theory is that a topos can be considered alternatively as a space in itself, or as a category in which every object is a space. The former is referred to as a *petit topos*, while the latter is a *gros topos*. A typical example of this would be for instance the topos of smooth spaces **Smooth**, which contains (among other things) all manifolds, as a gros topos, while the topos of the site of opens on a topological space $\mathrm{Sh}(\mathrm{Op}(X))$ would be an example of a petit topos.

Theorem 4.11.10 *For any topos* $\mathbf{H}$*, the slice category given by one of its object* $\mathbf{H}_{/X}$ *is itself a topos.*

This construction allows us to bridge the gros and petit topos, in that given a space X in a gros topos $\mathbf{H}$, its corresponding petit topos will be $\mathbf{H}_{/X}$.

"Also in 1973 Grothendieck says the objects in any topos should be seen as espaces etales over the terminal object of the topos, in a generalized sense that includes saying any orbit of a group action lies "etale" over a fixed point. "

As a topos, a Grothendieck topos has a subobject classifier Ω, which, as an object of a sheaf category, will be itself a sheaf. From the properties of a sheaf, we can also look at its various properties.

Any sheaf topos has an object with the property of a subobject classifier, given by the sheaf of principal sieves. That is, for any object U of the site,

$$\Omega(U) = \{S | S \text{ is a sieve on } U\} \tag{4.177}$$

$$\Omega(f) : \Omega(U) \to \Omega(V),\ \Omega(g)(S) = S\Big|_g = \{h | g \circ h \in S\} \tag{4.178}$$

To show the subobject classifier, we first need to show the existence of a sheaf morphism from the terminal sheaf to this sheaf, $\top : 1 \to \Omega$. This is simply given by the components

$$\top_X : 1(X) \quad \to \quad \Omega(X) \tag{4.179}$$

$$* \quad \mapsto \quad t(X) \tag{4.180}$$

Example 4.11.2 *In the sheaf topos* $\mathbf{Set} \cong \mathrm{Sh}(\mathbf{1})$

Sheaf topos is Cartesian closed, internal hom from this monoidal structure

Theorem 4.11.11 *In a Grothendieck topos, all coproducts are disjoint, in the sense that for any two subobjects of the same object,* $\iota_1, \iota_2 : X_1, X_2 \hookrightarrow Y$*, their intersection (pullback) is empty (the initial object) if their union (pushout) is isomorphic to the coproduct. For the diagram* $F = X_1 \hookrightarrow Y \hookleftarrow X_2$*,*

$$\lim_I F \cong 0 \to \mathrm{colim}_I(F) \cong X_1 + X_2 \tag{4.181}$$

Example 4.11.3 *Given the site of the interval category* $\tilde{\mathbf{2}} = \{0 \to 1\}$*, with coverage* $\{0, 1, \{0, 1\}\}$ *(?), the corresponding sheaf topos is the* Sierpinski topos*,*

$$\mathrm{Sh}(1 \to 2) \tag{4.182}$$

4.11.3 Lawvere-Tierney topology

[47]

Definition 4.11.4 *On a topos* **H**, *a* Lawvere-Tierney topology *is given by a morphism* j *on the subobject classifier, ie*

$$j : \Omega \to \Omega \tag{4.183}$$

which obeys the following properties :

- $j \circ \top = \top$
- $j \circ j = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

As a morphism on Ω corresponds to a subobject of it, by

$$\mathrm{Sub}(\Omega) \cong \mathrm{Hom}_{\mathbf{H}}(\Omega, \Omega) \tag{4.184}$$

Definition 4.11.5 *For a subobject* U *of* X*, the* closure operator *induced by* j *is a morphism*

$$\overline{(-)}_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X) \tag{4.185}$$

$$U \mapsto \overline{U}_X \tag{4.186}$$

which obeys

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(X,\Omega) & \xrightarrow{\cong} & \mathrm{Sub}(X) \\ {\scriptstyle \mathrm{Hom}_{\mathbf{C}}(1,j)} \downarrow & & \downarrow {\scriptstyle \overline{(-)}_X} \\ \mathrm{Hom}_{\mathbf{C}}(X,\Omega) & \xrightarrow{\cong} & \mathrm{Sub}(X) \end{array}$$

Example 4.11.4 *The basic example of a Lawvere-Tierney topology is in the topos of sets, where it is simply the identity map, obeying trivially all the properties of that operator. The associated subobject for this is the truth subobject* $\top$.

The closure of any subobject is

Analog of Grothendieck topology for a topos? [47]

The term of a closure operator derives from its interpretation in the context of a sheaf on a topological space. If given a topological space (X, τ) we pick our sheaf on the site of its category of opens, the category $\mathrm{Op}(X)$ with the canonical coverage, and given a collection $C = \{U_i\}_{i \in I}$, the locality operator j maps C to the open sets covered by C.

$$j(C) = \{U \in \mathrm{Op}(X) | U \subseteq \bigcup_{i \in I} U_i\} \tag{4.187}$$

Properties :

- If $U \in C$, $U \in j(C)$
- $j(\mathrm{Op}(X)) = \mathrm{Op}(X)$
- $j(j(C)) = j(C)$
- $j(C_1 \cap C_2) \subseteq j(C_1) \cap j(C_2)$
- If C_1, C_2 are sieves, $j(C_1 \cap C_2) = j(C_1) \cap j(C_2)$
- If $C = S(U)$, $j(C)$ is also a sieve.

If C is a sieve, it is an element of $\Omega(U)$, the subobject classifier on the topos of presehaves on X.

Generalization : j is a map $j : \Omega \to \Omega$ on the subobject classifier of a topos, the *Lawvere-Tierney topology*, with properties

- $j \circ \top = \top$
- $j \circ j = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

j is a modal operator on the truth values Ω.

Example 4.11.5 *In the Grothendieck topos $E = \mathrm{Set}^{\mathrm{Op}(X)^{\mathrm{op}}}$ of presheaves on X a topological space, Classifier object $U \mapsto \Omega(U)$, the set of all sieves S on U, a set of open subsets $V \subseteq U$ such that $W \subseteq V \in S$ implies $W \in S$.*

Each open subset $V \subseteq U$ determines a principal sieve $\hat{V}$ consisting of all opens $W \subseteq V$

The map $\top_U : 1 \to \Omega(U)$ is the map that picks the maximal sieve $\hat{U}$ on U.

$$J(U) = \{S |\ S \text{ is a sieve on } U \text{ and } S \text{ covers } U\ \} \tag{4.188}$$

S covers U means

Adjoint of closure : interior?

4.11.4 Open subobjects

Open sets in terms of functions for Top : Given the Sierpinski space S, with set $\{0,1\}$ and topology $\{\varnothing, \{1\}, \{0,1\}\}$,

Given any object X in a topos, with the terminal morphism

$$!_X : X \to 1 \tag{4.189}$$

we say that X is *subterminal* if $!_X$ is a monomorphism. As the terminal object is the terminal sheaf mapping every object of the site to $\{\bullet\}$, this means that a subterminal object is a sheaf mapping every object to a subsingleton set, ie either empty or a singleton.

For $g_1, g_2 : Y \to X$,

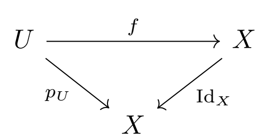
$$!_X \circ g_1 =!_X \circ g_2 \to g_1 = g_2 \tag{4.190}$$

Open sets : comodal objects of the interior comonad?

Gros topos / petit topos thing

Terminal object in $\mathbf{H}_X$: $\mathrm{Id}_X : X \to X$

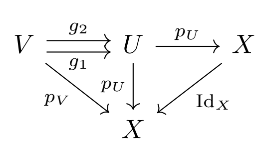
Map $!_U : p_U \to 1$, $p_U : U \to X$, a morphism $f : U \to X$ such that $\mathrm{Id}_X \circ f = p_U$, ie $f = p_U$



ie any terminal map $p_U \to 1$ is the map corresponding to $p_U : U \to X$

Monomorphism : for another object V, defining a bundle $p_V : V \to X$, with two morphisms $\overline{g}_1, \overline{g}_2 : p_V \to p_U$, corresponding to two morphisms $g_1, g_2 : V \to U$ where $p_V \circ g_i = p_U$. This is a monomorphism if

$$!_U \circ \overline{g}_1 =!_U \circ \overline{g}_2 \to \overline{g}_1 = \overline{g}_2 \tag{4.191}$$



or in terms of the original category [?]

$$p_U \circ g_1 = p_U \circ g_2 \to g_1 = g_2 \tag{4.192}$$

if p_V, p_U are monomorphisms,

Example 4.11.6 *In a Grothendieck topos, every bundle from an object of the site is subterminal.*

Proof 4.11.8 *Given a topos* $\mathbf{H} = \mathrm{Sh}(\mathbf{C})$*, for some object of the site* $U \in \mathbf{C}$*, we can embed it in* $\mathbf{H}$ *via the Yoneda embedding* よ(U)*. For any other object of the site* U'*, we have*

$$よ(U)(U') = \mathrm{Hom}_C(U', U) \tag{4.193}$$

If we have some morphism $f :$ よ$(U) \to X$ *for some object* X*, in the slice category* $\mathbf{H}_{/X}$*,*

$$Z \rightrightarrows \mathrm{Hom}(-, U) \to X \tag{4.194}$$

Open subobjects : modal types of j?

4.11.5 Factorization system

In any topos, we have an orthogonal factorization system such that for any morphism $f : X \to Y$, this can be factorized uniquely as

$$f = e \circ m \tag{4.195}$$

For $e \in E$, the class of all epimorphisms, and $m \in M$, the class of all monomorphisms. This is the *(epi, mono)-factorization system.*

Theorem 4.11.12 *Any elementary topos admits an (epi, mono)-factorization system.*

Proof 4.11.9

Example 4.11.7 *Sets and the restriction, corestriction system*

4.12 Stalks and étale space

[47]

4.12.1 In a topological context

The notion of a stalk is meant to encode local behaviours in a sheaf. For instance, if we consider a structure sheaf on a topological space, like say some set of continuous maps $X \to \mathbb{R}$ with some specific properties, the stalk of that sheaf is the behaviour of those maps at a single point. The stalk is to be understood as some equivalence class of those functions that "behave similarly" at the given point.

If we considered for instance the discrete space X, for which all functions (as functions on sets) $X \to \mathbb{R}$ are continuous, the stalk at $x \in X$ would simply be the value at that point, as there are no further constraints on those functions. Any function with that value at that point will be part of the stalk.

However, if we take a more complex case such as smooth functions on a manifold, the local behavior of a function cannot be only reduced to its mere value at that point. It has slightly further "reaching" behaviors, such as the values of its derivatives etc.

To get the proper definition of a stalk, we need to look at the frame of opens of our space $\mathrm{Op}(X)$, and

Definition 4.12.1 *Given a sheaf of rings F on a topological space X, its stalk at x is given as the directed limit over the neighbourhood net of x*

$$F_x = \lim_{U \in N(x)} F(U) \tag{4.196}$$

Example 4.12.1 *As we discussed, the stalks for the structure sheaf of continuous functions of a discrete space $C(X, Y)$ for some space Y is Y itself.*

Proof 4.12.1 *In a discrete space, the neighbourhood net of a point is the poset given by every subset containing that point ordered by inclusion*

$$N(x) = \{S \subseteq X \mid x \in X\} \tag{4.197}$$

The associated directed set is given by reverting the arrows and applying the functor

In **Top**, consider an object B (the base space), and take the slice category $\mathbf{Top_{/B}}$, the category of bundles $\pi : E \to B$ over B.

If π is a local homeomorphism, ie for every $e \in E$, there is an open neighborhood U_e such that $\pi(U_e)$ is open in B, and the restriction $\pi|_{U_e} : U_e \to \pi(U_e)$ is a homeomorphism, then we say that $\pi : E \to B$ is an *étale space*, with $E_x = \pi^{-1}(x)$ the *stalk* of π over x.

For $\mathrm{Sh}(\mathbf{C}, J)$ a topos, if F is a sheaf on $(\mathbf{C}, J)$, the slice topos $\mathrm{Sh}(\mathbf{C}, J)/F$ has a canonical étale projection $\pi : \mathrm{Sh}(\mathbf{C}, J)/F \to \mathrm{Sh}(\mathbf{C}, J)$, a local homeomorphism of topoi, the étale space of F.

For any object $X \in \mathbf{C}$, $y(X)$ the Yoneda embedded object,

$$U(X) = \mathrm{Sh}(\mathbf{C}, J)/y(X) \tag{4.198}$$

Sections of π_F over $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ are in bijection with elements of $F(X)$.

If $(\mathbf{C}, J)$ is the canonical site of a topological space, each slice $\mathrm{Sh}(\mathbf{C}, J)/F$ is equivalent to sheaves on the étale space of that sheaf. In particular, $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ corresponds to the inclusion of an open subset.

4.12.2 In a sheaf

Stalks can be generalized from the case of sheaves on a topological space to the general case.

4.13 Topological topoi

While the category of topological spaces **Top** is *not* a topos, I feel like I should bring up some comments on this, as the notion of spaces in mathematics as a topological space is historically very deeply anchored, for various reasons such as history, ease of use, broadness, etc.

In terms of category, we have the category of topological spaces **Top**, with objects the class of all topological spaces, and morphisms the class of all continuous functions. From this definition, we can tell that **Top** is a concrete category, where the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is simply the functor associating the underlying set X of a topological space (X, τ), and it is faithful as we have

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y) = C(X, Y) \tag{4.199}$$

The injectivity to $\mathrm{Hom}_{\mathbf{Set}}(U(X), U(Y))$ implies that two different continuous functions lead to two different functions, which is trivially true since we defined our continuous functions to merely be a subset of the functions here.

The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has a left and right adjoint, the first giving us the reflective subcategory of discrete spaces, the second giving us the coreflective subcategory of trivial spaces.

[proof]

Adjoints

Theorem 4.13.1 *The initial and terminal objects of* **Top** *are the empty topological space* $\varnothing$ *and the singleton topological space* $\{\bullet\}$*, both of which have a unique topology :*

$$\tau_{\varnothing} = \mathcal{P}(\varnothing) = \{\varnothing\} \tag{4.200}$$

$$\tau_{\{\bullet\}} = \mathcal{P}(\{\bullet\}) = \{\varnothing, \{\bullet\}\} \tag{4.201}$$

Proof 4.13.1 *As the initial object is given by the empty set, for which there is only one possible function to any other function (the empty function), we only need to prove that this is always a continuous function, which is vacuously true since an empty function has an empty preimage.*

The same goes for the terminal object, as any set has a single morphism to the singleton, so that we only need to show it to be a continuous function. This is

Theorem 4.13.2 **Top** *admits a product and coproduct, which are given by the product topology and coproduct topology.*

Proof 4.13.2 *The product topology of* (X, τ_X) *and* (Y, τ_Y) *is given by the set* $X \times Y$ *equipped with the topology*

Theorem 4.13.3 **Top** *admits equalizers and coequalizers, which are given by the subspace topology and the quotient topology.*

Proof 4.13.3 *Given two topological spaces* X, Y *and two parallel morphisms between them,*

$$X \to Y \tag{4.202}$$

From this, we have that **Top** is both finitely complete and cocomplete. However, this is as far as we can go to make it a topos.

Theorem 4.13.4 **Top** *is not a balanced category.*

Proof 4.13.4 *In* **Top***, morphisms that are both mono and epi are continuous bijective functions, ie There are however continuous bijections which are not homeomorphisms. Take the topological spaces* $\mathbb{R}_{\text{can}}$*, the real line with the canonical topology, and* $\mathbb{R}_{\text{disc}}$*, the real line with the discrete topology. Take the continuous function which is the identity function on sets,*

$$f : \mathbb{R}_{\text{disc}} \to \mathbb{R}_{\text{can}} \tag{4.203}$$

As any map from a discrete space to a topological space is continuous, this function is continuous, as well as trivially bijective. However, the image of an open set $\{x\}$ *in* $\mathbb{R}_{\text{disc}}$ *is a closed set in* $\mathbb{R}_{\text{can}}$*, and therefore not a homeomorphism.*

From this we have that there cannot be a subobject classifier on **Top**.

The non-boolean structure of the poset of subobjects for **Top** is due to

The other obstruction to this is the lack of exponential object in **Top** :

Theorem 4.13.5 *There is no exponential object in* **Top**.

Proof 4.13.5 *To prove this we only need to find an object of* **Top** *that is not exponentiable, which is given by the fact that a topological space X is only exponentiable if the functor $X \times -$ preserves coequalizers, ie if we have a quotient map $q : Y \to Z$, for some space Y, then the function*

$$q \times \mathrm{Id}_X : Y \times X \to Z \times X \tag{4.204}$$

is also a quotient map.

"This functor always preserves coproducts, so this condition is equivalent to saying that $X \times -$ preserves all small colimits. This is then equivalent to exponentiability by the adjoint functor theorem."

A counter example to this is to take $\mathbb{Q}$ the space of rational numbers with the subspace topology from $\mathbb{R}$ and the quotient map from $\mathbb{Q}$ to the quotient space $\mathbb{Q}/\mathbb{Z}$, the rationals up to the equivalence

$$\forall k \in \mathbb{Z},\ q \sim q + k \tag{4.205}$$

with the appropriate quotient topology. This is roughly the space of all rational points on the circle S^1. Given this, we can show that the Cartesian product $\mathbb{Q} \times -$ does not preserve quotients, as we have that

$$q \times \mathrm{Id}_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \quad \to \quad (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} \tag{4.206}$$

$$(q_1, q_2) \quad \mapsto \quad (f(q_1), q_2) \tag{4.207}$$

is not a quotient map. To prove this,

Take the sequence

$$r_n = \begin{cases} 1 & n = 0 \\ \sqrt{2}/|n| & n > 0 \end{cases} \tag{4.208}$$

which is irrational for $n > 0$ and converges to 0. Take A_n to be an open subset of $[n, n+1] \times \mathbb{R}$ such that the closure of A is given by the boundary points

$$x \tag{4.209}$$

Due to this, **Top** is not a topos, quasi-topos or even Cartesian closed category, and is therefore not a particularly good category to perform categorical processes in.

There are several possible ways to try to find a compromise to have a topos of topological spaces. One is to take some subcategory that does form a topos [sober spaces idk]

A better behaved subcategory of **Top** is the category of compact Hausdorff spaces **CHaus**, which, in addition to being complete and cocomplete [proof?],

As compact Hausdorff spaces are rather limiting, we will instead use the broader category of compactly generated topological spaces.

Definition 4.13.1 *A topological space X is said to be* compactly generated *if it obeys one of the equivalent properties :*

- *For any space Y and function on the underlying sets $f : |X| \to |Y|$, then the lifts*

Convenient category of topological spaces : subcategory of Top such that

Every CW complex is an object **C** is Cartesian closed **C** is complete and cocomplete Optional : **C** is closed under closed subspaces in Top : if $X \in \mathbf{C}$ and $A \subseteq X$ is a closed subspace, then A belongs to **C**.

Theorem 4.13.6 *Every first-countable topological space is compactly generated.*

Proof 4.13.6

Another way is to find some other category that has a broad intersection with **Top** (ex smooth spaces).

Theorem 4.13.7 *There's an embedding of Δ-generated topological spaces in the topos of smooth spaces.*

Why **Top** isn't a topos : not Cartesian closed or localy Cartesian closed[48]

4.14 Geometry

The broad notion of "geometry" in a topos involved the use of so-called *geometric morphisms* (although in terms of the topos itself, those are actually functors).

The underlying notion of geometric morphisms can be understood from the point of view of Grothendieck topoi. Take two such topoi, with sites $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$. We will consider some morphism of sites $f : X \to Y$, inducing a functor by precomposition :

$$(-) \circ f : \mathrm{Sh}(Y) \rightarrow \mathrm{Sh}(X) \tag{4.210}$$
$$(F : Y^{\mathrm{op}} \rightarrow \mathbf{Set}) \mapsto (G : X^{\mathrm{op}} \rightarrow \mathbf{Set}) \tag{4.211}$$

ie for some element $y \in Y$, and a presheaf F, we have the map

$$F \circ f : Y \rightarrow \mathrm{Set} \tag{4.212}$$

Upon restriction to act on sheaves, this is our inverse image functor f^*, with the right adjoint to this being the direct image functor.

$$f_* : \mathrm{Sh}(X) \rightarrow \mathrm{Sh}(Y) \tag{4.213}$$
$$f^* : \mathrm{Sh}(Y) \rightarrow \mathrm{Sh}(X) \tag{4.214}$$

Example 4.14.1 *The basic example which gives the morphisms their names is the case of a sheaf over a topological space X, where the site is the category of opens* $\mathrm{Op}(X)$*, and a common type of sheaf is the sheaf of functions to some space A :* $\mathrm{Sh}(\mathrm{Op}(X)) = C(X, A)$*. A presheaf F is then a map*

$$F : \mathrm{Op}(X) \rightarrow \mathrm{Set} \tag{4.215}$$
$$U \mapsto C(U, A) \tag{4.216}$$

The site morphism is then given by some continuous function $f : X \rightarrow Y$, in which case the direct image functor is

and the inverse image functor is given by

$$f^* : \mathrm{Sh}(Y) \rightarrow \mathrm{Sh}(X) \tag{4.217}$$
$$F \mapsto F \circ f \tag{4.218}$$

ie given some open set $U_Y \in \mathrm{Op}(Y)$, with image $f(U_Y) = U_X$, we have

$$(f^* F)(U_Y) = F \circ f(U_Y) = F(U_X) \tag{4.219}$$

A morphism of site in this case is a continuous function $f : X \rightarrow Y$, which induces a functor on $\mathrm{Op}(X)$ *by restriction :*

$$f(U \in \mathrm{Op}(X)) = \tag{4.220}$$

$$f_* F(U) = F(f^{-1}(U)) \tag{4.221}$$

For two toposes E, F, a geometric morphism $f : E \to F$ is a pair of adjoint functors (f^*, f_*)

$$f_* : E \to F \quad (4.222)$$
$$f^* : F \to E \quad (4.223)$$

such that the left adjoint f^* preserves finite limits. f_* is the *direct image functor*, while f^* is the *inverse image functor*.

Example 4.14.2 *Another similar example to look at the basic functioning of the geometric morphisms is to consider two slice topos from the category of sets. Taking two sets X and Y, consider the slice topoi $\mathbf{Set}_{/X}$ and $\mathbf{Set}_{/Y}$.*

Example : One of the most common type of geometric morphism on a (Grothendieck) topos is the case of global sections. The site morphism involved is from whichever site we decide on our topos X to the trivial site $*$, so that our geometric morphism is between our topos and the topos of sets, $\mathbf{Set} = \mathrm{Sh}(*)$. The only site morphism available here is the constant functor

$$p : X \to * \quad (4.224)$$

which is a site morphism as any covering family of X is sent to $\mathrm{Id}_* : * \to *$, which is the only covering of 1. As the terminal category does not have much in the way of limits, we will have to show that this functor is filtered.

The induced functor is therefore some functor from the category of sets to our topos, so that for any object $F : X^{\mathrm{op}} \to \mathbf{Set}$ in our topos, and any object $x \in X$ in the site, the precomposition becomes

$$(-) \circ p : \mathrm{Set} \to \mathrm{Sh}(X) \quad (4.225)$$
$$(A : * \to \mathrm{Set}) \mapsto (A \circ p : X^{\mathrm{op}} \to * \to \mathbf{Set}) \quad (4.226)$$

ie for any "sheaf" $* \to$ A (a set), we obtain a sheaf on X simply giving us back this set.

there is only one morphism between two sites, $\mathrm{Id}_* : * \to *$. The induced functor on the sheaf is

$$F \circ \mathrm{Id}_* : * \quad (4.227)$$

direct image functor is

$$x \tag{4.228}$$

If f^* has a left adjoint $f_! : E \to F$, f is an essential geometric morphism.

Direct image functor :

$$f_* F(U) = F(f^{-1}(U)) \tag{4.229}$$

Global section : if $p : X \to *$, $*$ the terminal object of the site

Inverse image functor :

$$f^{-1} G(U) = G(f(U)) \tag{4.230}$$

Definition 4.14.1 *The category* Topos *is the category with as objects all topoi and as morphisms the geometric morphisms between those topoi.*

Definition 4.14.2 *If a geometric morphism $(f^* \dashv f_*)$ has a further right adjoint $f^!$,*

$$(f^* \dashv f_* \dashv f^!) : \mathbf{E} \begin{array}{c} \xrightarrow{\quad f^* \quad} \\ \xleftarrow{\quad f_! \quad} \\ \xrightarrow{\quad f^* \quad} \end{array} \mathbf{S}$$

such that $f^!$ is fully faithful, we say that it is a local geometric morphism.

Definition 4.14.3 *If a geometric morphism $(f^* \dashv f_*)$ has a further left adjoint $f_!$,*

we say that it is an essential geometric morphism.

Connection to image, orthogonal factorization

4.15 Subtopos

As the geometric morphism is the natural map between topoi, we need some kind of inclusion map to define the notion of a subtopos. This is given by the geometric embedding :

Definition 4.15.1 *A* geometric embedding $f : \mathbf{H}_S \hookrightarrow \mathbf{H}$ *is a geometric morphism for which the direct image functor f_* is full and faithful (so that $\mathbf{H}_S$ is a full subcategory).*

In particular, in a Grothendieck topos, a

Theorem 4.15.1 $\epsilon : f^* f_* \to \mathrm{Id}_{\mathbf{H}_S}$ *is an isomorphism.*

Proof 4.15.1

Reflective localization?

Example 4.15.1 *The initial topos* $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$ *is always a subtopos of any topos.*

Proof 4.15.2 *As there is only one map from* $\mathbf{H}$ *to* $\mathbf{1}$*, we can take this as a baseline.*

$$\Delta_* : \mathbf{H} \hookrightarrow \mathbf{1} \tag{4.231}$$

Then the left adjoint of this map is

$$\mathrm{Hom}_{\mathbf{H}}(L(*), X) \cong \mathrm{Hom}_{\mathbf{1}}(*, \Delta_*(X)) \tag{4.232}$$

so that there is only one map from $L(*)$ *to any object of the topos, ie it is the constant map to the initial object,* Δ_0*. Dually if we try to find its right adjoint, it will be the constant map to the terminal functor* Δ_1*.*

Theorem 4.15.2 *The slice category* $\mathbf{H}_{/X}$ *of a topos* $\mathbf{H}$ *is itself a topos and a subtopos of* $\mathbf{H}$*.*

Proof 4.15.3 *(Maclane thm 7.1) As the topos contains an initial object, any of its slice categories contains an initial and final object. The product in a slice category is simply the pullback in* $\mathbf{H}$*, which does always exist in a topos. [equalizer?] "the equalizer of two arrows X Y in E / B is clearly just the equalizer in E equipped with the evident map to B" The subobject classifier* $\Omega \times X \to X$ *Power object*

Definition 4.15.2 *For subterminal object* $U \in \mathbf{H}$*, the map* $o_U(V) = U \to V$ *defines a Lawvere-Tierney topology*

over topos, comma topos?

Definition 4.15.3 *A subtopos* $\iota : \mathbf{H}_j \hookrightarrow \mathbf{H}$ *with Lawvere-Tierney topology* j *is a* dense subtopos

Lawvere-Tierney topology j

Level of a topos : an essential subtopos $H_l \hookrightarrow H$ is a *level* of H.

"the essential subtoposes of a topos, or more generally the essential localizations of a suitably complete category, form a complete lattice"

"If for two levels $H_1 \hookrightarrow H_2$ the second one includes the modal types of the idempotent comonad of the first one, and if it is minimal with this property, then Lawvere speaks of "Aufhebung" (see there for details) of the unity of opposites exhibited by the first one."

4.16 Motivic yoga

[Keep?]

In addition to the geometric morphisms and their two induced functors, direct images f

[49, 50, 51] Six functor formalism

4.17 Localization

[52]

In some cases we wish to create a new category from an old one by "quotienting it" along some specific morphisms, at least up to isomorphisms. That is, we wish to declare that a given morphism is in fact an isomorphism, by adjoining an inverse morphism for all such morphisms.

4.17.1 Localization of a commutative ring

A non-categorical example of this is given by the notion of localizing a commutative ring.

If given some commutative ring $(R, +, \cdot)$, we speak of a *localization* at some subset $S \subseteq R$, denoted by $R[S^{-1}]$, if we give each of those elements a multiplicative inverse.

Example 4.17.1 *For the ring $\mathbb{Z}$, its localization at $S = \mathbb{Z} \setminus \{0\}$ is the field of rational numbers, $\mathbb{Q}$.*

As we are making every element of S invertible, we can also much more simply localize it at the value of its primes, since any rational numbers can be written out as

$$q = \frac{a}{b} = a \prod_i (\frac{1}{p_i})^{k_i} \tag{4.233}$$

So that in fact we can write it as $\mathbb{Q} = \mathbb{Z}[\mathbb{P}^{-1}]$. This is a general notion that can be applied,

Theorem 4.17.1 *For any localization $R[S^{-1}]$*

Example 4.17.2 *The localization of the ring of polynomials $k[x]$*

We can of course localize rings at narrower sets, such as for the case of $\mathbb{Z}[\frac{1}{2}^{-1}]$, the ring of half-integers, which is for instance the ring of values for particle spin in quantum mechanics.

More interesting for us is the case of smooth rings, such as $C^\infty(M)$

Theorem 4.17.2 *The localization of $C^\infty(M)$ at the set S_x of functions that do not vanish on x is the germs of smooth functions at x :*

$$S_x^{-1}C^\infty(M) \cong C_x^\infty(M) \tag{4.234}$$

Proof 4.17.1

4.17.2 Categorical localization

In the case of a category $\mathbf{C}$, we speak of a localization at a set $W \subseteq \mathrm{Mor}(\mathbf{C})$ of morphisms. The localized category $\mathbf{C}[W^{-1}]$ is then a larger category for which every morphism in W admits an inverse.

Definition 4.17.1 *A localization of a category $\mathbf{C}$ by a set of morphisms $\mathbf{C}[W^{-1}]$ is a category $\mathbf{C}[W^{-1}]$ equipped with a functor*

$$Q : \mathbf{C} \to \mathbf{C}[W^{-1}] \tag{4.235}$$

such that

$$\forall f : X \to Y \in W,\ Q(f) \in \mathrm{Iso}(X, Y) \tag{4.236}$$

A motivating example for this is algebraic cases. For instance, if we have a natural number object N, we can consider its automorphisms corresponding to additions by some number, ie

$$(+k) : N \quad \to \quad N \tag{4.237}$$

$$n \quad \mapsto \quad n + k \tag{4.238}$$

In terms of internal operations, this is given by all the maps $k : 1 \to N$ and the addition map $+ : N \times N \to N$, where the $(+k)$ map is then obtained by

$$N \xrightarrow{\langle \mathrm{Id}_N, k \circ !_N \rangle} N \times N \xrightarrow{+} N \tag{4.239}$$

$n \mapsto (n, k) \mapsto n + k$

These maps are not invertible, as there is (outside of $+0$) no inverse in N for each of these.

This does give us the definition on an *integers object* Z.

This process can be used for a wide variety of cases, identifying objects such as homotopy equivalent objects,

[53] For a category C and a collection of morphisms $S \subseteq \mathrm{Mor}(C)$, an object $c \in C$ is S-local if the hom-functor

$$C(-, c) : C^{\mathrm{op}} \to \mathrm{Set} \tag{4.240}$$

sends morphisms in S to isomorphisms in Set, so that for every $(s : a \to b) \in S$,

$$C(s, c) : C(b, c) \to C(a, c) \tag{4.241}$$

is a bijection

”localization of a category consists of adding to a category inverse morphisms for some collection of morphisms, constraining them to become isomorphisms”

Example 4.17.3 *The basic example of a localization is that of a commutative ring. localizing with prime* 2 *:* $\mathbb{Z}[1/2]$*, localization away from all primes :* $\mathbb{Q}$

Example 4.17.4 *Localization of* $\mathbb{R}[x]$ *away from* a *: rational functions defined everywhere except at* a

Localization at a class of morphisms W : reflective subcategory of W-local objects (reflective localization).

Localization of an internal hom : localization of the morphisms defined by $\prod_{X,Y}[X, Y]$?

Localization of a topos corresponds to a choice of Lawvere topology, localization of a Grothendieck topos to a Grothendieck topology.

Duality of a localization?

4.17.3 Reflective localization

Definition 4.17.2 *A localization* $\mathbf{C}[W^{-1}]$ *is* reflective *if the localization functor* Q *admits a fully faithful right adjoint, ie*

$$(Q \dashv T) : \mathbf{C} \to \mathbf{C}[W^{-1}] \tag{4.242}$$

Theorem 4.17.3 *Every reflective subcategory* $(\iota, T) : \mathbf{C} \to \mathbf{D}$ *is a reflective localization at the preimage of the isomorphism in*

4.18 Moduli spaces and classifying spaces

As we've seen in 4.7.2, for categories of presheaves on some underlying category **C**, there exists special presheaves called the *representable presheaves*, which are such that they can be written as a hom-functor

$$F = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{4.243}$$

with X the *representing object*. Those are in some sense the equivalent of the objects of **C** in our topos.

Representation of a functor F :

$$\theta : \mathrm{Hom}_{\mathbf{C}}(-, X) \xrightarrow{\cong} F \tag{4.244}$$

"As above, the object c is called a representing object (or often, universal object) for F, and the element e is called a universal element for F. Again, it follows from the Yoneda lemma that the pair (c,e) is determined uniquely up to unique isomorphism."

Fine v. coarse moduli space

Definition 4.18.1 *An object* $X \in [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ *is a* coarse moduli space

Example 4.18.1 *In the category of sheaves on manifolds,* $[\mathrm{SmoothMan}^{\mathrm{op}}, \mathbf{Set}]$*, The moduli space of k-differential forms* Ω^k *associates to every manifold* M *its set of k-differential forms* $\Omega^k(M)$ *Correspondence between the actual space of k-differential form* $\Omega^k(M)$ *as a sheaf itself and the internal hom* $[X, \Omega^k]$

Moduli space : space that is a

Classifying morphisms,

beep

4.19 Number objects

One benefit of topoi as a category is the guaranteed existence of a natural number object under broad circumstances[54], which is an internalization of the notion of positive integers. As we wish for our topoi to be a universe for mathematics, this is a fairly fundamental object.

Definition 4.19.1 *A* natural number object *for a topos is an object denoted* $\mathbb{N}$ *such that there exists the morphisms*

- *The morphism* $z : 1 \to \mathbb{N}$
- *The* successor morphism $s : \mathbb{N} \to \mathbb{N}$

such that for any diagram $q : 1 \to X$, $f : A \to A$, *there is a unique morphism* u

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "q"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbb{N} \arrow[d, "u"] \\ & A \arrow[r, "f"] & A \end{tikzcd}$$

The map $z : 1 \to N$ is the zero element of N, while the $s : N \to N$ is the successor morphism, analogous to the operation $s(n) = n + 1$.

This number object is in fact (isomorphic to) the algebra generated by the maybe monad $=$ Maybe(n), which in this context we will call the *successor monad* $S(n)$.

f defines a sequence, such that $a_0 = q$ and $a_{n+1} = f(a_n)$

Relation to maybe monad

Show that a morphism $\mathbb{N} \to A$ induces a diagram $A \to A \to A \to \ldots$, which induces a limit

$$\lim_f A \tag{4.245}$$

Theorem 4.19.1 *Every Grothendieck topos admits a natural number object.*

Remark 4.19.2 *Beware that a "natural number object" may be a poor model of natural numbers in some extreme corner cases. A good example of this is the initial topos* $\mathbf{1} = \mathrm{Sh}(\mathbf{0})$*, for which the natural number object is*

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{Id}_1"] \arrow[dr, "\mathrm{Id}_1"] & 1 \arrow[r, "\mathrm{Id}_1"] \arrow[d, "\mathrm{Id}_1"] & 1 \arrow[d, "\mathrm{Id}_1"] \\ & 1 \arrow[r, "\mathrm{Id}_1"] & 1 \end{tikzcd}$$

which does obey all appropriate properties for it, but is only really the trivial algebra $(\{0\}, +)$*.*

Likewise, topoi also induce a *real number object*

Definition 4.19.2 *A* real number object R *is a commutative ring object*

$$0 : 1 \to R \quad (4.246)$$
$$1 : 1 \to R \quad (4.247)$$
$$+ : R \times R \to R \quad (4.248)$$
$$\cdot : R \times R \to R \quad (4.249)$$

with an apartness relation

4.20 Pointed spaces

For each object X of a topos, we can define a family of associated pointed spaces given by a pair of a given point $p : 1 \to X$ and the space X. This is defined by the notion of pointed objects :

Definition 4.20.1 *A* pointed object *of a category with a terminal object* 1 *is an element of the coslice category* $\mathbf{C}^{1/}$

For brevity, the pointed objects $p : 1 \to X$ will be denoted as X_p. The point p is the *base point* of the pointed space X_p.

Theorem 4.20.1 *The morphisms of the category of pointed objects* $\mathbf{C}^{1/}$ *map every base point to another base point,*

$$\forall f : X_p \to Y_q,\ f \circ p = q \quad (4.250)$$

Proof 4.20.1 *By definition of morphisms in a coslice category.*

Maybe monad

Theorem 4.20.2 *The pointed category of a topos has a zero object.*

Proof 4.20.2 *From the universal property of the terminal object in the topos, there is only one pointed object* $1 \to 1$*. This object is terminal in* $\mathbf{H}^{1/}$ *as any morphism to it will correspond to the morphism* $f : X \to 1$*, which factors as*

$$\mathrm{Id}_1 = f \circ !_X \quad (4.251)$$

as there is only one such morphism, the object is terminal. It is also initial as any morphism from it will correspond to the morphism $g : 1 \to X$*, which factors as*

$$g \circ \mathrm{Id}_1 = !_X \quad (4.252)$$

meaning that $g = !_X$*, so that there is also only one such morphism.*

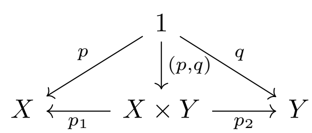
Hom-sets are pointed sets, pointed by the zero morphism.

Theorem 4.20.3 *Wedge sum*

Proof 4.20.3

Theorem 4.20.4 *If* $\mathbf{C}$ *has a product, the pointed category* $\mathbf{C}^{1/}$ *has a canonical tensor product, the smash product.*

Proof 4.20.4 *Given two objects* X, Y *with product* $X \times Y$*, and two pointed objects* X_p, Y_q*, the universal property of the product gives us*



4.21 Ringed topos

As with sheaves in general, we do not have to consider our topos to be exclusively set-valued, and we can give it a variety of other types. One of the most commonly used is that of a ringed topos.

Definition 4.21.1 *A* ringed topos $(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$ *is a topos* $\mathbf{H}$ *with a distinguished ring object* $\mathcal{O}_{\mathbf{H}}$

4.22 Integration

Ends?

For a commutative ring object in $\mathbf{H}$

5

Schemes, algebras and dualities

In our analysis of topos, a very common way to define them will be via duality, where rather than define our categories intrinsically, we define them as the duals of simpler categories. This is due to the duality of spaces as they are defined by their probes (as we can describe manifolds as the charts $\mathbb{R}^n \to M$), and spaces as they are defined by their values (as we can describe manifolds by their function spaces living on them, such as $C^\infty(M)$). While for manifolds it is commonly more useful to describe them with the former, ie as presheaves, we will deal also with many spaces for which the latter is more practical, ie as copresheaves.

For this, we will need to define what a dual is exactly in category theory.

[55]

5.1 Algebras

The notion of algebra has come about quite a few time throughout but we need now to figure out some rather general

Algebraic structures, rings, algebras, etc

linear categories?

One example we saw early on regarding the duality between spaces and algebras is that of **FinSet**, whose opposite category we saw was that of finite boolean algebras.

Copresheaf on a point?

For our case of looking at physical categories, one instructive example is that of C^∞-rings, which are roughly speaking analogues of the standard algebra of smooth functions on $\mathbb{R}^n$, $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Definition 5.1.1 *A C^∞-algebra (or smooth algebra, also called C^∞-ring) is a copresheaf on* **CartSp** *that preserves finite products.*

Definition 5.1.2 *The category of C^∞ algebras is the category of all such copresheaves and the natural transformations between them, ie*

$$C^\infty\mathbf{Alg} \subset [\mathbf{CartSp}, \mathbf{Set}] \tag{5.1}$$

Example 5.1.1 *As a basic example, we can look at the case of the assignment given by the representable functor*

$$h^X : \mathbf{CartSp} \to \mathbf{Set} \tag{5.2}$$

$$\mathbb{R}^n \mapsto \mathrm{Hom}_{\mathbf{CartSp}}(X, \mathbb{R}^n) \cong C^\infty(X, \mathbb{R}^n) \tag{5.3}$$

By definition this is the set of every smooth function from the Cartesian space X to $\mathbb{R}^n$, and a copresheaf. This is a C^∞ algebra since the hom functor preserves limits,

$$C^\infty(X, \mathbb{R}^n \times \mathbb{R}^m) \cong C^\infty(X, \mathbb{R}^n) \times C^\infty(X, \mathbb{R}^m) \tag{5.4}$$

where smooth maps to a product of Cartesian spaces are isomorphic to a pair of smooth maps to those Cartesian spaces.

$$f(x) = \mathrm{pr}_1(f(x)) \times \mathrm{pr}_2(f(x)) \tag{5.5}$$

We will see later on exactly how this

5.2 Affine varieties

Definition 5.2.1 *Given the affine space A^n, and a set of polynomials valued in A^n, an* affine variety *is given by the intersection of the sets of zeros of those polynomials.*

$$V(f_1, \ldots, f_n) = \{x \in A^n \mid \} \tag{5.6}$$

5.3 Schemes

[56, 57]

Definition 5.3.1 *The spectrum of a ring R is a topological space* $\mathrm{Spec}(R) = (P, Z)$*, with P the set of the prime ideals of R*

$$P = \{\mathfrak{p} \mid \mathfrak{p} \textit{ prime ideal of } R\} \tag{5.7}$$

and Z is the Zariski topology *on P, defined by the closed sets that, for any ideal I, its equivalent closed set is given by*

$$V(I) = \{\mathfrak{p} \in P \mid I \subseteq P\} \tag{5.8}$$

along with a sheaf of rings $\mathcal{O}_R$

Example 5.3.1 *For the ring of real numbers, $\mathbb{R}$, its only ideals are the trivial ideal* 0*, which is a prime ideal, and itself. There is then only two closed sets defined by the Zarinski topology, $V(\mathbb{R}) = \varnothing$ and $V(\{0\}) = \{0\}$, giving us the terminal topology.*

Example 5.3.2 *Take the ring of smooth functions on an underlying manifold, $C^\infty(M)$. An example of ideals is the set of functions vanishing on a region U, ie $f\big|_U = 0$:*

$$\forall g \in C^\infty(M),\ (fg)\big|_U = 0 \tag{5.9}$$

Let's note those ideals by $\mathfrak{m}_U$. Those ideals are related by $U' \subseteq U$ implying $\mathfrak{m}_U \subseteq \mathfrak{m}_{U'}$, since a function vanishes on U' if it vanishes on a larger set, so that those ideals cannot be maximal unless U is a single point $\{p\}$, which we will denote $\mathfrak{m}_p$.

We can show that those ideals are maximal as they are

the prime ideals are Whitney

Example 5.3.3 *Taking the ring of integers $\mathbb{Z}$, its ideals are given by the sets of multiples,*

$$k\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}\ n = km\} \tag{5.10}$$

such as the set of even numbers $2\mathbb{Z}$. The prime ideals are given by the multiples of the prime numbers, $p\mathbb{Z}$ for $p \in \mathbb{P}$ (in addition to the zero ideal), so that its spectrum is given by [Multiples or primes themselves?]

$$\mathfrak{p} = \mathbb{P} \cup \{0\} \tag{5.11}$$

5.4 Duality

In its most generality, a duality is simply an equivalence of categories between one category and the dual of another (in the sense of the opposite category) [58], which is the notion of dual adjunction we saw in 3.16.1 for the case of an equivalence.

Definition 5.4.1 *A duality is an equivalence between a category* $\mathbf{C}$ *and the dual of a category* $\mathbf{D}$*, that is, we have two functors going between each as*

$$S : \mathbf{C} \to \mathbf{D} \tag{5.12}$$
$$T : \mathbf{D} \to \mathbf{C} \tag{5.13}$$

where their composition in either order is naturally isomorphic to the identity functor on each category :

$$\eta : \mathrm{Id}_{\mathbf{C}} \to TS \tag{5.14}$$
$$\epsilon : \mathrm{Id}_{\mathbf{D}} \to ST \tag{5.15}$$

obeying that for every $X \in \mathbf{C}$ *and* $Y \in \mathbf{D}$

$$T\epsilon_X \circ \eta_{TX} = \mathrm{Id}_{TX} \tag{5.16}$$
$$S\eta_Y \circ \epsilon_{SY} = \mathrm{Id}_{SY} \tag{5.17}$$

but for the most part, we will be interested in more specific cases of dualities via certain methods.

Definition 5.4.2 *If two dual categories* $\mathbf{C}$*,* $\mathbf{D}$ *are concrete, ie with*

$$U : \mathbf{C} \to \mathbf{Set} \tag{5.18}$$
$$V : \mathbf{D} \to \mathbf{Set} \tag{5.19}$$

that are faithful and representable, $U \cong \mathrm{Hom}_{\mathbf{C}}(E_{\mathbf{C}}, -)$*,* $V \cong \mathrm{Hom}_{\mathbf{D}}(E_{\mathbf{D}}, -)$

[is concreteness and representability needed]

Definition 5.4.3 *A* concrete duality

[59]

Dualizing object :

Example 5.4.1 *The standard example of duality is that of dual vectors, with dualizable object* $I \cong k \in \mathbf{Vec}_k$*. The two functors are simply* $U = V = \mathrm{Hom}_{\mathrm{Vec}_k}(I, -)$*, mapping vector spaces to their set of points. Then a dualizable object is a vector space* X

$$T : \mathrm{Vec}_k^{op} \xrightarrow{\mathrm{Hom}_{\mathrm{Vec_k}}(-,\mathrm{X})} \mathrm{Vec}_k \tag{5.20}$$
$$S : \mathrm{Vec}_k^{op} \xrightarrow{\mathrm{Hom}_{\mathrm{Vec_k}}(-,\mathrm{X})} \mathrm{Vec}_k \tag{5.21}$$

$$US : \mathbf{Vec}_k \to \mathbf{Set} \tag{5.22}$$
$$VT : \mathbf{Vec}_k \to \mathbf{Set} \tag{5.23}$$

representable : $\exists X, Y \in \mathbf{Vec}_k$ *such that*

$$US = \mathrm{Hom}_{\mathbf{Vec}_k}(-, X) \quad (5.24)$$
$$VT = \mathrm{Hom}_{\mathbf{Vec}_k}(-, Y) \quad (5.25)$$

Representing elements : $\phi = \mathrm{Hom}_{\mathbf{Vec}_k}(X, X)$, $\psi = \mathrm{Hom}_{\mathbf{Vec}_k}(Y, Y)$, *space of square matrices*

nat :

$$\eta : \mathrm{Id}_{\mathbf{Vec}_k} \to TS \quad (5.26)$$
$$\epsilon : \mathrm{Id}_{\mathbf{Vec}_k} \to ST \quad (5.27)$$

Those are respectively the resolution of the identity and [trace?]

Contravariant representation?

Canonical isomorphism $\omega : |X| \to |Y|$ *via*

$$|X| \xrightarrow{X} USTX \quad (5.28)$$

Adjunction

$$(-) \otimes A \dashv (-) \otimes A^* \cong [A, -] \quad (5.29)$$

$$X^* \cong k \otimes X^* \cong [A, k] \quad (5.30)$$

[...]

In the context of presheaves, this situation is simply the case where

As a presheaf can be considered as the generalization of a space that can be probed by specific objects, a copresheaf can be considered as the generalization of a quantity on that space with values in the same objects.

The basic example for this is to consider the algebra of smooth functions on a Cartesian space, via the functor associating R-algebras to Cartesian spaces :

$$C^\infty : \mathbf{SmoothMan} \to \mathrm{R}\text{-}\mathbf{Alg} \quad (5.31)$$

Associating the algebra $C^\infty(M)$ to any manifold M. Adding the forgetful functor $U : R{-}\mathbf{Alg} \to \mathbf{Set}$, this gives us some copresheaf on the category of Cartesian spaces.

Theorem 5.4.1 *The opposite category of* **CartSp** *is the category of smooth real functions.*

$$\mathbf{CartSp}^{\mathrm{op}} \cong C^\infty \quad (5.32)$$

Proof 5.4.1 *Given the category of smooth algebras on Cartesian spaces, with objects the smooth algebras $C^\infty(\mathbb{R}^n)$ and morphisms the algebra homomorphisms, take the functor*

$$C^\infty(-) : \mathbf{CartSp} \to \mathbf{C}^\infty \tag{5.33}$$

We can define this functor via the contravariant hom-functor $\mathrm{Hom}(-, \mathbb{R})$. *On objects, this clearly sends any Cartesian space to their set of smooth functions, by definition of the morphisms in* **CartSp**. *As a contravariant functor, any morphism is sent to*

$$\mathrm{Hom}(f, \mathbb{R}) : \mathrm{Hom}(Y, \mathbb{R}) \to \mathrm{Hom}(X, \mathbb{R}) \tag{5.34}$$

So that we have a morphism $C^\infty(Y) \to C^\infty(X)$ defined by precomposition, ie for any $g \in C^\infty(Y)$, $g \circ f$

[Smooth algebras on $\mathbb{R}$ is enough since we can define products of functions and diagonal for $\mathbb{R}^n$]

Theorem 5.4.2 *Milnor duality :*

$$\mathrm{Hom}_{\mathrm{SmoothMan}}(X, Y) \cong \mathrm{Hom}_{\mathbf{CAlg}_{\mathbb{R}}}(C^\infty(Y), C^\infty(X)) \tag{5.35}$$

This copresheaf is furthermore a cosheaf. If we consider the restriction of our manifold to some submanifold $\iota : S \hookrightarrow M$,

Copresheaf : functor $F : \mathbf{C} \to \mathbf{Set}$

Cosheaf : copresheaf has for a covering family $\{U_i \to U\}$

$$F(U) \cong \varinjlim \left(\coprod_{ij} F(U_i \times_U U_j) \rightrightarrows \coprod_i F(U_i) \right) \tag{5.36}$$

Equivalence via Yoneda :

$$\mathrm{CoSh}(\mathbf{C}) \cong \tag{5.37}$$

The copresheaf assigns to each test space $U \in \mathbf{C}$ the set of allowed maps from A to U - U-valued functions on A

Example 5.4.2 *There is a duality between the algebra $\mathbb{R}$ and the single point space $\{0 < 1\}$*

Proof 5.4.2 *As $\mathbb{R}$ has only a single ideal,* 0*,*

$$\mathrm{Spec}(\mathbb{R}) \tag{5.38}$$

Its Zariski topology is simply given by the

Duality for functors and monads and comonads? Given functors between two presheaves, what is the associated functor on the two dual copresheaves?

Theorem 5.4.3 *Given a functor between two presheaves in a presheaf category,*

$$F : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D}) \tag{5.39}$$

there is a dual functor between their copresheaves

5.5 Q-categories

The duality between spaces and quantities is, when it comes to algebras, generally only well defined for the commutative cases. But in a modern context there are many cases where we want a non-commutative geometry, which is the dual of a non-commutative algebra.

Unlike the commutative case, there is no sheaf dual to that kind of algebra, but there is a more general construction

Definition 5.5.1 *A Q-*category *is given by a pair of adjoint functor*

$$(u^* \dashv u_*) : \overline{\mathbf{A}} \underset{- u_* \to}{\overset{\leftarrow u^* -}{\rightleftarrows}} \mathbf{A}$$

such that u^* is a fully faithful functor.

Theorem 5.5.1 *For a field k and the category of k-commutative algebras* $\mathbf{CAlg}_k$*, and*

5.6 Weil algebras and infinitesimal spaces

One of the duality we will exploit the most here is that given by the Weil algebras.

Definition 5.6.1 *A* Weil algebra *(or* Artinian ring*) is a commutative unital ring of the form*

$$A = R \oplus W \tag{5.40}$$

composed of a ring R, and a module of finite rank over R consisting of nilpotent elements, ie

$$\forall x \in W,\ \exists n \in \mathbb{N},\ x^n = 0 \tag{5.41}$$

The nilpotent elements are in the context of infinitesimal geometry interpreted to be "infinitesimal elements", where their nilpotency indicates their "smallness" in the fact that their product is formally zero, an algebraic version of the cutting off of higher order terms like $(dx)^2$.

In this manner, every ring is trivially a Weil algebra via

$$R \oplus \{0\} \tag{5.42}$$

Example 5.6.1 *The simplest non-trivial example of a Weil algebra is the ring of* dual numbers*, defined as the quotient of the real polynomial ring by the square*

$$\mathbb{R}[\varepsilon] = \mathbb{R}[X]/\langle X^2 \rangle \tag{5.43}$$

which has the graded structure

$$\mathbb{R}[\varepsilon] = \mathbb{R} \oplus W \tag{5.44}$$

where W contains the monomials of degree 1, for which $x^2 = 0$ by definition.

The relationship between this Weil algebra and infinitesimals can be seen by considering its behaviour on polynomial functions,

$$(x+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\varepsilon^k = x^n + nx^{n-1}\varepsilon + 0 \tag{5.45}$$

So that the polynomial of a dual number is equivalent to the pairing of the polynomial of the actual function on $\mathbb{R}$ and of its derivative,

$$\forall f \in \mathbb{R}[x],\ f(x+\varepsilon) = f(x) + \varepsilon f'(x) \tag{5.46}$$

Higher nilpotent

Category of smooth loci

Definition 5.6.2 *The category of formal Cartesian spaces is the category for which objects*

[...]

The copresheaf on the category of formal Cartesian spaces can then be interpreted as some space with structures upon it with values in a formal Cartesian space.

$$[X, \mathbb{R}^n \times] \tag{5.47}$$

Infinitesimal objects as left adjoints of the internal hom?

The basic example of a Weil algebra is the algebra of *dual numbers*, a vector space composed by a pair of real numbers, $\mathbb{R}[\varepsilon]/\varepsilon^2$

$$(x + \epsilon y)^2 = x^2 + xy\varepsilon \tag{5.48}$$

As a copresheaf?

Its formal dual is the spectrum

$$D = \mathrm{Spec}(\mathbb{R}[\varepsilon]/\varepsilon^2) \tag{5.49}$$

which is the first order infinitesimally thickened point in one dimension (also sometimes called a ”linelet”, an infinitesimally small line).

6

Example categories

[60]

For the consideration of the methods to be studied, we need to look at a few good examples of appropriate categories. We will mostly look at topos (the main focus of this), more specifically Grothendieck topos, as well as a few categories of some physical interest relating to quantum theory, which may differ from the topos structure, hopefully illuminating the differences with the more classical logic associated with classical mechanics.

Quantales? Topos of the sheaves of commutative algebras on Hilbert space? Effectus categories?

6.1 Spatial topoi

As the classic example of a sheaf is that of a sheaf on the set of opens of a topological space, let's first look at its topos, the full category of sheaves on that same set of opens, the sheaf category $\mathrm{Sh}(\mathrm{Op}(X))$ with the subcanonical coverage, simply written as $\mathrm{Sh}(X)$ for short. This is a *spatial topos*. This topos will contain for instance the structure sheaves of continuous functions to some topological space Y.

[...]where the restriction maps $\rho_{U,V}$ are simply given by the domain restriction

Restriction maps, gluing, locality

Theorem 6.1.1 *The initial object of a spatial topos is the empty set,* $よ(\varnothing)$.

Theorem 6.1.2 *The terminal object of a spatial topos is the whole space,* $よ(X)$

Interpretation as a structure sheaf from the computation of stalks?

Theorem 6.1.3 *Given a sheaf on a spatial topos,*

"The problem with excluded middle in topological models is that it may not hold continuously: e.g. a subspace $U \subseteq X$ and its complement $X \setminus U$ will together contain all the points of X, but the map from their coproduct $U + (X \setminus U)$ to X does not have a continuous section because the topology on the domain is different. Thus, we cannot expect to have the full law of excluded middle in a topological model"

6.2 Category of sets

The most basic topos (outside of the initial topos $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$) is the category of sets **Set**, with objects made from the class of all sets and morphisms the class of all functions.

In terms of a sheaf topos, sets can be defined as the sheaf on the terminal category : $\mathbf{Set} = \mathrm{Sh}(\mathbf{1})$, which is the functor from $\mathbf{1}^{\mathrm{op}} = \mathbf{1}$ to Set. As the set of all functions from $\{\bullet\}$ to any set X is isomorphic to X itself, the presheaves on **1** are easily seen to be isomorphic to **Set**. We only need to consider sieves on its unique object $*$, and as there is only one possible morphism there, there are only two possible sieves : the empty sieve $S_\varnothing$ which maps $*$ to the empty set, and the maximal sieve S_* which maps it to the singleton containing the identity map.

This allows us two possible topologies, the *chaotic topology* for the empty sieve and the *standard topology* for the maximal sieve.

For the chaotic topology, if we consider the category of presheaves on **1** (isomorphic to **Set**), our sheaves

The choice of the terminal category with the maximal sieve is in fact equivalent to **Set** being a spatial sheaf on the topological space of one point. Sets can therefore also be seen as functions to sets on a single point.

As the site only has a trivial coverage, there is only a fairly limited amount of assembly that we can do from it.

A variation of **Set** is **FinSet**, the category of finite sets. Trivially a full subcategory of **Set** via the obvious inclusion functor.

Theorem 6.2.1 *The category* FinSet *is a subtopos of* **Set***.*

Proof 6.2.1 *As we are only concerned with finite limits here, all basic arguments for* **Set** *being a topos apply here. All finite limits of finite sets in* **Set** *are also finite sets, as are all internal homs and the subobject classifier.*

For any set, sheaf on that set equivalent to Set? (with the standard coverage)

Theorem 6.2.2 *For any discrete category with the maximal coverage, the sheaf on that site is equivalent to* **Set**.

Proof 6.2.2 *For some discrete category* **n**, *if we consider its sheaf*

$$\mathrm{Sh}(\mathbf{n}) \tag{6.1}$$

for any such sheaf $F : \mathbf{n}^{\mathrm{op}} \to \mathbf{Set}$, *we have an equivalent sheaf which has empty image for any* $n \in \mathbf{n}$ *other than one :*

$$F'(\bullet_0) = \coprod_{i=0}^{n} F(\bullet_i) \tag{6.2}$$

6.2.1 Limits and colimits

As the prototype for the very notion of limits and colimits, **Set** has all the finite limits and colimits. In fact, it has all small limits and colimits, ie when the diagram itself is small enough to be a set.

Theorem 6.2.3 *The empty set* $\varnothing$ *is the initial object of* **Set**.

Proof 6.2.3 *We need to show that for any set* $X \in \mathrm{Obj}(\mathbf{Set})$, *there is a unique function* $f : \varnothing \to X$. *A function* $f : A \to B$ *is a subset of* $A \times B$ *obeying some properties, therefore we need to look at the set of subsets of* $\varnothing \times A$. *By properties of the Cartesian product,*

$$\varnothing \times A = \{\varnothing\} \tag{6.3}$$

There is therefore only one element to choose from, $\varnothing$, *which is indeed a function since it obeys (vacuously) the constraints on functions.*

Theorem 6.2.4 *Any singleton set* $\{\bullet\}$ *is a terminal object of* **Set**, *all isomorphic.*

Proof 6.2.4 *We need to show that for any set* $X \in \mathrm{Obj}(\mathbf{Set})$, *there is a unique function* $f : X \to \{\bullet\}$.

$$X \times \{\bullet\} \tag{6.4}$$

The terminal object is also the unique representable presheaf on **1**, simply given by $\mathrm{Hom}_{\mathbf{1}}(*, *)$, mapping the unique element of **1** to the set of the unique identity morphism of $*$, as we would expect.

As a category of sets, which are fundamentally defined by $\in$, **Set** has global elements $x : I \to X$. Those global elements are separators

Well-pointed topos

Theorem 6.2.5 *The product on* **Set** *is isomorphic to the Cartesian product.*

Proof 6.2.5 *The Cartesian product has by construction two projection operators,* pr_1 *and* pr_2*. Given two functions* $f_i : X \to Y_i$*, there is a natural function* $f = f_1 \times f_2$ *given by*

$$f(x) = (f_1(x), f_2(x)) \tag{6.5}$$

which obeys the universal property of the product.

Theorem 6.2.6 *The coproduct on* **Set** *is isomorphic to the disjoint union.*

Proof 6.2.6

Theorem 6.2.7 *Given two functions* $f, g : A \to B$*, the equalizer in* **Set** *is the subset* $C \subseteq A$ *on which those functions coincide,*

$$\mathrm{eq}(f, g) = \{c \in A | f(c) = g(c)\} \tag{6.6}$$

Theorem 6.2.8 *The equalizer of two functions* $f, g : A \rightrightarrows B$ *is the set of elements of* A *whose image agree :*

$$\mathrm{eq}(f, g) = \{x \in A | f(x) = g(x)\} \tag{6.7}$$

Theorem 6.2.9 *The coequalizer of two functions* $f, g : A \rightrightarrows B$ *is the quotient set on* A *by the equivalence relation*

$$x \sim y \leftrightarrow f(x) = g(y) \tag{6.8}$$

Proof 6.2.7

$$A \to B \to C \tag{6.9}$$

Definition 6.2.1 *The pullback of the cospan* $A \to C \leftarrow B$ *is the indexed set*

Theorem 6.2.10 *The pushout of the span* $A \leftarrow C \to B$ *is the*

Theorem 6.2.11 *Directed limit*

Given these, we can see that **Set** has all small limits and colimits.

6.2.2 Elements

Fairly obviously, given its status as the model for it, **Set** has global elements : $x : \{\bullet\} \to X$, corresponding to the functions

$$\forall x \in X,\ x(\bullet) = x \tag{6.10}$$

so that explicitly, $x = \{(\bullet, x)\}$ (this set can be shown to exist with the axiom of pairing). Those points are furthermore *separators*, in that the morphisms on **Set** are entirely defined by them

$$f = g \leftrightarrow \forall x : \{\bullet\} \to X,\ f \circ x = g \circ x \tag{6.11}$$

As it is also a non-degenerate topos, in that $\{\bullet\} \neq \varnothing$, this makes **Set** a well-pointed topos.

Set is boolean, two-valued, has split support

6.2.3 Subobject classifier

For **Set**, the subobject classifier is the set $\{\varnothing, \{\bullet\}\}$, also denoted by $\{0, 1\}$ or $\{\bot, \top\}$, corresponding to the two boolean valuations of a subobject : either being a subset or not being a subset. While we could define it as such, this is also simply the subobject classifier of the terminal topos, as this is also the closed sieve on the terminal category **1**, so that the subobject classifier is the functor

$$\Omega : \mathbf{1} \to \{\varnothing, \{\bullet\}\} \tag{6.12}$$

which, in the equivalence of sheaves $\mathbf{1} \to \mathbf{Set}$ with sets, is simply a set of two elements.

In terms of set theory, a subobject is a subset (up to isomorphism), that is

$$A \subseteq X \leftrightarrow \exists \chi_A : X \to \Omega \tag{6.13}$$

For a subset $\iota_S : S \hookrightarrow X$

$$\begin{array}{ccc} S & \xrightarrow{!_S} & 1 \\ \downarrow{\scriptstyle \iota_S} & & \downarrow{\scriptstyle \text{true}} \\ X & \xrightarrow{\chi_S} & \Omega \end{array}$$

The function χ_S is more typically called the characteristic function and uses the notation $\chi_S : U \to \mathbb{B}$

$$\chi_U(x) = \begin{cases} 0 \\ 1 \end{cases} \tag{6.14}$$

Internal hom : The set of functions

$$\begin{aligned} [X, Y] &= \{f | f : X \to Y\} && (6.15) \\ &= \{f \subseteq X \times Y | \forall x \in X,\ \exists y \in Y,\ (x, y) \in f \wedge ((x, y) \in f \wedge (x, z) \in f \to y = z)\} && (6.16) \end{aligned}$$

6.2.4 Natural number object

Natural number object : the natural number construction.

The maybe monad is simply

$$\text{Maybe}(X) = X \sqcup \{\bullet\} \tag{6.17}$$

Coproduct for natural number :

$$0 \sqcup 0 = \{\} \tag{6.18}$$

Lawvere-Tierney topology : some morphism $j : \mathbf{2} \to \mathbf{2}$

Properties : $j(\{\bullet\}) = \{\bullet\}$, $j(j(x)) = j(x)$, $j(a \wedge b) =$

$j(\{\bullet\}) = \{\bullet\}$ reduces the choice to $j = \text{Id}_\Omega$ and $j = \{\bullet\}$, the constant map.

For the identity map : Given a subset $\iota : S \hookrightarrow X$, with classifier $\chi_S : A \to \Omega$, the composition $j \circ \chi_S$ defines another subobject $\bar{\iota} : \overline{S} \hookrightarrow A$ such that s is a subobject of $\bar{\iota}$, $\bar{s}$ is the j-closure of s

Identity map closure : every object is its own closure. This is the *discrete topology*.

Constant map $j(x) = \{\bullet\}$: the composition $j \circ \chi_S$ is the "always true" characteristic function, which is just χ_A. The closure of a set S in A is the entire set A. This is the *trivial* or *codiscrete* topology.

Those are the only two allowed topologies in **Set**.

Relation to $\text{loc}_{\neg\neg}$: $\neg : \Omega \to \Omega$ is

$$\begin{aligned} \neg(\{\bullet\}) &= \varnothing && (6.19) \\ \neg(\varnothing) &= \{\bullet\} && (6.20) \end{aligned}$$

$\neg\neg$ is simply the identity on **Set**. The j-closure associated to it is the identity map.

Localization?

Power object

6.2.5 Closed Cartesian

That the category of sets is closed Cartesian simply stems from the usual construction of functions on sets in basic set theory. As functions from A to B are defined as a subset of the Cartesian product $A \times B$, this is guaranteed by the existence of the Cartesian product of sets and the axiom schema of specification.

Definition 6.2.2 *For any two sets A, B, the hom-set $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ is itself a set, by the traditional set definition of functions*

$$f : A \to B \leftrightarrow f = \{(a, b) \subseteq A \times B | f(a) = b\} \tag{6.21}$$

so that []

The evaluation map of a function in **Set** is given by the traditional formulation of function evaluations in set theory. For a function $f : A \to B$, its evaluation by an element $x \in A$ is the unique element y in Y for which $(x, y) \in R_f$. In set theoretical terms, this can be defined Russell's iota operator,

$$\begin{aligned} \mathrm{ev}(x, f) &= \iota y,\ x R_f y & (6.22) \\ &= \bigcup \{z \mid \{y \mid x R_f y\} \ = \{z\}\} & (6.23) \end{aligned}$$

Counit of the tensor product/internal hom adjunction?

$$S \times (-) \dashv [S, -] \tag{6.24}$$

currying

6.2.6 Internal objects

The category of **Set** contains just about all the typical internal objects that we would expect, as most basic structures in mathematics are defined using set theory. Any group in the traditional definition of the sense (as sets with some extra structure) have an equivalent internal group in **Set**, with the expected underlying set, and the same is true for internal rings, internal modules, etc etc.

We have in particular our various number objects with their associated internalized structures. The real number object $\mathbb{R}$ can be given an internal group structure with the addition morphism, it can be given an internal ring structure, etc.

6.3 The Sierpinski topos

The simplest non-trivial site is the one given by the interval category $\Delta[\vec{1}]$ in the simplex category, with a distinct initial object and terminal object. The Sierpinski topos is the presheaf category over that site :

$$\mathbf{SierpTop} = \mathrm{PSh}(\Delta[\vec{1}]) \tag{6.25}$$

Theorem 6.3.1 *The Sierpinski topos is equivalent to the sheaf topos over the Sierpinski topological space.*

Proof 6.3.1 *The Sierpinski site is given by the set of two elements $\{0,1\}$ with the topology $\{\varnothing, \{1\}, \{0,1\}\}$, with the poset structure The coverage of each ele-*

$$\begin{array}{c} \{0,1\} \\ | \\ \{1\} \\ | \\ \varnothing \end{array}$$

ment is given by

$$\varnothing : \varnothing \hookrightarrow \varnothing \tag{6.26}$$
$$\{1\} : \varnothing \hookrightarrow \{1\} \tag{6.27}$$
$$\{1\} \hookrightarrow \{1\} \tag{6.28}$$
$$\{0,1\} : \varnothing \hookrightarrow \{0,1\} \tag{6.29}$$
$$\{1\} \hookrightarrow \{0,1\} \tag{6.30}$$
$$\{0,1\} \hookrightarrow \{0,1\} \tag{6.31}$$

There's no cover of any open set in this topology that doesn't require the whole space itself,

Theorem 6.3.2 *The Sierpinski topos is isomorphic to the arrow category of* **Set** *:*

$$\mathbf{SierpTop} \cong \mathrm{Arr}(\mathbf{Set}) \tag{6.32}$$

Proof 6.3.2

Theorem 6.3.3 *The initial object of* **SierpTop**

Proof 6.3.3

6.4 Topos on the simplex category

As one of our example of a presheaf, the simplicial sets, given by the presheaves on the simplex category

$$X : \mathbf{\Delta} \to \mathbf{Set} \tag{6.33}$$

form a topos, as they are simply the presheaf category

$$\mathbf{sSet} = \mathrm{PSh}(\mathbf{\Delta}) \tag{6.34}$$

For each simplicial complex $\Delta[\vec{\mathbf{n}}]$, let's define the sets of n-simplices as

$$X_n = \Delta[\vec{\mathbf{n}}] \tag{6.35}$$

Forgetful functor to the various simplices?

Definition 6.4.1 *A* simplicial set X *is composed of a sequence of sets* $(X_n)_{n\in\mathbb{N}}$*, its set of* n*-simplices, represented by a totally ordered set (ordered simplices)*

For every injective map $\delta_i^n : [n-1] \to n$*, there's a map* $d_i^n : X_n \to X_{n1}$*, the* i*-th face map on* n*-simplices*

Theorem 6.4.1 *Simplicial sets as presheaves on the simplex category and as collections of sets are isomorphic*

Proof 6.4.1

Theorem 6.4.2 *The simplicial category* $\mathbf{\Delta}$ *is a concrete site.*

Proof 6.4.2 $\mathbf{\Delta}$ *has a terminal object, which is the*

Unlike more complex cases like sheaves over Cartesian spaces6.5, we will not have objects that do not "locally look like" the underlying category, as the category is locally finitely presentable : every object in it can simply be represented as colimits of [is that true]

[In case of non-injective map, pullback or something idk]

[Locally representable ergo locally presentable, means that we don't have any funny business?]

Initial and terminal object :

Theorem 6.4.3 *The initial simplicial set* 0 *is the simplicial set of no points.*

Theorem 6.4.4 *The terminal simplicial set* 1 *is the constant simplicial set mapping every simplicial category to the singleton, equivalent to a point and degenerate higher simplices.*

Proof 6.4.3 *As the terminal sheaf, it is simply the sheaf*

$$X(\Delta[\vec{\mathbf{n}}]) = \{\bullet\} \tag{6.36}$$

So that this is the simplicial set of a single point, for which the (degeneracy map?)

Theorem 6.4.5 *The product on* **sSet** *is the*

example :

Functors of **sSet** : to top, to simplicial complexes, etc

Concrete simplicial sets are simplicial complexes?

6.4.1 Limits and colimits

Theorem 6.4.6 *The initial object for simplicial sets is the empty simplicial set.*

Proof 6.4.4

Theorem 6.4.7 *The terminal object for simplicial sets is the points with every degenerate higher simplex mapped onto it.*

Proof 6.4.5

Theorem 6.4.8 *The product of two simplicial sets is given by the simplicial set where every k-simplex is given by*

$$(X \times Y)_k = X_k \times Y_k \tag{6.37}$$

and with face and degeneracy maps

$$d_i^{X\times Y}(x, y) = (d_i^X(x), d_i^Y(y)) \tag{6.38}$$

$$s_i^{X\times Y}(x, y) = (s_i^X(x), s_i^Y(y)) \tag{6.39}$$

Example 6.4.1 *The product of two intervals $I \times I$ is a square.*

Proof 6.4.6 *The non-degenerate components of an interval are two points, $\{\bullet_0, \bullet_1\}$, and a line $\{\ell\}$, with the face map*

$$s \tag{6.40}$$

Theorem 6.4.9 *The coproduct of two simplicial sets*

6.5 Category of smooth spaces

[61, 62]

A more geometric category for a topos is the topos of smooth spaces **Smooth**, which is defined as the sheaf over the category of smooth Cartesian spaces,

$$\mathbf{Smooth} = \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{Smooth}}) \tag{6.41}$$

The category of smooth spaces can be given a variety of sites equivalently, such as the site of smooth manifolds SmoothMan, the site of open subsets of $\mathbb{R}^{\mathrm{n}}$, or the site of $\mathbb{R}^n$, all with morphisms being the smooth maps between such objects. As we will see, all those possible sites end up producing the same topos, and we will typically just use the category of Cartesian spaces **CartSp** where the only objects are $\mathbb{R}^n$, as this will make for the simpler site. This stems from the fact that the other sites can be constructed by limits and colimits of each other that we do not have to worry too much. While the objects are $\mathbb{R}^n$, this is of course only up to diffeomorphism so that this will include all manners of contractible domains like open balls.

The coverage of this site is slightly tricky. The most obvious cover is simply the coverage by open sets (diffeomorphic to $\mathbb{R}^n$ in our case). While we can construct a sheaf over this coverage (and it will in fact lead to an equivalent topos), there are coverages with better properties.

Definition 6.5.1 *A* good open cover *is an open cover for which any finite intersection of open sets is contractible, ie a good open cover $\{f_i : U_i \to X\}$ of X*

$$\int \prod_{i \in I,\ X} U_i \cong \star \tag{6.42}$$

Example 6.5.1 *The typical cover of the circle, S^1, is usually composed by two lines, U_N, U_S, where each line relates to the circle via the stereographic projection at the north and south. This cover is however not a good open cover as the overlap of those open sets is the coproduct of two intervals,*

$$U_N \cap U_S \cong I + I \tag{6.43}$$

a good open cover of the circle will be for instance given by three

This is taken typically taken as the cover as, from algebraic topology, we know that such covers have better properties, as the Čech cohomology of the cover will be identical to that of the space itself.

Theorem 6.5.1 *Any smooth manifold admits a good open cover.*

This implies a homeomorphism to the open ball

Definition 6.5.2 *A good open cover* $\{f_i : U_i \to X\}$ *is a* differentially good open cover *if finite intersections of the cover are all diffeomorphic to the open ball.*

$$\prod_{i \in I,\ X} U_i \cong B^k \tag{6.44}$$

Theorem 6.5.2 *All three coverage of* $\mathbf{CartSp}_{\mathrm{smooth}}$ *lead to isomorphic sheaves :*

$$\mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{open}) \cong \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{good}) \cong \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{diff})$$

Therefore for our purpose we can pick the best behaved coverage.

Sheaves on the category of Cartesian spaces is best understood, in the context of geometry, as being *plots*, the more general version of what would be an atlas in the case of manifolds.

Definition 6.5.3 *A* plot *is a map between an open set of a Cartesian space* $\mathcal{O} \subseteq \mathbb{R}^n$ *and a topological space* X

From the Yoneda lemma, we have that for any sheaf $X \in \mathbf{Smooth}$ and Cartesian space U, we have the isomorphism

$$X(U) \cong \mathrm{Hom}_{[\mathbf{CartSp},\mathbf{Set}]}(よ(U), X) \tag{6.45}$$

$$よ(U) : \mathbf{CartSp}^{\mathrm{op}} \to \mathbf{Set} \tag{6.46}$$

$$O \mapsto \tag{6.47}$$

While this is a good intuitive way to understand the spaces probed by plots, it can be useful to know that in fact the topological space X itself is not necessary as a data to define a space cattaneo.

Theorem 6.5.3 *Given the set of transition functions on a manifold, the topological space can be reconstructed as*

$$M = \coprod O_i / \sim \tag{6.48}$$

where two points in $O_i \sqcup O_j$ *are equivalent if* $\tau_{ij}(x_i) = x_j$

This is what we do with the smooth sets topos, as we are only considering the existence of those maps (as the set $F(\mathbf{Cartsp})$), and the behaviour of those plots over overlapping regions.

Example 6.5.2 *Take the circle S^1, which we will defined a bit simplistically as a plot over $\mathbb{R}$. To avoid any issue of bad covers, let's take three different overlapping plots so that all overlaps are pairwise contractible. Just considering those plots, we get*

$$S^1(\mathbb{R}) = \{\varphi_1, \varphi_2, \varphi_3\} \tag{6.49}$$

In terms of an atlas, if we considered our circle as the interval $[0,1]$ with ends identified, we could cover it via the three intervals $(0,3/4)$, $(1/2,1)$ and $(3/4,1/2)$ (simply pick any map $\mathbb{R} \to (0,1)$ to appropriately rescale everything such as the arctan *map)*

[diagram]

with overlaps

$$1;2 \quad : \quad (1/2, 3/4) \tag{6.50}$$
$$1;3 \quad : \quad (0, 1/2) \tag{6.51}$$
$$2;3 \quad : \quad (3/4, 1) \tag{6.52}$$

with the transition maps

$$\tau_{12} \quad = \tag{6.53}$$
$$\tau_{13} \quad = \tag{6.54}$$
$$\tau_{23} \quad = \tag{6.55}$$

In terms of overlap, we have $U^+ \cap U^- \cong I \sqcup I$, so that we need to consider additionally the plot of that Cartesian space (slightly complicated by the non-connected aspect of it, but we can consider the open set of the line $(-1,0) \cup (0,1)$. While we can do it this is partly why we generally consider good open covers)

$$S^1(I \sqcup I) = \{\varphi^{\pm}\} \tag{6.56}$$

which maps this overlap region onto S^1. The inclusion of this overlap area is done as

$$\iota_+(x \in I \sqcup I) \quad = \quad x \tag{6.57}$$
$$\iota_-(x \in I \sqcup I) \quad = \quad 2 - x \tag{6.58}$$

The overlap in terms of the plot is that we map $(-1,0) \cup (0,1)$ to the interval I as

Those morphisms on **CartSp** *are mapped onto opposite mappings on* **Set** *:*

$$S^1(\iota_+) : \{\varphi^+, \varphi^-\} \to \{\varphi^\pm\} \tag{6.59}$$

If we take the less abstract case of a concrete sheaf to look at smooth spaces, considering CartSp is a concrete site, the concrete presheaf of Sh(CartSp) is the category of *diffeological spaces* DiffeoSp, where each global element $X : 1 \to$ DiffeoSp is a diffeological space.

[Diff is a quasitopos]

An important subcategory is also the category of smooth manifolds **SmoothMan**.

$$\mathbf{SmoothMan} \subseteq \mathbf{DiffeoSp} \subseteq \mathbf{Smooth} \tag{6.60}$$

SmoothMan is not itself a topos, as it lacks an exponential object (Hom-sets between manifolds such as $C^\infty(M, N)$ are not themselves manifolds, although they are close to it [63]), and the quotients or equalizers of manifolds are not themselves manifolds [examples]

Smooth manifolds are locally representable objects of **Smooth**. If $X : 1 \to$ **Smooth** is a concrete smooth space (diffeological space), it is locally representable if there xists $\{U_i \hookrightarrow X\}$, $U_i \in$ **Smooth** such that the canonical morphism out of the coproduct

$$\coprod_i U_i \to X \tag{6.61}$$

Is an effective epimorphism in **Smooth**.

$$\coprod_i U_i \times_X \coprod_j U_j \rightrightarrows \coprod_i U_i \to X \tag{6.62}$$

By commutativity of coproduct and pullback [prove it]

$$\coprod_{i,j} (U_i \times_X U_j) \rightrightarrows \coprod_i U_i \to X \tag{6.63}$$

Theorem 6.5.4

$$\mathbf{Smooth} \cong \mathrm{Sh}(\mathbf{SmoothMan}) \tag{6.64}$$

An important property of **Smooth** is that it contains a large proportion of **Top**, more specifically the category of

Status wrt top, delta generated top, etc

Due to this wide variety of physically important objects in **Smooth**, it will typically be (or at least some wider categories that we will define later) the topos serving as the setting for physics in general.

Subobject classifier : for any $U \in \mathbf{CartSp}$, $\Omega(U)$ is the set of subsheaves of h_U.

Finer diffeology same as subobject relation of sheaves?

Definition 6.5.4 *For a smooth space X, given a set F of parametrizations of X, the* diffeology generated by F *is the finest diffeology containing F.*

$$\langle F \rangle = \bigcap \mathcal{D} \tag{6.65}$$

Example 6.5.3 *For any diffeological space, the diffeology generated by the empty family $F = \varnothing$ is the discrete diffeology.*

Dimension

Definition 6.5.5 *For a smooth space X, the dimension of X is the infimum of the dimension of its generating family.*

$$\dim(X) = \inf_{\langle F \rangle = D} \dim(F) \tag{6.66}$$

Note that the topos of smooth sets is much larger than that of manifolds, including infinite dimensional manifolds, and contains quite a lot of spaces which do not have any obvious interpretation in those terms.

Example 6.5.4 *The* wire diffeology *is given by taking the standard diffeology on $\mathbb{R}^2$ [representative sheaf?] and only keeping the plots for which the parametrization factors through $\mathbb{R}$, ie for any parametrization $p : U \to \mathbb{R}^2$, for any $u \in U$, there exists an open neighbourhood $u \in V \subseteq U$ with a smooth map $F : V \to \mathbb{R}$ and a smooth curve $q : \mathbb{R} \to \mathbb{R}^2$ such that $p\big|_V = q \circ F$ [Same differential structure as $\mathbb{R}^2$ 1] The identity map is not a plot*

6.5.1 Limits and colimits

Proposition 6.5.1 *The initial object of* **Smooth** *is the constant functor*

$$\Delta_\varnothing : \mathbf{CartSp}_{\text{Smooth}} \to \mathbf{Set} \tag{6.67}$$

which maps every cartesian space to the empty set $\varnothing$.

The interpretation of this is that the initial object can be seen as the empty manifold, with the empty atlas.

Theorem 6.5.5 *The terminal object of* **Smooth** *is the constant functor*

$$\Delta_{\{\bullet\}} : \mathbf{CartSp}_{\mathrm{Smooth}} \to \mathbf{Set} \tag{6.68}$$

which maps every cartesian space to the singleton $\{\bullet\}$*.*

The interpretation of this is that the terminal object of **Smooth** is a *point*. This can be shown as the space obviously has a single point, since $p : \mathbb{R}^0 \to \{\bullet\}$, but every other plot factors through $\mathbb{R}^0$, since

as in particular the plot of points $p : \mathbb{R}^0 \to \{\bullet\}$ is the only one which is an isomorphism. This can be seen by the fact that this smooth space is generated by the diffeology $\mathbb{R}^0 \to \{\bullet\}$, by taking the pullbacks of every map through $\mathbb{R}^0$. ie for our further maps $\mathbb{R}^k \to \{\bullet\}$, those can all be generated through the unique map $\mathbb{R}^k \to \mathbb{R}^0$,

$$\mathbb{R}^k \xrightarrow{!_{\mathbb{R}^k}} \mathbb{R}^0 \xrightarrow{p} \{\bullet\} \tag{6.69}$$

Theorem 6.5.6

Important functors :

Definition 6.5.6 *The forgetful functor*

$$U_{\mathbf{Set}} : \mathbf{Smooth} \to \mathbf{Set} \tag{6.70}$$

is the functor mapping every smooth space to its plot of points.

6.5.2 Subcategories of smooth sets

As a sheaf topos, we have that all objects of the site correspond themselves to objects of the topos via the representable presheaves. Any Cartesian space $\mathbb{R}^k$ therefore has a corresponding smooth set via

$$X_{\mathbb{R}^k} = \mathrm{Hom}_{\mathbf{CartSp}}(-, \mathbb{R}^k) \tag{6.71}$$

such that their plots are given by

$$\mathrm{Plot}_{X_{\mathbb{R}^n}}(\mathbb{R}^l) = \mathrm{Hom}_{\mathbf{CartSp}}(\mathbb{R}^l, \mathbb{R}^k) = C^\infty(\mathbb{R}^l, \mathbb{R}^k) \tag{6.72}$$

and their transition functions are

The subcategory of concrete sheaves in **Smooth** is of particular importance as it represents the diffeological spaces.

Definition 6.5.7 *A* diffeological space

Theorem 6.5.7 *The category of diffeological spaces and of concrete smooth spaces are isomorphic.*

Example 6.5.5

Smooth manifolds : locally representable sheaves

6.5.3 Stalks

Stalks via family of open balls?

6.5.4 Non-concrete objects

As we've seen, the concrete sheaves in **Smooth** do not form the entire topos, leaving non-concrete sheaves.

[Coarse moduli spaces?]

The archetypal example of this is the smooth set of differential k-forms,

$$\Omega^k : \mathbf{CartSp} \to \mathbf{Set} \tag{6.73}$$
$$U \mapsto \Omega^k(U) \tag{6.74}$$

which associates to every Cartesian space the set of k-forms over that space. The value of objects is simply the set of all possible k-forms on that Cartesian space, while the morphisms $f : U_1 \to U_2$ induce a pullback

Theorem 6.5.8 *The universal smooth set of differential k-forms is a smooth set.*

Proof 6.5.1 *Let's take the function associating to any Cartesian space its set of differential k-forms (where we consider here only the objects of the category)*

$$\Omega^k : \mathbf{CartSp} \to \mathbf{Set} \tag{6.75}$$

To rise to a functor, the smooth maps between Cartesian space must obey functorial rules, ie for some smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$, we have a corresponding function (in opposite order for a presheaf)

$$\Omega^k(f) : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^m) \tag{6.76}$$

which is given by the pullback of differential form : any k-form of $\mathbb{R}^n$ *is mapped to a k-form of* $\mathbb{R}^m$ *via f :*

$$\Omega^k(f)(\omega) = f_*\omega \tag{6.77}$$

and we indeed have $\mathrm{Id}_* = \mathrm{Id}_{\mathbf{Set}}$*, with the composition law [...]*

Sheaf properties :

If we attempt to look at the set of "points" of this space, if that term can be applied here, that would be the plot of the terminal object in the site, $\mathbb{R}^0$. But of course, in the sense of the sheaf as described here, this will just be the set of all k-forms over the point $\Omega(\mathbb{R}^0)$, which will just include the zero section, so that if we try to consider this plot as the "point content" of the space, there is but a single point :

$$\Omega(\mathbb{R}^0) = \{0\} \tag{6.78}$$

As we would not really consider the elements of this space to be that single section, it is therefore important to be mindful of what the plots of the sheaf represent.

Global sections :

$$\Gamma(\Omega^k) = \mathrm{Hom}_{\mathbf{Smooth}}(1, \Omega) \tag{6.79}$$

$$\Omega^k(X) \cong [X, \Omega^k] \tag{6.80}$$

Specific k-form on $X : 1 \to [X, \Omega^k]$

Value of the k-form at a point :

6.5.5 Important objects

The category of smooth spaces contains most of the objects of importance in physics and other fields, so that it is useful to look at the various types of objects within it.

First, as a topos, it has a terminal object as we've seen (the constant sheaf 1 which maps all probes to a single element, the constant plot). From this and the coproduct, we can construct objects similar to sets as we wish (this is in fact what the discrete functor will be later on), and as with any topos, a natural number object in particular.

As the coverage is subcanonical, the Yoneda embedding makes any Cartesian space a smooth space via its representable presheaf,

$$U \mapsto \mathrm{Hom}_{\mathbf{CartSp}}(-, U) \tag{6.81}$$

As we have seen, any diffeological space is a smooth space, in fact every concrete smooth space is a diffeological space.

Manifolds

By the Cartesian closed character of the topos, for any pair of manifolds, the set of all smooth maps between them is itself a smooth space, ie

$$C^\infty(M, N) \in \mathbf{Smooth} \tag{6.82}$$

We can also ask for various additional properties on those exponential objects. For instance, for two objects X, Y which are internal vector spaces, we can ask for the subset X^Y that respects

Theorem 6.5.9 *Linear maps between two internal vector spaces are a smooth space.*

Proof 6.5.2

Full category of real vector spaces internal to smooth?

Important classes of non-concrete sheaves are the *moduli spaces*, which are sheaves giving back appropriate function spaces on a Cartesian space. For instance the moduli space of Riemannian metrics Met is a sheaf

$$\mathrm{Met} : \mathbf{CartSp}^{\mathrm{op}} \to \mathbf{Set} \tag{6.83}$$

$$U \subseteq \mathbb{R}^n \mapsto \tag{6.84}$$

where $\mathrm{Met}(U)$ is the set of all Riemannian metrics on U. For instance, as there is only one metric on a point (since the tangent bundle there is zero dimensional), we have

$$\mathrm{Met}(\mathbb{R}^0) = \{0\} \tag{6.85}$$

And there is only one component to the metric on the line which must also be positive, so its set of metric is that of the positive definite smooth functions.

Other moduli structures of importance are the moduli space of differential forms $\Omega^\bullet$,

$$[X, \Omega^\bullet] = \Omega^\bullet(X) \tag{6.86}$$

and for classical mechanics, the moduli space of symplectic forms omega

True for any section?

Theorem 6.5.10 *The moduli spaces of sections is a smooth space*

Moduli space of connections

Other moduli spaces of importance are gauge-related ones, such as the moduli space of principal bundles, but this will be looked at in a more general lens in 8.

6.5.6 Internal objects

Definition 6.5.8 *An internal group of* **Smooth** *is a* Lie smooth space*.*

A particular case of this is the common case of Lie groups, which is simply an internal group whose underlying object is a manifold.

A more general example of a Lie smooth space that is not a Lie group would be for instance the diffeomorphism group. Given the power object of a manifold M^M, the diffeomorphism group is the subobject of invertible endomorphisms

Theorem 6.5.11 *The diffeomorphism group is a smooth space.*

Proof 6.5.3

This gives $\mathrm{Diff}(M)$ a natural internal group structure by composition.

As with many categories, one of the fundamental internal ring of the topos is $\mathbb{R}$, the representable functor of $\mathbb{R}$ itself.

A particularly important class of internal objects for smooth spaces is that of internal R-modules. By default, our internal function spaces are of the form of internal homs, $[X, Y]$. If we wish to speak of some real-valued function, it is an object in $[X, R]$, but there is no notion of this space being a module. Maps $X \to R$ are not enriched as modules (there is no notion of adding or multiplying them), nor can we do so with elements of $[X, R]$.

This is where the notion of internal R-module comes in. An internal R-module is fairly clear from the other notions of internalization seen before, we simply have some object M which is an internal Abelian group $(M, +)$ along with some morphism $\rho : R \times M \to M$, the action of the ring object R on M.

Theorem 6.5.12 *Any internal hom $[X, R]$ to a ring object R defines an R-module object.*

Proof 6.5.4 *First we must show the structure of $[X, R]$ as an Abelian group object. As a ring object, we have some smooth map $+ : R \times R \to R$ and zero map $0 : 1 \to R$ [...] We need to find some equivalent morphisms on $[X, R]$. For the addition, this is*

$$+ : [X, R] \times [X, R] \to [X, R] \tag{6.87}$$

With some implicit braiding to move the product around, this is constructed using the adjunct of this morphism :

$$\tilde{+} : [X,R]\times[X,R]\times X \overset{\mathrm{Id}_{[X,R]\times[X,R]}\times\Delta_X}{\longrightarrow} [X,R]\times[X,R]\times X\times X \overset{\mathrm{ev}_{X,R}\times\mathrm{ev}_{X,R}}{\longrightarrow} R\times R \overset{+}{\longrightarrow} R$$

whose right adjunct is

$$+ : [X, R] \times [X, R] \to [X, R] \tag{6.88}$$

and similarly for the additive inverse, this is some morphism $[X, R] \to [X, R]$, given by

$$\tilde{-} : [X, R] \times X \overset{\mathrm{ev}_{X,R}}{\longrightarrow} R \overset{-}{\longrightarrow} R \tag{6.89}$$

with likewise its right adjoint. The zero morphism is given by

$$\tilde{-} : [X, R] \times X \overset{\mathrm{ev}_{X,R}}{\longrightarrow} R \overset{-}{\longrightarrow} R \tag{6.90}$$

proof of associativity

proof of unitality proof of invertibility

Does $[X, R]$ also form an algebra?

Basic example : The real line itself is equivalent to the algebra over a point? $[1, R]$

6.5.7 Integration

As a category of locally Euclidian spaces which contains its own functions, one notion that we would like to see expressed in **Smooth** is that of integration, and more generally of distributions. If we have our space of functions $[X, R]$, our real line object R and some subspace of X in its power object X^Ω, we would like to have some notion of associating those functions over those subspaces to a value.

[64]

6.6 Category of classical mechanics

The exact category to give to classical mechanics is somewhat controversial, due to the difficulties of finding an appropriate notion of morphisms. If we pick the most obvious candidate (symplectic manifolds and symplectomorphisms), all morphisms have to preserve the symplectic form. For two symplectic spaces (P_1, ω_1) and (P_2, ω_2), the map $f : (P_1, \omega_1) \to (P_2, \omega_2)$ implies

$$f^* \omega_2 = \omega_1 \tag{6.91}$$

[...]

An alternative category is given by the Weinstein symplectic category, whose objects are symplectic manifolds

alternative : the *category of Poisson manifolds*?

Definition 6.6.1 *A* Poisson manifold (P, π) *is a manifold* P *equipped with a Poisson bivector* $\pi \in \Gamma(\bigwedge^2 P)$

Poisson bracket :

$$\{f, g\} = \langle df \otimes dg, P \rangle \tag{6.92}$$

Definition 6.6.2 *An* ichtyomorphism *is a smooth map preserving the Poisson bivector :* $f^* \pi = \pi$

From this, the category of Poisson manifolds **Poiss** is the category with Poisson manifolds as objects and ichtyomorphisms as morphisms. In terms of the smooth topos, we have to look at the moduli space of Poisson bivectors :

$$\begin{aligned} \Pi(-) : \mathbf{CartSp} &\to \mathbf{Set} && (6.93)\\ \mathbb{R}^n &\mapsto && (6.94) \end{aligned}$$

which is the set of all Poisson bivectors over $\mathbb{R}^n$. A choice of a Poisson manifold is therefore given by the internal hom of our underlying manifold and the moduli space of Poisson bivectors :

$$P(X) = [X, P] \tag{6.95}$$

Slice topos $\text{Smooth}_{/\Omega^2}$?

Poisson manifold : locally representable concrete object?

What are the morphisms in $\text{Smooth}_{/\Omega^2}$

6.6.1 Logic

The logic of classical mechanics is tied to the logic of measurement of observables. If we have some classical theory, with a Poisson manifold

Example : phase space of a point particle in n dimensions $\mathbb{R}^{2n}$, with the Poisson bracket

If we have some observable

$$f_o : \mathbb{R}^{2n} \to \mathbb{R} \tag{6.96}$$

Inversely, f_o selects a subset of the Poisson manifold. The statement that the measurement m_o is in the Borel subset $\Delta_o \subseteq \mathbb{R}$ is equivalent to a subobject of **Poiss**

$$S_o = f_o^{-1}(\Delta_o) \tag{6.97}$$

Limits and colimits :

$$S_{o_1} \sqcup S_{o_2} \tag{6.98}$$

What is the topos

Logic and presheaf etc

[65, 66, 67]

6.7 Categories for quantum theories

[68, 69]

To contrast with the topos we have seen for classical mechanics, we need to also consider an appropriate category to look at the behaviour of quantum mechanics, to see if specific categorical properties have some important things to tell us about quantum systems.

There are for this quite a lot of different categories we can look at. These are :

- The category of Hilbert spaces
- The slice category of a Hilbert space
- The spectral presheaf of a Hilbert space
- The Bohr topos

Those categories can all be used for quantum mechanics with various results [etc]

[70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85]

6.7.1 Quantum mechanics as a symmetric monoidal category

The basic formulation of quantum mechanics in terms of category theory is to simply look at the categories of its main objects, which are Hilbert spaces and C^*-algebras.

Definition 6.7.1 *The category* **Hilb** *of Hilbert spaces has as its objects Hilbert spaces and as morphisms bounded linear maps between two Hilbert spaces.*

The condition of bounded linear maps is here to guarantee the existence of a dual on every operator, which will be of use for us, as otherwise linear maps are not guaranteed a dual in the infinite dimensional case, for instance by picking the linear map on $L^2(\mathbb{R}^n)$

$$f(\psi) = \psi(0) \tag{6.99}$$

which is indeed linear etc, but corresponds to a "dual vector" $\chi = \delta_0$, which is not an $L^2(\mathbb{R}^n)$ Hilbert space, but part of the rigged Hilbert space of the position operator, $\Phi^*_{\hat{x}}$.

[proof]

Hilb can be entirely defined in categorical terms[86], so that we will look at this in a bit more detail to get some feel of the categorical structures involved here.

If we were to say them all at once, then **Hilb**, the category of (complex) Hilbert spaces with bounded linear operators, is defined as :

- A dagger compact symmetric monoidal category with duals, $(\mathbf{Hilb}, \otimes, I, \dagger, ^*)$
- The monoidal unit I is a separator
-

The monoidal symmetric part is simply enough the tensor product $\otimes$, which is symmetric by interchange, and the monoidal unit is $\mathbf{C}$. The dagger $\dagger$ corresponds to the adjoint on operators, and the compact closedness means that any object X has a dual object X^*, with the unit and counit

$$\eta : \tag{6.100}$$

[...]

Interpretations :

The monoidal unit I can be shown to be equivalent to the complex numbers. First we take the hom-set

$$\mathrm{End}(I) = \mathrm{Hom}_{\mathbf{Hilb}}(I, I) \tag{6.101}$$

and show that it has some scalar structure.

In terms of quantum mechanics, we have that for a given object $\mathcal{H}$, the Hilbert space vectors are given by morphisms $\psi : \mathbb{C} \to \mathcal{H}$, as the underlying field is the free Hilbert space here and therefore the appropriate object for generalized elements. Dually, maps of the form $\omega : \mathcal{H} \to \mathbb{C}$ form dual vectors on the Hilbert space

Dagger

In addition to all we've seen, we can also consider the enrichment of the category **Hilb** over Banach spaces, turning it into a *C^*-category.* This enrichment means that for any three Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$,

[...]

Zero map, sum of maps, scalar product of maps, involution

Definition 6.7.2 *Two subobjects $\iota_1 : \mathcal{H}_1 \hookrightarrow \mathcal{H}$, $\iota_2 ! \mathcal{H}_2 \hookrightarrow \mathcal{H}$ are said to be* orthogonal *if $\iota_1^* \iota_2 = 0$.*

Theorem 6.7.1 *The duality of an operator defines a canonical splitting of monomorphisms in that for any inclusion map $\iota : \mathcal{H}' \hookrightarrow \mathcal{H}$,*

$$\iota : \tag{6.102}$$

Proof 6.7.1 *This is the equivalent of requiring orthogonal projectors in Hilbert spaces, where a projection is orthogonal if for $\psi \in \mathcal{H}$,*

$$\langle P\psi, (1 - P)\psi' \rangle = 0 \tag{6.103}$$

meaning in categorical terms that the adjoint map $P\psi$

Those properties are in fact those of a C^*-algebra. As the notion of a C^*-algebra will be quite important later on for our other quantum categories, it is good to keep in mind.

Theorem 6.7.2 *The C^* algebra of an object $\mathcal{H} \in$ **Hilb** is its hom-object with the Banach space enrichment.*

In addition to C^*-algebras, we will also need to look at the von Neumann algebras of our Hilbert spaces

Definition 6.7.3 *A von Neumann algebra (or W^*-algebra) is a C^*-algebra A that admits a predual, a complex Banach space A_* with an isomorphism of complex Banach spaces*

$$* : A \to (A_*)^* \tag{6.104}$$

6.7.2 Slice category of Hilbert spaces

For a short detour in terms of interpreting quantum theory in a categorical manner, we will need to look at slice categories of **Hilb**. If we pick some Hilbert space $\mathcal{H}$, the slice category $\mathbf{Hilb}_{/\mathcal{H}}$ will be the appropriate setting to talk about this specific Hilbert space.

The objects of $\mathbf{Hilb}_{/\mathcal{H}}$ are the (bounded) linear maps from any Hilbert space to $\mathcal{H}$. For our interest later on, this contains in particular all the subspace maps from all subspaces of $\mathcal{H}$. As the slice category of an enriched category, it is furthermore an enriched slice category,

Hilb is a category in which every monomorphism is split, so every object of our slice category can be give by an appropriate retract, a projection.

[unique by orthogonal projections?]

6.7.3 Daseinisation

While monoidal categories are a perfectly serviceable setting for dealing with quantum mechanics, it has a few issues making it unsuitable for this analysis. In some sense it corresponds to the construction of an actual "quantum object" with an existence independent of measurement, giving it fairly problematic properties from a logical perspective (this is the content of the Kochen-Specker theorem). Due to this it also famously fails to be a topos, which is the main object we are concerned with here.

To deal with those problems, we have to deal with the *Daseinisation*[79] of the category, where rather than deal with some quantum object directly, we only consider its measurements in some context.

The simplest way to consider a measurement in quantum mechanics is to look at the projectors P of the theory. If we ignore the wider case of positive operator-valued measure and only look at projection-valued measure (we will assume no additional source of uncertainty beyond quantum theory), every measurement in a quantum theory can be modelled by this. If a measurement is associated with an observable A with spectrum $\sigma(A)$, and of projection-valued measure

$$E : \Sigma(\sigma(A)) \rightarrow \text{Proj}(\mathcal{H}) \tag{6.105}$$
$$\Delta \mapsto E(\Delta) \tag{6.106}$$

The Born rules is that the probability of the measurement lying in some measurable subset of the spectrum Δ is

$$P(X \in \Delta|\psi) = \langle \psi, E(\Delta)\psi \rangle \tag{6.107}$$

After said measurement the system will collapse to the state $E(\Delta)\psi$. Our logic is that a system is indeed such that $X \in \Delta$ if it was last measured to be so. The creation of a *context* from there is to consider the set of all measurements composed from commutative operators so that they can be said to be both true at the same time in a manner consistent with classical logic. If we have another measurement derived from an observable A' with a projection-valued measure E', the two PVM commute, in the sense that for any two measurable subsets of their spectra, $\Delta \subset \sigma(A), \Delta' \subset \sigma(A')$, we have

$$E(\Delta)E(\Delta') = E(\Delta')E(\Delta) \tag{6.108}$$

Meaning that if we have done a first measure $E(\Delta)$ (meaning $x \in \Delta$), and a second measure $E'(\Delta')$ ($x' \in \Delta'$), a third measure of the original quantity will yield the same result :

First measurement : Collapse

$$\psi \rightarrow \frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \tag{6.109}$$

Second measurement : Collapse $E(\Delta)\psi$ to $E'(\Delta')E(\Delta)\psi$

$$\frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \rightarrow \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|} \tag{6.110}$$

Third measurement :

$$\begin{aligned}
P(X \in \Delta| \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}) &= \langle \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}, E(\Delta)\frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|} \rangle \\
&= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2} \langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)E(\Delta)\psi \rangle \\
&= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2} \langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)\psi \rangle \\
&= 1
\end{aligned} \tag{6.111}$$

Therefore in a context, we can say that the measured values are "real" in that they do not depend on the measurement.

As the identity and the zero projector both commute with every operator, they are a part of every context.

We will furthermore need the notion of ordering of projectors, which corresponds to the ordering of the lattice in quantum logic, ie we say that two projectors P_1, P_2 are ordered if

$$P_1 \leq P_2 \leftrightarrow \mathrm{im}(P_1) \subseteq \mathrm{im}(P_2) \tag{6.112}$$

or equivalently, $P_1 P_2 = P_2 P_1 P = P_1$. This means

Example : Given a projection-valued measure P and a measurable set of its spectrum Δ, with some subset $\Delta' \subseteq \Delta$, by the rules

$$E(\Delta') = E(\Delta' \cap \Delta) = E(\Delta')E(\Delta) \tag{6.113}$$

We therefore have $E(\Delta') \leq E(\Delta)$.

In terms of interpretation, this means that for $P \leq P'$, P' is *weaker* : we only know that our state is in some subspace larger than for P. This can be seen in the case of projection-valued measures on some interval, where the weaker statement is $x \in [a-\varepsilon_1, b+\varepsilon_2]$ compared to the more precise statement $x \in [a, b]$. The best one could find is in fact the 1-dimensional projector, as no projector is smaller than that (except for the zero projector which cannot provide any information), and corresponds to the measurement of the exact state. Due to this, two 1-dimensional projectors are never ordered, unless they are the same

$$\forall P, P',\ \dim(\mathrm{im}(P)) = \dim(\mathrm{im}(P')) = 1 \to (P \leq P' \leftrightarrow P = P') \tag{6.114}$$

Properties :

$$\forall P,\ 0 \leq P \tag{6.115}$$

$$\forall P,\ P \leq \mathrm{Id} \tag{6.116}$$

The point of daseinisation is to consider measurements in general not as projectors in the category of Hilbert spaces, but spread onto all possible contexts that a system may have by considering the closest approximation of that measurement in a given context. This approximation is given by the narrowest projection that is superior to our projector, ie for all the projectors P' in the context, we wish to find the one such that $P \leq P'$, and for any other projector P'' which also obeys $P \leq P''$, $P' \leq P''$. This projector is denoted by, for a context V, $\delta(P)_V$, the V-support of P. In terms of lattice notation, this is given by

$$\delta(P)_V = \bigwedge \{P' \in \text{proj}(V) \mid P \leq P'\} \tag{6.117}$$

As Id is always part of every context and the supremum of any context, we are always guaranteed to have such a projector more precise or equal to the identity projector, which merely informs us that the state is in the Hilbert space at all and nothing more. If $P \in \text{proj}(V)$, $\delta(P)_V = P$.

Example of a subset again

We will need to consider the approximation of $E(\Delta)$ in every possible contexts

$$P \to \{\delta(E(\Delta))_V | V \in \mathcal{V}(\mathcal{H})\} \tag{6.118}$$

Why is this a sheaf? Contexts are ordered

(in the usual category of compact symmetric monoidal objects etc of quantum logic) is transformed to a (clopen) sub-object $\delta(P)$ of the spectral presheaf in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$

Kochen-Specker theorem : equivalent to the presheaf on the category of self-adjoint operator has no global element

Take a C^*-algebra (von Neumann?) A.

Subcategory of commutative subalgebras ComSub(A) is the poset wrt inclusion maps

for any operator (self-adjoint?) A, let W_A be the spectral algebra. W_A is the boolean algebra of projectors $E(A \in \Delta)$ that projects onto the eigenspaces associated with the Borel subset Δ of the spectrum $\sigma(A)$. $E[A \in \Delta]$ represents the proposition $A \in \Delta$

Spectral theorem : for all Borel subsets J of the spectrum of $f(A)$, the spectral projector $E[f(A) \in J]$ for $f(A)$ is equal to the spectral projector $E[A \in f^{-1}(J)]$ for A. In particular, if $f(\Delta)$ is a Borel subset of $\sigma(f(A))$, since $\Delta \subseteq f^{-1}(f(\Delta))$,

$$E[A \in \Delta] \leq E[A \in f^{-1}(f(\Delta))] \tag{6.119}$$

$$E[A \in \Delta] \leq E[f(A) \in f(\Delta)] \tag{6.120}$$

This means $f(A) \in f(\Delta)$ is weaker than $A \in \Delta$. $f(A) \in f(\Delta)$ is a *coarse graining* of $A \in \Delta$.

If $A \in \Delta$ has no truth value defined, $f(A) \in f(\Delta)$ may have for some f

Relations between two logical systems here :

First, any proposition corresponding to the zero element of the Heyting algebra should be valued as false, $\nu(0_L) = 0_{T(L)}$.

If $\alpha, \beta \in L$, $\alpha \leq \beta$, then α implies β. Ex : $A \in \Delta_1$, $A \in \Delta_2$, $\Delta_1 \subseteq \Delta_2$. Valuation should be $\nu(\alpha) \leq \nu(\beta)$ (monotonicity).

If $\alpha \leq \alpha \vee \beta$, $\beta \leq \alpha \vee \beta$, then $\nu(\alpha) \leq \nu(\alpha \vee \beta)$ and $\nu(\beta) \leq \nu(\alpha \vee \beta)$, and therefore

$$\nu(\alpha) \vee \nu(\beta) \leq \nu(\alpha \vee \beta) \tag{6.121}$$

Not as strong as $\nu(\alpha) \vee \nu(\beta) = \nu(\alpha \vee \beta)$. For instance for $A = a_1$, $A = a_2$, the projection operator for both of these proposition projects on the 2D span of the eigenvectors, not their union.

Similarly,

$$\nu(\alpha \wedge \beta) \leq \nu(\alpha) \wedge \nu(\beta) \tag{6.122}$$

Exclusivity : a condition and its complementation cannot both be totally true :

$$\alpha \wedge \beta = 0_L \wedge \nu(\alpha) = 1_{T(L)} \to \nu(\beta) \leq 1_{T(L)} \tag{6.123}$$

Unity condition : $\nu(1_L) = 1_{T(L)}$

Take the boolean subalgebra W of the lattice $P(H)$ of projection operators. Forms a poset under subalgebra inclusion. W is a poset category.

Take the set $\mathcal{O}$ of all bounded, self-adjoint operators on $\mathcal{H}$. Spectral representation :

$$A = \int_{\sigma(A)} \lambda dE^A_\lambda \tag{6.124}$$

$\sigma(A) \subseteq \mathbb{R}$ the spectrum of A, $\{E^A_\lambda | \lambda \in \sigma(A)\}$ a spectral family of A.

$$E[A \in \Delta] = \int_\Delta dE^A_\lambda \tag{6.125}$$

for Δ a borel subset of $\sigma(A)$. If a belongs to the discrete spectrum of A, the projector ontop the eigenspace with eigenvalue a is

$$E[A = a] := E[A \in \{a\}] \tag{6.126}$$

for $f : \mathbb{R} \to \mathbb{R}$ any bounded Borel function,

$$f(A) = \int_{\sigma(A)} f(\lambda) dE^A_\lambda \tag{6.127}$$

Categorification of $\mathcal{O}$: Objects are elements of $\mathcal{O}$, morphisms from B top A if an equivalence class of Borel functions $f : \sigma(A) \to \mathbb{R}$ exists such that $B = f(A)$, ie

$$B = \int_{\sigma(A)} f(\lambda) dE^A_\lambda \tag{6.128}$$

Definition 6.7.4 *The spectral algebra functor $W : \mathcal{O} \to W$ is*

- *Objects mapped $W(A) = W_A$, W_A is the spectral algebra of A*
- *Morphisms : if $f : B \to A$, then $W(f) : W_B \to W_A$ is the subset inclusion of algebras $i_{W_B W_A} : W_B \to W_A$.*

Spectral algebra for $B = f(A)$ is naturally embedded in the spectral algebra for A since $E[f(A) \in J] = E[A \in f^{-1}(J)]$ for all Borel subsets $J \subseteq \sigma(B)$

$$i_{W_{f(A)} W}(E[f(A) \in J]) = E[A \in f^{-1}(J)] \tag{6.129}$$

[...]

Category $\mathcal{O}_d$ of discrete spectra self-adjoint operators

Definition 6.7.5 *The spectral presheaf on $\mathcal{O}_d$ is the contravariant functor $\Sigma : \mathcal{O}_d \to$* **Set**

- *$\Sigma(A) = \sigma(A)$ (spectrum of A)*
- *if $f_{\mathcal{O}_d} : B \to A$, so that $B = f(A)$, then $\Sigma(f_{\mathcal{O}_d}) : \sigma(A) \to \sigma(B)$ is defined by $\Sigma(f_{\mathcal{O}_d})(\lambda) = f(\lambda)$ for all $\lambda \in \sigma(A)$*

Works because on discrete spectrum $\sigma(f(A)) = f(\sigma(A))$.

$$\Sigma(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d}) = \Sigma(f_{\mathcal{O}_d}) \circ \Sigma(g_{\mathcal{O}_d}) \tag{6.130}$$

global section : function γ that assigns for every object of the site an element γ_A of the topos, such that if $f : B \to A$, then $H(f)(\gamma_A) = \gamma_b$.

For the spectral functor, a global section / element is a function that assigns to each self-adjoint operator A with a discrete spectrum a real number $\gamma_A \in \sigma(A)$, such that if $B = f(A)$, then $f(\gamma_A) = \gamma_B$.

Kochen-Specker theorem : if $\mathrm{Dim}(\mathcal{H}) > 2$, there are no global sections of the spectral presheaf.

Continuous case

By Gelfand duality, the presheaf topos PSh(ComSub(A)) contains a canonical object, the presheaf

$$\Sigma : C \mapsto \Sigma_C \tag{6.131}$$

which maps a commutative C^*-algebra $C \hookrightarrow A$ to (the point set underlying) its Gelfand spectrum Σ_C.

[87]

von Neumann algebras

projection

To formalize this idea, we will need to use the notion of von Neumann algebra. While we could merely use C^* -algebras, there will be a difficulty if we do so : the projectors of C^*-algebras do not form a complete lattice, ie there may be subsets $S \subseteq \text{proj}(A)$ which lack a lower or upper bound.

An example for this would be the algebra of compact operators $K(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$

Every projection in $K(\mathcal{H})$ has finite rank :

$$\dim(\text{im}(P)) < \infty \tag{6.132}$$

If we consider an infinite dimensional Hilbert space, like $L^2(\mathbb{R})$, consider this subset : $\{P_i\}_{i\in\mathbb{N}}$, such that P_i maps to an i-dimensional subspace, and we have

$$\text{im}(P_i) \subset \text{im}(P_{i+1}) \tag{6.133}$$

Union of these is dense in $\mathcal{H}$?

$$\overline{\bigcup_{i\in\mathbb{N}} \text{im}(P_i)} = \mathcal{H} \tag{6.134}$$

Supremum : $\sup(\{P_i\}) = I$, but I is not a compact operator.

[Why it happens? Relation to topology]

This possible lack of supremum and infimum would lead to the absence of disjunctions and conjunctions in our category [cf logic chapter]. While not tragic (this would only affect infinite conjunctions of propositions), we will try to keep things complete.

To insure the completeness of the lattice, we will use instead von Neumann algebras

Definition 6.7.6 *A* von Neumann algebra *A is a C^*-algebra with a predual A_*, a Banach space dual to A.*

Weak operator topology : The basis of neighbourgoods of 0 given by sets of the form

$$U(x, f) = \{A \in L(V, W) | f(A(x)) < 1\} \tag{6.135}$$

for $x \in V$, $f \in W^* = \mathrm{Hom}_{\mathrm{TVS}}(W, k)$. A sequence of operators (A_n) converges to A iff $(A_n(x))$ in the weak topology on W.

von Neumann algebras are closed in weak operator topology : any limit of net converges.

[...]

Definition 6.7.7 *An Abelian von Neumann algebra*

Definition 6.7.8 *For any Abelian von Neumann algebra over $\mathcal{B}(\mathcal{H})$, there exists a self-adjoint operator generating it as [...]*

6.7.4 The Bohr topos

[88]

A broader notion of categorical quantum theory is given by the *Bohr topos*. Rather than only consider the spectral presheaf of spectra associated with each projector,

As with the case of the spectral presheaf, we start out with the [Partial?] C^*-algebra A of our quantum system, and define its poset of commutative subalgebras ComSub(A). This can be given as the functor

$$\mathcal{C} : C^*\mathbf{Alg} \to \mathbf{Poset} \tag{6.136}$$

[Image of morphisms]

Furthermore, we consider the functor sending any poset to its Alexandrov topology

$$\mathrm{Alex} : \mathbf{Poset} \to \mathbf{Top} \tag{6.137}$$

Definition 6.7.9 *Given a quantum system defined by its C^*-algebra A, its* Bohr site *is*

$$B(A) = \mathrm{Alex}(\mathcal{C}(A)) \tag{6.138}$$

Definition 6.7.10 *The* Bohr topos *of a C^*-algebra A is the ringed Grothendieck topos on the Bohr site,*

$$\mathrm{Bohr}(A) = (\mathrm{Sh}(\mathrm{Alex}(\mathcal{C}(A))), \underline{A}) \tag{6.139}$$

with $\underline{A}$ the tautological copresheaf

$$\mathrm{Bohr}_{\neg\neg}(A) = (\mathrm{Sh}_{\neg\neg}(\mathrm{Alex}(\mathcal{C}(A))), \underline{A}) \tag{6.140}$$

Definition 6.7.11 *The Bohrification of A is the tautological sheaf*

$$\underline{A} \in \mathrm{Sh}(\mathrm{ComSub}(A)) \tag{6.141}$$

which maps any commutative subalgebra C to its underlying set

$$\underline{A}(C) = |C| \tag{6.142}$$

and morphisms of the inclusions $C \subseteq D$ to the injection of sets

$$\underline{A}(C \subseteq D) = |C| \hookrightarrow |D| \tag{6.143}$$

Theorem 6.7.3 *The Bohrification of A is an internal C^*-algebra of the topos, with morphisms*

$$0 : \underline{1} \rightarrow \underline{A} \tag{6.144}$$
$$+ : \underline{A} \times \underline{A} \rightarrow \underline{A} \tag{6.145}$$
$$\cdot : \tag{6.146}$$

Relationship between Bohr topos and the spectral presheaf? Is it a specific sheaf in the dual topos?

[89]

6.7.5 The finite dimensional case

Take the case $\mathbb{C}^n$ of the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$, which is for instance used in quantum computing. The C^*-algebra is just

$$C^*(\mathcal{H}) = L(\mathcal{H}, \mathcal{H}) \tag{6.147}$$

(denoted $L(\mathcal{H})$ for short), with operator composition as its algebraic operation and complex conjugate as involution, as all finite-dimensional linear maps are bounded. If we pick a specific basis, this is the algebra of $n \times n$ matrices on $\mathbb{C}^n$ with matrix multiplication.

The algebra $L(\mathcal{H})$ is also a von Neumann algebra [proof]

any subalgebra is a von Neumann subalgebra

The projections of this algebra are formed by the orthogonal projections, as any oblique projection would not be self-adjoint, classified by the Grassmannians of the space,

$$\bigoplus_{i=0}^{n} \mathrm{Grass}(i, \mathcal{H}) \tag{6.148}$$

$$P = I_r \oplus 0_{d-r} \tag{6.149}$$

An operator A will simply be one of the linear map $A \in L(\mathcal{H})$

A context here is an Abelian von Neumann subalgebra of $L(\mathcal{H})$. The category of contexts $\mathcal{V}(L(\mathcal{H}))$, equivalently a set of commuting matrices

"An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator."

Theorem 6.7.4 *Any abelian von Neumann algebra on a separable Hilbert space is $*$-isomorphic to either*

- $\ell^\infty(\{1, 2, \ldots, n\})$
- $\ell^\infty(\mathbb{N})$
- $L^\infty([0, 1])$
- $L^\infty([0, 1] \cup \{1, 2, \ldots, n\})$
- $L^\infty([0, 1] \cup \mathbb{N})$

There is therefore some surjection from the self-adjoint operators to commutative von Neumann algebras :

$$f : \mathcal{B}_{\mathrm{sa}}(\mathcal{H}) \to \mathcal{V}(W^*(\mathcal{H})) \tag{6.150}$$

Spectral theorem :

Theorem 6.7.5 *For a bounded self-adjoint operator, there is a measure space (X, Σ, μ) and a real-valued essentially bounded measurable function f on X and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that*

$$U^\dagger T U = A \tag{6.151}$$

$$[T\varphi](x) = f(x)\varphi(x) \tag{6.152}$$

and $\|T\| = \|f\|_\infty$

Finite dimensional :

Theorem 6.7.6 *There exists eigenvalues* $\{\lambda_i\}$ *(ordered by value by* i*) of* A *and eigen subspaces* $V_i = \{\psi \in \mathcal{H} | A\psi = \lambda_i \psi\}$ *such that*

$$\mathcal{H} = \bigoplus_{i=1}^{n} V_j \tag{6.153}$$

Theorem 6.7.7 *For self-adjoint* A*, there exists an orthonormal basis of eigenvectors of* A*.*

Theorem 6.7.8 *For a self-adjoint operator* A *with respect to an orthogonal matrix, there exists an orthogonal matrix* T *such that* $T^{-1}AT$ *is diagonal.*

Theorem 6.7.9 *For a self-adjoint operator* A*, there exists different eigenvalues* $\{\lambda_i\}$*,* $i \leq j \to \lambda_i \leq \lambda_j$*, and eigen subspaces,*

$$W_i = \{\psi \in \mathcal{H} \mid A\psi = \lambda_i \psi\} \tag{6.154}$$

Let P_i *be the orthogonal projection of* $\mathcal{H}$ *onto* W_i*, then*

- $\mathcal{H}$ *is an orthogonal direct sum of* W_i *:* $\mathcal{H} = \bigoplus_{i=1}^{n} W_i$*, and* $W_i \perp W_j$ *for* $i \neq j$
- $P_i P_j = \delta_{ij} P_i$ *and* $\mathrm{Id}_{\mathcal{H}} = \sum_i P_i$
- $A = \sum_i \lambda_i P_i$

Theorem 6.7.10 *For a normal operator* A *(ie, commutes with its adjoint), there is a spectral resolution of* A*.*

Spectrum in finite dimension :

$$\sigma(A) = \tag{6.155}$$

For our observable A,

$$A = \sum_i \lambda_i^m P_i \tag{6.156}$$

The commutative algebra generated is that which is spanned by those projective operators, ie

$$\forall B \in \mathrm{ComSub}(A),\ \exists \{c_i\} \in \mathbb{C}^k,\ B = \sum_i^m c_i P_i \tag{6.157}$$

All those operators are commutative, simply by the commutativity of the projectors between themselves.

Example of two operators with the same commutative subalgebra : any two operators with the same projectors but different eigenvalues

Alternatively : define them entirely by sets of projectors (up to a scale?), ie some subset of commutative projector (between 0 and n)

The Gelfand spectrum of a von Neumann algebra is the unique measurable space we define

"The predual of the von Neumann algebra B(H) of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm ——A——= Tr(—A—). The Banach space of trace class operators is itself the dual of the C*-algebra of compact operators (which is not a von Neumann algebra)."

Self-duality in finite dimension due to every operator being trace-class

Spectral measure [90] (1) :

For (X, Ω) a Borel space, a spectral measure is

$$\Phi : \Omega \to \mathcal{B}(\mathcal{H}) \tag{6.158}$$

- $\Phi(U)$ is an orthogonal projection for all U, $\Phi(U)^2 = \Phi(U) = \Phi(U)^*$
- $\Phi(\varnothing) = 0$ and $\Phi(X) = \mathrm{Id}$
- $\Phi(U \cap V) = \Phi(U)\Phi(V)$
- For a sequence (U_i) of pairwise disjoint Borel subsets,

$$\Phi(\bigcup_i U_i) = \sum_i \Phi(U_i)$$

(convergence wrt strong operator topology)

[...]

Spectral measure for finite dimensional case : Given the Abelian von Neumann algebra generated by

$$A = \sum_i \lambda_i P_i \tag{6.159}$$

with functions

$$f(A) = \sum f(\lambda_i) P_i \tag{6.160}$$

$$\int f d\mu_\psi = \langle \psi, f(A)\psi \rangle \tag{6.161}$$
$$= \sum_i f(\lambda_i)\langle \psi, P_i \psi \rangle \tag{6.162}$$
$$= \sum_i f(\lambda_i)\|P_i\psi\|^2 \tag{6.163}$$

measure is the counting measure

$$\mu_\psi = \sum_i \|P_i \psi\| \delta_{\lambda_i} \tag{6.164}$$

Gelfand dual :

- The space is the discrete space $\sigma(A)$
- The sigma-algebra is the discrete sigma algebra given by $\mathcal{P}(\sigma(A))$
- The measure is the counting measure

The spectral presheaf is then the presheaf

$$\underline{\Sigma} : \mathcal{V}(\mathrm{VNA}(\mathcal{H}))^{\mathrm{op}} \to \mathbf{Set} \tag{6.165}$$

which maps

Decomposition of operators : Given a set of n 1-dimensional orthogonal projectors, $\{P_i\}$, $P_i P_j = 0$,

The two-dimensional case

The simplest case we can use is the one-dimensional case, $\mathbb{C}$, but having only a single state in its projective Hilbert space, is a bit too trivial (its only projection is the identity, and therefore its underlying category is the terminal category), so let's look at $\mathbb{C}^2$.

To classify its orthogonal projectors, let's look at the Grassmannians of various dimensions for $\mathbb{C}^2$:

- $\mathrm{Gr}_0(\mathbb{C}^2) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^2) \cong \mathbb{C}P^1$
- $\mathrm{Gr}_2(\mathbb{C}^2) = \{\mathbb{C}^2\}$

The zero and two dimensional cases are simple enough, the zero-dimensional projection operator being the zero operator 0, with Abelian von Neumann algebra the trivial algebra $\{0\}$, and the two-dimensional projection operator is the identity map $\mathrm{Id}_{\mathbb{C}^2}$, with Abelian von Neumann algebra the scaling matrices, $c\mathrm{Id}_{\mathbb{C}^2}$

The one-dimensional case will contain most of the cases of interest. for some point $p \in \mathbb{C}P^1$, ie a point on the Riemann sphere $p \in S^2$, $p = (\theta, \phi)$, there is a projector to that line in the complex plane.

Given any self-adjoint operator $\mathcal{B}_{\mathrm{sa}}(\mathbb{C}^2)$, the finite-dimensional spectral theorem tells us that the Hilbert space can be decomposed into orthogonal subspaces $\{W_i\}$ which each contain one or more of the eigenvectors of the operator. As there can only be as many orthogonal spaces as the sum of their dimension being inferior or equal to the total dimension, this will only allow the trivial case (Just the 0-dimensional subspace), a single 1-dimensional subspace, two 1-dimensional subspace, or a single 2-dimensional subspace. The first case is simply the projector 0, corresponding only to the 0 operator. The second case is, for the choice of a point (θ, ϕ) on the Riemann sphere,

$$A = \lambda P_{(\theta,\phi)} \tag{6.166}$$

The third case is

$$A = \lambda_1 P_{(\theta_1,\phi_1)} + \lambda_2 P_{(\theta_1,\phi_2)} \tag{6.167}$$

And the last case is a diagonal operator,

$$A = \lambda \mathrm{Id}_{\mathbb{C}^2} \tag{6.168}$$

The Abelian von Neumann algebras are therefore classified by those two points on the Riemann sphere,

$$((\theta_1, \phi_1), (\theta_2, \phi_2)) \to \mathrm{VNA}(\mathbb{C}^2) \tag{6.169}$$

The category of contexts is therefore such that

- The trivial von Neumann algebra is included in all algebras
- The scaling von Neumann algebra is not included in any other algebra?
- The von Neumann algebra constructed from a single one dimensional projection $P_{(\theta,\phi)}$ is included in any von Neumann algebra constructed from two one-dimensional projections, as long as they share that projection.

Diagram of the category

$\mathrm{VNA}(P_{(\theta,\phi)}) \leq \mathrm{VNA}(P_{(\theta,\phi)}, P_{(\theta',\phi')})$

Approximation of a projection : For any projection P, there is only two possible cases :

- The projection is 0, and the V-

Kochen-Specker : $\mathbb{C}^2$ is not concerned by this.

The three-dimensional case

To have a case that is actually covered by the big quantum theorems properly, we will have to consider the case of the Hilbert space $\mathbb{C}^3$. This is for instance the case given by massive spin 1 particles.

The classification of projectors is much the same as previously, thanks to the duality of Grassmannians,

- $\mathrm{Gr}_0(\mathbb{C}^3) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_2(\mathbb{C}^3) = \mathrm{Gr}_{3-2}(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_3(\mathbb{C}^3) = \{\mathbb{C}^3\}$

The orthogonal subspaces of an operator will be

1. The empty subspace 0
2. One 1-dimensional subspace
3. Two 1-dimensional subspace
4. Three 1-dimensional subspace
5. One 2-dimensional subspace
6. One 1-dimensional subspace and one 2-dimensional subspace
7. One 3-dimensional subspace

As before, the first and last case are trivial, consisting of the trivial subspace and the whole subspace.

6.7.6 The infinite-dimensional case

For a Bohr topos with a more interesting structure, such as a differential cohesive structure that we will need later on10, let's consider instead a simple infinite dimensional case, of the theory of a quantum particle in one dimension,

$$\mathcal{H} = L^2(\mathbb{R}, \ell) \tag{6.170}$$

with ℓ the Lebesgue measure and the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^\dagger \psi_2 \mu_\ell \tag{6.171}$$

where two functions ψ, ψ' are identified if they have the same inner product with all other functions, ie up to differences on a set of measure zero.

Due to its much more complex nature, the full classification of projection operators, and therefore contexts, is not gonna be attempted here, so that only a few representative examples will be look at here.

The basic operator we will investigate is the position operator, which is a multiplication operator of the constant function, $\hat{x} = \hat{M}_x$, acting on wavefunctions as

$$\hat{x}\psi(x) = x\psi(x) \tag{6.172}$$

Theorem 6.7.11 *The operator $\hat{x}$ is unbounded.*

Proof 6.7.2 *The norm of the operator is given by*

$$\|\hat{x}\| = \sup_{\|\psi\|=1} |\rangle \psi, \hat{x}\psi \rangle| \tag{6.173}$$

The supremum of the expectation value. If we pick a rather boring L^2 state such as the Heaviside function of width and height 1 centered at a given parameter x_0, we have a collection of wavefunctions of expectation value x_0 for any value of $x_0 \in \mathbb{R}$, so that

$$\|\hat{x}\| = \infty \tag{6.174}$$

Theorem 6.7.12 *The spectrum of $\hat{x}$ is $\mathbb{R}$.*

Proof 6.7.3

$\hat{x}$ is both a

Theorem 6.7.13 *The position operator $\hat{x}$ is self-adjoint.*

Proof 6.7.4

Theorem 6.7.14 *The projection-valued measure for the position operator is, for some subset $S \subseteq \mathbb{R}$ of the spectrum $\sigma(\hat{x})$, given by*

$$\hat{P}_S \psi = \hat{M}_{\chi_S} \psi \tag{6.175}$$

Proof 6.7.5

From this we can see that the projection (and therefore measurement) of the position on a subset of $\mathbb{R}$ is identical up to a set of measure 0.

1-dimensional classification : every projector is part of the set of all 1-dimensional subspaces of $\mathcal{H}$, is it the projective limit $\mathcal{C}P^\infty$? The Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, classifier of U(1) bundles

Kuiper's theorem?

Examples of operators [projectors?] with continuous spectrum

As a continuous operator, $\hat{x}$ does not have an eigenbasis[91] (outside of the more general case of the Gelf'and triple rigged Hilbert space), but we can instead compute its projection-valued measure.

One hierarchy of projections we can do for this is to consider the projections for the localization of the particle (we are assuming the non-relativistic case here where the particle *can* be localized). If the particle ψ is entirely localized in some interval $[a, b]$, that is,

$$\int_a^b \psi^\dagger(x) x \psi(x) dx = 1 \tag{6.176}$$

then we will denote this by the projector operator $P_{[a,b]}$. This selects wavefunctions of support $[a, b]$ (almost everywhere). Given the projection-valued measure of the position operator μ_x, this projection is given by

$$P_{[a,b]} = \int_a^b \hat{\mu}_x \approx \int_a^b dx \ |x\rangle\langle x| \tag{6.177}$$

with $|x\rangle\langle x|$ the standard physicist notation for the projection on the rigged Hilbert space basis.

Those projectors obey the following properties. Being the multiplication operator M_x, for some Borel subset $I \subseteq \mathbb{R}$, its spectral measure is given by the multiplication operator

$$P_I = M_{\chi_I} \tag{6.178}$$

Therefore, for a set of measure zero like the singleton, we have

$$P_{[a,a]} = 0 \tag{6.179}$$

Since wavefunctions are only defined modulo sets of measure 0.

For some subinterval $[a, b] \subset [a', b']$, we have

$$\begin{aligned} P_{[a,b]} P_{[a',b']} \psi = \chi_{[a,b]} \chi_{[a',b']} \psi & \quad (6.180) \\ = \chi_{[a,b] \cap [a',b']} \psi & \quad (6.181) \\ = \chi_{[a,b]} \psi & \quad (6.182) \end{aligned}$$

Hence $P_{[a,b]} P_{[a',b']} = P_{[a',b']} P_{[a,b]} = P_{[a,b]}$

$$\neg P_{[a,b]} = P_{[0,1] \setminus [a,b]} \tag{6.183}$$

Given two intervals, $[a, b]$, $[c, d]$, we have the following operations on $P_{[a,b]}$ and $P_{[c,d]}$.

Heyting algebra of the projectors as a homomorphism of algebra from the interval algebra of $\mathbb{R}$?

Power set boolean algebra for $\mathbb{R}$?

For any two subsets of $\mathbb{R}$, A, B, we have the standard boolean algebra, ie

$$\begin{aligned} A \leq B &\rightarrow A \subseteq B & (6.184) \\ A \wedge B &= A \cap B & (6.185) \\ A \vee B &= A \cup B & (6.186) \\ \neg A &= \mathbb{R} \setminus A & (6.187) \\ A \rightarrow B &= A^c \cup B & (6.188) \\ \top &= \mathbb{R} & (6.189) \\ \bot &= \varnothing & (6.190) \end{aligned}$$

From the properties of the indicator function χ_A, we have the following algebra homomorphism

$$\begin{aligned} P_{A \wedge B} \psi &= \chi_{A \cap B} \psi & (6.191) \\ &= \chi_A \chi_B \psi & (6.192) \\ &= P_A P_B \psi & (6.193) \\ &= P_B P_A \psi & (6.194) \end{aligned}$$

$$\begin{aligned} P_{A \vee B}\psi &= \chi_{A\cup B}\psi & (6.195) \\ &= (\chi_A + \chi_B - \chi_A\chi_B)\psi & (6.196) \\ &= (P_A + P_B - P_A P_B)\psi & (6.197) \\ & & (6.198) \end{aligned}$$

$$\begin{aligned} P_{\neg A}\psi &= \chi_{\mathbb{R}\setminus A}\psi & (6.199) \\ &= (\chi_{\mathbb{R}} - \chi_A)\psi & (6.200) \\ &= (P_{\mathbb{R}} - P_A)\psi & (6.201) \\ &= P_A^{\perp}\psi & (6.202) \end{aligned}$$

$$\begin{aligned} P_{A \to B}\psi &= \chi_{A^c\cup B}\psi & (6.203) \\ &= (\chi_{A^c} + \chi_B - \chi_{A^c}\chi_B)\psi & (6.204) \\ &= (P_{\mathbb{R}} - P_A + P_B - (P_{\mathbb{R}} - P_A)P_B)\psi & (6.205) \\ &= (P_A^{\perp} + P_B - P_A^{\perp}P_B)\psi & (6.206) \end{aligned}$$

For a fuller picture of the daseinisation of the theory, we also need some non-commutative operator with the position, the natural choice being the momentum operator.

$$\hat{p} = -i\hbar\frac{d}{dx} \tag{6.207}$$

Domain [91]

$$\mathrm{Dom}(\hat{p}) = H^{0,1} = \{\psi \in L^2([0,1]) \mid \psi' \in L^2([0,1])\} \tag{6.208}$$

Projection valued measure?

Interaction in the Heyting algebra

If we consider our position operator $\hat{x}$ as the generator of an Abelian von Neumann algebra,

We now have two commutative von Neumann algebras, V_x and V_p. For any projector in either of those, their support is simply themselves

$$\delta_{V_x}(P_x) = \bigwedge\{P \in V_x \mid P_x \leq P\} \tag{6.209}$$

as $P_x \in V_x$ and P_x is its own greatest lower bound.

Now if we wish to compare a momentum projector in a position context, we need to figure out the interaction of the two measurements. By the Paley-Wiener theorem, if we perform any measurement localizing the wavefunction to some compact subset, giving a wavefunction of compact support after collapse, then the resulting momentum space wavefunction is holomorphic, and therefore analytic. If it is equal to zero on any open set, then it is identically zero, meaning that the spectrum will always be the full momentum space.

This result does not generalize to the full subset algebra of $\mathbb{R}$, however, as it is possible to have wavefunctions where the support is not the entire real line nor a dense subset of it, but neither does the momentum space wavefunction[92].

As the momentum is merely a multiplicative operator on the Fourier transform of the wavefunction, given some initial measurement on the subset I, so that our wavefunction is some L^2 function with essential support in I

[...]

$$P_{[p_1,p_2]}P_{[x_1,x_2]} \tag{6.210}$$

$$\delta_{V_x}(P_p) = \bigwedge\{P \in V_x \mid P_p \leq P\} \tag{6.211}$$

7

Logic

[93, 94, 95]

One element of interest of topoi as a good foundation for math is that there is a connection between topos and logical theories. Given some category $\mathbf{C}$, there exists some equivalence with a type theory $L(\mathbf{C})$, which, given some appropriate constraints on the category, can be interpreted as a logic on this category.

7.1 Logic and order structures

To look at this mapping, first let's look at how logics, in the broad sense, are composed. First, let's look at the "bare" logic of propositions. This is composed by a set of symbols for propositional variables

$$P = \{p_1, p_2, \ldots\} \tag{7.1}$$

the standard n-ary functions on propositions, ie $\neg$, $\wedge$, $\vee$ and $\to$, and parenthesis for bracketing, with the syntax

- Any propositional variable p_i is a proposition.
- For any proposition α, there is a proposition $\neg\alpha$.
- For any two propositions α, β, there is a proposition for any binary function, ie $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \to \beta$

Sequent calculus

An example of application of the calculus can be done with some basic application of logical operators with a single hypothesis,

Theorem 7.1.1 ***Modus ponens*** *: Given the proposition $p \to q$ and p, we have q :*

$$\frac{\Gamma_1 \vdash \Delta_1, p \to q \qquad \Gamma_2 \vdash \Delta_2, p}{\Gamma_1, \Gamma_2 \ \vdash \Delta_1, \Delta_2, q}$$

Proof 7.1.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(I)}{\Gamma_1, p \vdash p}\,(WL)}{p, \Gamma_1 \vdash p}\,(PL)}{p, \Gamma_1 \vdash p, \Delta_1}\,(WR)}{p, \Gamma_1 \vdash \Delta_1, p}\,(PR) \dfrac{x}{x}\,(I)}{p \to q, \Gamma_1 \vdash \Delta_1, p}\,(\to L)$$

Theorem 7.1.2 *Given the equivence $p \leftrightarrow q$, this entails $p \to q$ and $q \to p$*

Proof 7.1.2 *Simply by*

Theorem 7.1.3 *If two propositions are equivalent, $p \leftrightarrow q$, they can prove the same proposition, ie we have*

$$\frac{\Gamma, p \vdash \Delta, A \qquad (p \leftrightarrow q) \vdash}{\Gamma, q \vdash \Delta, A}$$

Proof 7.1.3

$$\frac{\Gamma, p \vdash \Delta, A \qquad (p \leftrightarrow q) \vdash}{\Gamma, q \vdash \Delta, A}$$

The correspondence between a logical theory and an algebra on an order structure is known as the *Lindenbaum-Tarski algebra* of the theory.

Theorem 7.1.4 *For any type of logical system with an associated sequent calculus including at least [sequent rules], where we identify equivalent propositions, $\alpha \sim \beta$ meaning that $\alpha \to \beta \wedge \beta \to \alpha$ is true, then there is a corresponding [order?]*

Proof 7.1.4 *The mapping f is done using the translation that any proposition α corresponds to an element of the poset, any sequent of the form*

$$\pi \vdash \alpha \tag{7.2}$$

corresponds to an order relation $f(\pi) \leq f(\alpha)$

[...]

For this to be the appropriate [order],

The mapping is reflexive : comes from the axiom of identity, $A \vdash A$ [Add extra context] The mapping is antisymmetric : if $A \vdash B$ and $B \vdash A$, $\vdash A \cong B$ [what does it mean on a lattice level]

Transitive : $A \vdash B$, $B \vdash C$, then $A \vdash C$ (cut rule?)

In addition to this bare logic, we will also involved typed entities. For instance, in the case of sets, every

Definition 7.1.1 *In a logical theory, a (first-order) signature is composed of*

- *A set Σ_0 of* sorts
- *A set Σ_1 of* function

Example 7.1.1 *For classical logic, its signature is given by*

Definition 7.1.2 *For a category* **C** *with finite products, and a signature Σ, a* Σ-structure *M in* **C** *defines :*

- *A function between sorts in Σ_0 and objects in* **C**
- *A function between functions in Σ_1 and morphisms in* **C**
- *A function between relations and subobjects*

Connection between logic and order structures, relation to subobject order structures

[internal v. external logic]

Support object, h-propositions

$$\text{isProp}(A) = \prod_{x:A} \prod_{y:A} (x = y) \tag{7.3}$$

difference between h-propositions and propositions as types?

[96]

Logic from types, logic from topos, Heyting algebra [97] The subobjects of objects X in a topos $\mathbf{H}$ form a Heyting algebra, with operations $\cap, \cup, \rightarrow$ the partial ordering $\subseteq$ and the greatest and smallest elements 1_A, 0_A.

The language $L(\mathbf{H})$ of a topos is a many-sorted first-order language having the objects $X \in H$ as types for the terms of $L(\mathbf{H})$, there is a type operator τ which assigns to any term of $L(\mathbf{H})$ an object $\tau(p)$ of $\mathbf{H}$ called the type of p.

- 0_H is a constant term of type 1.
- For any object A of E, there is a countable number of variables of type A
- For any map $f : A \rightarrow B$, there is an "evaluation operator" $f(-)$ for terms of type A to terms of type B : p of type $A \Rightarrow f(p)$ of type B
- For any ordered pair (A, B) of $\mathbf{H}$, there is an ordered pair operator $\langle -, - \rangle$
- For any subobject $M : A \rightarrow \Omega$, there is a unary "membership-predicate $(-) \in M$ for elements of A. $x \in M$ is an atomic formula provided $x \in A$.
- The propositional connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$ are allowed for new formulas
- For any object A and variable $x \in A$, the quantifier $\exists x \in A$ and $\forall x \in A$ are allowed

Definition 7.1.3 *Two objects $x, y \in A$ are equal if*

$$x = y \leftrightarrow \langle x, y \rangle \in \Delta_A \tag{7.4}$$

$\Delta_A : A \times A \rightarrow \Omega$ *the diagonal operator*

Unique equality :

$$(\exists! x \in A)\phi(x) \leftrightarrow \exists x \in A, \forall y \in A, (\phi(y) \leftrightarrow x = y) \tag{7.5}$$

Membership :

$$x \in y \leftrightarrow \langle y, x \rangle \in (\mathrm{ev} : PA \times A \rightarrow \Omega) \tag{7.6}$$

For $x \in A$ and $F \in B^A$,

$$F(x) = (\mathrm{ev} : B^A \times A \rightarrow B)\langle F, x \rangle \tag{7.7}$$

For any map $f : A \rightarrow B$ with exponential adjoint $\overline{f} : 1 \rightarrow B^A$, we define an element $f_e = \overline{f}(0_e) \in B^A$ which represents f internally.

Example 7.1.2 *Some of the most barebones "logics" one can have for a category are the initial category* $\mathbf{0}$ *and the terminal category* $\mathbf{1}$*. The initial category has no types and therefore no terms or propositions, corresponding to the empty logic. The terminal category has a single type (the unit type), and only one term given by the identity, and a single proposition* $* \to *$*, which corresponds to the truth statement. As all the limits in the terminal category are its single object, this transfers to its category of subobjects, meaning that every logical construction there is also* $1 \to 1$*, so that this is the trivial logic, for which any proposition is true (this can be seen as stemming from the initial and terminal object being the same here, and therefore not having a notion of falsehood different from truth).*

Example 7.1.3 *A standard example of an internal logic is given by a group category, in the sense of a groupoid of one element with endomorphisms isomorphic to* G*. Having only one object, there is only one associated type, the group type* G*, and only one subobject relation, which is* $\mathrm{Id}_G : G \hookrightarrow G$*.*

Lawvere-Tierney topology as locality modality

7.2 Internal logic

[98, 99]

From the Lindenbaum-Tarski correspondence between logic and algebra, we also have the converse of being able to associate a logic to every algebra on an order structure, at least for a broad enough definition of "logic".

[proof?]

This means in particular that in a category, for any object $X \in \mathbf{C}$, we have some logic associated with the "type" of X given by the poset of its subobjects.

Objects as types

These "logics" will depend on the properties of the category. In particular, we will not go very far in terms of a logic without a notion of finite products [correspondence to what?]

Syntatic category, category of contexts?

Meaning of the implication wrt Heyting implication

7.2.1 Lawvere theory

[100, 101, 102]

An important part of the logical structure of a category is given by its algebraic structure. In a logical theory, given some set X, we define the n-tuples of that set X^n, with $X^0 = \{\bullet\}$. An n-ary function is then some function $f : X^n \to X$.

As functions $X^0 = \{\bullet\} \to X$ are isomorphic to X, the nullary functions simply represent constant values.

Universal algebra :

Once a set of such operations has been defined, an algebra is defined by a set of *equational laws*

example : commutativity

$$\forall x, y \in X,\ x * y = y * x \tag{7.8}$$

specific relations :

$$1() + 1() = 2() \tag{7.9}$$

Example 7.2.1 *A group is a algebra given by a set G and three functions, the binary multiplication $\cdot$, the unary inverse $(-)^{-1}$, and the nullary neutral element e*

Generally the smallest amount of structure we can ask of a theory to give something deserving of the name logic is a finite product. Without this, we would not even be able to define [equivalence, multiple propositions in sequents?]

A logical theory associated with an object for which finite products are defined with its subobjects is called a *Lawvere theory*

Definition 7.2.1 *Given the skeletal category of finite sets, denoted by*

$$\aleph_0 = \mathrm{sk}(\mathbf{FinSet}) \tag{7.10}$$

equipped with the finite coproduct defined by disjoint union, we define its opposite category $\aleph_0^{\mathrm{op}}$, with an associated finite product.

As every element of $\aleph_0$ is a coproduct of 1, we have that

$$\mathrm{Hom}_{\aleph_0}(n, m) \cong \mathrm{Hom}_{\aleph_0}(1, m)^n \tag{7.11}$$

so that every morphism in $\aleph_0$ is an n-fold product of coproduct injections. Its opposite category is then such that every object is an n-fold product of 1

As we've seen, the opposite category for finite sets are finite boolean algebras.

Definition 7.2.2 *For a small category* $\mathbf{L}$ *with strictly associative finite products, we say that it is a* Lawvere theory *if there exists an identity on objects functor* $I : \aleph_0^{\mathrm{op}} \to \mathbf{L}$ *that preserves finite products strictly.*

As the functor is an identity on objects, the interpretation of a Lawvere theory is that there exists some distinguished object X of the category (corresponding to $I(1)$), called the *generic object* (or *generating object*), and every other object of L is simply a power of L, via

$$I((\coprod 1)^{\mathrm{op}}) = \prod I(1) \tag{7.12}$$

so that we have that any object of $\mathbf{L}$ is simply X^n. In particular, X^0 is the initial object.

L is *one-sorted.*

Example 7.2.2 *The simplest such Lawvere theory is*

Example 7.2.3 *For a given group* G *considered as a category of one object with morphisms* $\mathrm{Mor}(G) = G$*, Lawvere theory of groups*

Definition 7.2.3 *A model of a Lawvere theory is a finite product preserving functor*

$$M : \mathbf{L} \to \mathbf{C} \tag{7.13}$$

for some category $\mathbf{C}$ *with finite products.*

7.2.2 Internal logic

From this, we assign a type theory, and hopefully a logic, to a given category.

Definition 7.2.4 *In a category* $\mathbf{C}$ *with finite products, and a logical signature* Σ*, a* Σ-structure *on* $\mathbf{C}$ *is given by the following three functions :*

- *A function* $M : \Sigma \to \mathbf{C}$ *which assigns for every finite list of* Σ*-sorts an object of* $\mathbf{C}$*, such that*

$$M(S_1, \ldots, S_n) = M(S_1) \times \ldots \times M(S_n) \tag{7.14}$$

  and

$$M() = 1 \tag{7.15}$$

- *A function assigning to every function symbol* $f : S_1 \times \ldots \times S_n \to T$ *an arrow*

$$M(f) : M(S_1, \ldots, S_n) \to M(T) \tag{7.16}$$

- *A function assigning to every relation* $R(S_1, \ldots, S_n)$ *a subobject*

$$M(R) \hookrightarrow M(S_1, \ldots, S_n) \tag{7.17}$$

7.2.3 Internal logic of a topos

As categories with fairly rich structures, topoi typically have fairly well-behaved associated logical theories. From its properties, we can already guess that it will be a Lawvere theory, contain function types, etc

Theorem 7.2.1 *Any internal logic of an elementary topos* **H** *is an intuitionistic higher order logic.*

Proof 7.2.1 *Left and right "and" elimination :* $(p \wedge q) \to p$, $(p \wedge q) \to q$ *: projection functions*

Axiom of simplification : $p \to (q \to p)$*. Distributivity :*

$$\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r)) \tag{7.18}$$

[Since every morphism has a pullback in a topos, every morphism is a display morphism and therefore corresponds to a predicate]

$$[\![X]\!] = X \tag{7.19}$$

7.3 Mitchell-Benabou language

In the case of a category with a subobject classifier, there is a simpler way of expressing the internal logic of a category, as the subobjects of a given object can be simply given by morphisms $X \to \Omega$. This isomorphism is the *Mitchell-Benabou language* of the category.

Theorem 7.3.1 *In the internal logic of a category, a proposition can equivalently be described by a characteristic morphism.*

Example 7.3.1 *Internal logic of natural numbers*

7.4 Modal logic

The internal logic of a category will translate monadic and comonadic operators in terms of modalities.

[...]

It has been historically a point of contention between analytic and continental philosophers as to whether or not there was a point to modalities. Part of this contention relies on the reducibility of modal logic (at least the main type of modal logic of use back then, S4/S5 modal logic) to standard predicate logic via the Kripke semantics.

Our goal here will not be to judge of its *necessity*, but rather of its utility.

7.5 The internal logic of Set

The internal language $L(\mathbf{Set})$ will roughly correspond to classical logic (as applied to sets). More precisely, in terms of a set theory, it corresponds to *bounded Zermelo set theory*, which is similar to ZFC set theory, but with bounded comprehension and choice, ie 1

- For any predicate ϕ with only bounded quantifiers ($\forall x \in A$, $\exists x \in B$), and a set B, then $\{x \in B \mid \phi(x)\}$ is a set
- Bounded choice?

This is due to the lack of tools to talk of unbounded quantification over all objects internally [but see stack semantics].

In terms of its ”fundamental” operations, this means that any atomic predicate will be of the form $x \in A$, including the equality operation :

$$A = B \leftrightarrow \forall x \in A,\ x \in B \wedge \forall x \in B,\ x \in A \tag{7.20}$$

So that any proposition will fundamentally depend on the morphisms $x : 1 \to A$ (this simply stems from the category being well-pointed).

Many-sorted first order language having the objects of the topos as types

Boolean algebra of subsets : for $X \in \mathbf{Set}$, we consider the boolean algebra of subobjects $\mathrm{Sub}(X)$ with the correspondences

Boolean algebra	Set operator
$a, b, c, \ldots$	$A, B, C \subseteq X$
$\wedge$	$\cap$
$\vee$	$\cup$
$\leq$	$\subseteq$
0	$\varnothing$
1	X
$\neg A$	$X \setminus A = A^c$
$A \to B$	$(X \setminus A) \cup B = A^c \cup B$

Table 7.1: Caption

$\to$: every element except the elements of A that aren't also elements of B.

Boolean algebra identities :

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{7.21}$$
$$A \cup B = B \cup A \tag{7.22}$$
$$A \tag{7.23}$$

[...]

A basic example of statement in our topos is given by the truth morphism $\top : 1 \to \Omega$, which trivially factors through itself,

$$1 \xrightarrow{\mathrm{Id}_1} 1 \xrightarrow{\top} \Omega \tag{7.24}$$

which corresponds to the trivial statement

$$\vdash \top \tag{7.25}$$

Simply stating that truth is always internally valid. Conversely, the falsity morphism $\bot : 1 \to \Omega$, representing falsehood, should not have any such factoring, ie

$$1 \xrightarrow{f} 1 \xrightarrow{\top} \Omega \tag{7.26}$$

such that $\top \circ f = \bot$. As there is only one endomorphism on 1, this can only be the identity map, and therefore is only true if $\top = \bot$. This would however mean that, as $\top$ is the classifying map of 1 and $\bot$ that of 0 as subobjects of 1, by pullback this would mean that both 0 and 1 are the same object, which is not the case in **Set**.

Therefore, we can write that falsity is indeed false,

$$\nvdash \bot \tag{7.27}$$

but on the other hand, the negation of falsity, $\neg\bot$, is true, as

[...]

More generally, we can derive this

Theorem 7.5.1 *If the negation of a morphism is true, ie*

$$A \xrightarrow{f} \Omega \xrightarrow{\neg} \Omega \tag{7.28}$$

factors through 1 *:*

$$A \xrightarrow{f'} 1 \xrightarrow{\top} \Omega \tag{7.29}$$

such that $\neg \circ f = \top \circ f'$, then this morphism is false, ie there is no such factoring of f through $\top$.

Proof 7.5.1

Axioms :

Negation : for $p : A \hookrightarrow X$, ie

$$A \overset{p}{\hookrightarrow} X \xrightarrow{\chi_A} \Omega \tag{7.30}$$

The negation of the set $\neg A$ is such that $\neg p : \neg A \hookrightarrow X$ factors through the negation and p,

$$\chi_{\neg A} = \chi_A \circ \neg \tag{7.31}$$

Statement with context : $p, p \to q \dashv q$: slice category $\mathbf{Set}_{/(p:A\to X)\times[]}$.

Localization modality : $\bigcirc_j$ for $j = \mathrm{Id}_\Omega$:

[...]

An example of internal logic we can derive in **Set** is by looking at the natural number object **Set** as an internal ring. If we just look at $\mathbb{N}$ and $\{\bullet\}$, we have the integer type $\mathbb{N}$ and the unit type 1, so that

The successor function $s : \mathbb{N} \to \mathbb{N}$ is an integer term with a free integer variable, ie we have

$$n : \mathbb{N} \dashv s(n) : \mathbb{N} \tag{7.32}$$

We also have the subobjects

Proof of induction?

Theorem 7.5.2 *Given a sequence given by $1 \xrightarrow{q} A\overset{f}{A}$, with*

$$\begin{aligned} a_0 &= q & (7.33) \\ a_{n+1} &= f(a_n) & (7.34) \end{aligned}$$

If a property ϕ on an element of A is true for a_0 and if $\phi(a_n) \to \phi(a_{n+1})$, then ϕ is true for a subobject of A such that $P \cong N$ (or something) 1

7.6 The internal logic of a spatial topos

Logic of $\mathrm{Sh}(\mathbf{X})$, $\mathrm{Sh}(\mathbf{CartSp}_{\mathbf{Smooth}})$

Locality modality j

7.7 The internal logic of smooth spaces

As a logical system, the topos of smooth sets has as types the various smooth sets

Proposition : subspaces $S \hookrightarrow X$

Subobject classifier : Ω is the sheaf associating to any $U \subseteq \mathbb{R}^n$ such that

$$\Omega(U) = \{S | S \text{ is a } J\text{-closed sieve on } U\} \tag{7.35}$$

$$\Omega(f) = f^* \tag{7.36}$$

$\top : 1 \to \Omega :$ maximal sieve on each object

Points in $\Omega :$ all the maps $1 \to \Omega$, all the J-closed sieves on $\mathbf{R}^0$.

7.8 The internal logic of classical mechanics

Internal logic for Poisson manifolds

Symmetric monoidal category with projection

In this context, a configuration is a subobject $f : 1 \to \text{Phase}$ for a given phase space.

let's consider the maps from configurations to the real numbers, [1]

A particular Poisson structure we can give is the trivial Poisson structure,

$$\{f, g\} = 0 \tag{7.37}$$

If we consider the map $\mathbb{R} \to P$ corresponding to this trivial structure on the real line, we will get our local example of a real line object. Statements about the measurements of a classical system can therefore be understood as morphisms from

Are the measurement given by $\mathbb{R}$ with the trivial Poisson structure

types given by morphisms to R?

A basic type of proposition on R is the notion of a measurement being in some interval. If we want to associate some interval $I \hookrightarrow R$ to a configuration in Phase, this is given by a morphism to the power object,

$$\text{Phase} \to \Omega^R \tag{7.38}$$

such that for a given configuration, we have the following diagram commute

Example 7.8.1 *If we only look at observables that give out a well-defined value in R, this is the case where the observable only gives us singleton values, so that in fact*

For many practical circumstances, we will want to know more specifically propositions about values. If we have a phase space Phase and some point in that phase space $(x, p) : 1 \to \text{Phase}$

7.9 The internal logic of quantum mechanics

There are three possible internal logics that we can consider for quantum mechanics here. If we consider it as a symmetric monoidal category, this is a form of *linear logic*, what we will call *quantum linear logic*. If we consider a given Hilbert space $\mathcal{H}$, the logic of the slice category $\mathbf{Hilb}_{\mathcal{H}}$ is von Neumann's *quantum logic*. And finally, we will look at the internal logic of the Bohr topos that we have constructed.

beep

[103]

7.9.1 Linear logic

[104]

The category of Hilbert spaces and linear logic are not quite like the other ones that we have looked into so far, not forming a topos. As we do not have a subobject classifier here, we will not be able to use a Mitchell-Benabou language. But we can still perform that translation using the basic translation as a type theory.

If we pick a given Hilbert space $\mathcal{H}$ as a reference, propositions are given by monomorphisms $\iota : W \hookrightarrow \mathcal{H}$ (up to isomorphisms). In the category of Hilbert spaces, all monomorphisms are split, meaning that there exists a retraction P_W

$$W \overset{\iota_W}{\hookrightarrow} \mathcal{H} \xrightarrow{P_W} W \tag{7.39}$$

such that $P_W \circ \iota_W = \text{Id}_W$. This is the notion we've seen before for the state of a system depending on a projection of the Hilbert space via some measurement operator.

The category of Hilbert spaces, like the category of vector spaces, has an initial and terminal object that are the same, the *zero object* 0, corresponding to the Hilbert space $\mathbb{C}^0$. Being the subobject of any Hilbert space, the unique map $0 : 0 \to \mathcal{H}$ has an interpretation as a proposition for any Hilbert space,

Interpretation of daggers logically

Interpretation as resource logic (Petri nets), dynamic logic

[105]

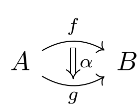
8

Higher categories

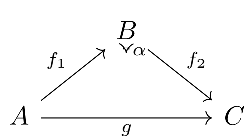
To generalize categories, we can introduce the concept of *higher categories*, in which in addition to objects and morphisms, we also introduce the possibility of transformations between any two objects of the same type, called a k-morphism.

Definition 8.0.1 *A k-morphism is defined inductively as an arrow for which the source and target is a $(k-1)$-morphism, and an object is a 0-morphism.*

Therefore the morphisms we saw so far are 1-morphisms (arrows between 0-morphisms or objects), 2-morphisms are morphisms between two 1-morphisms, and so forth. In diagrammatical terms, we will express higher order morphisms by arrows between arrows, such as the 2-morphism $\alpha : f \to g$



While all 2-morphisms can be expressed this way, called a *globular 2-morphism*, for brevity we will also define 2-morphisms on slightly more complex diagrams.



This is called a *simplicial 2-morphism*, and it is simply the globular 2-morphism for $\alpha : f_2 \circ f_1 \to g$. Similarly we have the *cubical 2-morphism*

[diagram]

which is the globular 2-morphism for $f_2 \circ f_1 \to g_2 \circ g_1$.

A basic example of this is that just as functors can be considered as a morphism in the category of categories **Cat**, so can natural transformations be considered as 2-morphisms in the 2-category **Cat**. Let's show it explicitly :

Theorem 8.0.1 *Considered as a 2-category, 2-morphisms in* **Cat** *are natural transformations.*

Proof 8.0.1 *As a 2-morphism is formally a function between two morphisms, which are functors in* **Cat***, all we need to do is to show naturality of those functions. An object of a category in* **Cat** *is given by a functor from the terminal category* $\mathbf{1}$*,* $\Delta_X : \mathbf{1} \to \mathbf{C}$*, so that a mapping from an object in* $\mathbf{C}$ *to one in* $\mathbf{D}$ *is given by the commutative triangle*

$$F \circ \Delta_X = \Delta_Y \tag{8.1}$$

Morphisms between two components in that sense are given by 2-morphisms between those morphisms, ie for $f : X \to Y$ *in* $\mathbf{C}$*, we have some 2-morphism*

$$\eta_f : \Delta_X \to \Delta_Y \tag{8.2}$$

and composition of those morphisms are given by vertical composition. For $f : X \to Y$ *and* $g : Y \to Z$*, we have*

$$\eta_{g \circ f} = \eta_g \circ \eta_f \tag{8.3}$$

The action of a functor on such morphisms, $F(f) : F(X) \to F(Y)$*, is that of whiskering the 2-morphism by that functor :*

$$F\eta_f : (F \circ \Delta_X) \to (F \circ \Delta_Y) = F(X) \to F(Y) \tag{8.4}$$

The components of a natural transformation are given by whiskering with that constant functor,

$$\eta_X = \Delta_X \eta \tag{8.5}$$

We therefore just need to show the naturality condition, $\eta_Y \circ F(f) = G(f) \circ \eta_X$ *:*

$$\eta_Y \circ F(f) \quad = \quad \eta \tag{8.6}$$

"The objects in the hom-category C(x,y) are the 1-morphisms in C from x to y, while the morphisms in the hom-category C(x,y) are the 2-morphisms of C that are horizontally between x and y."

Arrow category? Over category?

Example 8.0.1 *In the category* **Top***, homotopies between two continuous functions are* 2*-morphisms.*

From this we can define the notion of n-category to be categories with morphisms up to level n, with the appropriate rules regarding composition and identity.

Definition 8.0.2 *An n-category* $\mathbf{C}$ *is given by a sequence of n classes of k-morphisms* $(\mathrm{Mor}_k(\mathbf{C}))_{0\leq k\leq n}$*, such that for every $k > 0$, we have two functions*

$$s_k, t_k : \mathrm{Mor}_k(\mathbf{C}) \to \mathrm{Mor}_{k-1}(\mathbf{C}) \tag{8.7}$$

and a function of composition of k-morphisms,

$$\circ_k : \mathrm{Mor}_k(\mathbf{C}) \times \mathrm{Mor}_k(\mathbf{C}) \to \mathrm{Mor}_k(\mathbf{C}) \tag{8.8}$$

which have to obey the condition that and such that for every $k > 0$, for every $k-1$-morphism α, there exists a k-morphism Id_α *for which* $s_k(\mathrm{Id}_\alpha) = t_k(\mathrm{Id}_\alpha) = \alpha$

We can technically extend this definition to include cases for $k = 0$, in which case the only remaining class is $\mathrm{Mor}_0 = \mathrm{Obj}$, ie 0-categories are equivalent to sets.

The concept of an ∞-category is for the case where the set of classes of k-morphism is of infinite cardinality (typically countable).

We will also define more specifically the concept of (n, k)-categories :

Definition 8.0.3 *An (n, k)-category for $n, k \in \mathbb{N} \cup \{\infty\}$ is an n-category for which every l-morphism for $k < l \leq n$ possesses an inverse, ie for any k-morphism* $\alpha \in \mathrm{Mor}_k(\mathbf{C})$*, $k > 1$, then there exists another k-morphism α^{-1} which obeys*

$$\alpha \circ \alpha^{-1} = \mathrm{Id}_{t(\alpha)} \tag{8.9}$$

$$\alpha^{-1} \circ \alpha = \mathrm{Id}_{s(\alpha)} \tag{8.10}$$

Example 8.0.2 *A group G interpreted as a category* $\mathbf{G}$ *(*$\mathrm{Mor}(\mathbf{G}) \cong G$*) is a* $(1, 0)$*-category.*

Example 8.0.3 *The category of topological spaces with continuous maps as morphisms and homotopie equivalences between any two continuous map is a* $(2, 1)$*-category.*

Gauge example?

"An (n,r)-category is an r-directed homotopy n-type." Ex : a $(0, 0)$-category is isomorphic to a set (the set of all objects), a $(1, 0)$-category is a groupoid, a $(1, 1)$-category is a category

$(\infty, 0)$: ∞-groupoid (∞, ∞) :

Descent to negative degrees : $(-1, 0)$-category : truth values $(-2, 0)$-category : Point

n-truncation : a category is n-truncated if it is an n-groupoid

Truncation

Definition 8.0.4 *The n-truncation functor τ_n is an endofunctor on an ∞-category mapping all morphisms of order $k \geq n$ to the identity k-morphism.*

An n-truncated category therefore acts as the equivalent n-category. For the case of a 0-truncation, this is simply a basic category.

8.1 Groupoids

[106, 107]

The prototypical ∞-category is ∞**Grpd**, the ∞-category of ∞-groupoids, along with its various truncations, the k-groupoids,

$$k\mathbf{Grpd} \cong \tau_k \infty \mathbf{Grpd} \tag{8.11}$$

where in particular, the 0-groupoids, simply being given by 0-morphisms, are just sets, and the 1-groupoids are the usual notion of groupoids.

There are a number of different ways that we can consider the various levels of groupoids

First, any ∞-category with all invertible k-morphisms (an $(\infty, 0)$-category) can be embedded in ∞**Grpd**. Therefore any such diagram can be considered as such, which we will write either as their pure diagram

$$\mathcal{G} = \{\} \tag{8.12}$$

Another method is the traditional groupoid notation $\mathcal{G} : G_1 \rightrightarrows G_0$, where G_0 is the set of objects, G_1 the set of morphisms, and s, t are the source and target.

This method can be extended to the ∞-groupoid case by having an infinite sequence of k-morphisms

$$G_0$$

Given an ∞-group,

Delooping

8.1.1 Truncation

The k-truncation of an n-groupoid is the equivalent k-groupoid

Theorem 8.1.1 *The k-truncation of an ∞-category is a functor.*

Proof 8.1.1 *As the k-truncation sends every k'-morphism, $k < k'$, to the identity k'-morphism,*

Theorem 8.1.2 *The n-truncation monad has an adjoint modality of the n-connected monad.*

Definition 8.1.1 *If we have some left adjoint of the n-truncation functor,*

$$\mathrm{Hom}_{\mathbf{C}}() \tag{8.13}$$

The n-truncation monad is given by the

8.2 Homotopy category

There exists a weaker notion than that of a higher category which is that of a category with weak equivalences, where rather than any higher morphisms, we define classes of morphisms which are weak equivalences, a special class of morphisms which are not strictly speaking isomorphisms (they do not admit an inverse such that $f \circ f^{-1} = \mathrm{Id}$), but we consider them to have a weak inverse which we will consider to be something akin to an equivalence.

Definition 8.2.1 *A category $\mathbf{C}$ with weak equivalences is given by some subcategory $\mathbf{W} \hookrightarrow \mathbf{C}$ which contains all isomorphisms and which obeys the two out of three property : for any two composable morphisms f, g in $\mathbf{C}$, if two out of $\{f, g, g \circ f\}$ are in $\mathbf{W}$, then so is the third.*

In particular, if we have composable morphisms of the types $f : X \to Y$ and $g : Y \to X$ in $\mathbf{W}$, then we can say that X and Y are weakly equivalent

[...]

In terms of n-categories, weak equivalences are meant to represent the notion of 2-morphisms to the identity. That is

[...]

Definition 8.2.2 *A category* $\mathbf{C}$ *is* homotopical *if in addition to being a category with weak equivalences, the subcategory* $\mathbf{W}$ *also obeys the two out of six property : for any sequence of composable morphisms,*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \tag{8.14}$$

if $h \circ g$ *and* $g \circ f$ *are in* $\mathbf{W}$*, then so are* f, g, h *and* $h \circ g \circ f$*.*

The homotopy category $\mathrm{Ho}(\mathbf{C}, \mathbf{W})$ of a category with weak equivalences $(\mathbf{C}, \mathbf{W})$ is the localization of that category along the weak equivalences, ie for any morphism $f \in \mathbf{W}$, we adjoin an actual inverse f^{-1} to transform the weak equivalence into an equivalence.

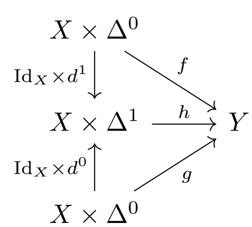
Example 8.2.1 *For the category of topological spaces* **Top***, we take the subcategory of weak equivalences to be that of continuous maps which are bijective on the path components, ie for* $f : X \to Y$*,* $f \in \mathbf{W}$ *if*

$$f_* : \pi_0(X) \to \pi_0(Y) \tag{8.15}$$

[Higher homotopy groups?]

Another common example of a homotopy category is given by the simplicial sets, which are in some sense a ”skeletal” version of topological spaces. This requires us to look at homotopies in simplicial sets first :

Definition 8.2.3 *A simplicial homotopy between two simplicial morphisms* $f, g : X \to Y$ *means that the following diagram exists (using the isomorphism* $X \times \Delta^0 \cong X$*) :*



Keeping the skeletal topological space interpretation, the simplicial homotopy is given by a morphism ”parametrized” by Δ^1, for which

Interpreting simplicial sets as skeletal topological spaces, we can see for instance that the interval Δ_1

Example 8.2.2 *The category of simplicial sets* **sSet** *can be turned into a homotopy category by taking its simplicial homotopy equivalence morphisms. For a simplicial morphism* $f : X \to Y$*, we say that it is a simplicial homotopy equivalence if there is a weak inverge* $g : Y \to X$

In addition to weak equivalences to reflect the notion of homotopy equivalence, we can also look at two additional notions from homotopy theory, which are fibrations and cofibrations.

As a reminder, those notions in topology are

Definition 8.2.4 *A fibration is a continuous map $p : E \to B$ such that every space X satisfies the homotopy lifting property.*

Definition 8.2.5 *A cofibration is a continuous map $i : A \to X$ such that for every space Y*

"good subspace embedding". Example of use : given a subspace $\iota : S \hookrightarrow X$, and a map $f : S \to R$, is there an extension of f to X?

Example 8.2.3 $0 \hookrightarrow X$ *is a cofibration*

Example 8.2.4 *A homeomorphism is a cofibration*

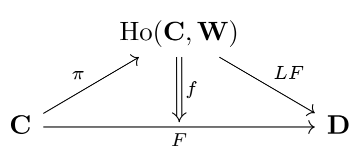
Definition 8.2.6 *A fibration in a category with weak equivalences is a*

Definition 8.2.7 *A* model category *is a category* $\mathbf{C}$ *with three classes of morphisms* $\mathbf{W}, \mathbf{Fib}, \mathbf{Cofib}$ *where* $(\mathbf{C}, \mathbf{W})$ *is a category with weak equivalences, and*

Similarly to how any category can be considered an ∞-category by simply considering all higher morphisms to be identities, we can also consider any category to be a category with weakly equivalences by simply picking W to be only isomorphisms, or a homotopy category by

in which case the homotopy projection is simply the skeletal category $\mathrm{Ho}(\mathbf{C}, \mathbf{W}) \cong \mathrm{Sk}(\mathbf{C})$

Definition 8.2.8 *Given a category with weak equivalences* $(\mathbf{C}, \mathbf{W})$ *and some functor* $F : \mathbf{C} \to \mathbf{D}$*, its* left derived functor LF *is the right Kan extension of* F *along the projection* $p : \mathbf{C} \to \mathrm{Ho}(\mathbf{C}, \mathbf{W})$



In a category with weak equivalences, the notions of limits and colimits do not typically preserve weak equivalences. ie if two diagram functors F, F' are weakly equivalent ($F(X) \cong_w F'(X)$ for all X), it is however not guaranteed that the limits from such a diagram are.

Example : pushout for $1 \leftarrow S^n \to D^{n+1}$ and $1 \leftarrow S^n \to 1$. The first colimit is S^{n+1} while the other is 1, which are not weakly equivalent in homotopy top

Definition 8.2.9 *A* homotopy limit *of a functor $F : I \to \mathbf{C}$ with diagram of shape I is the right derived functor of the limit functor* $\lim_I : [I, C] \to C$

$$\mathrm{holim}_{\mathrm{I}} F = (R \lim_I) F \tag{8.16}$$

Theorem 8.2.1 *Any discrete limit is homotopy invariant under [some condition] :*

$$\lim_{\mathbf{n}} F = \mathrm{holim}_{\mathbf{n}} F \tag{8.17}$$

Definition 8.2.10

Homotopy limit

Homotopy categories can be described by ∞-categories (2-categories, in fact), via

Theorem 8.2.2 *Given a homotopy category* $(\mathbf{C})$

8.3 Intervals, loops and paths

The interaction of categories of spaces like topoi and ∞-categories or homotopy categories is what gives rise to concepts of homotopy. To talk of a notion of homotopy in our spaces (at least the type of homotopy we typically associate to spaces), we need to be able to talk about some notion of a path. Obviously our main idea behind such notions is that of the standard path in geometry, a map from the unit interval to a space

$$\gamma : [0, 1] \to X \tag{8.18}$$

But we will need to keep things somewhat general.

Definition 8.3.1 *An interval object I is some object containing two points, the boundary points*

$$1 \sqcup 1 \underset{\iota_0}{\overset{\iota_1}{\rightrightarrows}} I \tag{8.19}$$

This is a pretty broad definition but we also typically depend that it be contractible.

Definition 8.3.2 *A* contractible interval object *is an interval object*

Typical examples of interval objects are the 1-simplex in the category of simplicial sets, where its boundary points are the maps $\vec{1} :\to \Delta[1]$, the unit interval $[0,1]$ for a variety of spatial categories, or the standard interval object $0 \to 1$ in the category of categories.

A path space object is then to be interpreted as the maps from this interval object to another space, where a path will be some element of the exponential object X^I

Definition 8.3.3

$$X \xrightarrow{S} \mathrm{Path}_X \overset{(d_0,d_1)}{\longrightarrow} X \times X \tag{8.20}$$

Loop space object and suspension object

Theorem 8.3.1 *The localization of a category by its*

Definition 8.3.4 *Path ∞-groupoid*

Definition 8.3.5 *In an ∞-category with a terminal object 1, a* suspension object ΣX *of X is the homotopy pushout*

$$\Sigma X = X +^h_1 1 \tag{8.21}$$

Theorem 8.3.2 *Fiber and cofiber of loop and suspension*

[108]

homotopy circle

Boring homotopy theory : interval object is terminal?

Example 8.3.1 *If we pick as our line object the terminal object 1, the path space object X^I is isomorphic to X itself, with*

$$X \to X^I \to X \times X \tag{8.22}$$

[...]

Trivial model category?

8.4 Line object

Given a Lawvere theory T, if it contains the theory of Abelian groups,

Example 8.4.1 *In the topos of smooth spaces, the line object is the Yoneda embedding of the real line, $R = よ(\mathbb{R})$*

Proof 8.4.1

8.5 Stable homotopy theory

Definition 8.5.1 *Given an $(\infty, 1)$-category with finite limits,*

[109]

8.6 Postnikov etc

Example 8.6.1 *To find the Postnikov tower for the sphere S^2, let's look at some of its homotopy sequence :*

$$\pi_0(S^2) = 0 \tag{8.23}$$
$$\pi_1(S^2) = 0 \tag{8.24}$$
$$\pi_2(S^2) = \mathbb{Z} \tag{8.25}$$
$$\pi_3(S^2) = \mathbb{Z} \tag{8.26}$$
$$\pi_4(S^2) = \mathbb{Z}_2 \tag{8.27}$$
$$\pi_5(S^2) = \mathbb{Z}_2 \tag{8.28}$$

Whitehead tower

A useful example of this in physics is given by the delooping of the orthogonal group, $\mathbf{B}\mathrm{O}(n)$. As the orthogonal group has an exactly known periodic homotopy group,

$$\pi_{8k+0}(\mathrm{O}(n)) = \mathbb{Z}_2 \tag{8.29}$$
$$\pi_{8k+1}(\mathrm{O}(n)) = \mathbb{Z}_2 \tag{8.30}$$
$$\pi_{8k+2}(\mathrm{O}(n)) = 0 \tag{8.31}$$
$$\pi_{8k+3}(\mathrm{O}(n)) = \mathbb{Z} \tag{8.32}$$
$$\pi_{8k+4}(\mathrm{O}(n)) = 0 \tag{8.33}$$
$$\pi_{8k+5}(\mathrm{O}(n)) = 0 \tag{8.34}$$
$$\pi_{8k+6}(\mathrm{O}(n)) = 0 \tag{8.35}$$
$$\pi_{8k+7}(\mathrm{O}(n)) = \mathbb{Z} \tag{8.36}$$

First Stiefel-Whitney class, second Stiefel-Whitney, first fractional Pontryagin class, second frac. Pontr. class, etc

orthogonal group, special orthogonal group, spin group, string group, etc

Orthogonal structure, orientation, spin structure, string structure, etc

8.7 Homotopy localization

In addition to the localization of every possible weak isomorphism to form the homotopy category $\mathrm{Ho}(\mathbf{C})$, another possibility is to perform this localization at a smaller set of such weak isomorphisms.

Definition 8.7.1 *Given an object X in a model category $(\mathbf{C}, W, \mathrm{Fib}, \mathrm{CoFib})$, its* homotopy localization *is its localization at the set of morphisms W_X of the form*

$$Y \times X \xrightarrow{\mathrm{pr}_1} Y \tag{8.37}$$

$$W_X = \tag{8.38}$$

"In homotopy theory, for example, there are many examples of mappings that are invertible up to homotopy; and so large classes of homotopy equivalent spaces"

8.8 Adjunctions and monads

From the definition of an n-category, we can rewrite definitions for adjunctions and monads internally to a category. In particular, we can think of the definition we've seen in terms of the category of categories, **Cat**.

Given this, we can not only define adjunctions and monads within that language, but in fact generalize it for any 2-category or higher n-category.

Definition 8.8.1 *In a 2-category $\mathcal{K}$, an adjunction*

$$\eta, \mu : L \vdash R : X \to Y \tag{8.39}$$

is a pair of 1-morphisms $L : B \to A$ and $R : A \to B$ and 2-morphisms $\eta : \mathrm{Id} \to RL$ and $\mu : LR \to \mathrm{Id}$ which obey the triangle identities :

Working in the 2-category **Cat**, we end up with the appropriate definition for adjunctions.

Example 8.8.1 *A non-***Cat** *example of this is to pick the homotopy category of topological spaces, in which case we can look at the* suspension-loop adjunction*,*

We can furthermore also define monads and comonads in this context

Definition 8.8.2 *A* monad *in a 2-category is a triple (T, η, μ) over an object X given by an endomorphism $T : X \to X$, a 2-morphism $\eta : \mathrm{Id}_X \to T$, and a 2-morphism $\mu : T^2 \to T$, such that [...]*

Theorem 8.8.1 *Given two adjunctions*

$$\eta, \mu : L \vdash R : C \rightarrow D \tag{8.40}$$
$$\eta', \mu' : L' \vdash R' : C' \rightarrow D' \tag{8.41}$$

and two 1*-morphisms* $F : C \to C'$ *and* $G : D \to D'$*, the following* 2*-morphisms are isomorphic :*

$$\zeta : L'G \xrightarrow{L'G\eta} \tag{8.42}$$

Definition 8.8.3 *Given two such* 2*-morphisms as* ξ *and* ζ*, we say that* ξ *and* ζ *are* mates *under adjunction, written* $\xi \dashv \zeta$*.*

8.9 ∞-sheaves

Just as categories have sheaves as functors to the category of sets, ∞-categories have ∞-sheaves as functors to the category ∞**Grpd**.

Before we get into ∞-sheaves, however, let's first look at the simpler case of stacks, or groupoid stacks more specifically (throughout we will simply refer to groupoid stacks as stacks, as we will not consider any other category for valuation)

Definition 8.9.1 *A* pre-stack *is a groupoid-valued presheaf, ie a functor*

$$S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Grpd} \tag{8.43}$$

Definition 8.9.2 *A* stack *is a prestack*

Example 8.9.1 *A traditional example of a stack is given by the stack of similar triangles[110, 111]. Take the set of all possible triangles, with sides* (a, b, c)*. To only consider similar triangles, ie up to equivalence* $(a, b, c) \cong (\alpha a; \alpha b, \alpha c)$*, we will take triangles of a constant perimeter :*

$$a + b + c = 2 \tag{8.44}$$

The space of all such values is therefore some two-dimensional subspace of $[0, 2]^3 \subseteq \mathbb{R}^3$*, and more specifically this is a simplex.*

For each point of this simplex, we have an associated triangle, and we attach to each of those point the group of symmetries of those triangle. The structure thus formed is a groupoid where each point is a group. The groups involved are typically the symmetric groups S_2 *(isoceles triangles),* S_3 *(equilateral triangles) as well as the reflection symmetry* $\mathbb{Z}_2$

Example 8.9.2 *The simplest non-trivial example of a smooth stack is that of an* orbifold*, which are spaces modelled locally by quotients of open subsets of* $\mathbb{R}^n$ *by some finite group action. The simplest example of this being a line acted on by* $\mathbb{Z}_2$*, which resembles the half-line* $[0,\infty)$*.*

As in the context of a stack we are instead interested in mapping a Cartesian space to a groupoid, we will instead look at the action of the functor H *(for half line) on* $\mathbb{R}$*.*

We are

[112]

Definition 8.9.3 *A* ∞*-presheaf* F *is given by a functor*

$$F : \mathbf{C}^{\mathrm{op}} \to \infty\mathbf{Grpd} \tag{8.45}$$

Definition 8.9.4 *A* ∞*-sheaf* F *is an* ∞*-presheaf*

Example 8.9.3 *If we consider a set* X *as an* ∞*-sheaf on the terminal site of* 0*-type, and we consider its quotient under group action by some group object* G*, via the action*

$$\rho : G \times X \to X \tag{8.46}$$

8.10 ∞-topos

Definition 8.10.1 *An* ∞*-category* $\mathbf{H}$ *is an* ∞*-topos if*

Example 8.10.1 *As in the case of* 1*-topoi, two basic examples of* ∞*-topoi are given by the initial* ∞*-topos,* $\mathrm{Sh}_\infty(\mathbf{0})$*, which is the initial* ∞*-category, and the terminal topos* $\mathrm{Sh}_\infty(\mathbf{1})$*, which is* $\infty\mathbf{Grpd}$.

8.10.1 Principal bundles

Definition 8.10.2 *A* G-principal bundle P *over an object* X *is a pullback of a morphism* $X \to \mathbf{B}G$ *and (some point?)* $1 \to \mathbf{B}G$*.*

The only non-trivial morphism of the pullback is the projection $\pi : P \to X$.

[107, 113]

8.11 Simplicial homotopy

A simple case of a model category is given by the model category of simplicial sets, **sSet**, the Quillen model category.

Given the category **sSet**, its model structure is defined

Cylinder functor

$$\begin{array}{ccccc} X \cong X \times \Delta[0] & \xrightarrow{\mathrm{Id}_X \times \delta^1} & X \times \Delta[1] & \xleftarrow{\mathrm{Id} \times \delta^0} & X \times \Delta[0] \cong X \\ & \searrow^{f} & \downarrow{\scriptstyle \eta} & \swarrow_{g} & \\ & & Y & & \end{array}$$

8.12 Smooth groupoids

The higher categorical equivalent of the category of smooth spaces is the ∞-topos of smooth ∞-groupoids over the site of Cartesian spaces, which are smooth sets along with their homotopy equivalences. This will allow us to treat the notion of groups without internalization in the context of smooth spaces, such as what we would need for gauge theories.

If we look at those spaces once more in the context of their analogies with diffeological spaces, rather than mapping our probes to some sets of atlases, we are mapping them to some ∞-groupoid. All of our previous diffeological spaces are simply sheaves that are non-empty only for the discrete objects of the ∞-groupoid (in the sense of being 0-truncated, ie all their higher order morphisms are just the identities). Equivalently this can be done via the composition of the smooth set and the embedding of **Set** into ∞**Grpd**.

Definition 8.12.1 *A* pre-smooth groupoid *is a presheaf of groupoids on the site* **CartSp***,*

$$X : \mathbf{CartSp}^{op} \to \mathbf{Grpd} \tag{8.47}$$

The most basic kind of non-0-truncated smooth groupoid is a 1-truncated smooth groupoid of a single point, which simply corresponds to a group, which as a sheaf is given by

$$G(\mathbb{R}^0) = G \tag{8.48}$$

Another example is the *classifying space* of a group. This will be in the context of a smooth groupoid a diffeological space for which the k-homotopy groups will be equal to G.

Example 8.12.1 *The circle S^1 is the classifying space of the group $\mathbb{Z}$. As a smooth groupoid, this is given by considering the sheaves on $\mathbb{R}^1$ (for instance using the stereographic projection)*

A less trivial example for this is the notion of a G-manifold, ie a manifold M equipped with a smooth group action by the group G, such as the pair of the line manifold $L \cong \mathbb{R}$ and the one-dimensional translation group $T \cong \mathbb{R}$ (as a group).

$$\rho : M \times G \to M \tag{8.49}$$

Action groupoid

Definition 8.12.2 *An orbifold X is a topological space with an atlas of orbifold charts (U, G, ϕ), where U is an open set of X, ϕ is a continuous map from some connected open subset $\mathcal{O} \subseteq \mathbb{R}^n$ to U, and G is a finite group acting smoothly and effectively on $\mathcal{O}$, such that ϕ is a G-equivariant map.*

Example 8.12.2 *The simplest non-trivial orbifold is the real line reflected along $\mathbb{Z}_2$. Take the unique chart $(\mathbb{R}, \mathbb{Z}_2, \phi)$, with the action the typical $\mathbb{Z}_2$ involution :*

$$g_{-1} \in \mathbb{Z}_2,\ g_{-1}(x) = -x \tag{8.50}$$

Topologically the reflected real line is simply $[0, \infty)$, and its orbifold chart is given by

$$\phi : \tag{8.51}$$

Theorem 8.12.1

Example 8.12.3 *The localization of the circle S^1 is homotopy equivalent to the simplicial circle $\Delta_1/\partial\Delta_0$*

Proof 8.12.1

$$S^1 = R/\mathbb{Z} \tag{8.52}$$

Localization :

$$\mathrm{loc}_R(S^1) \cong \mathrm{loc}_R(R/\mathbb{Z}) \tag{8.53}$$

Left adjoint due to reflective localization, preserves colimits, therefore commutes with the coequalizer

$$\mathrm{loc}_R(S^1) \cong 1/\mathbb{Z} \tag{8.54}$$

Stack?

8.12.1 Principal bundle

In the case of a smooth groupoid, the principal bundle is the usual notion of a G-principal bundle.

The basic example is that of a manifold M, ie some locally representable smooth 0-type, with G some Lie group of the same type. If we consider some open cover $\{U_i \to M\}$, and its Čech groupoid $C(\{U_i\})_\bullet$,

8.12.2 Connections

Having defined principal bundles, we are now able to define (gauge) connections in this formalism.

Connections have many equivalent definitions, but the one that will interest us here is that of a transport operator. A connection will give us an isomorphism between two bundles after transport along a curve. If we have a curve $\gamma : [0,1] \to M$ on some manifold, and a principal bundle $\pi : P \to M$ a G-principal bundle, the transport along γ is some homomorphism

$$\mathrm{tra}_\nabla(\gamma) : P_{\gamma(0)} \to P_{\gamma(1)} \tag{8.55}$$

In terms of categories, our curve is an object in the path space object $\gamma : 1 \to [I, M]$

Groupoidal property of the transport

In the context of connections, we will usually need more information than that given by the fundamental groupoid (we don't expect all homotopic paths between two points to have the same transport property, as this is prevented by curvature), but less than that of the path space object : the transport along a path and then its reverse should in fact be isomorphic to the null path. The groupoid we are interested in is the *path groupoid*.

Definition 8.12.3 *A curve* $\gamma : 1 \to [I, M]$ *has* sitting instants

Definition 8.12.4 *The* path groupoid $P(M)$ *of a manifold* M *is*

Generalization to smooth ∞-groupoids [114]:

Definition 8.12.5 *The* path ∞-groupoid $P_\infty(X)$ *of a smooth* ∞*-groupoid* X

Gravitational connection?

tratritra

8.13 Cohomology

As ∞-categories will typically encode data relating to the homotopy of their objects, there is some natural notion of (co)homology emerging from it,

Definition 8.13.1 *Given some ∞-topos $\mathbf{H}$, the cohomology of $X \in \mathbf{H}$ with values in $A \in \mathbf{H}$ is the set of connected components of the hom-∞-groupoid :*

$$H(X; A) = \pi_0 \mathrm{Hom}_{\mathbf{H}}(X, A) \tag{8.56}$$

and its degree n cohomology is given by the n-fold delooping of A :

$$H^n(X; A) = \pi_0 \mathrm{Hom}_{\mathbf{H}}(X, A_n) \tag{8.57}$$

Definition 8.13.2 *The Eilenberg-MacLane object*

Example 8.13.1 *In the $(\infty, 1)$-category of topological spaces* Top *with homotopies, the Eilenberg-MacLane object $K(\mathbb{Z}, 1)$ is the circle S^1.*

Proof 8.13.1 *As the circle is connected, we have $\pi_0(S^1) = \{\bullet\}$, and its higher homotopy groups*

$$(0, \mathbb{Z}, 0, 0, \ldots) \tag{8.58}$$

Example 8.13.2 *If we take the $(\infty, 1)$-category of topological spaces* Top *with homotopies, and the Eilenberg-MacLane object $K(\mathbb{Z}, 1)$ as its values, the singular cohomology is*

Example 8.13.3 *The cohomology of the simplicial circle in the category of simplicial sets, $\Delta_1 / \partial \Delta_0$*

Homology? Dold-Kan correspondence

8.14 Higher order logic

The associated internal logic of a higher category theory is that of a type theory with homotopy types

9
Subjective logic

The lower level of Hegel's logic is what he calls the *subjective logic*, which is roughly comparable to propositional or predicate logic or type theory. The subjective logic is not about the concepts and notions themselves, but about their relations. That is, we can speak of propositions and objects without ever giving specific examples of those.

The basic objects in the objective logic are *concepts* and *moments* [is that the subjective logic actually].

The concepts are meant in Hegel to represent thoughts (as this is primarily an idealist philosophy), notions, etc, which can include those of actual physical objects. This would fit notions of mathematical objects well enough, although whether they fit perfectly in the framework of a category, who knows.

Immanence $\approx$ internal logic?

For any two concepts, we can look at morphisms as relations between those, following the semantics of morphisms as predicates

The general properties of an object in a category defined by the slice category of that object?

Relations between concepts as morphisms

The formalization of qualities of an object are given by the concept of *moments* of an object, given by some "projection operator" $\bigcirc : C \to C$, such that

$$\bigcirc \bigcirc X \cong \bigcirc X \tag{9.1}$$

If we reduce the object X to merely the qualities given by $\bigcirc$, there is nothing left to remove so that any subsequent projection will be isomorphic to it. In categorical term, we also demand that the projection given as $X \to \bigcirc X$ be, within the category of qualities $\bigcirc$, an equivalence :

$$\bigcirc(X \to \bigcirc X) \in \mathrm{core}(X) \tag{9.2}$$

In terms of types, this is an idempotent monad $(\bigcirc, \eta, \mu)$, where the projection $X \to \bigcirc X$ is the unit of the monad,

$$\eta_X^{\bigcirc} : X \to \bigcirc X \tag{9.3}$$

and the isomorphism is given by the multiplication map

$$\mu_X^{\bigcirc} : \bigcirc^2 X \xrightarrow{\cong} \bigcirc X \tag{9.4}$$

Dually, we can also define idempotent comonads $(\square, \epsilon, \delta)$, where we have some counit

$$\epsilon_X^{\square} : \square X \to X \tag{9.5}$$

and the comultiplication map which is an equivalence

$$\delta_X^{\square} : \square^2 X \xrightarrow{\cong} \square X \tag{9.6}$$

Definition 9.0.1 *A* moment *on a type system/topos* $\mathbf{H}$ *is either an idempotent monad* $\bigcirc$ *or comonad* $\square$*.*

If we keep the monad or comonad status of the moment ambiguous, we will denote it by the functor Δ.

As usual for idempotent (co)monads, they are defined via some triple $(\bigcirc, \eta, \mu)$ for monads, with the unit

$$\eta^{\bigcirc} : \mathrm{Id}_{\mathbf{H}} \to \bigcirc \tag{9.7}$$

and the multiplication map (an isomorphism, for an idempotent monad)

$$\mu^{\bigcirc} : \bigcirc \overset{\cong}{\to} \bigcirc\bigcirc \tag{9.8}$$

and for the idempotent comonad $(\square, \epsilon, \delta)$, with the counit

$$\epsilon^{\Box} : \Box \to \mathrm{Id}_{\mathbf{H}} \tag{9.9}$$

and the comultiplication map (an isomorphism, for an idempotent comonad)

$$\delta^{\Box} : \Box\Box \xrightarrow{\cong} \Box \tag{9.10}$$

We will denote by $\mathbf{H}_{\bigcirc}$ the Eilenberg-Moore category of a monad $\bigcirc$, and by $\mathbf{H}_{\Box}$ that of a comonad $\Box$. As an idempotent (co)monads, this Eilenberg-Moore category is a (co)reflective subcategory of the original topos,

$$(T_{\bigcirc} \dashv \iota_{\bigcirc}) : \mathbf{H}_{\bigcirc} \underset{\iota_{\bigcirc}}{\overset{T_{\bigcirc}}{\rightleftarrows}} \mathbf{H}$$

$$(\iota_{\Box} \dashv T_{\Box}) : \mathbf{H}_{\Box} \underset{T_{\Box}}{\overset{\iota_{\Box}}{\rightleftarrows}} \mathbf{H}$$

where the adjoint of the inclusion and its (co)reflector, $(T_{\bigcirc} \dashv \iota_{\bigcirc})$ or $(\iota_{\Box} \dashv T_{\Box})$, gives the moment via the usual composition of monads and comonads from adjoint pairs,

$$\bigcirc = \iota_{\bigcirc} \circ T_{\bigcirc} \tag{9.11}$$

$$\Box = \iota_{\Box} \circ T_{\Box} \tag{9.12}$$

Those Eilenberg-Moore categories are fully faithful subcategories of the original topos.

The unit of the monad $\eta_{\bigcirc}$ gives us a mapping from the original object to the object in this subcategory

[Free algebra?]

Therefore, for a monadic or comonadic moment, we have both the embedding of the (co)reflective subcategory, $\mathbf{H}_{\Box} \hookrightarrow \mathbf{H}$, and the (co)reflection $\mathbf{H} \twoheadrightarrow \mathbf{H}_{\Delta}$

As any (co)reflective subcategory can be understood as the localization of the original category by some subcategory $\mathbf{W}$ of weak equivalences, there is some corresponding set of morphisms S for which $\mathbf{H}_{\Delta}$ is the localization of $\mathbf{H}$.

As we are projecting our objects down to some of their moments, we can define some notion of similarity. For any two objects in $\mathbf{C}$, we say that they are Δ-similar if their modality is isomorphic,

$$X \cong_{\Delta} Y \leftrightarrow \Delta X \cong \Delta Y \tag{9.13}$$

[Is this the weak equivalence of the localization?]

”we may naturally make sense of ”pure quality” also for (co-)monads that are not idempotent, the pure types should be taken to be the “algebras” over the monad.”

A particular moment that we will see more in details later but that will be of great importance throughout is the one given by the smallest possible subtopos, which is just the terminal topos $\mathbf{1} \cong \mathrm{Sh}(\mathbf{0})$. Being the smallest possible subtopos, it corresponds in some sense to an object with no qualities left beyond being an object, what we will call its quality of *being*. This can be seen in the sense that the category only contains a single object and a single relation (the identity), so that all objects are identical under this modality. This will be defined in more details in 10.1, but I include it here as it will be used for a few basic constructions of moments.

Example 9.0.1 *The adjunction* Even $\dashv$ Odd *is an opposition of the form* $\Box \dashv \bigcirc$

From the form of the monads and comonads, where the mapping of the basic type X to their moment ΔX for some moment Δ is either given by the unit (return operation) for the monad

$$X \to \Delta X \tag{9.14}$$

or the counit (extend operation) for the comonad

$$\Delta X \to X \tag{9.15}$$

we will also call monads *successive moments*, or s-moments, and comonads as *preceding moments*, or p-moments, and we will denote them by default as $\bigcirc$ and $\Box$ respectively, with Δ for a moment of unstated type.

As monads	As moments	Notation
Comonad	p-moment	$\Box$
Monad	s-moment	$\bigcirc$

Table 9.1: Caption

Semantics : any p-moment is given by the embedding of $\mathbf{H}_\Box \hookrightarrow \mathbf{H}$

Example 9.0.2 *The basic example of a monad that we've seen for the inclusion* $\mathbb{Z} \hookrightarrow \mathbb{R}$ *of*

$$\text{Ceiling} \dashv \text{Floor} \tag{9.16}$$

is a pair of an idempotent monad and comonad. The unit of Ceiling *is given by the order relation in* $\mathbb{R}$

$$x \in \mathbb{R} \to \text{Ceiling}(x) \tag{9.17}$$

or $x \leq \text{Ceiling}(x)$ *for short, so that the integral moment of* x *is indeed literally* successive *here, ie it is the following element. Likewise for* Floor*,*

$$\text{Floor}(x) \to x \tag{9.18}$$

it is preceding.

Definition 9.0.2 *Given a moment* $\bigcirc$*, for* $X \in \mathbf{H}$*,* $\bigcirc_X$ *is a moment on* $\mathbf{H}_{/X}$ *sending* $p : E \to X$ *to* $\bigcirc_X E \to X$ *via the pullback*

$$\begin{array}{ccc} \bigcirc_X E & \longrightarrow & \bigcirc E \\ \downarrow & & \downarrow \bigcirc p \\ X & \xrightarrow{\eta_X^{\bigcirc}} & \bigcirc X \end{array}$$

Definition 9.0.3 *Given two moments,* Δ *and* $\bigstar$*, we can partially order them as*

$$\Delta \prec \bigstar \tag{9.19}$$

if $\bigstar\Delta = \Delta$

That is indeed a partial order as we have $\Delta \prec \Delta$ (by idempotence), $\Delta \prec \bigstar$ and $\bigstar \prec \Delta$ implying $\Delta \cong \bigstar$, as we have [?]

$$\begin{aligned} \bigstar\Delta &\cong \Delta && (9.20) \\ &\cong \Delta\bigstar\Delta && (9.21) \\ &\cong && (9.22) \end{aligned}$$

Theorem 9.0.1 *Given two s-moments,* $\bigcirc_1$ *and* $\bigcirc_2$*,*

$$\bigcirc_1 \prec \bigcirc_2 \tag{9.23}$$

Any $\bigcirc_1$*-modal type is a* $\bigcirc_2$*-modal type.*

Proof 9.0.1 *An object* X *is a* $\bigcirc_1$*-modal type if we have the isomorphism*

$$\eta_X^{\bigcirc_1} : X \overset{\cong}{\to} \bigcirc_1 X \tag{9.24}$$

If we apply $\bigcirc_2$*'s unit to it, the naturality square is given by*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^{\bigcirc_2}} & \bigcirc_2 X \\ {\scriptstyle \eta_X^{\bigcirc_1} X}\downarrow & & \downarrow{\scriptstyle \bigcirc_2 \eta_X^{\bigcirc_1}} \\ \bigcirc_1 X & \xrightarrow[\eta_{\bigcirc_1 X}^{\bigcirc_2}]{} & \bigcirc_2 \bigcirc_1 X \end{array}$$

From the moment inclusion, we have that $\eta_{\bigcirc_1 X}^{\bigcirc_2}$ has an inverse, and from the fact that X is a $\bigcirc_1$-modal type, so does $\eta_X^{\bigcirc_1} X$.

Therefore, $\eta_X^{\bigcirc_2}$ has an inverse given by

$$(\eta_X^{\bigcirc_1})^{-1} \circ (\eta_{\bigcirc_1 X}^{\bigcirc_2})^{-1} \circ \bigcirc_2 \eta_X^{\bigcirc_1} \tag{9.25}$$

making it an isomorphism.

Theorem 9.0.2 *For two s-moments, $\bigcirc_1$ and $\bigcirc_2$,*

$$\bigcirc_1 \prec \bigcirc_2 \tag{9.26}$$

the corresponding reflective subcategories are as

$$\mathbf{H}_{\bigcirc_1} \underset{\iota_{\bigcirc_1}}{\overset{T_{\bigcirc_1}}{\rightleftarrows}} \mathbf{H}_{\bigcirc_2} \underset{\iota_{\bigcirc_2}}{\overset{T_{\bigcirc_2}}{\rightleftarrows}} \mathbf{H}$$

such that the composition [...]

Proof 9.0.2

[Are moments Cartesian (co)monads??? Only some? The negations?]

[Not gonna be all since $\circledast$ is not a Cartesian monad : $1 \times_1 Y = Y \neq X$]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X^{\bigcirc}}\downarrow & & \downarrow{\scriptstyle \eta_Y^{\bigcirc}} \\ \bigcirc X & \xrightarrow[\eta^{\bigcirc}(f)]{} & \bigcirc Y \end{array}$$

9.1 Unity of opposites

[115]

As we've seen previously 3.22.18.8, we can define a specific notion of adjunction for (co)monads. This can be done in a few different ways, either directly at the level of the pair of monad and comonad, or via a string of adjoint functors.

First, we can define them as traditional adjoint with some additional properties. Let's first look at the case of a monad and comonad

$$(\bigcirc \dashv \Box) : \mathbf{H} \underset{\bigcirc}{\overset{\Box}{\rightleftarrows}} \mathbf{H}$$

with the associated unit and counit of this adjunction,

$$\eta : \mathrm{Id}_{\mathbf{H}} \to \Box\bigcirc \tag{9.27}$$
$$\epsilon : {\bigcirc}\Box \to \mathrm{Id}_{\mathbf{H}} \tag{9.28}$$

In addition to the (co)monads being adjoint as functors, we ask that their respective units and counits be adjoint in a more specific sense.

$$\eta^{\bigcirc} : \mathrm{Id}_{\mathbf{H}} \to \Box\bigcirc \tag{9.29}$$
$$\epsilon^{\Box} : \bigcirc\Box \to \mathrm{Id}_{\mathbf{H}} \tag{9.30}$$

$$(\Box \dashv \bigcirc) : \mathbf{H} \underset{\Box}{\overset{\bigcirc}{\rightleftarrows}} \mathbf{H}$$

We can also define them as an adjoint triple. If we have three adjoint functors

$$(F \dashv G \dashv H) : \mathbf{C} \;\begin{array}{c}\xrightarrow{\;F\;}\\ \xleftarrow{\;G\;}\\ \xrightarrow{\;H\;}\end{array}\; \mathbf{D}$$

where $F, H : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, we can define a pair of monads $\Delta, \bigstar$ on $\mathbf{C}$

$$\Delta = G \circ F : \mathbf{C} \to \mathbf{C} \tag{9.31}$$
$$\bigstar = G \circ H : \mathbf{C} \to \mathbf{C} \tag{9.32}$$

The two adjunctions imply that G preserves all limits and colimits in D

Gives rise to an adjoint pair of a monad GF and a comonad GH on $\mathbf{C}$

$$(GF \dashv GH) : \mathbf{C} \underset{GF}{\overset{GH}{\rightleftarrows}} \mathbf{C}$$

and dually, the adjoint pair of the comonad FG and the monad HG on $\mathbf{D}$

$$(FG \dashv HG) : \mathbf{D} \underset{FG}{\overset{HG}{\rightleftarrows}} \mathbf{D}$$

Theorem 9.1.1 *For $F \dashv G \dashv H$, F is fully faithful iff H is.*

F being fully faithful is equivalent to $\eta : \mathrm{Id} \to GF$ being a natural isomorphism.

H being fully faithful is equivalent to $\varepsilon : GH \to \mathrm{Id}$ being a natural isomorphism.

GF is isomorphic to the identity if GH is

$F \dashv G \dashv H$ is a fully faithful adjoint triple in this case. "This is often the case when D is a category of "spaces" structured over C, where F and H construct "discrete" and "codiscrete" spaces respectively."

If we denote a pair of idempotent adjoint monad $\bigcirc$ and comonad $\Box$, with the associated reflective and co-reflective categories

$$(T_\Box \dashv \iota_\Box) : \mathbf{H}_\Box \underset{T_\Box}{\overset{\iota_\Box}{\rightleftarrows}} \mathbf{H}$$

$$(FG \dashv HG) : \mathbf{H}_\bigcirc \underset{FG}{\overset{HG}{\rightleftarrows}} \mathbf{H}$$

From the adjunction, we have that $\mathbf{H}_\Box \cong \mathbf{H}_\bigcirc$ (for $\mathbf{H}$ one of the category in the adjoint triple of functor, this is the other category). Given either the adjunction $(\bigcirc \dashv \Box)$ or $(\Box \dashv \bigcirc)$, the adjoint triple will therefore be either $T_\Box \ldots$

The opposite of a moment $\bigcirc$ is a moment $\Box$ such that they form either a left or right adjunction, ie :

$$(\Box \dashv \bigcirc) : \mathbf{H}_\bigcirc \cong \mathbf{H}_\Box \begin{array}{c} \xhookrightarrow{\quad \iota_\bigcirc \quad} \\ \xleftarrow{T_\Box \cong T_\bigcirc} \\ \xhookrightarrow{\quad \iota_\Box \quad} \end{array} \mathbf{H}$$

or

We will denote the unity of opposites $\Box \dashv \bigcirc$ as a unity of a preceding to a successive moment, or *ps* unity, and $\bigcirc \dashv \Box$ as an *sp* unity.[8]

Theorem 9.1.2 *A ps unity* $(\Box \dashv \bigcirc)$ *defines an essential subtopos.*

$$(\Box \dashv \bigcirc) : \mathbf{H}_\bigcirc \cong \mathbf{H}_\Box \begin{array}{c} \xhookrightarrow{\quad \iota_\bigcirc \quad} \\ \xleftarrow{T_\Box \cong T_\bigcirc} \\ \xhookrightarrow{\quad \iota_\Box \quad} \end{array} \mathbf{H}$$

with

$$\Box = \iota_\Box \circ T \tag{9.33}$$

$$\bigcirc = \iota_\bigcirc \circ T \tag{9.34}$$

Proof 9.1.1 *The monad and comonad are adjoint endofunctors, that*

Interpretation : two different opposite "pure moments", level of a topos

The reflector and coreflector are the same, $T_\Box = T_\bigcirc$. Any object in the category is mapped to the same object in the category of "pure moment" regardless of the moment (this is their ”unity”). The inclusion map however will not be the same, and will typically be quite different, although it remains possible that a given object share the same moments for both.

Examples of *ps*-unities : even-odd adjunction, being-nothing, flat-sharp

Theorem 9.1.3 *An sp-unity* $(\bigcirc \dashv \Box)$ *defines a bireflective subcategory*

$$(\bigcirc \dashv \Box) : \mathbf{H}_{\bigcirc} \cong \mathbf{H}_{\Box} \begin{array}{c} \xleftarrow{\quad T_{\bigcirc} \quad} \\ \xhookrightarrow{\iota_{\Box} \cong \iota_{\bigcirc}} \\ \xleftarrow{\quad T_{\Box} \quad} \end{array} \mathbf{H}$$

with

$$\Box = \iota \circ T_{\Box} \tag{9.35}$$

$$\bigcirc = \iota \circ T_{\bigcirc} \tag{9.36}$$

The inclusion maps of the reflection/coreflection are such that $\iota_{\Box} \cong \iota_{\bigcirc}$, they are ”the same objects”

”one pure moment, but two opposite ways of projecting onto it.”

This means that for any object X in $\mathbf{H}$, its projected moment for either adjoint (say $\Box$) will be equivalent to the projection by the other adjoint of another object. If we take project our object, $\Box X = \iota_{\Box} \circ T_{\Box} X$, this maps it to some object in the bireflective subcategory before mapping it back to $\mathbf{H}$. As $\iota_{\Box} \cong \iota_{\bigcirc}$, this bireflection corresponds similarly to some object $Y \in \mathbf{H}$ for which $\Box X \cong \bigcirc Y$, given by $Y = \iota_{\bigcirc} T_{\Box} X$ (this is some pure $\bigcirc$ moment object in $\mathbf{H}$). So we have

$$\forall X \in \mathbf{H},\ \exists Y \in \mathbf{H},\ \Box X \cong \bigcirc Y \tag{9.37}$$

and using a similar argument, we have

$$\forall X \in \mathbf{H},\ \exists Y \in \mathbf{H},\ \bigcirc X \cong \Box Y \tag{9.38}$$

This property gives us the following

Theorem 9.1.4 *For an sp-unity, we have*

$$\Box \bigcirc X \cong \bigcirc X \tag{9.39}$$

$$\bigcirc \Box X \cong \Box X \tag{9.40}$$

Proof 9.1.2 *Using the idempotence of the moments,*

$$\Box \bigcirc X \cong \Box \Box Y \tag{9.41}$$

$$\cong \Box Y \tag{9.42}$$

$$\cong \bigcirc X \tag{9.43}$$

and dually, $\bigcirc \Box X = \Box X$

Theorem 9.1.5 *For a ps-unity, we have*

$$\square \bigcirc X \cong \square X \quad (9.44)$$
$$\bigcirc \square X \cong \bigcirc X \quad (9.45)$$

Proof 9.1.3 *by the triangle identity,*

$$\square \bigcirc X \cong \iota_\square T \iota_\bigcirc T \quad (9.46)$$
$$\cong \iota_\square T \quad (9.47)$$

Examples of *sp*-unities : ceiling-floor, shape-flat

Ceiling-floor : the similar objects under both modalities are simply given by $\lfloor x \rfloor$ and $\lceil x \rceil$, both mapped to integers and given the pure moment by including them back in $\mathbf{R}$. That is, for some real $x = k + \delta$, $\delta \in [0,1]$ (and $\neq 0$), we have

$$\square x = \lceil x \rceil = k = \lfloor k \rfloor \quad (9.48)$$

for any $k \in \mathbb{Z}$, we have $\iota(k)$ being the same

$$\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor \quad (9.49)$$
$$\lfloor \lceil x \rceil \rfloor = \lceil x \rceil \quad (9.50)$$

Shape-flat :

$$\int \flat X = \flat X \quad (9.51)$$
$$\flat \int X = \int X \quad (9.52)$$

Pieces of points are the points, points of the pieces are the pieces

spaces among retractive spaces? zero-vector bundles among all vector bundle?

There is an additional case of unity of opposites we can consider, where the adjoint functors form an ambidextrous adjunction, ie both of the outer functors of the adjoint triple are identical (or isomorphic, anyway)

$$(T \dashv \iota \dashv T) \quad (9.53)$$

in which case the monad and comonad are isomorphic,

$$L \cong R \quad (9.54)$$

a unique modality that we will call $\natural$.

Theorem 9.1.6 *In an ambidextrous adjunction,* $\eta \circ \epsilon \cong \mathrm{Id}$.

Frobenius monads and comonads

Definition 9.1.1 *Classical modality (quintessential localizations)*

Example 9.1.1 *The name of classical modality stems from its application as a map on mixed quantum-classical types in information theory, where the mixed type is given by a bundle of Hilbert spaces over a set,* $\prod_{s:S} \mathcal{H}_s \to S$. *The classical modality in this case factors through the underlying set via*

$$T_\natural(\prod_{s:S} \mathcal{H}_s \to S) \cong S \tag{9.55}$$

and sent back via

$$\iota_\natural S \cong (\prod_{s:S} \mathbb{C}^0 \to S) \cong (\mathbb{C}^0 \times S \to S) \tag{9.56}$$

The T *map is both reflective and coreflective.*

$$\mathrm{Hom}_{\mathbf{C}}((X \to S), (\{0\} \times S' \to S')) \cong \mathrm{Hom}_{\mathbf{Set}}(S, S') \tag{9.57}$$

which is true as every bundle map $(f, \phi) : (X \to S) \to (\{0\} \times S' \to S')$ *has to obey*

$$\begin{array}{ccc} X & \xrightarrow{f} & S' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_S} \\ S & \xrightarrow{\phi} & S' \end{array}$$

So that $f = \phi \circ \pi$, *the bundle morphisms* $(f, \phi) = (\phi \circ \pi, \phi)$ *are entirely determined by the underlying functions of the set. And dually,*

$$\mathrm{Hom}_{\mathbf{C}}((S' \times \{0\} \to S'), (X \to S)) \cong \mathrm{Hom}_{\mathbf{Set}}(S', S) \tag{9.58}$$

$$\begin{array}{ccc} S' & \xrightarrow{f} & X \\ {\scriptstyle \mathrm{Id}_{S'}}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\phi} & S \end{array}$$

giving $\phi = \pi \circ f$, *so that the bundle map* $(f\phi) = (f, \pi \circ f)$ *[?]*

Definition 9.1.2 *A subtopos is an* essential subtopos *if the inclusion map* $\iota : \mathbf{S} \hookrightarrow \mathbf{H}$ *is an essential geometric morphism, ie in addition to* ι *being a full and faithful functor with a left adjoint* ι^* *that preserves finite limits,* ι^* *furthermore has a left adjoint* $\iota_!$

Theorem 9.1.7 *A ps-unity* $(\Box \dashv \bigcirc)$ *defines an essential subtopos.*

Proof 9.1.4 *As a set of two moments, we have*

Theorem 9.1.8 *For an sp-unity* $(\bigcirc \dashv \Box)$*, we have*

$$\eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \cong \eta^{\bigcirc}_{\Box X} \tag{9.59}$$

Proof 9.1.5 *Given an object* X*, if we look at the component* $\epsilon^{\Box}_X : \Box X \to X$*, its naturality square under* $\eta^{\bigcirc}$ *is given by*

$$\begin{array}{ccc} \Box X & \xrightarrow{\eta^{\bigcirc}_{\Box X}} & \bigcirc\Box X \\ {\scriptstyle \epsilon^{\Box}_X}\downarrow & & \downarrow{\scriptstyle \eta^{\bigcirc}(\epsilon^{\Box}_X)} \\ X & \xrightarrow{\eta^{\bigcirc}_X} & \bigcirc X \end{array}$$

As $\eta^{\bigcirc}(\epsilon^{\Box}_X) : \bigcirc\Box X \to \bigcirc X$ *is an equivalence for an sp-unity, we have*

$$\eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \cong \eta^{\bigcirc}_{\Box X} \tag{9.60}$$

9.2 Types

As with monads in general, we define the various types associated with modalities. So that for $\bigcirc$, we define the modal types to be the ones for which $\eta^{\bigcirc}$ is an isomorphism, and submodal types if it is a monomorphism, and likewise for the comodality $\Box$, we define comodal types if $\epsilon^{\Box}$ is an isomorphism, and the supcomodal types if it is a monomorphism.

Theorem 9.2.1 *For an sp-unity, the modal types and the comodal types are identical.*

Proof 9.2.1 *If* X *is a modal type, we have* $X \cong \bigcirc X$*. By the properties of the sp-unity,*

$$\begin{aligned} X &\cong \bigcirc X && (9.61) \\ &\cong \Box \bigcirc X && (9.62) \\ &\cong \Box X && (9.63) \end{aligned}$$

and likewise for a comodal type,

$$\begin{aligned} X &\cong \Box X && (9.64) \\ &\cong \bigcirc\Box X && (9.65) \\ &\cong \bigcirc X && (9.66) \end{aligned}$$

So that we can simply talk of modal types in general for *sp*-unities.

9.3 Morphisms

Definition 9.3.1 *Given a modality $\bigcirc$ on* $\mathbf{C}$*, we say that a morphism $f : X \to Y$ is* $\bigcirc$-modal *if*

Definition 9.3.2 *Given a modality $\bigcirc$ on* $\mathbf{C}$*, we define the class of morphisms $f : X \to Y$ to be* $\bigcirc$-étale*[116] or* $\bigcirc$-closed*[117] the morphisms for which the naturality square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X^{\bigcirc}}\downarrow & & \downarrow{\scriptstyle \eta_Y^{\bigcirc}} \\ \bigcirc X & \xrightarrow[\bigcirc f]{} & \bigcirc Y \end{array}$$

is a pullback :

$$X \cong \bigcirc X \times_{\bigcirc Y} Y \tag{9.67}$$

For comodalities, with the comodality naturality square being a pushout?

$$\begin{array}{ccc} \Box X & \xrightarrow{\Box f} & \Box Y \\ {\scriptstyle \epsilon_X^{\Box}}\downarrow & & \downarrow{\scriptstyle \epsilon_Y^{\Box}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

$$Y \cong X +_{\Box X} \Box Y \tag{9.68}$$

9.4 Negation

In addition to oppositions, monads and comonads can also have *negations*. The negation of a moment will be, if it exists, an operator which removes specifically the attributes of a given moment. This is given for a monad $\bigcirc$ and comonad $\Box$ by the fiber and cofiber of the unit and counit, with a specific basepoint $p : 1 \to X$ for the fiber :

$$\overline{\bigcirc}_p X = \mathrm{Fib}_p(X \to \bigcirc X) \tag{9.69}$$

$$\overline{\Box} X = \mathrm{Cofib}(\Box X \to X) \tag{9.70}$$

For the negation of a monad, this corresponds to the following pullback :

$$\begin{array}{ccc} \overline{\bigcirc}_p X & \xrightarrow{!_{\overline{\bigcirc} X}} & 1 \\ {\scriptstyle p^*\eta^{\bigcirc}_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X \end{array}$$

The rough meaning of this is that we are looking for a subobject of X for which the monad $\bigcirc$ will map everything onto a single point, and vice versa, the negation of the modal type $\bigcirc X$ is a single point. One way of this is easy enough to show, as this is the fiber of an isomorphism, by [X], this is simply the terminal object:

$$\begin{aligned} \overline{\bigcirc}_p \bigcirc X &\cong \mathrm{Fib}_p(\bigcirc X \to \bigcirc \bigcirc X) && (9.71) \\ &\cong \mathrm{Fib}_p(\mathrm{Id}_X) && (9.72) \\ &\cong 1 && (9.73) \end{aligned}$$

Furthermore, $\epsilon^{\overline{\bigcirc}_p}_{\bigcirc X} \cong p$, since we have $\bigcirc p \cong p$ as both 1 and $\bigcirc X$ are isomorphic under $\bigcirc$, and the left and right morphisms are isomorphic since the top and bottom morphisms are isomorphisms.

$$\begin{array}{ccc} \overline{\bigcirc}_p \bigcirc X & \xrightarrow{\cong} & 1 \\ {\scriptstyle \epsilon^{\overline{\bigcirc}_p}_{\bigcirc X}}\downarrow & & \downarrow{\scriptstyle p} \\ \bigcirc X & \xrightarrow{\cong} & \bigcirc^2 X \end{array}$$

p^*

The other way around, $\bigcirc\overline{\bigcirc}_p X$ is not necessarily true however. But if we have the identity $\bigcirc 1 \cong 1$ (for instance if the monad preserves products), and that the monad preserves the fiber (via preserving pullbacks, or products again and the equalizer), we get furthermore

$$\begin{aligned} \bigcirc\overline{\bigcirc}_p X &\cong \bigcirc\mathrm{Fib}_p(X \to \bigcirc X) && (9.74) \\ &\cong \mathrm{Fib}_{\circ p}(\bigcirc X \to \bigcirc \bigcirc X) && (9.75) \\ &\cong \mathrm{Fib}_{\circ p}(\mathrm{Id}_{\bigcirc X}) && (9.76) \\ &\cong 1 && (9.77) \end{aligned}$$

$$\begin{array}{ccc} \bigcirc\overline{\bigcirc} X & \xrightarrow{!_{\overline{\bigcirc} X}} & 1 \\ \downarrow & & \downarrow{\scriptstyle \bigcirc p} \\ \bigcirc X & \xrightarrow{\cong} & \bigcirc X \end{array}$$

So that $\bigcirc\overline{\bigcirc}X \cong \overline{\bigcirc}_p \bigcirc X \cong 1$, which is the semantics that we would like : each modality preserves mutually exclusive moments of the object.

While the modality depends on a specific point, it can occur that it is independent of a specific choice of point, or that this dependence is shifted to the category by instead using the category of pointed objects $\mathbf{H}^{1/}$ (or as we will see later on, by picking connected objects in the homotopic case). In this case, we will simply denote it by $\overline{\bigcirc}$.

Similarly for the comonad $\square$, we have this identity easily enough in one case by idempotence,

$$\overline{\square}\square X \cong \mathrm{Cofib}(\square\square X \to \square X) \tag{9.78}$$
$$\cong 1 \tag{9.79}$$

$$\begin{array}{ccc} \square X & \xrightarrow{\epsilon_X} & X \\ \downarrow{\scriptstyle !_{\square X}} & & \downarrow \\ 1 & \longrightarrow & \overline{\square} X \end{array}$$

If we have both the comonad and its negation acting on an object,

$$\square\overline{\square}X = \square\mathrm{Cofib}(\square X \to X) \tag{9.80}$$
$$= \mathrm{Cofib}(\square\square X \to \square X) \tag{9.81}$$
$$= \mathrm{Cofib}(\mathrm{Id}_{\square X}) \tag{9.82}$$
$$= 1 \tag{9.83}$$

We are left with the terminal object with no specific properties. Equivalently, $\square\overline{\square} = \bigcirc_*$, the modality of being that we will see later on.

If we are in the context of a unity of opposites, we have the two cases of an *sp*-unity, $\bigcirc \dashv \square$, and a *ps*-unity : $\square \dashv \bigcirc$. As right adjoints preserve limits and left adjoints colimits, we have naturally that for an *sp*-unity,

$$\bigcirc 1 \cong 1 \tag{9.84}$$
$$\bigcirc \mathrm{Fib} \cong \mathrm{Fib} \bigcirc \tag{9.85}$$
$$\square \mathrm{Cofib} \cong \mathrm{Cofib} \square \tag{9.86}$$

meaning that the property of the negation $\square\overline{\square} = \circledast$ is guaranteed, and for a *ps*-unity, $\square 1 \cong 1$, meaning that additional conditions are required for it to guarantee it [CHECK IT], such as the preservation of the fiber and cofiber.

To do this, we need to find a map from the category to the subcategory containing only objects that the moment map to the terminal object. Given the counit $\Box X \to X$, this is the cofiber :

Definition 9.4.1 *The negation of a comonadic moment $\Box$ is given object-wise by the cofiber of its counit :*

$$\overline{\Box} X = \mathrm{Cofib}(\Box X \to X) \tag{9.87}$$

Or as a limit, it is the pullback of the cospan $1 \leftarrow \Box X \to X$ *:*

$$\begin{array}{ccc} \Box X & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle !_{\Box X}}\downarrow & & \downarrow \\ 1 & \longrightarrow & \overline{\Box} X \end{array}$$

To define : composition of a pullback, fiber, cofiber with a natural transformation, (co)monad

Limit as a natural transformation in the category of functors $[I, C]$

$$\eta : \Delta_{\lim F} \to F \tag{9.88}$$

For fiber : arrow category, slice category C_*, coslice category?

Diagram for coslice categories :

$$\begin{array}{ccc} \mathbf{H}^{X/} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow {\scriptstyle \Delta_X} \\ [I, \mathbf{H}] & \longrightarrow & \mathbf{H} \end{array}$$

Theorem 9.4.1 *The negation of a monad defines a comonad, and the negation of a comonad defines a monad.*

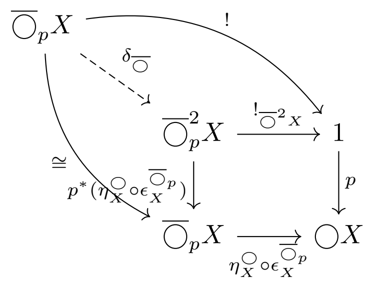
Proof 9.4.1 *The fiber naturally defines a counit*

$$p^* \eta_X^{\bigcirc} = \epsilon_X^{\overline{\bigcirc}_p} : \overline{\bigcirc}_p X \to X \tag{9.89}$$

and we can construct a comultiplication map via a second pullback, using the composite map

$$\eta_X^{\bigcirc} \circ \epsilon_X^{\overline{\bigcirc}_p} : \overline{\bigcirc}_p X \to \bigcirc X \tag{9.90}$$

where, using the universal property of the pullback, we can define a unique map



$$\delta_{\overline{\bigcirc}} : \overline{\bigcirc}_p X \to \overline{\bigcirc}_p^2 X \tag{9.91}$$

obeying the property

$$\delta_{\overline{\bigcirc}} \circ p^*(\eta_X^{\bigcirc} \circ \epsilon_X^{\overline{\bigcirc}_p}) \cong \mathrm{Id}_{\overline{\bigcirc}_p X} \tag{9.92}$$

To define a multiplication map, it needs to obey

$$\overline{\bigcirc}_p(\epsilon_X^{\overline{\bigcirc}_p}) \circ \delta_X^{\overline{\bigcirc}} \cong \mathrm{Id} \cong \delta_X^{\overline{\bigcirc}} \circ \epsilon_{\overline{\bigcirc}_p X}^{\overline{\bigcirc}_p} \tag{9.93}$$

Alt : if the negation commutes with the fiber,

$$\overline{\bigcirc}_q \overline{\bigcirc}_p X \cong \mathrm{Fib}_p(\overline{\bigcirc}_q X \to \overline{\bigcirc}_q \bigcirc X) \tag{9.94}$$

Since $\overline{\bigcirc}_q \bigcirc X \cong 1$,

$$\overline{\bigcirc}_q \overline{\bigcirc}_p X \cong \mathrm{Fib}_p(\overline{\bigcirc}_q X \to 1) \tag{9.95}$$

[...] For a comonad $\square$*, the cofiber naturally defines a unit,*

$$!^*_{\square X} \epsilon_X^{\square} \eta_X^{\overline{\square}} : X \to \square X \tag{9.96}$$

As in the case of an *sp*-unity, the modal types are in some sense "the same", ($\bigcirc \dashv \square$), we also have that $\square \bigcirc X = \bigcirc X$, ie there is

Determinate negation : if ($\bigcirc \dashv \square$) is such that $\bigcirc 1 \cong 1$ and $\square \to \bigcirc$ is epi, there is determinate negation

For determinate negation, we have

$$\bigcirc \overline{\square} \cong 1 \tag{9.97}$$

Dually we can also define the determinate negation of a monadic moment, but while a cofibration only involves the pushout $1 \leftarrow \square X \to X$, where the morphism $\square X \to 1$ is unique, the fibration is the pullback $1 \to X \leftarrow \bigcirc X$, which depends on a specific choice of a point $1 \to X$ in the object. This means that this negation will either be defined if the object contains a single point, if we are given a specific choice of a point, if the result is independent from the choice of basepoint, or if we allow more flexible negations such as the homotopy fiber of a connected object.

Definition 9.4.2 *The determinate negation of a monadic moment*

$$\overline{\bigcirc} X = \mathrm{Fib}(X \to \bigcirc X) \tag{9.98}$$

$$\Box \overline{\Box} = \circledast \tag{9.99}$$

Show that the intersection of subcategories is something

Example 9.4.1

Definition 9.4.3 Determinate negation *of a unity of opposite moments* $\bigcirc \dashv \Box$ *if* $\Box, \bigcirc$ *restrict to* 0*-types and*

- $\bigcirc * \cong *$
- $\Box \to \bigcirc$ *is an epimorphism.*

"For an opposition with determinate negation, def. 1.14, then on 0-types there is no $\bigcirc$-moment left in the negative of $\Box$-moment"

$$\bigcirc \overline{\Box} \cong * \tag{9.100}$$

Proof 9.4.2 *By left adjoints preserving colimits,*

$$\bigcirc \overline{\Box} X = \bigcirc \mathrm{cofib}(\Box X \to X) \cong \mathrm{cofib}(\Box X \to \bigcirc X) \tag{9.101}$$

Since $\Box X \to \bigcirc X$ *is epi, which is preserved by pushout, this is an epimorphism from the terminal object, therefore the terminal object itself.*

As we have that there is no $\bigcirc$ moment left in the negative of $\Box$, we also have dually that the negation of $\bigcirc$ is just the identity on such a type :

Theorem 9.4.2 *For a determinate negation on an sp-unity* $(\bigcirc \dashv \Box)$*, we have*

$$\overline{\bigcirc}_p \overline{\Box} X \cong \overline{\Box} X \tag{9.102}$$

Proof 9.4.3 *The modal negation diagram on* $\overline{\Box} X$ *is*

$$\begin{array}{ccc}
\overline{\bigcirc}_p \overline{\square} X & \xrightarrow{!_{\overline{\bigcirc}\overline{\square}X}} & 1 \\
{\scriptstyle p^*\eta^{\bigcirc}_{\overline{\square}X}} \downarrow & & \downarrow {\scriptstyle p} \\
\overline{\square} X & \xrightarrow[\eta^{\bigcirc}_{\overline{\square}X}]{} & \bigcirc \overline{\square} X
\end{array}$$

as we have furthermore the identity $\bigcirc_p \overline{\square} X \cong 1$, p is an isomorphism here so that as pullbacks preserves isomorphisms, $p^*\eta^{\bigcirc}_{\overline{\square}X}$ is as well, giving us

$$\overline{\bigcirc}_p \overline{\square} X \cong \overline{\square} X \tag{9.103}$$

[...]

From all these identities, we can draw the following tables :

	$\bigcirc$	$\square$	$\overline{\bigcirc}$	$\overline{\square}$
$\bigcirc$	$\bigcirc^2 \cong \bigcirc$	$\bigcirc\square \cong \square$	$\bigcirc\overline{\bigcirc}$	$\bigcirc\overline{\square}$
$\square$	$\square\bigcirc \cong \bigcirc$	$\square^2 \cong \square$	$\square\overline{\bigcirc}$	$\square\overline{\square}$
$\overline{\bigcirc}$	$\overline{\bigcirc}\bigcirc \cong 1$	$\overline{\bigcirc}\square$	$\overline{\bigcirc}^2$	$\overline{\bigcirc}\overline{\square}$
$\overline{\square}$	$\overline{\square}\bigcirc$	$\overline{\square}\square \cong 1$	$\overline{\square}\overline{\bigcirc}$	$\overline{\square}^2$

Table 9.2: For an *sp*-unity

	$\bigcirc$	$\square$	$\overline{\bigcirc}$	$\overline{\square}$
$\bigcirc$	$\bigcirc^2 \cong \bigcirc$	$\bigcirc\square \cong \bigcirc$	$\bigcirc\overline{\bigcirc}$	$\bigcirc\overline{\square}$
$\square$	$\square\bigcirc \cong \square$	$\square^2 \cong \square$	$\square\overline{\bigcirc}$	$\square\overline{\square}$
$\overline{\bigcirc}$	$\overline{\bigcirc}\bigcirc \cong 1$	$\overline{\bigcirc}\square$	$\overline{\bigcirc}^2$	$\overline{\bigcirc}\overline{\square}$
$\overline{\square}$	$\overline{\square}\bigcirc$	$\overline{\square}\square \cong 1$	$\overline{\square}\overline{\bigcirc}$	$\overline{\square}^2$

Table 9.3: For a *ps*-unity

9.4.1 de Rham modalities

In addition to the negations of modalities, we will also consider their duals, the *de Rham* modalities, which are simply given by the fiber of the counit for a comonad, and cofiber of the unit for a monad.

$$\widetilde{\bigcirc} X = \mathrm{Cofib}(X \to \bigcirc X) \tag{9.104}$$
$$\widetilde{\square} X = \mathrm{Fib}_p(\square X \to X) \tag{9.105}$$

Those types of modalities will mostly be of use in the ∞-categorical case, where some specific properties of such categories make them more relevant, but they will however still function as intended in the 1-categorical case. To see their relationship to negation, we will have to look at the 1-categorical case of the loop and suspension functors :

$$\Omega X \cong 1 \times_X 1 = \mathrm{eq}(1 \rightrightarrows X) \tag{9.106}$$

$$\Sigma X \cong 1 +_X 1 \tag{9.107}$$

Another way to interpret the negations and de Rham modalities of a given moment is to consider them as functors between the category of pointed objects on the topos, $\mathbf{H}^{1/}$, and the topos itself, as the cofiber and fiber define respectively a pointed object from an object and vice versa.

$$(\overline{\bigcirc} \dashv \overline{\square}) : \mathbf{H}^{1/} \underset{\overline{\square}}{\overset{\overline{\bigcirc}}{\rightleftarrows}} \mathbf{H}$$

$$(\tilde{\bigcirc} \dashv \tilde{\square}) : \mathbf{H}^{1/} \underset{\tilde{\bigcirc}}{\overset{\tilde{\square}}{\rightleftarrows}} \mathbf{H}$$

As the pointed objects form a monoidal category, we have a natural structure of a tensor product given by the smash product and a zero object given by the identity $1 \to 1$.

9.5 Sublation

Sublation (or *Aufhebung* in the original German), levels of a topos

§180 The resultant equilibrium of coming-to-be and ceasing-to-be is in the first place becoming itself. But this equally settles into a stable unity. Being and nothing are in this unity only as vanishing moments; yet becoming as such is only through their distinguishedness. Their vanishing, therefore, is the vanishing of becoming or the vanishing of the vanishing itself. Becoming is an unstable unrest which settles into a stable result.

§181 This could also be expressed thus: becoming is the vanishing of being in nothing and of nothing in being and the vanishing of being and nothing generally; but at the same time it rests on the distinction between them. It is therefore inherently self-contradictory, because the determinations it unites within itself are opposed to each other; but such a union destroys itself.

§182 This result is the vanishedness of becoming, but it is not nothing; as such it would only be a relapse into one of the already sublated determinations, not the resultant of nothing and being. It is the unity of being and nothing which has settled into a stable oneness. But this stable oneness is being, yet no longer as a determination on its own but as a determination of the whole.

§183 Becoming, as this transition into the unity of being and nothing, a unity which is in the form of being or has the form of the onesided immediate unity of these moments, is determinate being.

Unity of opposites, like idempotent monads themselves, can be ordered. As idempotent monads and comonads can be classified by inclusion, being described by (co)reflective subcategories, we can also do so for the adjunctions.

In the case of a *ps*-unity, if we have some original adjunction ($\boxed{\spadesuit} \dashv \textcircled{\clubsuit}$), we say that another adjunction ($\boxed{\heartsuit} \dashv \textcircled{\diamondsuit}$) is of a *higher level* if we have both $\boxed{\spadesuit} \prec \boxed{\heartsuit}$ and $\textcircled{\clubsuit} \prec \textcircled{\diamondsuit}$, which we denote diagrammatically by

[diagram]

In other words, we have

$$\boxed{\heartsuit}\boxed{\spadesuit} \cong \boxed{\spadesuit} \quad (9.108)$$
$$\textcircled{\diamondsuit}\textcircled{\clubsuit} \cong \textcircled{\clubsuit} \quad (9.109)$$

Likewise for an *sp*-unity, the adjunction ($\textcircled{\spadesuit} \dashv \boxed{\clubsuit}$) has a higher level adjunction ($\textcircled{\heartsuit} \dashv \boxed{\diamondsuit}$) if we have $\textcircled{\spadesuit} \prec \textcircled{\heartsuit}$ and $\boxed{\clubsuit} \prec \boxed{\diamondsuit}$

$$\textcircled{\heartsuit}\textcircled{\spadesuit} \cong \textcircled{\spadesuit} \quad (9.110)$$
$$\boxed{\diamondsuit}\boxed{\clubsuit} \cong \boxed{\clubsuit} \quad (9.111)$$

Furthermore, if we have

Definition 9.5.1 *Given some adjoint moments* ($\boxed{\spadesuit} \dashv \textcircled{\clubsuit}$)*, we say that a second pair of adjoints moments* ($\boxed{\heartsuit} \dashv \textcircled{\diamondsuit}$) *is a* sublation *of the first adjunction if we have the equalities*

$$\boxed{\heartsuit}\boxed{\spadesuit} \cong \boxed{\spadesuit} \quad (9.112)$$
$$\textcircled{\diamondsuit}\textcircled{\clubsuit} \cong \textcircled{\clubsuit} \quad (9.113)$$

In other words, the objects of pure Δ moments of the category are also of pure $\bigstar$ moment.

Definition 9.5.2 *For a sublation* ($\boxed{\spadesuit} \dashv \textcircled{\clubsuit}$) *to* ($\boxed{\heartsuit} \dashv \textcircled{\diamondsuit}$)*, we say that this sublation is a* left *(resp.* right*) sublation if in addition to the properties of a sublation, we also have that* $\boxed{\spadesuit} \prec \boxed{\heartsuit}$ *(resp.* $\spadesuit \prec \diamondsuit$*)*

$$x \quad (9.114)$$

Theorem 9.5.1 *Any sublation of an sp-unity is automatically a left and right sublation.*

Proof 9.5.1

Theorem 9.5.2 *For an sp-opposition, the sublation* $(\Delta_1 \dashv \Delta_2)$ *to* $(\bigstar_1 \dashv \bigstar_2)$, *with unit and counit* $(\eta^{\Delta_1}, \epsilon^{\Delta_1})$ *and* $(\eta^{\bigstar_1}, \epsilon^{\bigstar_2})$,

$$\eta_X^{\Delta_1} : X \quad \to \quad \Delta_1 X \tag{9.115}$$

$$\epsilon_X^{\Delta_2} : \Delta_2 X \quad \to \quad X \tag{9.116}$$

$$\eta_X^{\bigstar_1} : X \quad \to \quad \bigstar_1 X \tag{9.117}$$

$$\epsilon_X^{\bigstar_2} : \bigstar_2 X \quad \to \quad X \tag{9.118}$$

then the unit and counit of the original opposition factor through their sublated moments :

$$\eta_X^{\Delta_1} : X \quad \to \quad \bigstar_1 X \to \Delta_1 X \tag{9.119}$$

$$\epsilon_X^{\Delta_2} : \Delta_2 X \quad \to \quad \bigstar_2 X \to X \tag{9.120}$$

Proof 9.5.2 *Looking at the commutative square of* $\eta^{\bigstar} : \mathrm{Id} \to \bigstar_1$,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^{\bigstar}} & \bigstar_1 X \\ {\scriptstyle \eta_X^{\Delta_1}}\downarrow & & \downarrow{\scriptstyle \bigstar_1 \eta_X^{\Delta_1}} \\ \Delta_1 X & \xrightarrow[\eta_{\Delta_1 X}^{\bigstar}]{} & \bigstar_1 \Delta_1 X \end{array}$$

From $\bigstar_1 \Delta_1 \cong \Delta_1$, *we have the existence of an inverse natural transformation [inverse specifically to* $\eta^{\bigstar}$ *?]*

$$\alpha : \bigstar_1 \Delta_1 \to \Delta_1 \tag{9.121}$$

So that, applying it to our commutative diagram,

$$\alpha \bigstar_1 \eta_X^{\Delta_1} \eta_X^{\bigstar} = \alpha \eta_{\Delta_1 X}^{\bigstar} \eta_X^{\Delta_1} \tag{9.122}$$

$$\alpha \bigstar_1 \eta_X^{\Delta_1} \eta_X^{\bigstar} = \eta_X^{\Delta_1} \tag{9.123}$$

Furthermore, we can apply the monad $\bigstar_1$ *to the unit as [whiskering?]*

$$(\bigstar_1 \eta^{\Delta_1})_x : \bigstar_1 X \to \bigstar_1 \Delta_1 X \tag{9.124}$$

Then given the composition

$$(\bigstar_1 \eta^{\Delta_1} \eta^{\bigstar_1}) : X \xrightarrow{\eta_X^{\bigstar_1}} \bigstar_1 X \xrightarrow{\bigstar_1 \eta_X^{\Delta_1}} \bigstar_1 \Delta_1 X \cong \Delta_1 X \tag{9.125}$$

$$\bigstar_1 \Delta_1 \tag{9.126}$$

$$\begin{array}{ccc} X & \xrightarrow{\eta^{\Delta_2}} & \Delta_2 X \\ {\scriptstyle \eta_X^{\Delta_1}}\downarrow & & \downarrow{\scriptstyle \Delta_2 \eta_X^{\Delta_1}} \\ \Delta_1 X & \xrightarrow[\eta^{\Delta_2}_{\Delta_1 X}]{} & \Delta_2 \Delta_1 X \end{array}$$

If the unit/counit is epi/mono, does that generalize to the sublation?

Theorem 9.5.3 *For an sp-unity* $(\bigcirc \dashv \Box)$ *where the unit* $\eta_X^{\bigcirc} : X \to \bigcirc$ *is an epimorphism [on* X*?],*

Proof 9.5.3 *If* $\eta_X^{\bigcirc}$ *is epi, we have*

$$g_1 \circ \eta_X^{\bigcirc} = g_2 \circ \eta_X^{\bigcirc} \to g_1 = g_2 \tag{9.127}$$

$$g_1 \circ \eta_X^{\bigcirc'} = g_2 \circ \eta_X^{\bigcirc'} \tag{9.128}$$

Action of sublation on negations?

$$\spadesuit \tag{9.129}$$

[118]

10
Objective logic

[119]

“These many different things stand in essential reciprocal action via their properties; the property is this reciprocal relation itself and apart from it the thing is nothing”

As there will be many notations for very similar concepts of different types, we will require the following convention :

- Unless a more specific unambiguous symbol exists, monads will be denoted by $\bigcirc_{(-)}$, with its subscript to differentiate it
- Similarly, comonads will be denoted by $\Box_{(-)}$, with its subscript to differentiate it
- A generic topos $\mathbf{H}$
- The terminal object is 1
- The initial object is 0

This to avoid circumstances such as $*$ to represent both an object, functor, category and monad.

A trivial opposition we have in the objective logic is

$$\mathrm{Id} \dashv \mathrm{Id} \tag{10.1}$$

This is an opposition defined by the triple of endofunctors

$$(\mathrm{Id} \dashv \mathrm{Id}) : \mathbf{H} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathbf{H}$$

Representing three identity functors, composing into the two identity monad and comonad, with the subtopi being **H** itself. (Moment of identity?)

10.1 Being and nothingness

"Being, pure being, [...] it has no diversity within itself nor any with a reference outwards"

"Nothing, pure nothing: it is simply equality with itself, complete emptiness"

The most basic type of moments are the monads of *being* (*Sein*) and *nothingness* (*Nichts*). This can be seen easily enough on a topos level by considering that the smallest subtopos is the initial topos $\mathrm{Sh}(\varnothing) \cong \mathbf{1}$. Let's consider the unique functor from our topos to it, the constant functor on this subtopos,

$$\Delta_* : \mathbf{C} \to \mathbf{1} \tag{10.2}$$

which maps every object of $\mathbf{C}$ to the unique object $*$ of the terminal category $\mathbf{1}$. This corresponds to the *unit type* in type theory term, and this is what is referred to as *The One* (*Das Eins*) in objective logic.

There are only two possibilities here : either the underlying opposition is a *ps*-unity, with Δ_* its (co)reflector, or it is an *sp*-unity, in which case we have some adjoint string $(\Delta_* \dashv F \dashv \Delta_*)$ where Δ_* is the same on both sides (since there is no other functor between those categories), making it an ambidextrous adjunction.

If we try to look at the *sp*-unity case,

$$\mathrm{Hom}_{\mathbf{1}}(\Delta_* X, *) = \mathrm{Hom}_{\mathbf{H}}(X, F(*)) \tag{10.3}$$
$$\mathrm{Hom}_{\mathbf{H}}(F(*), Y) = \mathrm{Hom}_{\mathbf{1}}(*, \Delta_*(Y)) \tag{10.4}$$

ie the hom-set of functions to and from $F(*)$ only have one element. As any topos has at least an initial and terminal object, any object outside of the initial and terminal object should have at least two such morphisms (the identity morphism and the morphism from the initial object/to the terminal object), and if it is either the initial object or the terminal object, it should have at least one morphism to or from any other object. Therefore there is no such adjunction, unless we are dealing with the initial topos itself, in which case this adjunction is simply the identity.

For a *ps*-adjunction, we give this functor a left and right adjoint, which as we will see are the constant functors of the initial and terminal object, so that we will denote them as Δ_0 and Δ_1, forming the adjoint cylinder

$$(\Delta_0 \dashv \Delta_* \dashv \Delta_1) : \mathbf{1} \; \begin{matrix} \xhookrightarrow{\Delta_0} \\ \xleftarrow{\Delta_*} \\ \xhookrightarrow{\Delta_1} \end{matrix} \; \mathbf{H}$$

An easy way to see this is via the adjunction of hom-sets :

$$\mathrm{Hom}_{\mathbf{C}}(\Delta_0(*), X) \cong \mathrm{Hom}_{\mathbf{1}}(*, \Delta_* X) \tag{10.5}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between $\Delta_0(*)$ and any object X, making it the initial object of the topos. Similarly,

$$\mathrm{Hom}_{\mathbf{1}}(\Delta_*(X), *) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Delta_1(*)) \tag{10.6}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between any object X and $\Delta_1(*)$, making it the terminal object of the topos, confirming our choice of those adjoints as constant functors.

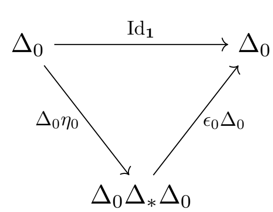
We can also look at this in terms of the unit and counit of the adjunction. First if we look at the adjunction $\Delta_0 \dashv \Delta_*$, as adjoint functors, they are equipped with the unit and counit natural transformations,

$$\eta_0 \quad : \quad \mathrm{Id}_{\mathbf{1}} \Rightarrow \Delta_* \circ \Delta_0 \tag{10.7}$$

$$\epsilon_0 \quad : \quad \Delta_0 \circ \Delta_* \Rightarrow \mathrm{Id}_{\mathbf{C}} \tag{10.8}$$

As there is only one endofunctor on $\mathbf{1}$, eta_0 is simply the identity natural transformation.

they have to obey the triangle identities



In terms of components, this means that for any object $X \in \mathbf{C}$ (and the only object $*$ in $\mathbf{1}$),

$$\mathrm{Id}_{\Delta_0(*)} = \epsilon_{\Delta_0(*)} \circ \Delta_0(\eta_*) \tag{10.9}$$
$$\mathrm{Id}_{\Delta_*(X)} = \Delta_*(\epsilon_X) \circ \eta_{\Delta_*(X)} \tag{10.10}$$

We have the identities $\Delta_*(X) = *$, and any component of the counit can only be the identity morphism on $*$, so that

$$\mathrm{Id}_{\Delta_0(*)} = \epsilon_{\Delta_0(*)} \circ \mathrm{Id}_{\Delta_0(*)} \tag{10.11}$$
$$\mathrm{Id}_* = \mathrm{Id}_* \tag{10.12}$$

The second line is trivial, but the first line tells us that ϵ is the identity on $\Delta_0(*)$

for any object $X \in \mathbf{C}$, there exists an object $\Delta_*(X) \in \mathbf{1}$ and a morphism $\epsilon_X : \Delta_0 \circ \Delta_*(X) \to X$ such that for every object in $\mathbf{1}$ (so only for $*$), and every morphism $f : \Delta_0(*) \to X$, there exists a unique morphism $g : * \to \Delta_*(X) = *$ with $\epsilon_X \circ \Delta_0(g) = f$.

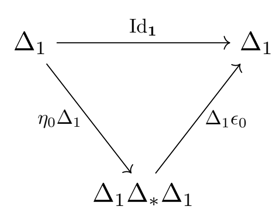
The unique morphism g is simply the identity function Id_1, and our natural transformation at c is simply $\epsilon_X : \Delta_0(*) \to X$. We therefore need the constraint

$$\epsilon_X \circ \Delta_0(\mathrm{Id}_1) = f \tag{10.13}$$

However, as g can only be one function, we cannot have more than one such possible morphism f, and we need exactly one to map to Id_1. This means that the empty functor Δ_0 maps the single object of 1 to the initial object 0 in $\mathbf{C}$ (if the category contains one), as its name indicates, justifying our notation of the constant functor Δ_0.

[Do the other triangle?]

Conversely, the right adjoint Δ_1 has to obey



Δ_1 maps the unique object of the terminal category to the terminal object 0 of $\mathbf{C}$, if this object exists.

Therefore, we have an adjoint triple $(\Delta_0 \dashv \Delta_* \dashv \Delta_1)$ between our topos and the initial topos, forming a *ps* unity of opposites, giving rise to an adjoint modality that we will write as ($\boxslash \dashv \circledast$), with $\boxslash$ the modality of nothingness (or empty comonad) and $\circledast$ the modality of being (or unit monad), defined by

$$\boxslash = \Delta_0 \circ \Delta_* \tag{10.14}$$
$$\circledast = \Delta_1 \circ \Delta_* \tag{10.15}$$

where the unit and counit of the adjunction are given by those that we have seen, the unit of the monad $\circledast$ and the counit of the comonad $\boxslash$

$$\eta^{\circledast} : \quad \mathrm{Id}_{\mathbf{H}} \to \Delta_1 \circ \Delta_* \tag{10.16}$$
$$\epsilon^{\boxslash} : \Delta_0 \Delta_* \to \mathrm{Id}_{\mathbf{1}} \tag{10.17}$$

Forming the adjunction

$$(\boxslash \dashv \circledast) : \mathbf{1} \begin{array}{c} \xhookrightarrow{\;\Delta_0\;} \\ \xleftarrow{\;\Delta_*\;} \\ \xhookrightarrow{\;\Delta_1\;} \end{array} \mathbf{H}$$

As a *ps*-unity, we find what we expect : by necessity, only one projection, and two different inclusions of this in the topos.

In terms of their modal action, the empty monad maps any object of the category to its initial element,

$$\boxslash(X) = 0 \tag{10.18}$$

and any morphism to the identity on the initial object

$$\boxslash(f : X \to Y) = \mathrm{Id}_0 \tag{10.19}$$

while the unit monad maps any object of the category to its terminal element,

$$\circledast(X) = 1 \tag{10.20}$$

and any morphism to the identity on the terminal object

$$\circledast(f : X \to Y) = \mathrm{Id}_1 \tag{10.21}$$

Relation to $X \to 1$, where does Ω factor in? Is it the ”truth” diagram $X \to 1 \to \Omega$, with the semantics that this is the map of the subobject $X \hookrightarrow X$?

Something like $\vdash \forall X,\ X = X$?

[Stack semantics?]

The unit and empty modality also have a variety of alternative interpretations in terms of other adjunctions. For instance, given the dependent adjunction of the reader monad and coreader comonad, from the base change $f : X \to Y$

$$(\Diamond_f \dashv \Box_f) : \mathbf{H}_{/X} \underset{\prod_f}{\overset{\sum_f}{\begin{array}{c}\hookrightarrow \\ \xleftarrow{f^*} \\ \hookrightarrow\end{array}}} \mathbf{H}_{/Y}$$

the adjunction of those two monads will correspond to the base change $0_1 : 0 \to 1$, as we have $\mathbf{H}_{/1} \cong \mathbf{H}$ and $\mathbf{H}_{/0} \cong \mathbf{1}$. [corresponds to Ex falso quodlibet?]

As a duality of hom and tensor product

In terms of localization, the unit modality is the case of the localization by every morphism of the topos, so that

$$\circledast \cong \mathrm{loc}_{\mathrm{Mor}(\mathbf{H})} \tag{10.22}$$

ie every morphism becomes an equivalence so that every two objects are isomorphic.

Likewise, the empty modality is the colocalization [...]

This notion can be formalized internally by the terminal object, as this is the only object for which any other object has a single relation to.

Now if we would like to look at the properties of this being, we can

In the Hegelian sense : $\circledast$ maps every object of $\mathbf{H}$ to a single object (they all share the same characteristic of existence, "pure being", and there is nothing differentiating them in that respect, no further qualities). The image of any two objects under this modality are identical, as there are no other characteristics to differentiate them, and the only relation they can have is that of the identity. This is also true of the empty object []

"In its indeterminate immediacy it is equal only to itself. It is also not unequal relatively to an other; it has no diversity within itself nor any with a reference outwards"

Conversely, $\boxslash$ maps every object to nothing, the opposition of being.

The unit of the monad and counit of the comonad are given in terms of components by the typical morphisms of the terminal and initial object,

$$\epsilon_X : \quad X \to \circledast X \cong 1 \tag{10.23}$$

$$\eta_X : 0 \cong \boxslash X \to X \tag{10.24}$$

Both of those adjoint functors roughly reflect the fact that each has to map elements to a single element and morphisms between that element and every other element to a single morphism.

The composition of the unit and counit give us the *unity of opposites* for being and nothingness

$$0 \to X \to 1 \tag{10.25}$$

"there is nothing which is not an intermediate state between being and nothing."

An alternative interpretations of this modality is given by the opposition of the dependent sum and dependent product on the empty context

$$\sum_{\varnothing}(-) \vdash \prod_{\varnothing}(-) \tag{10.26}$$

Cartesian product v. internal home adjunction of the unit type

$$((-) \times \varnothing) \dashv (\varnothing \to (-)) \tag{10.27}$$

Negation in categories : internal hom to the initial object : $\neg = [-, \varnothing]$

Examples of those two modalities on a topos will not give us very different results overall, as they all roughly have the same behaviour. For **Set** for instance,

$$\forall A \in \mathrm{Obj}(\mathbf{Set}),\ \boxed{\varnothing}\,(A) = \varnothing \tag{10.28}$$

$$\forall f \in \mathrm{Mor}(\mathbf{Set}),\ \boxed{\varnothing}\,(f) = \mathrm{Id}_{\varnothing} \tag{10.29}$$

$$\forall A \in \mathrm{Obj}(\mathbf{Set}),\ \circledast\, A = \{\bullet\} \tag{10.30}$$

$$\forall f \in \mathrm{Mor}(\mathbf{Set}),\ \circledast\,(f) = \mathrm{Id}_{\{\bullet\}} \tag{10.31}$$

[...]

As the examples we have given thus far do not have particularly varied definitions for the initial and terminal object (being mostly concrete categories where $I = \varnothing$ and $T = \{\bullet\}$, the modality of being and nothingness do not offer particularly more insight in those topos. Smooth sets simply get mapped to the empty space and the one point space, etc

"In geometric language these are categories equipped with a notion of discrete objects and codiscrete objects."

As any further adjoint would have to be a new functor from **H** to **1**, they would simply be just the functor Δ_* again, so that the further adjoints would simply be the same monads again. There is therefore no further moments at this level.

$\circledast$ is monoidal :

$$\circledast(X) \times \circledast(Y) \cong 1 \times 1 \cong 1 \cong \circledast(X \times Y) \tag{10.32}$$

[...]

The types involved in the initial opposition are fairly easy to work through : For $\circledast$, we only have a single modal type, which is the terminal object 1, and for $\boxslash$, the only modal type is the terminal object, 0. The submodal types of $\circledast$ are the subterminal objects, the objects for which the unique morphism $X \to 1$ is a monomorphism. By topoi having strict initial objects, there is only one submodal type for $\boxslash$, which is the initial object itself.

Subterminal objects : serve to detect open sets?

$\circledast_X$ as a map to pointed objects?

Adjoint functions :

$$f : \boxslash X \to Y \tag{10.33}$$

$$\tilde{f} : X \to \circledast Y \tag{10.34}$$

Every morphism is $\circledast$-closed?

10.1.1 Negations

As the unit monad $\circledast$ is a right adjoint, it preserves limits, and therefore admits a negation with all the appropriate properties.

Theorem 10.1.1 *The negation of the monad of being is the identity monad :*

$$\overline{\circledast} = \mathrm{Id} \tag{10.35}$$

Proof 10.1.1 *The computation is simple enough, as*

$$\begin{aligned}
\overline{\circledast}(X) &= \mathrm{Fib}(X \to \bigcirc_* X) && (10.36)\\
&= \mathrm{Fib}(X \to 1) && (10.37)\\
&= \mathrm{Fib}(!_X) && (10.38)\\
&= X \times_* * && (10.39)\\
&= X && (10.40)
\end{aligned}$$

This is independent of the choice of basepoint, so that the negation of being is therefore the identity.

This indeed obeys $\overline{\circledast}\circledast = \circledast\overline{\circledast} = \circledast$.

This relates well enough to the interpretation of "the determinate negation containing the part of the structure that is trivialized by the unit $X \to \bigcirc X$", as the unit monad removes all structure from the object, the trivialized part is all of it. Being the identity functor, its further adjoints are all itself, so that nothing further of interest can be gotten.

In terms of pointed objects, this simply transforms any pointed object into its equivalent non-pointed object, ie it is the forgetful functor between the pointed category $\mathbf{H}^{1/}$ and the topos $\mathbf{H}$.

$$\overline{\circledast} : \mathbf{H}^{1/} \to \mathbf{H} \tag{10.41}$$

On the other hand, the modality of nothingness's cofibration does not give us exactly a determinate negation :

Theorem 10.1.2 *The negation of the comonad of nothingness is the maybe monad :*

$$\overline{\boxslash} = \text{Maybe} \tag{10.42}$$

Proof 10.1.2

$$\begin{aligned} \overline{\boxslash} &= \text{Cofib}(\Box_{\varnothing} X \to X) & (10.43)\\ &= X +_{\Box_{\varnothing} X} 1 & (10.44)\\ &= X +_{\varnothing} 1 & (10.45)\\ &= X + 1 & (10.46) \end{aligned}$$

While the determinate [?] negation is well-defined, it is not an idempotent monad (since the components of its multiplication map is not an isomorphism except in the degenerate case where we have the initial topos, as $\mu_0 : 2 \to 1$ is clearly not), as it simply adds a new element to any object.

$$\text{Maybe}^2 X = (\text{Maybe} X) \sqcup \{\bullet\} = X \sqcup \{\bullet_1, \bullet_2\} \tag{10.47}$$

$$\overline{\boxslash}\,\boxslash\, X = 1 \tag{10.48}$$

$\Box_{\varnothing}\overline{\boxslash} = 0$

One way it can be understood is that there cannot be any property that we remove from an object to make it into the unit type, as there is no object to simplify here, therefore the only way to get us to a unit type is to add it.

As the maybe monad adds a new copy of the terminal object, it will clearly not preserve either the initial or terminal object,

$$\overline{\boxed{\varnothing}}0 \cong 1 \tag{10.49}$$
$$\overline{\boxed{\varnothing}}1 \cong 2 \tag{10.50}$$

Unless we are in the initial topos, so that it does not preserve either limits or colimits, meaning that it has no further adjoints.

There is little point to the de Rham modalities here [?], but they are as follow :

$$\tilde{\circledast}X = \mathrm{Cofib}(X \to 1) \tag{10.51}$$
$$= X +_1 1 \tag{10.52}$$
$$= \mathrm{coeq}(X + 1 \rightrightarrows 1) \tag{10.53}$$

[$0 \times X \cong X$ because strict initial object]

$$\tilde{\boxed{\varnothing}}X = \mathrm{Fib}_p(0 \to X) \tag{10.54}$$
$$= 0 \times_X 1 \tag{10.55}$$
$$= \mathrm{eq}(0 \times 1 \rightrightarrows X) \tag{10.56}$$
$$= \mathrm{eq}(0 \rightrightarrows X) \tag{10.57}$$

As this is a strict initial object, only itself can map into it, so that the equalizer must be 0, so that the de Rham modality for the empty modality is simply itself.

Decomposition using *ps*-hexagon?

10.1.2 Algebra

As a monad, we can try to give the unit monad an associated algebra. For a given element X, and a morphism $x : \circledast X \cong 1 \to X$, ie a point of X, we need to have the commutation

$$\eta_X \circ x = \mathrm{Id}_X \tag{10.58}$$

Unfortunately, as η_X is the unique map to the terminal object, it cannot be true unless X is itself the terminal object [proof]. So our only possible algebra will be $(1, \mathrm{Id}_1)$, as we'd expect since our Eilenberg-Moore category is the terminal category.

Free algebra : algebra on 1 with morphism $\mathrm{Id}_1 : 1 \to 1$, trivial algebra

Coalgebra of the empty comonad $\boxed{\varnothing}$: given X and a morphism $f : X \to \boxed{\varnothing}X \cong 0$

If there is no such map : empty coalgebra? Except on $\varnothing$, the cofree coalgebra, which is the coalgebra with

10.1.3 Logic

In terms of logic, those two monads will correspond to modalities sending each object (as types) to either the unit type or the empty type. As a modal type theory, we are therefore simply sending the logic of our ambient topos to that of the initial category. So first let's look at its internal logic.

The initial category $\mathrm{Sh}(\varnothing) \cong \mathbf{1}$ has a rather barebone structure as a logical system. Its subobject classifier, initial and terminal object are all the same object, $*$, meaning that both the truth and falsehood morphisms are also the unique morphism Id_*, and the only subobject category is $\mathrm{Sub}(*)$, which is simply $*$ itself (*both* as the object itself and the terminal object). Trivially, any map factors through the terminal object and its truth map, meaning that any proposition is true in there. The unique proposition is simply the one defined by the subobject $* \hookrightarrow *$, which due to the rather collapsed logic could be either the truth or falsehood morphism.

$$\vdash \top \tag{10.59}$$

Likewise, any limit and colimit in the poset of subobjects will still be $*$, making any operation on propositions still "true" (or false, depending on the viewpoint). This is the trivial logic, in which any proposition is true.

In terms of the ambient topos $\mathbf{H}$, any proposition $p : X \to \Omega$ is therefore mapped to this proposition,

$$\Delta_*(p : X \to \Omega) = \top/\bot : * \to * \tag{10.60}$$

And likewise, any negation is given by the same formula

$$\Delta_*(\neg p : X \to \Omega) = \top/\bot : * \to * \tag{10.61}$$

Going back to the ambient category, this gives us, depending on the modality,

$$\boxslash(p : X \to \Omega) = 0 \hookrightarrow 0 \tag{10.62}$$
$$\circledast(p : X \to \Omega) = 1 \hookrightarrow 1 \tag{10.63}$$

The first modality maps any proposition to the $0 \hookrightarrow 0$ proposition, which is the "trivially true" proposition, both false and true but with no subobject that could actually be used to test it. The other is simply the true proposition $\top$.

Interpretation of the unit $X \to 1$ as a proposition

10.1.4 Interpretation

If we try to apply Hegel's notions onto all this, this is the property of every object of our topos simply being an object, a property shared by all objects. Similarly, in the equivalence in type theory, this is just given by the type judgement : any object X is a type.

$$\vdash X : \mathrm{Type} \tag{10.64}$$

[subobject classifier as a type of type?]

[Logical equivalence?]

From the notion that the being is somewhat related to the belonging to the type universe, logically this corresponds to its relation with the subobject classifier?

[...]

If we start the construction of the system with Being, we have merely some unknown topos $\mathbf{H}$, our domain of ideas, along with a subtopos $\mathbf{1}$, which is the moment of being for those ideas. All objects of our unknown topos will only manifest at one there, and therefore cannot be differentiated. If we were to assume that we were done, our topos would therefore be $\mathbf{1}$ itself. Due to the collapse of the logic to the trivial logic however, there is no difference between an object as

If we try to consider $\mathbf{1}$

What is the "nothingness" of this idea of being?

Stack semantics? If we pick our first undifferentiated universe of discourse $\mathbf{1}$, the sheaf over this category is $\mathrm{Sh}(\mathbf{1}) \cong \mathbf{Set}$, and its fundamental fibration is

$$x \tag{10.65}$$

Kripke-Joyal semantics : "Object has property P" : predicate P, "Object does not have property P"

The subobject classifier of the initial topos is $*$ itself, with $\mathrm{Sub}(*) \cong \mathbf{1} \cong \mathrm{Hom}(*, *)$. Its negation is given by the terminal object, also $*$, into the subobject classifier, therefore the negation of any property is also the property.

10.2 Necessity and possibility

Before looking further into the "standard" sublation of the ground opposition, let's briefly look at another direction to generalize.

The interpretation of being and nothingness as the duality between the dependent product and sum on the empty context gives us a possibility of generalization in this direction, in which we simply generalize to an arbitrary context.

As we've seen before, the context Γ of the internal logic corresponds to the slice category $\mathbf{C}_\Gamma$. If we wish to change our context, this is done via a display morphism $f : X \to Y$ which induces the functor

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{10.66}$$

The ground that we've seen is done on the empty context, which is given by the terminal object 0. The corresponding context is that of the display morphism $!_0 : 0 \to 1$, changing the context from 0, falsity, to 1, truth. The corresponding contexts are $\mathbf{C}_1 \cong \mathbf{C}$ and $\mathbf{C}_0 = \mathbf{1}$, giving the base change functor

$$f^* : \mathbf{C} \to \mathbf{1} \tag{10.67}$$

which is exactly the functor that we used as its basis.

The interpretation in this sense is therefore that $\circledast$ is adding the context

For a morphism $f : X \to 1$ (what context is that?) :

$$f^* : \mathbf{C} \to \mathbf{C}_{/X} \tag{10.68}$$

Adjoints : $(f_! \dashv f^* \dashv f_*)$

$$(\textstyle\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}_{/X} \underset{\sum_f}{\overset{\prod_f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow f^* \longrightarrow}}}} \mathbf{H}_{/Y}$$

for $f : X \to 1$, we have $\mathbf{H}_{/Y} \cong \mathbf{H}$

[...]

Theorem 10.2.1 $\prod_f$ *is a right adjoint and therefore preserves limits, in particular the terminal object :*

$$\prod_f 1 = 1 \tag{10.69}$$

Theorem 10.2.2 $\sum_f$ *is a left adjoint and therefore preserves colimits, in particular the initial object :*

$$\sum_f 0 = 0 \tag{10.70}$$

From this, we can see that each of those adjunctions is a sublation of the ground.

Base change f^* preserves limits and colimits because toposes are stable under pullbacks?

so $f_* f^* = \prod_f f^* = \Box_f$ is such that

$$\Box_f 0 = \prod_f 0 \tag{10.71}$$

Adjoint modality : $(f_! f^* \dashv f_* f^*)$

Writer comonad and reader monad

Possibility comonad and necessity monad $(\Box \dashv \Diamond)$

[...]

The interpretation of this modality in terms of standard modal logic (the modality of necessity and possibility) can be understood using the Kripke semantics of modal logic.

interpretation of the ground in this context : $\bigcirc_*$ sends every proposition to true and $\Box_\varnothing$ to false. Every proposition is *possibly* true and none are *necessarily* true.

10.3 Cohesion

[120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132]

geometry

cohesion in rome

Differential Cohesive Type Theory (Extended Abstract)

Precohesive Toposes over Arbitrary Base Toposes “Quantity is the unity of these moments of continuity and discreteness”

Qualität/Quality, Etwas/Something, Die Endlichket/Finitude, Etwas und ein Anderes/Something and another

10.3.1 Continuity

There are other directions in which to sublate the initial opposition $(\boxslash \dashv \circledast)$ to find higher ones. To find some resolution of $(\boxslash \dashv \circledast)$ to some higher adjunction $(\boxslash' \dashv \circledast')$, we need some adjunction obeying the property

$$\boxslash'\boxslash = \boxslash \quad (10.72)$$
$$\circledast'\circledast = \circledast \quad (10.73)$$

In other words, each of these preserve the initial and terminal object

$$\boxslash' 0 = 0 \quad (10.74)$$
$$\circledast' 1 = 1 \quad (10.75)$$

We will look here more specifically at a *right sublation*, with the additional property

$$\circledast'\boxslash \cong \boxslash \quad (10.76)$$

From the properties of the empty comonad, this simply means that the sublated monad preserves the terminal product :

$$\circledast' 0 \cong 0 \quad (10.77)$$

[Is there a possible left sublation?

$$\boxslash'\circledast \cong \circledast \quad (10.78)$$

Left sublation must preserve the unit monad

$$\boxslash' 1 \cong 1 \quad (10.79)$$

This is true of $\sharp$ but is it the first possible left sublation?]

which is the exact property of $\mathbf{H}_{\circledast'}$ being a dense subtopos. Fortunately there is a natural choice for this, given by this theorem

Theorem 10.3.1 *The smallest dense subtopos of a topos is that of local types with respect to double negation* $\mathrm{loc}_{\neg\neg}$*. (Johnstone 02, corollary A4.5.20)*

Proof 10.3.1

From this, we have that the natural sublation of the ground opposition can be constructed from the localization by the double negation. This is called the *sharp modality*, denoted by $\sharp$.

$$\circledast' = \sharp = \mathrm{loc}_{\neg\neg} \tag{10.80}$$

To understand the role of the sharp modality here, let's consider why a double negation is not typically already localized. This is true in any boolean topos (by definition), including **Set**, but if we look at any topos with a more topological aspect to it, such as **Smooth** or **sSet**, we will see some issues with it.

Consider some topological space X (or its closest equivalent in some topological topos) and a subspace $\iota : U \hookrightarrow X$. We can try to consider its negation in X by using the naive complement in X, simply picking its complement as a set, with the appropriate subspace topology on this complement. In this case, we do have the law of excluded middle we'd expect from a boolean topos :

$$U + \neg U \cong X \tag{10.81}$$

Implying that there is some isomorphism between the two, but this is not the case. There is generally no continuous map

$$U + \neg U \to X \tag{10.82}$$

as can be seen by looking at this example. Let's take U some open subset of X and U^c its (point-wise) complement, in the case of U^c containing a non-empty boundary ∂U. For instance, $X = (0,1)^2$ and U some open disk

[diagram]

Since those two objects are meant to be homeomorphic, the image of any open set in one is an open set in the other. For any neighbourhood V of X containing a point of the boundary ∂U, by the properties of a boundary point, this neighbourhood will overlap both subspaces. For continuity, we should have an open preimage, but the preimage in U^c will contain the boundary ∂U which overlaps with the preimage of V, and therefore fails to be open in the subspace topology.

So if we look at a topological space in logical terms, its "negation" cannot merely be the point-wise complement, but only the pseudo-complement given by

$$\neg U = \mathrm{Int}(X \setminus U) \tag{10.83}$$

While the preimage of an open set does end up in an open set [prove], this means that the law of excluded middle does not hold, since the disjoint union of an open subspace and its complement will not even contain every point of X, and on a more spatial level, it will also lack every open set straddling both subspaces, making it disconnected along that missing boundary.

Likewise, the double negation (whose equivalence to the identity implies excluded middle, see) will not generally be an equivalence :

$$\neg\neg U = \mathrm{Int}(X \setminus \mathrm{Int}(X \setminus U)) \tag{10.84}$$

If we take a fairly simple case like a closed disk in the plane, its pseudo-complement will be the exterior of the disk minus the circle itself, and its double negation is the interior of this disk. Typically, any double negation will get rid of any boundary in this context.

As we can see, both from the law of excluded middle and the double negation, the issue is always due to the "spatial structure". If we try to consider a space X as merely the sum of its parts (via the coproduct), we are simply missing out on the spatial cohesion along the division of those part. This is understandable simply enough due to the fact that if we take some originally connected space and split it into multiple subspaces, the coproduct of any number of those subspaces will not be connected, but connectedness should be preserved by homeomorphisms. The open sets meant to connect those regions have gone missing in the process.

From the laws of Heyting algebras however, we do always have the inclusion

$$U \hookrightarrow \neg\neg U \tag{10.85}$$

In logical terms, the proposition $P : U \hookrightarrow X$ implies its double negation. A property that is true "continuously" in U will be true in $\neg\neg U$.

The localization of the double negation should tell us that in some sense we may be trying to get back to the naive negation where we consider the actual point-wise complement. As usual, this localization defines a reflective subcategory

$$(T \dashv \iota) : \mathbf{H}_{\neg\neg} \underset{\iota}{\overset{T}{\rightleftarrows}} \mathbf{H}$$

where the idempotent monad is the composition $\sharp = \iota \circ T$. To look a bit more at the behavior of the sharp modality, let's look at the Eilenberg-Moore category of our modality.

Theorem 10.3.2 *The subtopos by the localization of the double negation is boolean.*

Proof 10.3.2 *As the double negation is a local operator $\neg\neg : \Omega \to \Omega$, the sheaf topos $\mathrm{Sh}_{\neg\neg}(\mathbf{H})$ is boolean if $\neg\neg = q(U)$ for a subterminal object. [j sheaves, q is the quasi-closed local operator]*

$$\Omega \xrightarrow{\cong} \Omega \times 1 \xrightarrow{\mathrm{Id}_\Omega \times u \times u} \Omega \times \Omega \times \Omega \xrightarrow{\to \times \mathrm{Id}_\Omega} \Omega \times \Omega \xrightarrow{\to} \Omega \tag{10.86}$$

$u : U \to 1$, $q(0)$ *is double negation*

The subtopos $\mathbf{H}_\sharp$ is therefore a *Boolean topos*. It can be understood in terms of the existence of a complement for any subobject, where every object can be split exactly in two parts by a subobject, one part which contains every point of A, and another part which contains every other point. [etc etc]

As a boolean topos, $\mathbf{H}_\sharp$ is typically either the topos of sets **Set** or some variant thereof, such as some ETCS variant of it. We will generally assume that we are using **Set** here, but most of the properties used here should generalize to any boolean topos.

There are some obvious counter examples, such as taking the initial topos **1**, which is its own double negation subtopos, [This? ref]

If we do actually pick **Set** to be our underlying Eilenberg-Moore category, the most natural choice for the adjoint functors of our monad $\sharp$ is given by the global section functor,

$$\Gamma : \mathbf{H} \to \mathbf{Set} \tag{10.87}$$
$$X \mapsto \mathrm{Hom}_{\mathbf{H}}(1, X) \tag{10.88}$$

Theorem 10.3.3 *The global section functor is a localization by the double negation.*

Proof 10.3.3 *As we are localizing the double negation morphism $\neg\neg : \Omega \to \Omega$, we have that $\neg\neg \in W$,*

Given the double negation morphism for some subobject $\iota : U \hookrightarrow X$,

$$\neg\neg U \to U \tag{10.89}$$

the global section functor gives us

$$\Gamma(\neg\neg U) \to \Gamma(U) \tag{10.90}$$

If this functor has a right adjoint, we will call the codiscrete functor, CoDisc.

Definition 10.3.1 *A* locally local topos *is a topos for which the global section functor admits a right adjoint.*

$$(\Gamma \dashv \mathrm{Codisc}) : \mathbf{Set} \underset{\mathrm{Codisc}}{\overset{\Gamma}{\rightleftarrows}} \mathbf{H}$$

The adjunction tells us that

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(X), S) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{CoDisc}(S)) \tag{10.91}$$

The codiscrete functor will therefore transform any set S into a space for which the hom-set from any other object will be isomorphic to that of the set of functions between X and S. This is a property that is similar to the trivial topology : any function on the points of the underlying set of that space lifts to a "continuous" function in the topos.

On a more "topological" description, we can look at the coverage of $\sharp X$. As an adjunction of geometric morphisms, $(\Gamma \dashv \mathrm{CoDisc}) \cong (f^* \dashv f_*)$, with respect to the terminal topos $\mathbf{Set} \cong \mathrm{Sh}(\mathbf{1})$,

Embedding geometric morphism

For any subobject $[U] \hookrightarrow \mathrm{CoDisc}(X)$, or in subobject classifier term, $\chi_U : X \to \Omega$, the closure map is given by

$$\chi_{\overline{U}} = j \circ \chi_U \tag{10.92}$$

Show that $\mathrm{CoDisc}(X)$ is a $\neg\neg$-sheaf of $\mathbf{H}$,

Monomorphism $U \hookrightarrow X$ is dense if $\overline{U} = X$, ie $\chi_{\overline{U}} \cong \top \circ !_X$.

$$\chi_{\overline{U}} \quad = \quad \neg \circ \neg \circ \chi_U \tag{10.93}$$

[As a j sheaf all images are dense in X and therefore]

[Compute the Grothendieck topology]

In other words, every non-empty subobject of $\mathrm{CoDisc}(X)$ is dense, which is another characteristic of the codiscrete topology : $\overline{U} = X$. In particular, if $\mathbf{H}$ is a Grothendieck topos, its Grothendieck topology for $\sharp X$ is the trivial one, with simply $\{0 \to \sharp X, \sharp X \to \sharp X\}$(?)

Theorem 10.3.4 *A codiscrete space has the same point content as the set it is generated from*

$$\Gamma \circ \mathrm{Codisc} \circ \Gamma \cong \Gamma \tag{10.94}$$

Proof 10.3.4 *Triangle identity*

Our sharp modality therefore simply maps a space X to its set of points $\Gamma(X)$, before mapping it back to the codiscrete space composed of these points. As the type of space with the strongest topology, $\sharp X$ does not obey the law of excluded middle or have a double negation isomorphic to the identity, and in fact in some sense is maximally in violation of these. In terms of our motivating example with trivial topologies, we have (for $U \neq \varnothing$)

$$\neg\neg U = \mathrm{Int}(X \setminus \mathrm{Int}(X \setminus U)) \quad (10.95)$$
$$= \mathrm{Int}(X \setminus \varnothing) \quad (10.96)$$
$$= X \quad (10.97)$$
$$(10.98)$$

and

$$U \vee \neg U = U \cup \mathrm{Int}(X \setminus U) \quad (10.99)$$
$$= U \quad (10.100)$$

Given some (strict) subobject $U \hookrightarrow \sharp X$

$$\neg\neg U = 0 \quad (10.101)$$

[$\neg\neg X = X$ otoh]

and $U \vee \neg U = U$. It is in some sense the worst at reconstructing the initial space by negation of subobjects.

Measuring the violation of the excluded middle : for any X with the canonical embedding $X \hookrightarrow \neg_X \neg_X X$, how

Therefore, under the sharp modality, if we split our object into subobjects, there will be no loss of "spatial" information, as would be the typical case from what we expect.

Property under the local operator?

From this subtopos, we can determine an important property of the sharp modality : if we have take any two spaces $X, Y \in \mathbf{H}$ and look at their underlying sets, then any morphism between those sets

$$f : \Gamma(X) \to \Gamma(Y) \quad (10.102)$$

can be lifted to a continuous function $X \to \sharp Y$. Via adjointness,

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(X), \Gamma(Y)) \cong \mathrm{Hom}_{\mathbf{C}}(X, \mathrm{CoDisc} \circ \Gamma(Y)) \quad (10.103)$$

This property marks it as the sharp modality being similar to the trivial topology. In particular, any morphism $S \to \sharp X$ that does not have a corresponding factoring through X via $S \to X \to \sharp X$ corresponds to a discontinuous function to X.

Sublation by $\sharp$ always exists for any topos?

Theorem 10.3.5 *A topos* $\mathbf{H}$ *is boolean if and only if it only has a single dense subtopos,* $\mathbf{H}_{\neg\neg}$*, for which* $\mathbf{H}_{\neg\neg} \cong \mathbf{H}$.

Properties of $\sharp$:

As a reflector preserving all limits (by being right adjoint to $\flat$), $\sharp$ defines a Lawvere-Tierney topology. That is, for any $X \in \mathbf{H}$, we have some closure operator

$$j_\sharp : \Omega \xrightarrow{\chi_{\sharp\top} \circ \sharp} \Omega \tag{10.104}$$

For the truth map $\top : 1 \to \Omega$, the sharp truth map $\sharp\top : 1 \to \sharp\Omega$

This is the Lawvere-Tierney topology defined by the constant truth value morphism, ie

$$x \tag{10.105}$$

Proof that this is the codiscrete one

[...]

Relation with the internal hom :

Theorem 10.3.6 *The internal hom of two objects has the global section of the hom-set :*

$$\Gamma([X,Y]) = \mathrm{Hom}_{\mathbf{H}}(X,Y) \tag{10.106}$$

Proof 10.3.5

$$\begin{aligned} \Gamma([X,Y]) &= \mathrm{Hom}_{\mathbf{H}}(1,[X,Y]) && (10.107)\\ &= \mathrm{Hom}_{\mathbf{H}}(1\times X,Y) && (10.108)\\ &= \mathrm{Hom}_{\mathbf{H}}(X,Y) && (10.109) \end{aligned}$$

Locally local topos has a NNO?

Sharp on bundles : $\sharp_X$?

10.3.2 Discreteness

As we've seen, the natural (right) sublation of the ground is given by the localization by the double negation, $\mathrm{loc}_{\neg\neg}$, the sharp modality $\sharp$. To get the full sublation of the ground adjunction, we will need also the existence of an adjoint modality, called the *flat modality* $\flat$.

We can decompose it as some adjoint functor pair, with the right functor simply being Γ. The left adjoint to the global section functor we will call the *discrete* functor Disc, which give us the adjoint cylinder

$$(\mathrm{Disc} \dashv \Gamma) : \mathbf{H}_\sharp \underset{\mathrm{Disc}}{\overset{\Gamma}{\rightleftarrows}} \mathbf{H}$$

We will ask that this adjunction be a geometric morphism, so that Disc also preserves all finite limits.

As $\sharp$ can be understood as the space generated from the set of its points with a trivial topology, $\flat$ is the space generated from the set of its points with a discrete topology, ie as a topos it is entirely determined by its points, and is simply the coproduct of every point.

$$\flat X \cong \coprod_{s:\Gamma(X)} 1 \tag{10.110}$$

This stems from the properties of the double negation topos, for which we have the property that

$$\forall S \in \mathbf{H}_\sharp,\ S \cong \coprod_{x \in \mathrm{Hom}(1,S)} 1 \tag{10.111}$$

Since for any point $p \in \mathrm{Hom}(1, S)$, we can decompose its object with the law of excluded middle as

$$p + \neg_S p = S \tag{10.112}$$

which leads by induction[?] to the identity[What if the remaining object $\neg^n S$ is pointless but not initial?]. We can therefore get

Theorem 10.3.7 *If the base topos' objects are coproducts indexed by their own hom-sets,*

$$S \cong \coprod_{x \in \mathrm{Hom}(1,S)} 1 \tag{10.113}$$

Then the left adjoint of the global section functor is the functor of locally constant sheaves LConst

$$\mathrm{LConst}(S) = \coprod_{s \in S} 1 \tag{10.114}$$

Proof 10.3.6 *As left adjoints preserve limits,*

$$\begin{aligned} \mathrm{LConst}(S) &= \mathrm{LConst}(\coprod_{s \in S} 1) && (10.115) \\ &= \coprod_{s \in S} \mathrm{LConst}(1) && (10.116) \end{aligned}$$

And since we assumed that finite limits are preserved,

$$\mathrm{LConst}(S) = \coprod_{s \in S} 1 \tag{10.117}$$

As we are in a topos, the category is extensive, so that this coproduct of terminal objects is just a collection of disjoint points, giving us the intuition of this modality sending an object to its equivalent discrete topology.

Dually to the codiscrete case, we also have that discrete objects obey the universal property that any function from a discrete space is continuous, ie that

$$\mathrm{Hom}_{\mathbf{H}}(\mathrm{Disc}(S), X) \cong \mathrm{Hom}_{\mathbf{H}_\sharp}(S, \Gamma(X)) \tag{10.118}$$

Topos localized by $\neg\neg$, the new topos is $H_\sharp$. The opposition $\sharp \dashv \flat$ is the ground topos of $H_\sharp$

the law of excluded middle for the flat case?

$$\flat(U + \neg_X U) \tag{10.119}$$

Definition 10.3.2 *A topos that admits a geometric morphism* $(\flat \dashv \sharp)$ *is called a* local topos

[133, 134]

10.3.3 Quality

If the flat modality admits a further adjunction, we will call its left adjoint the *shape modality*, $\int$. This will allow us to define the full notion of cohesion properly by adding some appropriate requirements to it. The shape modality roughly corresponds to the topological notion of connected objects, so that first we need to define what it means for an object to be connected in a topos. If we take a connected object to be the lack of decomposition into the disjoint sum of other objects, as they are for topological spaces, we will show that this definition gives out that result :

Definition 10.3.3 *An object X in a topos $\mathbf{H}$ is* connected *if* $\mathrm{Hom}_{\mathbf{H}}(X, -)$ *preserves finite coproducts.*

[connection to the definition of extensive category?]

Theorem 10.3.8 *If an object is connected, it cannot be expressed as the coproduct of more than one non-empty subobject.*

Proof 10.3.7

Definition 10.3.4 *A topos is* locally connected *if every object is the coproduct of connected objects* $(X_i)_{i\in I}$ *:*

$$X \cong \coprod_{i\in I} X_i \tag{10.120}$$

Theorem 10.3.9 *The index set I is unique up to isomorphism*

Proof 10.3.8

Definition 10.3.5 *The connected component functor maps objects of a locally connected topos to its index set of connected objects :*

$$\Pi_0(X) \cong \Pi_0(\coprod_{i\in I} X_i) \cong I \tag{10.121}$$

Theorem 10.3.10 *The connected component functor is indeed a functor*

Proof 10.3.9

Theorem 10.3.11 *The connected component functor is left adjoint to the discrete object functor.*

Proof 10.3.10 *As a discrete object is given by*

$$\mathrm{Disc}(S) \cong \coprod_{s\in S} 1 \tag{10.122}$$

with $\Pi_0(1) \cong 1$ *(as from its universal property, it cannot have more than one coprojection), we have therefore that*

$$\Pi_0(\mathrm{Disc}(S)) \cong S \tag{10.123}$$

As an adjoint triple of modalities, we therefore have two pairs of units and counits. First is the pair for $(\flat \dashv \sharp)$, a *ps*-opposition, which are componentswise

$$\epsilon^\flat_X : \flat X \rightarrow X \tag{10.124}$$

$$\eta^\sharp_X : X \rightarrow \sharp X \tag{10.125}$$

which give us relations of objects and their discrete and codiscrete equivalents, and the pair for $(\int \dashv \flat)$

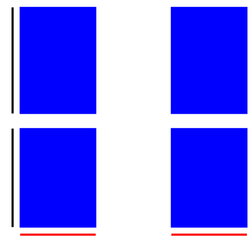


Figure 10.1: Example of a product of two spaces with two connected components

$$\epsilon^{\flat}_X : \flat X \quad \to \quad X \tag{10.126}$$
$$\eta^{\int}_X : X \quad \to \quad \int X \tag{10.127}$$

which deal with the relations of objects, their points and their connected components, which is an *sp*-opposition.

As an *sp*-opposition, both $\int$ and $\flat$ have the same modal type, which are the *discrete objects*, similarly to spaces of discrete topologies, with opposite ways to project on it.

Those are called respectively the *pieces-to-points* transformations and the *points-to-pieces* transformation, which are meant to signify that the first one deals with how the counit maps connected components of the space to

For most of our cases here, the base topos will be consider will be the topos of sets, **Set**, so that it is best to understand cohesion in terms of functors to sets. The archetypical cohesion is done using the global section functor Γ :

10.3.4 Cohesiveness

The mere adjoint triple $(\int \dashv \flat \dashv \sharp)$ does not fully define what we wish to have for the notion of cohesiveness, as many properties that would seem intuitively important for a spatial object may still be lacking. For instance, we would typically imagine that the connected components of a space obey basic algebraic rules. Take some space $(I + I)$, two intervals, and take its product space $(I + I) \times (I + I)$. From the preservation of the coproduct, if we have $\int I \cong 1$, we have $\int(I + I) \cong 1 + 1$, a space with two components, and we would expect the product of spaces to translate to a product of those components,

$$\int((I + I) \times (I + I)) = 4 \tag{10.128}$$

In particular, we want the conservation of the terminal object, the empty product, as if we are meant to interpret $\int$ as the connected components of an object, we would also like that $\int 1 \cong 1$. While guaranteed by

Counterexample 10.3.1 *The topos of G-sets is connected but not strongly connected.*

Proof 10.3.11 *The category of G-sets is given by the objects of sets X with a group action $G \times X \to X$ Forgetful functor :*

$$\Gamma(X, G) = X \tag{10.129}$$

Also while we typically try to admit spaces more broadly than what is defined by point set topology, we will ask that there is a minimum of cohesion : every connected component of a space should have at least one point. This is given by the condition that the point to piece transform is an epimorphism :

$$\flat X \to X \to \textstyle\int X \tag{10.130}$$

Definition 10.3.6 *A topos* **H** *is* cohesive *over a base topos* **B** *if it is equipped with the geometric morphisms*

$$(f^* \dashv f_*) : \mathbf{H} \underset{f_* \longrightarrow}{\overset{\longleftarrow f^*}{\rightleftarrows}} \mathbf{B}$$

and obeys the following properties :

- *It is a locally connected topos : there is a further left adjoint $(f_! \dashv f^*)$ and every object is a coproduct of connected objects.*
- *It is connected : $f_!$ preserves the terminal object*
- *It is strongly connected : $f_!$ preserves finite products.*
- *It is local : there is a further right adjoint $(f_* \dashv f^!)$*

Together these form the adjoint string that we have seen, which in the specific notation of cohesion gives us (for $\mathbf{B} \cong \mathbf{Set}$)

$$(\Pi_0 \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc}) : \mathbf{H} \begin{array}{c} \xrightarrow{\Pi_0} \\ \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\mathrm{Codisc}} \end{array} \mathbf{Set}$$

giving us the corresponding adjoint triple of cohesive modalities,

$$(\textstyle\int \dashv \flat \dashv \sharp) \tag{10.131}$$

$$\textstyle\int = \Gamma \circ \Pi_0 \tag{10.132}$$

$$\flat = \Gamma \circ \mathrm{Disc} \tag{10.133}$$

$$\sharp = \Gamma \circ \mathrm{Codisc} \tag{10.134}$$

In addition to this basic definition, it is common to ask for a few additional conditions.

Theorem 10.3.12 *If the terminal object* 1 *is connected,* $\mathrm{Hom}_{\mathbf{H}}(1, -)$ *preserves coproducts, then the point content of an object is given by any given subobject and its negation.*

$$\Gamma(A + \neg_X A) \cong \Gamma(X) \tag{10.135}$$

Proof 10.3.12 *From the preservation of coproducts, we have that*

$$\Gamma(A + \neg_X A) \cong \Gamma(A) + \Gamma(\neg_X A) \tag{10.136}$$

Preservation of pushout requires the geometric morphism to be surjective idk

Similarly $\flat(A + \neg_X A) \cong \flat(X)$?

Vague : $\flat$ maps to the points of X, and every point of $\int X$ is an image of one of those points. (point to pieces transform)

To give it its proper meaning of being about the connected components of

$$(\Pi_0 \dashv \mathrm{Disc}) : \mathbf{H} \xrightarrow{\Pi_0} \mathbf{Set} \xrightarrow{\mathrm{Disc}} \mathbf{H} \tag{10.137}$$

$$\mathrm{Hom}_{\mathbf{Set}}(\Pi_0(X), A) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{Disc}(A)) \tag{10.138}$$

The hom-set of functions from our space to the discrete space from a set A is isomorphic to the set of functions from $\Pi_0(X)$ to that set.

Interpretation : Π_0 send each element - subobject in the same "connected component" to a different point.

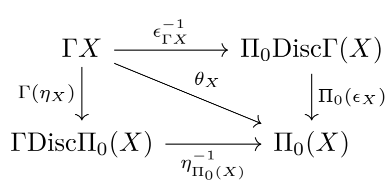
Connected object : X is a connected object if the hom-set functor preserves coproducts

Shape preserves connected objects?

Shape modality : $\int = \mathrm{Disc} \circ \Pi_0$. *sp*-unity, so $\int$ and $\flat$ share the same space : $\int X$ is a discrete space : $\flat \int = \int$

Commutating diagram for adjoint quadruples :

for $(\Pi_0 \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc})$, $X \in \mathbf{H}$ and $S \in \mathbf{Set}$,



and

$$\begin{array}{ccc} \mathrm{Disc}S & \xrightarrow{\eta \mathrm{Disc} S} & \mathrm{CoDisc}\Gamma\mathrm{Disc}(S) \\ {\scriptstyle \mathrm{Disc}(\epsilon_S^{-1})}\downarrow & \searrow^{\phi_S} & \downarrow{\scriptstyle \mathrm{CoDisc}(\eta_S^{-1})} \\ \mathrm{Disc}\Gamma\mathrm{CoDisc}(S) & \xrightarrow[???]{} & \mathrm{CoDisc}(S) \end{array}$$

Another intuitive property that we could ask of a topos is that its codiscrete objects are connected, that is

$$\int \sharp X \cong 1 \tag{10.139}$$

This is not generally true (**Set** is cohesive and does not obey it), not even for a sufficiently cohesive topos, but it is true in a category called codiscretely connected[126]

Definition 10.3.7 *A cohesive topos is* codiscretely connected *if for some set S, the unique map $\Pi_0 \mathrm{CoDisc} S \to \{\bullet\}$ is a monomorphism.*

This property is equivalent to it, as we can show that

Theorem 10.3.13 *A cohesive topos is codiscretely connected if and only if* $\int \mathrm{CoDisc}(2) \cong 1$

property

Theorem 10.3.14 *For any codiscrete object $\sharp X$, this object is connected and contractible :*

$$\int \sharp X \cong 1,\ \forall Y,\ \int (\sharp X)^Y \cong 1 \tag{10.140}$$

Proof 10.3.13 *Given a set S and its associated codiscrete object* $\overline{S} = \mathrm{CoDisc}(S)$,

[...]

Hierarchy :

Topos (every topos has a terminal geometric morphism with adjoint?), sheaf topos (geometric morphism?)

splitting : existence of a further left / right adjoint :

local topos (codisc adjoint), locally connected topos ()

essential topos? (???)

Theorem 10.3.15 *If a topos has a site with an initial object*

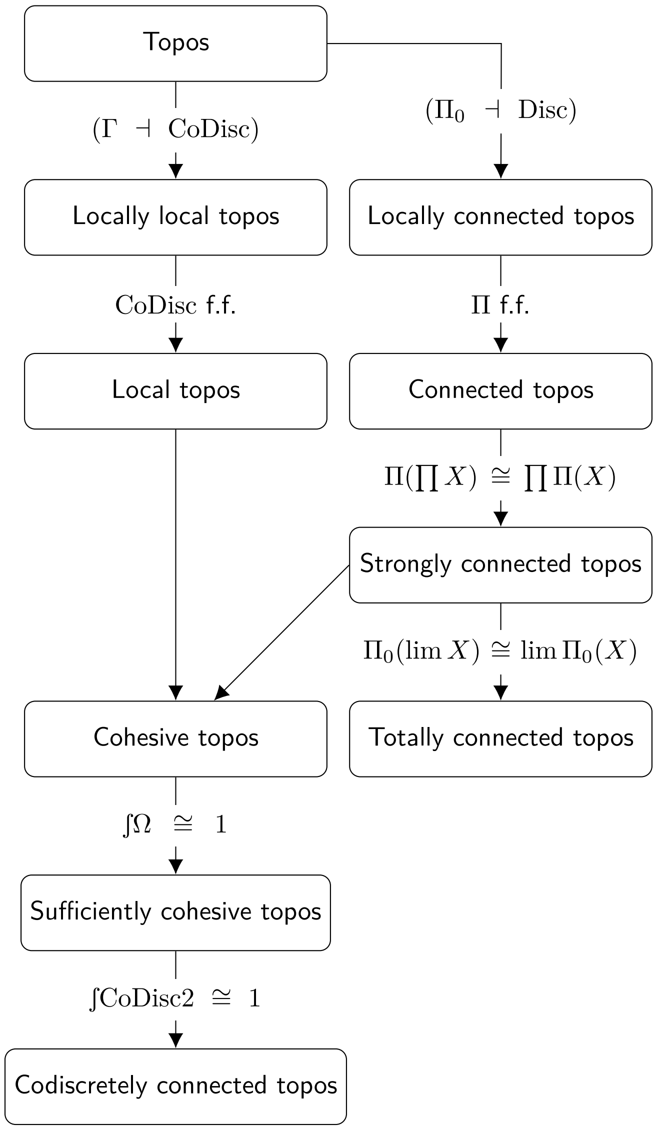


Figure 10.2: Hierarchy of cohesive topoi

Theorem 10.3.16 *If a topos has a site with an initial and terminal object, it is both locally local and locally connected.*

Proof 10.3.14

Cohesive site

Definition 10.3.8 *Given a cospan composed of a point $x : 1 \to X$ and the counit of the $(\flat \dashv \sharp)$ adjunction, $\epsilon^{\flat}_X : \flat X \to X$, the* de Rham flat modality $\flat_{\mathrm{dR}}$ *is given by the pullback*

Interpretation : $\bar{\flat}\flat = \Box_*$, ie the de Rham flat modality negates the flat modality. The de Rham flat modality of a discrete space is the terminal object.

For co-concrete objects, $\flat X \to X$ is an epimorphism

Nullstellensztz, the point to piece map is epi, equivalent to mono map from discrete to codiscrete spaces

Example 10.3.1 *As we have that for the real line, $\flat\mathbb{R} \cong |\mathbb{R}|$, the discrete space with the cardinality of $\mathbb{R}$, given some point $x : 1 \to \mathbb{R}$ and the inclusion map $|\mathbb{R}| \to \mathbb{R}$, its de Rham flat modality is given by the dependent product*

$$\bar{\flat}\mathbb{R} = |\mathbb{R}| \times_{\mathbb{R}} 1 = 1 \tag{10.141}$$

which does give us $\flat\bar{\flat}\mathbb{R} = 1$

The de Rham flat modality does not seem to be of much use, however it will reveal itself to be much more important later on for the case of higher category theory, and will earn its name of "de Rham" there.

Definition 10.3.9 *Given a morphism $f : X \to Y$, its $\int$-closure $c_{\int} f$ is given by the pullback*

$$c_{\int} f = Y \times_{\int Y} \int X \tag{10.142}$$

"Morphisms that are $\int$-closed may be identified with the total space projections of locally constant inf-stacks over Y"

Decomposition of morphisms into $\int$-equivalences and $\int$-closed

$$\tilde{X} = D^{\int_1}(X, 1) = \sum_{x:X} \eta(x) = \eta(1) \tag{10.143}$$

Theorem 10.3.17

$$\flat(\int X \to Y) \cong \flat(X \to \flat Y) \tag{10.144}$$

Proof 10.3.15

Adjuncts :

$$f : \textstyle\int(X) \to Y \tag{10.145}$$

$$\tilde{f} : X \to \flat(Y) \tag{10.146}$$

$$f : \flat(X) \to Y \tag{10.147}$$

$$\tilde{f} : X \to \sharp(Y) \tag{10.148}$$

10.3.5 Interpretation

As a sublation of the initial opposition, we are trying to find a larger domain of thinking for which we have a resolution of the opposition. Rather than the initial topos **1**, where the only object $*$ is both terminal and initial (in terms of internal logic, truth and falsehood are the same), we want some additional concepts which embody both of those notions as intermediaries. We need some *determinate* objects, whose properties differentiate them. In other words, given any property of the domain, we want to be able to define it as either *being* those properties or *not being* those properties, rather than the case we previously saw, where there is only one property

The sublation of the initial opposition corresponds here to the notion of *determinate being* :

The two basic notions involved with the initial adjoint ($\flat \dashv \sharp$) is to consider our objects as either collections of objects, or as a single whole. The unity of opposites involved therefore being the behaviour of objects with subobjects with respect to both those opposite notions.

[...]

When we take our initial opposition of being and nothingness, $0 \to X \to 1$, this is an opposition that has no information on our object. This is identically true

First, we need to look at what the notion of negation corresponds to here, which is that of *something and other*.

To consider the notion of objects having some kind of parts, we have first to assume the existence of more than one object.

“Something and other are, in the first place, both determinate beings or somethings.

Secondly, each is equally an other. It is immaterial which is first named and solely for that reason called something; (in Latin, when they both occur in a sentence, both are called aliud, or 'the one, the other', alius alium; when there is reciprocity the expression alter alterum is analogous). If of two things we call one A, and the other B, then in the first instance B is determined as the other. But A is just as much the other of B. Both are, in the same way, others. The word 'this' serves to fix the distinction and the something which is to be taken affirmatively. But 'this' clearly expresses that this distinguishing and signalising of the one something is a subjective designating falling outside the-something itself. The entire determinateness falls into this external pointing out; even the expression 'this' contains no distinction; each and every something is just as well a 'this' as it is also an other. By 'this' we mean to express something completely determined; it is overlooked that speech, as a work of the understanding, gives expression only to universals, except in the name of a single object; but the individual name is meaningless, in the sense that it does not express a universal, and for the same reason appears as something merely posited and arbitrary; just as proper names, too, can be arbitrarily assumed, given or also altered."

Those correspond to the notion of continuity and discreteness, as seen in many different discussions on the topic through history. This is the sort of notion we saw in the introduction 1, which we can also find in Iamblichus'

"The nature of the continuous and the discrete for all that is, that is to say for the whole structure of the cosmos, may be conceived in two ways: there is the discrete through juxtaposition and through piling up, and the continuous through unification and through conjunction.

In accordance with the essence of magnitude the cosmos would be conceived as one and would be called solid, spherical, and fused together, extended and conjoined; but, again, according to the form and concept of plurality the ordering, disposition, and joining together of the whole would be thought of as, we may say, being constructed of so many oppositions and similarities of elements, spheres, stars, kinds, animals, and plants. But in the case of the unified, division from the totality is without limit, while its increase is to a limited point; while conversely, in the case of plurality, increase is unlimited, but division limited."

Or Grassmann's Ausdehnungslehrer :

""Each particular existent brought to be by thought can come about in one of two ways, either through a simple act of generation or through a twofold act of placement and conjunction. That arising in the first way is the continuous form, or magnitude in the narrow sense, while that arising in the second way is the discrete or conjunctive form.

The simple act of becoming yields the continuous form. For the discrete form, that posited for conjunction is of course also produced by thought, but for the act of conjunction it appears as given; and the structure produced from the givens as the discrete form is a mere correlative thought. The concept of continuous becoming is more easily grasped if one first treats it by analogy with the more familiar discrete mode of emergence. Thus since in continuous generation what has already become is always retained in that correlative thought together with the newly emerging at the moment of its emergence, so by analogy one discerns in the concept of the continuous form a twofold act of placement and conjunction, but in this case the two are united in a single act, and thus proceed together as an indivisible unit. Thus, of the two parts of the conjunction (temporarily retaining this expression for the sake of the analogy), the one has already become, but the other newly emerges at the moment of conjunction itself, and thus is not already complete prior to conjunction. Both acts, placement and conjunction, are thus merged together so that conjunction cannot precede placement, nor is placement possible before conjunction. Or again, speaking in the sense appropriate for the continuous, that which newly emerges does so precisely upon that which has already become, and thus, in that moment of becoming itself, appears in its further course as growing there.

The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can also be regarded as continuous, and the continuous as discrete. The discrete may be regarded as continuous if that conjoined is itself again regarded as given, and the act of conjunction as a moment of becoming. And the continuous can be regarded as discrete if every moment of becoming is regarded as a mere conjunctive act, and that so conjoined as a given for the conjunction." "

and Hegel [...]

Any adjoint modality $\Box \dashv \bigcirc$ that includes the modalities $\varnothing \dashv *$, ie $\varnothing \subset \Box$, $* \subset \bigcirc$, formalizes a more determinate being (Dasein)

10.3.6 Types

From the adjoint modalities, we can define the various (co)modal types involved.

Definition 10.3.10 *An object in a cohesive topos is* concrete *if the unit of the adjunction* ($\Gamma \dashv$ CoDisc) *is a monomorphism, ie is a* $\sharp$-submodal type*, so that*

$$\eta^{\sharp}_X : X \hookrightarrow \sharp X \tag{10.149}$$

Definition 10.3.11 *An object in a cohesive topos is* codiscrete *if the unit of the adjunction* ($\Gamma \dashv$ CoDisc) *is an isomorphism, ie is a* $\sharp$-modal type*, so that*

$$X \cong \sharp X \tag{10.150}$$

This notion of concreteness can be understood in the sense that for an actual topological space, (X, τ), given a coarser topology (X, τ'), we have that the identity function (as sets) lifts to a continuous function on topological spaces, and is therefore an injective map.

Counterexample 10.3.2 *For the moduli space of* 1*-forms, as it has a single point, we have*

$$\sharp\Omega \cong 1 \tag{10.151}$$

If it were concrete, for any two morphisms $\omega_1, \omega_2 : \mathbb{R} \to \Omega$*, corresponding to two* 1*-forms on* $\mathbb{R}$*, we should have*

$$! \circ \omega_1 =! \circ \omega_2 \to \omega_1 = \omega_2 \tag{10.152}$$

which is always true for the terminal morphism, but as we have more than one 1*-form on* $\mathbb{R}$*, that cannot be true.*

As the name implies, a concrete object in a cohesive topos corresponds to a concrete sheaf :

Theorem 10.3.18 *The sharp object* $\sharp X$ *is a concrete sheaf.*

Proof 10.3.16 *Its underlying set is simply* $|\sharp X| = \Gamma(X)$*, and we have [...] Define as* V*-separated objects?*

$$V = \Gamma^{-1}(\mathrm{iso}(\mathbf{Set})) \tag{10.153}$$

In terms of what we saw about concrete categories3.17, the largest possible subcategory of $\mathbf{H}$ that is concrete is the category of all concrete objects. This is because for any functor $L : \mathbf{C} \to \mathbf{D}$ with a right adjoint R, the unit $X \to RLX$ is a monomorphism iff U is faithful on morphisms with target X, so Γ (forgetful functor) is faithful, therefore concrete

Proof 10.3.17 *By composing with the unit,*

$$\mathrm{Hom}_{\mathbf{C}}(X', X) \to \mathrm{Hom}_{\mathbf{D}}(LX', LX) \cong \mathrm{Hom}_{\mathbf{C}}(X', LRX) \tag{10.154}$$

function is injective etc

The interpretation of a concrete space is that, if we consider our spaces from both the lens of their points $\Gamma(X)$ and their algebra of subobjects Ω^X, the locale given by Ω^X can be entirely generated by $\Gamma(X)$

$X \hookrightarrow \sharp X \cong \mathrm{CoDisc}(\Gamma(X))$

$X \to X$ is mono obviously, and so on a coarser topology, remains continuous (conserve the mono idk). This is true for every concrete space

$\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$, $\Gamma(X) \cong \mathrm{Hom}(1, X)$

For every subobject $S \hookrightarrow X$, we have a characteristic morphism $\chi_S : X \to \Omega$, and we have the mapping

$$\Gamma(\chi_S) : \Gamma(X) \to \Gamma(\Omega) \tag{10.155}$$

Only makes sense if $\Gamma(\Omega)_{\mathbf{Set}}$?

Concretification :

$$\text{conc} : X \mapsto \text{im}(\eta^{\sharp}_X) \tag{10.156}$$

$$\mathbf{H} \underset{\iota_{\text{conc}}}{\overset{\text{conc}}{\rightleftarrows}} \mathbf{H}_{\text{conc}} \rightleftarrows \mathbf{Set}$$

Theorem 10.3.19 *The subcategory of concrete objects is a quasitopos.*

Proof 10.3.18

Dually to the concrete and discrete objects, we also have the types associated with the flat modality, the coconcrete and codiscrete types :

Definition 10.3.12

points are defined entirely by the locale? So can have ”not enough points”

Definition 10.3.13 *If a morphism has a concrete codomain, ie $f : X \to Y$ is such that X is a concrete object, it is called* intensive*.*

This is the generalization of the notion used in thermodynamics, where the function is determined entirely by its value at points. We could see this as considering our function as associating values to points

For any subobject $S \hookrightarrow X$, the

Concrete objects and separated presheaves

Copresheaves of algebras?

Definition 10.3.14 *An object in a cohesive topos is* discrete *if the counit*

$$X \cong \flat X \tag{10.157}$$

From this definition, we have that any discrete object is the image of the set $\Gamma(X)$ via the discrete functor Disc. As a property, this gives us

$$\text{Hom}_{\mathbf{Set}}(S, \Gamma(X)) \cong \text{Hom}_{\mathbf{H}}(\text{Disc}(S), X) \tag{10.158}$$

So that the hom-set of any discrete object Disc(S) to any object X is the hom-set of all functions from their underlying sets, which is a property similar to that of the discrete topology.

Discrete objects are roughly speaking an inclusion of sets in the topos, in that they have no "cohesion"

Theorem 10.3.20 *The closure of any subobject of a discrete object is itself.*

Theorem 10.3.21 *A discrete object is concrete.*

Proof 10.3.19 *From the epimorphism of the points to pieces transformation* $\flat \to \int$,

Theorem 10.3.22 *Any subobject of a discrete object is discrete[?]*

Proof 10.3.20 *For it to be discrete, we need to have the property that for any object* Z,

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(U), \Gamma(Z)) \cong \mathrm{Hom}_{\mathbf{H}}(U, Z) \tag{10.159}$$

If we have some monomorphism $U \hookrightarrow X$*, as we have* $X \cong \flat X$*, we also have by the monomorphic morphism,*

$$\mathrm{Hom}_{\mathbf{H}}(Z, \flat X) \to \mathrm{Hom}_{\mathbf{H}}(Z, U) \tag{10.160}$$

is injective

$$\flat(U \hookrightarrow X) \cong \flat U \hookrightarrow X \tag{10.161}$$

counit : $\flat U \to U$*, two out of three property? Composition* $\epsilon^{\flat}_{U} \circ \iota_U$ *:*

$$\flat U \to U \hookrightarrow X \tag{10.162}$$

modality :

$$\flat(\flat U \to U \hookrightarrow X) \cong (\flat U \xrightarrow{\cong} \flat U \hookrightarrow X) \tag{10.163}$$

global section functor is fully faithful on concrete objects, therefore preserves monomorphisms :

$$\Gamma(U) \hookrightarrow \Gamma(X) \tag{10.164}$$

$$\mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(X)) \hookrightarrow \mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(U)) \tag{10.165}$$

Definition 10.3.15 *For an intensive morphism, we have [some isomorphism idk]*

Extensive objects : maximally non-concrete codomain, ie X is

From Lawvere : example of extensive quantity as $M(X) = L_R(R^X, R)$, smooth linear functionals on the ring object R

the integration/end/eval of extensive and intensive quantity :

$$\int_X : R^X \times M(X) \to R \tag{10.166}$$

Does it have to be wrt a ring object?

[64]

Theorem 10.3.23 *For a cohesive sheaf topos on a concrete site* $\mathbf{C}$*, every locally*

Type associated to $\int$? Same as discrete objects?

10.3.7 Morphisms

$\sharp$-étale morphism? For some function $f : X \to Y$, with modal map $\sharp f : \sharp X \to \sharp Y$,

$\int$-étale

10.3.8 Infinitesimal cohesive topos

One particular special case regarding cohesion is the case where the points to pieces transform is an equivalence,

$$\flat \cong \int \tag{10.167}$$

In other words, for every connected component of a topos, there is a unique point in that piece. As the left adjoint of a functor being an equivalence implies that its right adjoint also is, this leads to the collapse of the cohesion adjoint triple ($\int \dashv \flat \dashv \sharp$) into a single modality $\natural$

$$\int = \Pi_0 \circ \Gamma = \mathrm{Disc} \circ \Gamma = \flat \tag{10.168}$$

This ambidextrous modality manifests as a bireflective subcategory,

$$(T_\natural \dashv \iota_\natural \dashv T_\natural) : \mathbf{H}_\natural \cong \mathbf{1} \underset{\longleftarrow T_\natural \text{---}}{\overset{\longleftarrow T_\natural \text{---}}{\hookrightarrow \iota_\natural \longrightarrow}} \mathbf{H}_{\mathrm{inf}}$$

As both the inclusion and (co)reflector are ambidextrous, they preserve all limits and colimits, and so does $\natural$. This means in particular that

$$\natural 1 = 1 \tag{10.169}$$

The notion of such spaces being infinitesimal comes from the general idea that infinitesimal spaces in the context of a topos are really just single points with a ”halo” of pointless regions around them, such as the infinitesimal disks [...]

Beware of overextending this idea however as this is not always an appropriate interpretation. **Set** is such an ”infinitesimal cohesive topos”, but has no real extension around its points (although it could be understood as the topos over the infinitesimally thickened point of order 0, or the dual to the trivial Weil algebra that is just $\mathbb{R}$, ie just a point). They also include such objects as the dual of non-commutative algebras or some moduli spaces.

Another way to interpret them is via the notion of *quality types.*

Definition 10.3.16 *A fully faithful functor f^**

”Hence from a more geometrical point of view, an object in a quality type is a particular simple kind of space with ‘degenerate’ components, or, if you prefer, a space with ‘thick’ or ‘coarse’ points which in turn can be viewed as a minimal vestige of cohesion: when a set is a space with no cohesion, an object in a quality type is a space with almost no cohesion.”

”The claim that quality types intend to model the philosophical concept of quality reoccurs at several places in Lawvere’s writings though the concrete connection still needs to be spelled out. Some motivation is provided in Lawvere (1992) where it is linked to the negation of quantity as a logical category of being that is indifferent to non-being (cf. Hegel’s Science of Logic) e.g. whereas the temperature varies continuously or “indifferently” below and above zero degree, the same transition makes a crucial difference for the phase or “qualitative being” of water.”

”Intuitively, an intensive quality is compatible with the points of its domain spaces and an extensive quality with the connected components.”

Nullstellensatz implies that infinitesimal objects as a category is closed under arbitrary subobjects and is thus epi reflective

If co-reflective as well, the geometric map(?) p (Γ?) factorizes as

$$p = qs \tag{10.170}$$

q has an adjoint string of length two because the left and right adjoints of q^* are isomorphic (see the ”classical modality” $\natural$)

s has adjoint length 3 due to a lack of codiscreteness, left adjoint of q doesn't preserve product.

q is the *quality type*, s is the *intensive quality* because it is compatible with right adjoint points functor, while an extensive quality is a map to a quality type compatible with the pieces aspect of p. Example : s is an extensive quality in the category of infinitesimal spaces

[135]

We have seen previously some examples of quality types, such as the universal moduli space of k-forms, Ω^k, the moduli space of Riemannian metrics Met, or the moduli space of symplectic structures ω.

Example of quality type :

10.3.9 Negation

[116]

Anti-sharp

Being a monad which is a right adjoint, the sharp modality preserves limits, and therefore admits a negation. To find the negation of the sharp modality, we need to find the fiber of its unit.

$$\bar{\sharp}_p X = \mathrm{Fib}_p(X \to \sharp X) \tag{10.171}$$

$$\begin{array}{ccc} \bar{\sharp}_p X & \xrightarrow{!_{\bar{\sharp} X}} & 1 \\ \downarrow & & \downarrow p \\ X & \xrightarrow{\eta^\sharp_X} & \sharp X \end{array}$$

$$\bar{\sharp}_p X \cong X +_{\sharp X} 1 \tag{10.172}$$

In terms of products : the equalizer of $X \times 1 \cong X$ by the morphisms $p : 1 \to \sharp X$ (which is equivalent to some point in X by etc), and $\eta^\sharp_X$. For a given point p, we are looking for an object which, set-wise, is the

$$\Gamma(p, \eta^\sharp_X) = \{(x, \bullet) \in X \times 1 \mid p(\bullet) = \eta^\sharp_X(x)\} \tag{10.173}$$

So that we are given an object whose only element is p. As far as the point content of an object goes, this means that every negation of continuity has the same content (a unique point). This is about what we would expect since continuity and its negation would reduce any object to a bare object, $\sharp(\bar{\sharp}(X)) = 1$.

However, there are additional informations left after the negation of the continuity. If we look for instance at an infinitesimal object X, $\sharp X \cong 1$, as the negation will merely be $X \times_1 1 = X$, any infinitesimal object (or "quality type") will be preserved. This means that such objects as infinitesimally thickened points, coarse moduli spaces, etc will remain invariant under this.

Theorem 10.3.24 *Any anti-sharp modality is infinitesimal.*

Proof 10.3.21 *By the definition of the negation, we have*

$$\sharp\bar{\sharp}_p X = 1 \tag{10.174}$$

meaning that it only has a single point, $\Gamma(\bar{\sharp}_p X) \cong \{\bullet\}$.

For a broader example we can look at the case of a space with infinitesimal extension, like the Cahier topos.

Theorem 10.3.25 *The negation of the continuity modality on a Cahier topos at a point* $p : 1 \to X$ *gives its infinitesimally thickened point.*

This is the general behavior of the anti-sharp modality, which is to give the "quality" of the space at that point, ie the maximally non-concrete part of that space around that point.

Behaviour on a Q-category?

Theorem 10.3.26 *Given a* Q*-category*

$$\mathbf{H} \begin{array}{c} \xleftarrow{\quad \Pi_0 \quad} \\ \xhookrightarrow{\quad \mathrm{Disc} \quad} \\ \xleftarrow{\quad \Gamma \quad} \end{array} \mathbf{H}_{\mathrm{th}}$$

where the subcategory $\mathbf{H}$ *is cohesive but with trivial infinitesimals,* $\mathbf{H}_{\mathrm{inf}} \cong \mathbf{Set}$*, and the thickened topos has some nilpotent ideal* I*, the anti-sharp modality gives us [...]*

Proof 10.3.22

What about some moduli space, ie some space M for which the internal hom $[X, M]$ is a moduli space of some sort, such that for the point, $[1, M] \cong 1$?

Moduli space defined by some family of objects such that for any object $B \in \mathbf{H}$, there is a surjective morphism $\pi : M \to B$, such that every fiber $M_b = \pi^{-1}(b)$ for $b \in B$ is also in the family of objects.

If M has a single point, it is infinitesimal

[Does the quality type being a quality (at a point) relate to the fact that a fine moduli space cannot have non-trivial automorphisms]

Universal property with this object : for any object Y such that there is a morphism $f : Y \to X$, there exists a unique function $\beta : Y \to \overline{\sharp} X$ such that

Since the exponential functor $(-)^X$ is a right adjoint, we have the preservation of pullbacks (and therefore fibers), so that $(\overline{\sharp}_p Y)^X$ defines the diagram

$$\begin{array}{ccc} (\overline{\sharp}_p Y)^X & \xrightarrow{!} & 1 \\ \downarrow & & \downarrow^{p} \\ Y^X & \xrightarrow{\eta^{\sharp}_{Y^X}} & \sharp(Y^X) \end{array}$$

Use adjunction with the product?

anti-sharp of a concrete object?

Theorem 10.3.27 *The anti-sharp modality of a concrete object is the terminal object[or subterminal?].*

Proof 10.3.23 *As a concrete object X has a monomorphism to its sharp modality, we have that since pullbacks preserve monomorphisms,*

$$\begin{array}{ccc} \overline{\sharp}_p X & \xhookrightarrow{!_{\overline{\sharp} X}} & 1 \\ {\scriptstyle p^* \eta^{\sharp}_X}\downarrow & & \downarrow^{p} \\ X & \xhookrightarrow[\eta^{\sharp}_X]{} & \sharp X \end{array}$$

$\overline{\sharp} X$ is subterminal.

Anti-flat

The flat modality, $\flat$, being a right adjoint to $\int$, also admits a negation.

$$\overline{\flat} = \mathrm{Cofib}(\flat X \to X) \tag{10.175}$$

with the pullback diagram

$$\begin{array}{ccc} \flat X & \xrightarrow{\eta^{\sharp}_X} & X \\ {\scriptstyle !_X}\downarrow & & \downarrow^{p} \\ 1 & \xrightarrow{p} & \overline{\flat} X \end{array}$$

Furthermore, also being in an *sp*-unity with $\int$, and the map $\flat \to \int$ being an epimorphism by definition of a cohesive topos, it is furthermore a determinate negation, so that we have

$$\int \bar{\flat} X \cong 1 \tag{10.176}$$

So that the anti-flat modality of an object is always connected.

As a quotient :

$$\bar{\flat} \cong \mathrm{coeq}() \tag{10.177}$$

$$\bar{\flat} = X / \flat X \tag{10.178}$$

In the case of a topological space, this would be something akin to the quotient of a topological space

$$\bar{\flat} X = \flat X +_X 1 \tag{10.179}$$

As $\flat X$ is already a coproduct, we are considering the space constituted by

$$\coprod_{x \in \Gamma(X) + \{\bullet\}} 1 \tag{10.180}$$

where we quotient out the relation given by $\flat X \to X$. As there is always an element of this coproduct which is such that $!_{\flat X}()$

$$\bar{\flat} X = \mathrm{coeq}(!_{\sharp X}, \eta^{\flat}_X) \tag{10.181}$$

Is it just $\bar{\flat} X \cong 1$??? It should probably include the ”lost” cohesion, so only 1 for discrete spaces

Topological example : if $X = \mathbb{R}$ or something, with $\flat \mathbb{R} \to \mathbb{R}$ the surjection of a finer topology to the original, we identify every point of $\flat \mathbb{R}$ with $\mathbb{R}$, singleton topology, only one

If it must preserve anything, it must be non-concrete.

Example 10.3.2 *Given the smooth space of k-forms Ω^k, its anti-flat modality is*

$$\bar{\flat} \Omega^k \quad = \quad 1 +_{\Omega^k} 1 \tag{10.182}$$

Sum of two infinitesimal spaces?

$$\begin{aligned}\bar{\flat}(X+Y) &= \mathrm{Cofib}(\flat(X+Y) \to (X+Y)) & (10.183)\\ &= \mathrm{Cofib}((\flat X + \flat Y) \to (X+Y)) & (10.184)\\ &= \mathrm{Cofib}(2 \to (X+Y)) & (10.185)\end{aligned}$$

For a concrete object, $X \hookrightarrow \sharp X$, we have the adjunct

$$\flat X \xrightarrow{\flat \eta^{\sharp}_X} \flat\sharp \xrightarrow{\epsilon^{\flat}_X} X \tag{10.186}$$

$$\flat(\int X \to Y) \cong \flat(X \to \flat Y) \tag{10.187}$$

Anti-shape

The last negation is that of the shape modality, given by

$$\bar{\int}_p X = \mathrm{Fib}_p(\int X \to X) \tag{10.188}$$

represented by the pullback diagram

$$\begin{array}{ccc} \bar{\int} X & \xrightarrow{!_{\bar{\int} X}} & 1 \\ \downarrow & & \downarrow p \\ X & \xrightarrow{\eta^{\int}_X} & \int_p X \end{array}$$

As a left adjoint, this modality is not guaranteed to have a proper negation, in the sense that it is not guaranteed to preserve limits. This will in fact be an issue in the higher categorical case, where the higher categorical version of the shape modality will not preserve homotopy pullbacks.

Theorem 10.3.28 *The*

This is the common usage of the fiber of a morphism, so that the anti-shape modality at a point p will give the connected component at that point.

$$\bar{\int} X = X \times_{\int X} 1 \tag{10.189}$$

This simply gives us the space back if it is connected, as we would expect. If we have multiple connected components, $X \to \int X$ is essentially a bundle over the discrete space of its points [prove it? Is it epi?], and the choice of point in $\int X$ is simply the choice of the connected component we will consider. In other words, if X is expressed as a coproduct of connected components,

$$X \cong \coprod_{i \in \int X} X_i \tag{10.190}$$

Then for a choice of point $p \in X_i$, we have

$$\bar{\int} X \cong X_i \tag{10.191}$$

[Is de Rham different from the anti-flat and anti-shape modalities?]

In addition to the negations of our various modalities, cohesion also comes with the notion of *de Rham* flat and shape modalities. Unlike the case of the negation, which is the cofiber of flat and the fiber of shape, it is the fiber of flat and the cofiber of shape :

$$\flat_{\mathrm{dR},p} X = \mathrm{Fib}_p(\flat X \to X) = \flat A / A \tag{10.192}$$

$$\int_{\mathrm{dR}} X = \mathrm{Cofib}(X \to \int X) \tag{10.193}$$

Properties wrt flat and shape?

$$\flat \flat_{\mathrm{dR},p} X = \tag{10.194}$$

[dR-shape : universal cover?]

[dR-flat : infinitesimal structure around all points?]

Modal hexagon

$$\begin{aligned} X &\cong \int X \times_{\int \bar{\flat} X} \bar{\flat} X && (10.195) \\ &\cong \int X \times_1 \bar{\flat} X && (10.196) \\ &\cong \int X \times \bar{\flat} X && (10.197) \end{aligned}$$

de Rham modalities

Definition 10.3.17 *The de Rham flat modality* $\tilde{\flat}_p$

$$\tilde{\flat}_p X = \mathrm{Fib}_p(\epsilon^\flat : \flat X \to X) \tag{10.198}$$

Tangent space???

Definition 10.3.18 *The de Rham shape modality* $\tilde{\int}$

$$\tilde{\int} X = \mathrm{Cofib}(\eta^{\int} : X \to \int X) \tag{10.199}$$

10.3.10 Algebra

If the topos is cohesive, the modalities for cohesion are all monoidal.

Theorem 10.3.29 *The sharp, flat and shape modalities are monoidal.*

Proof 10.3.24 *As* Γ *is both a left and right adjoint, it preserves limits and colimits, and* CoDisc *as a right adjoint preserves limits.*

$$\begin{aligned}
\sharp(X \times Y) &= \mathrm{CoDisc} \circ \Gamma(X \times Y) && (10.200)\\
&= \mathrm{CoDisc}(\Gamma(X) \times \Gamma(Y)) && (10.201)\\
&= \mathrm{CoDisc}(\Gamma(X)) \times \mathrm{CoDisc}(\Gamma(Y))) && (10.202)\\
&= \sharp X \times \sharp Y && (10.203)
\end{aligned}$$

And since Γ *and* Disc *are both left and right adjoints, this follows naturally with a similar proof.*

$$\flat(X \times Y) = \flat X \times \flat Y \tag{10.204}$$

The case for the shape modality follows from the same property of Disc *and the preservation of products for* Π_0 *in a cohesive topos.*

$$\int(X \times Y) = \int X \times \int Y \tag{10.205}$$

10.3.11 Logic

As we saw early on, one basic property of cohesion is that the law of excluded middle applies to varying degrees depending on the modalities involves.

Theorem 10.3.30 *For a discrete object, the law of excluded middle holds :*

$$U + \neg_X U \cong X \tag{10.206}$$

Proof 10.3.25 *As we have $X \cong \flat X$, where $\flat$ preserves colimits, we have*

$$\Gamma(U + \neg_X U) \cong \Gamma(X) \tag{10.207}$$

Definition 10.3.19 *A proposition $p : A \hookrightarrow X$ is* discretely true *if in the pull-back*

$$\begin{array}{ccc} \sharp A\big|_X & \longrightarrow & \sharp A \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\eta_X} & \sharp X \end{array}$$

$\sharp A\big|_X \to X$ is an isomorphism

Proposition that is true over discrete spaces.

Theorem 10.3.31 *If p*

Is there a way to classify truths by their modalities, and is there a semantics of flat ”collections are ensembles of parts” v. sharp ”collections are wholes”?

What is the appropriate method? If we have some predicate $p[X]$ for some space X, ie a dependent type(?), what is the condition for the predicate being ”as a whole” v. ”as a collection”?

Predicates of type Y are related to morphisms $p : X \to Y$, its valuation is given by specifying

Is the modality to be applied to the free term of the predicate or the predicate as a whole?

Ex. of part properties : ”U is a subobject of X”, ”the point x has value $f(x)$”, ”Has a number of elements”

Ex. of whole properties : ”X has total volume V”, ”two magnitudes are congruent”

Relation to intensive v. extensive properties? (this is for $\sharp$ and $\overline{\sharp}$ though $\flat$ isn't involved)

Let's check :

”X has n points” :

$$\Gamma(X) = n \tag{10.208}$$

Replace by :

$$\Gamma(\flat X) = n \tag{10.209}$$

Type :

$$\vdash \tag{10.210}$$

"X has n connected components" :

"X is constant over connected components"

Moduli space???

"X has volume V" : first pick a volume structure vol : $X \to \mathrm{Vol}$, then take the coend idk with the constant function to $1 : 1 \to R$,

$$\int_{x:X} \mathrm{vol} \times 1 \tag{10.211}$$

Careful : example of X having a volume V, this is a proposition which involves another space $\mathbb{R}$

Is it something like $p(\sharp X) \dashv p(X)$

For vol : a volume of X is given by some value $1 \to \mathbb{R}$

Integration map :

$$\int_X : [X, \mathbb{R}] \times [X, \mathbb{R}]^* \quad \to \quad \mathbb{R} \tag{10.212}$$

"the general sheaves for this base level are commonly called "codiscrete" or "chaotic" objects within the big category of Being, and the subtopos of them may be called "pure Becoming". The negative objects for this level are commonly called "discrete" and the subcategory of them deserves to be styled "non Becoming"."

10.3.12 Dimension

Definition 10.3.20 *A discrete object $X \cong \flat X$ has dimension zero.*

Lebesgue covering dimension? Poincaré's definition?

"Having described a basic framework, we can now return to the question of the intrinsic meaning of "one-dimensionality" of an object within such a framework. The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level. Thus a zero-dimensional space is just a discrete one (there are several answers, not gone into here, to the objection which general topologists may raise to that) and dimension one is the Aufhebung of dimension zero. Because of the special feature of dimension zero of having a components functor to it (usually there is no analogue of that functor in higher dimensions), the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of of components of the skeleton at that dimension of the space, or more pictorially: if two points of any space can be connected by anything, then they can be connected by a curve. Here of course by "curve" we mean any figure in (i.e. map to) the given space whose domain is one-dimensional."

"Continuing, two-dimensional spaces should be those negating a subtopos which itself contains both the one-dimensional spaces and the identical-but-opposite sheaves which the one-dimensional spaces negate."

10.3.13 Cohesion on sets

Set is trivially a cohesive topos, but it fails to have the stronger property of being *sufficiently cohesive*. If we consider the case where its base topos is itself, then we need to investigate the adjoint cylinder from its functor of global sections, ie

$$(\mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc}) : \mathbf{Set} \overset{\xhookrightarrow{\mathrm{Disc}}}{\underset{\xhookrightarrow{\mathrm{Codisc}}}{\xleftarrow{\Gamma}}} \mathbf{Set}$$

The global section functor, as the hom functor from the terminal object, is simply the identity on sets, as sets are entirely defined by their points. As the adjoints of the identity are always the identity, any adjunction string is simply going to be more identity functor. Our adjunction for cohesion is therefore

$$(\mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}}) \tag{10.213}$$

And its corresponding monads are just the identity monad.

$$(\mathrm{Id} \dashv \mathrm{Id} \dashv \mathrm{Id}) \tag{10.214}$$

From there, every property of cohesion is fulfilled simply by every limit and colimit being preserved by the identity functor and every unit and counit being appropriately mono or epi, as they are always isomorphisms.

The behaviour of Disc is about what we would expect for discrete objects, sending an object to the coproduct of terminal objects over its point content, in other words a set with the same cardinality. However, the codiscrete objects, being the same, does not seem to follow what we would expect of a codiscrete object, of being "one whole" in some sense, although they do obey the more decisive property of every function on its points lifting to a continuous function. Connected components are likewise about what we would expect since every point is its own component in the "topology" of a set. We have that every piece (a single point) has a point, and every point has its piece, and that this is a sublation of the ground by $\sharp 0 = 0$. That pieces of powers are powers of pieces is obviously true again due to the identity :

$$\Pi_0(X^{\mathrm{Disc}(S)}) = (\Pi_0 X)^S = X^S \tag{10.215}$$

The issue that prevents it from being a more intuitive cohesive topos is that of the connectedness of the subobject classifier, 2, as trivially, $\Pi_0 2 = 2$, and not 1 as we would need to. This stems from an obstruction found in [136], saying that a localic topos cannot both have a shape modality over sets that preserves the product and have a connected subobject classifier. [proof?]

This means that we cannot embed any object into some connected or contractible space, and in fact, there is no connected space outside of 1, meaning that its topology is fairly uninteresting as we would guess. In other words, there's no equivalence of our own "standard" Euclidian space of a dimension superior to 0, some simply connected space that can contain multiple disjoint objects.

As we've seen in the case of **Set**, there are two possible Lawvere-Tierney closure operators we could try over sets, and the $\mathrm{loc}_{\neg\neg}$ closure is the discrete topology, in the sense that every subset $S \subseteq X$ is its own closure, including singletons $\{\bullet\}$. No point is "in contact" with another, we can entirely separate a given element from the whole. The other closure operator is the trivial one $j(x) = 1$, giving us the *chaotic* topology on 1.

Equivalence between the lawvere-tierney topology and Grothendieck topology, chaotic grothendieck topology, collapse of sets into triviality?

As $\neg\neg = \mathrm{Id}_\Omega$, the localization does not do anything and the smallest dense subtopos is simply itself, there are no levels in between Set and the ground.

The global section functor $\Gamma(-) \cong \mathrm{Hom}_{\mathbf{Set}}(1, -)$ is simply the identity, as

$$\Gamma(S) \cong \mathrm{Hom}_{\mathbf{Set}}(1, S) \cong S \tag{10.216}$$

Since the hom-set of the point to a set is of the same cardinality as the set itself, and

$$\Gamma(f) \tag{10.217}$$

The left adjoint functor Disc here will work out as

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Disc}(-), -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, -) \tag{10.218}$$

while the right adjoint is

$$\mathrm{Hom}_{\mathbf{Set}}(-,-) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{10.219}$$

As far as the objects go, these are identities, so that the discrete and codiscrete objects of sets are the same objects (they have the same point content). However, the actions on morphisms does change, and most importantly, on the Lawvere-Tierney topology of the topos. For the morphism $j : \Omega \to \Omega$, $j \in \mathrm{Hom}_{\mathbf{Set}}(2,2)$, we have

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(-),-) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{10.220}$$

The left adjoint modality $\flat \dashv \sharp$

"Note that in this example, the “global sections” functor $S \to Set$ is not the forgetful functor $Set/U \to Set$ (which doesn't even preserve the terminal object), but the exponential functor $\Pi U = \mathrm{Hom}(U, -)$. This is the direct image functor in the geometric morphism $Set/U \to Set$, whereas the obvious forgetful functor is the left adjoint to the inverse image functor that exhibits S as a locally connected topos.”

Negation of sharp modality :

$$\bar{\sharp} X = X \sqcup_{\sharp X} * \tag{10.221}$$
$$= X \sqcup_{\sharp X} * \tag{10.222}$$

Is it just $*$? The negation of the moment of continuity (maximally non-concrete object)

From the definition of the cohesion, we have that every sheaf of **Set** is in fact infinitesimal, but only in the trivial sense. While they do obey $\flat \cong \int$, this is only due to the pieces being actual points without any "infinitesimal extension" to them.

10.3.14 Cohesion of simplicial sets

Simplicial sets can be seen as the simplest model of a "space" that has non-trivial cohesion, as there is a notion of "cohesion" between points as instantiated by the higher simplices between them.

The global section functor for simplicial sets is simply given by the 0-simplices.

$$\Gamma(X) = \mathrm{Hom}_{\mathbf{sSet}}(1, X) = X_0 \tag{10.223}$$

Theorem 10.3.32 *The discrete simplicial sets* $\mathrm{Disc}(S)$ *are the simplicial sets whose only non-degenerate simplices are those of dimension zero.*

Proof 10.3.26 *The left adjoint of* Γ *is as usual the set-tensoring functor, so that*

$$\mathrm{Disc}(X) = \coprod_{i \in \Gamma(X)} 1 \tag{10.224}$$

$$= \coprod_{i \in X_0} 1 \tag{10.225}$$

Disjoint sum of simplices etc

In other words, the discrete simplicial sets are simply a collection of disjoint points, as we would expect from a discrete space.

Theorem 10.3.33 *The codiscrete simplicial sets* CoDisc *are the simplicial sets which have as degree* k*-faces the set of* $(k+1)$*-tuples*

Proof 10.3.27 *For a simplicial set* X *and a set* S*,*

$$\mathrm{Hom}_{\mathbf{H}}(X, \mathrm{CoDisc}(S)) = \mathrm{Hom}_{\mathbf{Set}}(\Gamma(X), S) \tag{10.226}$$

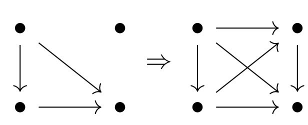
which means that for every function between the vertices of X *and* S*, there is a corresponding morphism. In particular, for simplices* Δ_n*, we have that there is a face map*

For the face map of n*, we have a corresponding hom-set* $\Gamma(X) \times S$*, ie every point is connected to every other point (including itself)*

Likewise for degeneracy map?

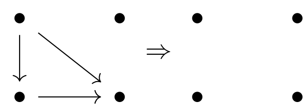
In other words, the codiscrete simplicial sets are the *maximally connected simplicial set* of that set. Every pair of point is connected by a line, every trio by a face, etc. This is also the appropriate intuition we would have for a "codiscrete" combinatorial space, where every point is "connected to" every other point.

The sharp modality is therefore the monad that will "complete" a simplicial set to its maximally connected version.



(10.227)

while the flat modality will forget all cohesion and simply return a set of disconnected points.



(10.228)

Theorem 10.3.34 *The connected component functor is given by the set of vertices X_0 quotiented by the relation that if any two vertices are connected by an edge, they belong to the same component, given as the coequalizer of X_0 by the degeneracy maps of each point of an edge*

$$X_1 \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} X_0 \longrightarrow \Pi_0(X)$$

This simply means that the connected components are the components of a graph as far as the vertices and edges are concerned. The shape modality is then simply the discrete space generated from those components.

$$\int X = \tag{10.229}$$

As the codiscrete space of a simplicial set is its maximally connected version, any simplicial set with at most one edge between two vertices will be concrete

[prove equivalence]

Any non-concrete simplicial sets will therefore correspond to pairs of vertices connected by more than one edge.

The submodal types are given, for the sharp modality, by the concrete simplicial sets, which are

Theorem 10.3.35 *A concrete simplicial set is a simplicial set for which every pair of points has at most one edge.*

Co-concrete? $\flat X \to X$ is an epimorphism, ie a surjection on every component $\eta^\flat_{X,n}$:.

Theorem 10.3.36 *The infinitesimal objects of the simplicial topos are the degenerate simplices of a single point and more than one edge between the two.*

Proof 10.3.28 *If we take some simplicial set with $X_0 = \{\bullet\}$ and $|X_1| \geq$, then we have*

$$\flat X = \int X = \Delta_0 \tag{10.230}$$

The simplest non-trivial case of this is to consider some single vertice with n edges to itself. This is the simplicial circle for $n = 1$, and additional such edges will make it the simplicial version of a bouquet of circles.

Theorem 10.3.37 *The bouquet of circle is not a concrete object.*

Proof 10.3.29

Morphisms into the bouquet object?

Is it an interesting moduli object? $[X, S_n]$:

$$[X, S_n]_n = \hom_{\mathbf{sSet}}(X \times \delta[\vec{\mathbf{n}}], S_n) \tag{10.231}$$

10.3.15 Cohesion of a spatial topos

Spatial topos always has disconnected truth values hence not a sufficiently cohesive topos cf Lawvere

10.3.16 Cohesion of smooth spaces

As a Grothendieck topos, the cohesiveness of smooth spaces relies on the cohesiveness of its site, the category of Cartesian spaces.

First, **CartSp** has the terminal object ($\mathbb{R}^0$), and every object $\mathbb{R}^n$ admits global sections (at least $0 : \mathbb{R}^0 \to \mathbb{R}^n$). **CartSp** is also cosifted in that it has finite products, as we have

$$\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1 + n_2} \tag{10.232}$$

We therefore just have to prove that it is a locally connected site with its topology of differentiably good open covers, ie for any object $X \in$ **CartSp**, any covering sieve on X is a connected subcategory of $\mathbf{CartSp}_{/X}$.

Theorem 10.3.38 **Smooth** *is sufficiently connected.*

Proof 10.3.30

[...]

As a cohesive site, **CartSp** has the global section functor Γ which simply assigns to each open set of $\mathbb{R}^n$ its set of points

$$\Gamma(U_{\mathbf{CartSp}}) = U_{\mathbf{Set}} \tag{10.233}$$

and as its left adjoint the functor $\Gamma^* \cong \mathrm{LConst} \cong \mathrm{Disc}$ which associate to any set the smooth space composed by an equinumerous number of disconnected copies of $\mathbb{R}^0$,

$$\mathrm{Disc}(X) = \coprod_{x \in X} \mathbb{R}^0 \tag{10.234}$$

This is equivalently the fine diffeology on a set, given by a diffeology where the only plots are the locally constant maps.

Its right adjoint is given by the coarse diffeology on a set

Right adjunction :

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(U), X) = \mathrm{Hom}_{\mathbf{CartSp}}(U, \mathrm{CoDisc}(X)) \tag{10.235}$$

Every possible function (as a set) between some Cartesian space U and our codiscrete space X correspond to a valid plot.

In terms of intuition, this means that every point is "next to" every other point in some sense. For instance, given the probe $[0,1]$, there is a smooth curve for every possible combination of points, ie the set of smooth curves is just $X^{[0,1]}$.

There is therefore no meaningful way to separate points (as we would expect from the smooth equivalent of the trivial topology).

Given these two functors, we can construct our two modalities. Our flat modality is

$$\flat = \mathrm{Disc} \circ \Gamma \tag{10.236}$$

which first maps a smooth space to its points, and then to the coproduct of $\mathbb{R}^0$ over those points, giving us the discrete space $\flat X$, switching the space X to its finest diffeology.

The sharp modality $\sharp$ is

$$\flat = \mathrm{Codisc} \circ \Gamma \tag{10.237}$$

which maps a smooth space to its points and then to the

Are all smooth spaces in the Eilenberg category diffeological spaces?

Theorem 10.3.39 *Given a concrete smooth space[necessary?], $X \hookrightarrow \sharp X$, its sharp modality is the coarse diffeology.*

Proof 10.3.31

Theorem 10.3.40 *Given a concrete smooth space[necessary?], $X \hookrightarrow \sharp X$, its flat modality is the fine diffeology.*

"Every topological space X is equipped with the continuous diffeology for which the plots are the continuous maps."

Extensive and intensive quantities

10.3.17 Cohesion of classical mechanics

As a slice topos of the topos of smooth sets, the topos of classical mechanics $\mathbf{H}_{/\omega}$ is trivially cohesive.

10.3.18 Cohesion of quantum mechanics

There will not be any obvious cohesive structure on the usual category of quantum mechanics, as the category of Hilbert spaces is not a topos, but we can try to define such a structure on the Bohr topos of a theory.

The terminal object in the Bohr topos is given by the constant sheaf which maps all commutative operators in a context to a spectrum of a single element, the singleton $\{\bullet\}$ in **Set**. This is the spectral presheaf which has a spectrum of a single value for all

Global section functor : hom-set of

$$\Gamma : \mathbb{C}^0 \to \tag{10.238}$$

10.4 Elastic substance

A further sublation we can get is that of $(\int \dashv \flat)$, the opposition given by

$$(\textstyle\int \dashv \flat) : \mathbf{H}_\sharp \underset{\longleftarrow \Gamma \longrightarrow}{\overset{\longleftarrow \Pi_0 \longrightarrow}{\hookrightarrow \mathrm{Disc} \longrightarrow}} \mathbf{H}$$

As $(\int \dashv \flat)$ is an *sp*-unity, any sublation of it will be automatically a left and right sublation. We will call this sublation $(\Im \dashv \&)$, and from the properties described, we will have

$$\Im\textstyle\int \cong \int \tag{10.239}$$
$$\&\flat \cong \flat \tag{10.240}$$
$$\&\textstyle\int \cong \int \tag{10.241}$$
$$\Im\flat \cong \flat \tag{10.242}$$

Those modalities will therefore not act on any space contracted to its connected components or that has lost its cohesion. In other words, it does not change discrete types. They are therefore in some sense modalities about the continuous structure of a space.

[137, 117]

From the factorization of unities of opposites, we have the

$$X \to \Im X \to \int X \tag{10.243}$$

There is some notion of a *partial* contraction of the entire space's components. If we are considering in particular the A^1-cohesion that we saw, this is a partial contraction of paths,

$$R \to \Im R \to \int R \cong 1 \tag{10.244}$$

None of the topos we have seen thus far are differentially cohesive, except in a trivial sense (all their objects are reduced, $\mathfrak{R}(X) = X$), but it is simple enough to extend them to be. This is generally done concretely by changing the site to include an infinitesimal structure. The basic example for this is the site of *formal Cartesian spaces*, **FormalCartSp**, which is the site with objects being open sets of $\mathbb{R}^n$ composed with an *infinitesimally thickened point*, D. As we will see, D is a point if we forget the elastic structure, $\mathfrak{R}(D) \cong *$, but in basic mathematical terms, this relates to Weil algebras, the algebras of infinitesimal objects, as seen in 5.6.

Definition 10.4.1 *The category of formal Cartesian spaces* **FormalCartSp** *is the full subcategory of smooth locus* $\mathbb{L}$ *of the form*

$$\ell A \cong \mathbb{R}^n \rtimes D \tag{10.245}$$

[Open subset $D(f) = \mathrm{Spec}(A) \setminus V(f)$, $V(f)$ all prime ideals containing f, then $D(f)$ is empty if f is nilpotent?]

Definition 10.4.2 *The* Cahier topos **Cahier** *is the sheaf topos over the site of formal Cartesian spaces :*

$$\mathbf{Cahier} = \mathrm{Sh}(\mathbf{FormalCartSp}) \tag{10.246}$$

with the coverage

Theorem 10.4.1 *The Cahier topos is cohesive*

Proof 10.4.1

[...]

As an *sp*-unity, we can construct the essential subtopos $\mathbf{H}_{\mathfrak{R}}$, the "reduced" topos

$$(\mathfrak{I} \dashv \&) : \mathbf{H}_{\mathfrak{R}} \underset{\longleftarrow T_{\&} \longrightarrow}{\overset{\longleftarrow T_{\mathfrak{I}} \longrightarrow}{\hookrightarrow \iota_{\&} \cong \iota_{\mathfrak{I}} \longrightarrow}} \mathbf{H}$$

In terms of essential geometric morphisms, this means that

$$i^! := T_{\&} \tag{10.247}$$

$$i_* := \iota \tag{10.248}$$

$$i^* = T_{\mathfrak{I}} \tag{10.249}$$

It is common to write out the reduced topos as the basic one, ie $\mathbf{H}_{\mathfrak{R}} := \mathbf{H}$, and the topmost topos here as an "augmented" topos, the infinitesimally thickened topos $\mathbf{H}_{\text{th}}$. Furthermore, by analogy with the cohesive case, we will call the inclusion ι the infinitesimal discrete functor Disc_{inf}, and the reflectors $T_{\mathfrak{I}}$ and $T_{\&}$ as the infinitesimal connected component functor Π_{inf} and the infinitesimal global section functor Γ_{inf}

$$\mathfrak{I} = \text{Disc}_{\text{inf}} \circ \Pi_{\text{inf}} \tag{10.250}$$

$$\& = \text{Disc}_{\text{inf}} \circ \Gamma_{\text{inf}} \tag{10.251}$$

Subtopos	Geometric	Inf.
$T_{\mathfrak{I}}$	i^*	Π_{inf}
ι	i_*	Disc_{inf}
$T_{\&}$	$i^!$	Γ_{inf}

Table 10.1: Notations

The reduced subtopos, unlike the boolean subtopos, does not seem to have any easy definition in terms of its own properties, so that we will have to look at some of its models. But first we can certainly deduce a few properties from its raw definition.

First, this reduced subtopos is the middle step in between the elastic topos and the cohesive subtopos, ie we have the following hierarchy

$$\mathbf{H}_{\sharp} \underset{\longleftarrow T_{\&} \longrightarrow}{\overset{\longleftarrow T_{\mathfrak{I}} \longrightarrow}{\hookrightarrow \iota_{\&} \cong \iota_{\mathfrak{I}} \longrightarrow}} \mathbf{H}_{\mathfrak{R}} \underset{\longleftarrow T_{\&} \longrightarrow}{\overset{\longleftarrow T_{\mathfrak{I}} \longrightarrow}{\hookrightarrow \iota_{\&} \cong \iota_{\mathfrak{I}} \longrightarrow}} \mathbf{H}$$

Chain the functors?

Theorem 10.4.2 *If $\Pi_{\text{inf}} X \cong 1$, then X is infinitesimal.*

Proof 10.4.2 *As we have that* $\Gamma_{\text{inf}} 1 \cong 1$ *[proof? Due to* $\& 1 \cong \& \flat 1 \cong \flat 1 \cong 1$*]*

$$\begin{aligned} \Gamma(X) &= \text{Hom}(1, X) && (10.252)\\ &= \text{Hom}(T_{\&} 1, X) && (10.253)\\ &= \text{Hom}(1, \Pi_{\text{inf}} X) && (10.254)\\ &= \text{Hom}(1, 1) && (10.255)\\ &= 1 && (10.256) \end{aligned}$$

So that X *only has a single point, and likewise, a single piece.*

This corroborates the interpretation of Π_{inf} as the contraction of infinitesimal paths.

Jet comonad

Definition 10.4.3 *For an object* X*, given the unit of the infinitesimal shape modality*

$$\eta^{\Im} : X \to \Im X \tag{10.257}$$

inducing the base change functors

$$\mathbf{H}_{/X} \underset{\eta^{\Im *}}{\overset{\eta^{\Im}_{*}}{\rightleftarrows}} \mathbf{H}_{\Im X} \tag{10.258}$$

Then the jet comonad *at* X Jet_X *is the map given by their composition*

$$\text{Jet}_X = \eta^{\Im *} \eta^{\Im}_{*} : \mathbf{H}_{/X} \to \mathbf{H}_{/X} \tag{10.259}$$

Left adjoint of jet comonad : forming infinitesimal disk bundle monad

Example 10.4.1 *The main interpretation of the jet comonad in our context is that of its action on bundles for smooth spaces. Given some bundle* $\pi : E \to X$ *in* $\mathbf{H}_{/X}$*, the action of the jet comonad is to send this bundle to its* ∞ *jet bundle.*

$$\text{Jet}_X(\pi) = J^{\infty} \pi \tag{10.260}$$

Theorem 10.4.3

The topos from that site is the *Cahier topos*,

$$\mathbf{Cahier} = \text{Sh}(\mathbf{FormalCartSp}) \tag{10.261}$$

[138, 139, 140]

Example 10.4.2 *As* **Set** *was already only trivially cohesive, so too is it only trivially differentially cohesive. We can understand the differential cohesion of sets as simply being from the trivial Weil algebra, so that our underlying site is just* $\mathbf{1} \rtimes \{0\}$*. Any set is therefore its own reduced element*

Theorem 10.4.4

$$\int(\Re X \to X) \tag{10.262}$$

is an equivalence

Proof 10.4.3

Definition 10.4.4 *In a Cartesian closed category, we say that an object* Δ *is* infinitesimal atomic *if the exponential object functor*

$$(-)^{\Delta} : \mathbf{H} \to \mathbf{H} \tag{10.263}$$

has a right adjoint $(-)^{1/\Delta}$*:*

$$\mathrm{Hom}_{\mathbf{H}}(X^{\Delta}, Y) = \mathrm{Hom}_{\mathbf{H}}(X, Y^{1/\Delta}) \tag{10.264}$$

Definition 10.4.5 *An object is a* tiny object

Theorem 10.4.5 *Any infinitesimal atomic object is tiny.*

Proof 10.4.4 *As the exponential* $(-)^{\Delta}$ *is a left adjoint, it preserves all colimits.*

Therefore $(-)^{\Delta}$ preserves colimits.

[141]

[Is there a nuance on the difference between set-wise (exponential) infinitesimal and synthetic infinitesimal as seen in the difference between Cahier topos infinitesimal v. hyperreal infinitesimal topos, where the infinitesimal region has (maybe) more than one point? Also infinitesimal Sierpinski topos?]

10.4.1 Synthetic infinitesimal geometry

KontsevichRosenbergNCSpaces

[142, 143, 144, 145, 146]

Probably the most common model of differential cohesion in topos theory is the one given by synthetic differential geometry.

Definition 10.4.6 *On a topos T with a ring object R, if there exists an object D such that*

$$D = \{x \in R \mid x^2 = 0\} \tag{10.265}$$

for which the canonical map

$$R \times R \quad \to \quad R^D \tag{10.266}$$

$$(x, d) \quad \mapsto \quad (\varepsilon \mapsto x + \varepsilon d) \tag{10.267}$$

is an isomorphism of objects.

Koch-Lawvere axioms

This axiom cannot work properly in the context of a boolean internal logic. Otherwise, if we picked two different "points" in D, given a function $g : D \to R$, we would have

$$x \tag{10.268}$$

10.4.2 Relative cohesion

Relative shape modality :

$$\int^{\mathrm{rel}} = \int X +_{\mathfrak{R}X} X \tag{10.269}$$

Relative flat modality :

$$\flat^{\mathrm{rel}} X = \flat X \times_{\mathfrak{I}X} X \tag{10.270}$$

$$(\int^{\mathrm{rel}} \dashv \flat^{\mathrm{rel}}) \tag{10.271}$$

(co)modal types : infinitesimal types (such that $\flat \to \mathfrak{I}$) is an equivalence

Relative shape preserves the terminal object

Relative sharp modality $\sharp^{\mathrm{rel}}$

10.4.3 Non-standard analysis

While a perfectly good model for differential structures in a topos, there are other alternative models to synthetic differential geometry outside of formal spaces.

As a space, infinitesimally thickened points are entirely structureless. Even beyond the single point, it also has a completely trivial mereology, so that its only differentiating aspect from a normal point is given by its sheaf structure.

Unlike actual points, there are multiple possible maps $X \to D$, including the core $\mathrm{Aut}(D)$

[Due to the nilpotence of the algebra?]

An alternative to this is to have invertible infinitesimals in the given algebra. There are a variety of models of such infinitesimals, such as the hyperreals, the generalized numbers or the asymptotic numbers

10.4.4 Differential cohesion of the Cahier topos

[147]

The various topos we have seen previously will not typically be models of differential cohesion, except in a trivial way, that is, where every object is its own reduced object

$$\Re X \cong X \tag{10.272}$$

The most common model used to define a differential cohesion in a "geometric" way is to use the Weil algebras we saw earlier5.6. To generalize this infinitesimally thickened point to a full on infinitesimal geometry, we will use, in addition to the Weil algebra, the smooth R-algebra associated with smooth spaces.

[...]

This gives us a sheaf topos on the site of formal Cartesian space **FormalCartSp**, where every object is the product of a Cartesian space with a Weil algebra :

$$\mathbb{R}^n \times \ell W \tag{10.273}$$

Theorem 10.4.6 *Isomorphism*

$$C^\infty(M, W) \cong C^\infty(M) \otimes W \tag{10.274}$$

via

$$(fw)(p) = f(p)x \tag{10.275}$$

[Is the "infinitesimally thickened point", ie an infinitesimal neighbourhood, guaranteed to be infinitesimal in the sense that $\flat \cong \int$, or is that specific to the Cahier topos, diff. with NSA]

10.4.5 The elastic Sierpinski topos

The simplest non-trivial case of an elastic topos is given by the infinitesimally thickened Sierpienski site, which is the presheaf on the total order $\Delta[\vec{\mathbf{2}}] = 0 \to 1 \to 2$. This is the non-spatial frame we saw in 4.5.4.

$$0 \longrightarrow 1 \longrightarrow 2$$

Theorem 10.4.7 *The presheaves on $\Delta[\vec{\mathbf{2}}]$ correspond to sequences of sets $A_2 \to A_1 \to A_0$.*

As a differential cohesion, we need to look at its Q-category, which is simply its pairing with the Sierpienski topos as a cohesive category.

$$(i_! \dashv i^* \dashv i_* \dashv i^!) : \mathrm{PSh}(\mathbf{\Delta}[\vec{\mathbf{3}}]) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \\ \xrightarrow{H} \end{array} \mathrm{PSh}(\mathbf{\Delta}[\vec{\mathbf{2}}])$$

In this context, the semantics we can give it is that 0 corresponds to a point, 1 correspond to an infinitesimally close point, and 2 is another point.

As an infinitesimal thickening, it requires

Theorem 10.4.8 *The infinitesimally thickened Sierpinski topos forms a Q-category with the Sierpinski topos.*

Proof 10.4.5

10.4.6 Crystalline cohomology

10.4.7 The standard limit

The derivative on a differential cohesive topos can be defined with the use of the jet comonad, but does this match up with the usual definition of derivatives with limits?

Function : map from the space to a ring object of the topos

Value at a point : stalk, net over that point

10.4.8 Negation

Interaction of elastic modalities with negations of cohesion?

$$\mathfrak{I}\bar{\sharp} \tag{10.276}$$

Colimit commutes with left adjoint :

$$\begin{aligned} \mathfrak{I}\bar{\flat}X &= \mathfrak{I}\mathrm{Cofib}(\flat X \to X) && (10.277) \\ &= \mathrm{Cofib}(\mathfrak{I}\flat X \to \mathfrak{I}X) && (10.278) \\ &= \mathrm{Cofib}(\flat X \to \mathfrak{I}X) && (10.279) \\ &= \mathrm{Cofib}(\flat\mathfrak{I}X \to \mathfrak{I}X) && (10.280) \\ & && (10.281) \end{aligned}$$

10.4.9 Interpretation

10.5 Solid substance

Given the opposition $\mathfrak{R} \dashv \mathfrak{I}$, a *ps*-unity, we seek another sublation, ie we want a further opposition $(\mathfrak{R}' \dashv \mathfrak{I}')$ obeying

$$\begin{aligned} \mathfrak{R}'\mathfrak{R} &= \mathfrak{R} && (10.282) \\ \mathfrak{I}'\mathfrak{I} &= \mathfrak{I} && (10.283) \end{aligned}$$

We will look here specifically at a left sublation, so that furthermore

$$\mathfrak{I}'\mathfrak{R} = \mathfrak{R} \tag{10.284}$$

We will call the opposition of this sublation the opposition of *solidity*, given by the bosonic modality $\rightsquigarrow$ and the rheonomic modality Rh,

$$(\rightsquigarrow \dashv \mathrm{Rh}) \tag{10.285}$$

where the sublation gives us the identities

$$\begin{aligned} \rightsquigarrow \mathfrak{R} &= \mathfrak{R} && (10.286) \\ \mathrm{Rh}\mathfrak{I} &= \mathfrak{I} && (10.287) \\ \mathrm{Rh}\mathfrak{R} &= \mathfrak{R} && (10.288) \end{aligned}$$

Meaning that those modalities have no effect on a reduced space and that the rheonomic modality has no effect on an infinitesimal space.

$$\Re X \to \overset{\rightsquigarrow}{X} \to X \tag{10.289}$$

Partial reduction of the quality type[?], from entirely reduced to only reduced as far as the infinitesimal component goes

$$x \tag{10.290}$$

Bosonic v. fermionic spaces

Definition 10.5.1 *A k Grassmann algebra on a k-vector space is an algebra defined by the anticommuting product*

$$x \wedge y = -y \wedge x \tag{10.291}$$

As this is a non-commuting algebra, the interpretation of its dual as a space is much more complicated. There is no obvious way to define a sober space from a non-commutative algebra[148], nor a locale[149], nor a sheaf[150], so that the geometric interpretation of that dual should be taken cautiously.

This dual space is the superpoint :

Definition 10.5.2 *The* superpoint *is the formal dual to the Grassmann algebra [...]*

What is its lattice structure

The general naming scheme of the opposition of solidity is meant to reflect Hegel's talking points on light and matter, not from the Science of Logic but Hegel's Encyclopaedia of the Philosophical Sciences [151].

"Light behaves as a general identity, initially in this determination of diversity, or the determination by the understanding of the moment of totality, then to concrete matter as an external and other entity, as to darkening. This contact and external darkening of the one by the other is colour."

meant to mirror the parallel that light relates to the bosonic moment as it is a bosonic particle while matter is of a fermionic nature.

This is probably the least convincing parallel in the hierarchy of nlab's objective logic here, as while the rigidity of matter can be connected to its fermionic nature, this is not something that can be deduced simply from the raw properties of fermionic spaces and is very much dependent on the model of physics that we use, which is a much more empirical fact than purely metaphysical. It is probably just included as solidity for purely aesthetic reasons, much like the I Ching notation for the classification of cohesive toposx

11

Higher order objective logic

We have done everything thus far in the context of category theory, but a generalization of this can be done in the context of ∞-categories. This will be useful as the most interesting parts of physics work best in the context of category theory where we use the full ∞-category context to account for gauge groups in a more natural way.

Most concepts are substituted easily enough. The domain of discourse is given by some ∞-topos, and the moments upon it are given by ∞-monads.

One of those construction is the notion of *accidence* : we say that a moment $\bigcirc$ is exhibited by a type J if $\bigcirc$ is a J-homotopy localization :

$$\bigcirc \cong \mathrm{loc}_J \tag{11.1}$$

which implies trivially that the object J has none of the qualities of $\bigcirc$:

$$\bigcirc J \cong * \tag{11.2}$$

In essence, a moment is exhibited by J if it contains none of the qualities of J [?]

Is there some property such that if $\Delta J = J$, then $\bigcirc \Delta X = 1$ or something

Homotopy localization : for an object $A \in \mathbf{C}$, take the class of morphisms W_A

$$X \times (A \overset{\exists !}{\to} *) : X \times A \xrightarrow{p_1} X \tag{11.3}$$

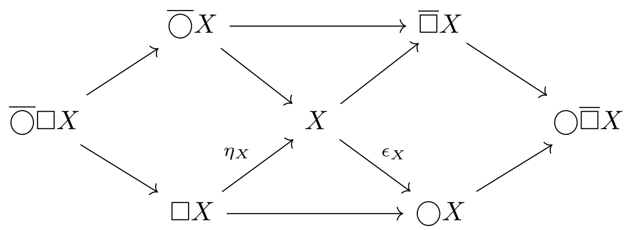
what means

”The idea is that if A is, or is regarded as, an interval object, then “geometric” left homotopies between morphisms $X \to Y$ are, or would be, given by morphisms out of $X \times A$, and hence forcing the projections $X \times A \to X$ to be equivalences means forcing all morphisms to be homotopy invariant with respect to A.”

Example using the real line for homotopy localization?

11.1 Negation

An aspect of note for higher order objective logic is the relations given by its adjunction and their determinate negations, $(\Box, \bigcirc, \overline{\Box}, \overline{\bigcirc})$. As we’ve seen in the 1-categorical case, various applications of this quadruple of modalities gives us the following commuting diagram in the case of an *sp*-unity 11.1.1



where both squares are pullbacks. In the ∞-categorical case however, we naturally replace the notion of determinate negation from the (co)fiber of the (co)modality to the homotopy (co)fiber of the ∞-(co)modality.

$$\overline{\bigcirc}X \cong \mathrm{HoFib}_p(X \to \bigcirc X) \tag{11.4}$$
$$\overline{\Box}X \cong \mathrm{HoCofib}(\Box X \to X) \tag{11.5}$$

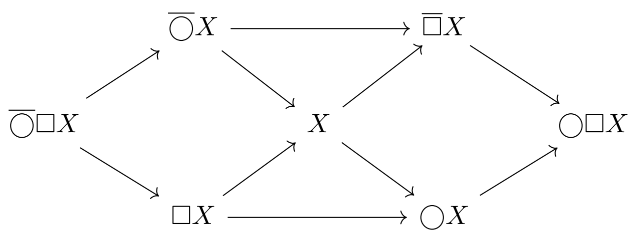
Fracturing for determinate negation, see the hexagonal diagram[116]

A special case of the application of negation in the ∞-categorical case is that for a stable object,

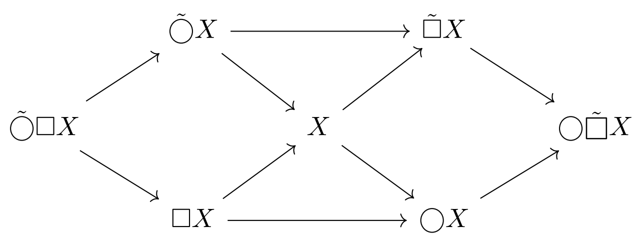
Theorem 11.1.1 *For a stable object X, any pushout square is also a pullback square.*

From this property, we get

Theorem 11.1.2 *For an sp-unity $(\bigcirc \dashv \Box)$ on a stable ∞-category* $\mathrm{Stab}(\mathbf{H})$*, the modal hexagon is equivalently*



or



and has the property that

- *both squares are homotopy pullbacks and pushouts*
- *the boundary sequences are long homotopy fiber sequences*

Proof 11.1.1

This allows us to do the decomposition of a stable type along one of those way :

$$X \cong (\bigcirc X) \underset{\bigcirc\overline{\Box}X}{\oplus} (\overline{\Box}X) \tag{11.6}$$

$$X \cong (\overline{\bigcirc}X) \underset{\overline{\bigcirc}\Box X}{\oplus} (\Box X) \tag{11.7}$$

$$X \cong (\bigcirc X) \underset{\bigcirc\tilde{\Box}X}{\oplus} (\tilde{\Box}X) \tag{11.8}$$

$$X \cong (\tilde{\bigcirc}X) \underset{\tilde{\bigcirc}\Box X}{\oplus} (\Box X) \tag{11.9}$$

11.1.1 Modal hexagons

In the case of an ∞-category, given a pointed object $p : 1 \to X$, we have the following long fiber sequence for some morphism $f : X \to Y$:

$$\ldots \to \Omega\mathrm{Fib}_p(f) \to \Omega X \to \Omega Y \to \mathrm{Fib}_p(f) \to X \to Y \tag{11.10}$$

This means that for a modality $\bigcirc$, we have the sequence

$$\ldots \to \Omega\overline{\bigcirc}_p X \to \Omega X \to \Omega \bigcirc X \to \overline{\bigcirc}_p X \to X \to \bigcirc X \tag{11.11}$$

If furthermore, X admits a delooping $\mathbf{B}X$, we can look at its sequence, using the identity $X \cong \Omega(\mathbf{B}X)$ [and $\mathbf{B}$ commutes with the fiber??? idk]

$$\ldots \to \overline{\bigcirc}_p X \to X \to \bigcirc X \to \overline{\bigcirc}_p(\mathbf{B}X) \to \mathbf{B}X \to \bigcirc\mathbf{B}X \tag{11.12}$$

From what we have seen, given an *sp*-unity $\bigcirc \dashv \square$, we have the following diagrams, for the modality

$$\overline{\bigcirc}_p X \xrightarrow{p^*\eta^{\bigcirc}_X} X \xrightarrow{\eta^{\bigcirc}_X} \bigcirc X$$

and the comodality

$$\square X \xrightarrow{\epsilon^{\square}_X} X \xrightarrow{!^*_X \epsilon^{\square}_X} \overline{\square} X$$

We can furthermore apply the modality and its negation to the comodality, and using the property of *sp*-unities, $\bigcirc\square \cong \square$

[REDO ALL]

[Pullback square : apply the negation of modality to the comodality fibration?]

Given the ptp transform, its naturality square with respect to $\epsilon^{\overline{\bigcirc}}$

$$\begin{array}{ccccc}
\overline{\bigcirc}_p \square X & \xrightarrow{\epsilon^{\overline{\bigcirc}}(\epsilon^{\square}_X)} & \overline{\bigcirc}_p X & \xrightarrow{\epsilon^{\overline{\bigcirc}}(\eta^{\bigcirc}_X)} & \overline{\bigcirc}_p \bigcirc X \cong 1 \\
{\scriptstyle \epsilon^{\overline{\bigcirc}_p}_{\square X}}\downarrow & & \downarrow{\scriptstyle \epsilon^{\overline{\bigcirc}_p}_X} & & \downarrow{\scriptstyle \epsilon^{\overline{\bigcirc}_p}_{\bigcirc X}} \\
\square X & \xrightarrow[\epsilon^{\square}_X]{} & X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X
\end{array}$$

or, replacing some of those morphisms,

$$\begin{array}{ccccc}
\overline{\bigcirc}_p \square X & \xrightarrow{\epsilon^{\overline{\bigcirc}}(\epsilon^{\square}_X)} & \overline{\bigcirc}_p X & \xrightarrow{!} & 1 \\
{\scriptstyle p^*\eta^{\bigcirc}_{\square X}}\downarrow & & \downarrow{\scriptstyle p^*\eta^{\bigcirc}_X} & & \downarrow{\scriptstyle p} \\
\square X & \xrightarrow[\epsilon^{\square}_X]{} & X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X
\end{array}$$

From 9.1.8, we have that

$$\eta^{\bigcirc}_{\Box X} \cong \eta^{\bigcirc}_{X} \circ \epsilon^{\Box}_{X} \tag{11.13}$$

so that we can see that both the right square and the whole square of this diagram are pullbacks, and therefore so is the left square by the pullback pasting law.

The naturality of the ptp transform with respect to $\eta^{\overline{\Box}}$ is given by

$$\begin{CD}
\Box X @>{\epsilon^{\Box}_{X}}>> X @>{\eta^{\bigcirc}_{X}}>> \bigcirc X \\
@V{\eta^{\overline{\Box}}_{\Box X}}VV @VV{\eta^{\overline{\Box}}_{X}}V @VV{\eta^{\overline{\Box}}_{\bigcirc X}}V \\
1 \cong \overline{\Box}\Box X @>>{\eta^{\overline{\Box}}(\epsilon^{\Box}_{X})}> \overline{\Box} X @>>{\eta^{\overline{\Box}}(\eta^{\bigcirc}_{X})}> \overline{\Box} \bigcirc X
\end{CD}$$

$$\begin{CD}
\Box X @>{\epsilon^{\Box}_{X}}>> X @>{\eta^{\bigcirc}_{X}}>> \bigcirc X \\
@V{!}VV @VV{\eta^{\overline{\Box}}_{X}}V @VV{\eta^{\overline{\Box}}_{\bigcirc X}}V \\
1 @>>{\eta^{\overline{\Box}}(\epsilon^{\Box}_{X})}> \overline{\Box} X @>>{\eta^{\overline{\Box}}(\eta^{\bigcirc}_{X})}> \overline{\Box} \bigcirc X
\end{CD}$$

Naturality for comodality fibration

$$\begin{CD}
\Box X @>{\epsilon^{\Box}_{X}}>> X @>{!^{*}_{X}\epsilon^{\Box}_{X}}>> \overline{\Box} X \\
@V{\eta^{\bigcirc}_{\Box X}}VV @VV{\eta^{\bigcirc}_{X}}V @VV{\eta^{\bigcirc}_{\overline{\Box} X}}V \\
\bigcirc\Box X @>>{\eta^{\bigcirc}(\epsilon^{\Box}_{X})}> \bigcirc X @>>> \bigcirc\overline{\Box} X
\end{CD}$$

Do $\bigcirc$ and $\overline{\Box}$ commute?

Right whiskering of $\eta^{\bigcirc}_{X}$?

So that this sequence can be connected to the triangle given by X, $\Box X$ and $\bigcirc X$ commute.

If we apply the negation of the modality to the cofibration $\Box X \to X \to \overline{\Box} X$, we get

$$\overline{\bigcirc}_{p}\Box X \to \overline{\bigcirc}_{p} X \to \overline{\bigcirc}_{p}\overline{\Box} X \cong \overline{\Box} X \tag{11.14}$$

Prove that

$$p^{*}\eta^{\bigcirc}_{X} \circ \overline{\bigcirc}_{p}(\epsilon^{\Box}_{X}) \cong p^{*}\eta^{\bigcirc}_{\Box X} \circ \epsilon^{\Box}_{X} \tag{11.15}$$

so that we have the square $\overline{\bigcirc}_p \Box X, \overline{\bigcirc}_p X, \Box X, X$

Furthermore, this square is a pullback. If we consider the pullback squares

$$\begin{array}{ccccc}
\overline{\bigcirc}_p \Box X & \xrightarrow{\overline{\bigcirc}_p(\epsilon^{\Box}_X)} & \overline{\bigcirc}_p X & \xrightarrow{!_{\overline{\bigcirc}_p X}} & 1 \\
{\scriptstyle p^*\eta^{\bigcirc}_{\Box X}}\downarrow & & \downarrow{\scriptstyle p^*\eta^{\bigcirc}_X} & & \downarrow{\scriptstyle p} \\
\Box X & \xrightarrow{\epsilon^{\Box}_X} & X & \xrightarrow{\eta^{\bigcirc}_X} & \bigcirc\Box X \cong \Box X
\end{array}$$

[show that it's commutative : top arrow is just !, bottom arrow is

$$\eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \cong \eta^{\bigcirc}_{\Box X} \tag{11.16}$$

this is just the fiber diagram of $\Box \to \bigcirc\Box$]

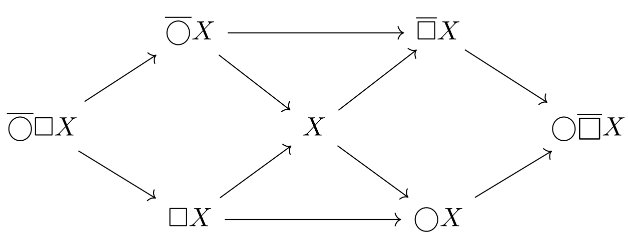
By the pasting law for pullbacks, this means that the left square is a pullback.

Other pullback :

$$\begin{array}{ccccc}
X & \xrightarrow{\overline{\bigcirc}_p(\epsilon^{\Box}_X)} & \overline{\Box} X & \xrightarrow{!_{\overline{\bigcirc}_p X}} & 1 \\
{\scriptstyle p^*\eta^{\bigcirc}_{\Box X}}\downarrow & & \downarrow{\scriptstyle p^*\eta^{\bigcirc}_X} & & \downarrow{\scriptstyle p} \\
\bigcirc X & \xrightarrow{\epsilon^{\Box}_X} & \bigcirc\overline{\Box} X & \xrightarrow{\eta^{\bigcirc}_X} & \Box X \cong \bigcirc\Box X
\end{array}$$

[...]

This means that given an *sp*-unity with determinate negation, we have the following commutative diagram :



where both squares are pullbacks. This is the *modal hexagon.* This means that we have

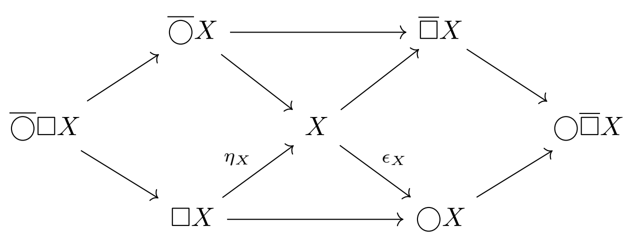
$$X \cong \Box X \times_{\overline{\bigcirc}\Box X} \overline{\bigcirc} X \tag{11.17}$$

In particular, if $\Box$ is entirely a $\bigcirc$ moment, so that

"However, this fiber depends on a chosen basepoint, so it only makes sense on types which have only one constituent (but possibly this constituent has (higher) equalities), or, thought of homotopically, have only one connected component. In this case,$\bigcirc T$ contains that part of the structure of T that is trivialized by $T \to \bigcirc T$. Note that $\Box$ and $\overline{\Box}$ (or the dual notions) do not form a unity of oppositions. However, for each object T , a sequence $\Box T \to T \to \overline{\Box} T$ exists and this sequence decomposes each T , in the sense that T could be reconstructed from its aspects under a moment and its negative, as well as their relation. This is not generally true for unities of oppositions."

Is there a relation between $\overline{\Box}$ and $\bigcirc$, and $\overline{\bigcirc}$ and $\Box$, for both *ps* and *sp* unities?

A useful identity that we will look at later on is given for *sp*-unities ($\bigcirc \dashv \Box$) by the following diagram



While the full utility of this diagram comes from the use of higher categories, it remains however valid in the 1-categorical case

Proof : $\bigcirc\overline{\Box} X$ is simply 1

Is $\overline{\bigcirc}\Box X \cong 1$?

$$\begin{aligned} \overline{\bigcirc}\Box X &= \mathrm{Fib}(\epsilon_{\Box X} : \Box X \to \bigcirc\Box X) && (11.18) \\ &= \mathrm{Fib}(\epsilon_{\Box X} : \Box X \to \Box X) && (11.19) \\ & && (11.20) \end{aligned}$$

Determinate negation as completion/residue v. modalities as localizations/torsions?

For the case of a *ps*-unity ($\Box \dashv \bigcirc$) :

Like the case of the *sp*-unity, we need to look at the modalities of the various fibrations. As usual, we have the two basic diagrams

$$\overline{\bigcirc}_p X \xrightarrow{p^* \eta_X^{\bigcirc}} X \xrightarrow{\eta_X^{\bigcirc}} \bigcirc X$$

$$\Box X \xrightarrow{\epsilon_X^{\Box}} X \xrightarrow{!_X^* \epsilon_X^{\Box}} \overline{\Box} X$$

Example 11.1.1

11.2 Being and nothingness

In the case of being and nothingness, the main change is simply to replace the terminal topos with the terminal ∞-topos, which is the topos defined by a single k-morphism for every level k connecting level $k-1$ morphisms to themselves. There is not a lot to say on this level, as most everything related to this unity is roughly similar to the 1-categorical case.

Theorem 11.2.1 *The ground opposition is given by the adjoint pair of ∞ modalities $\Box_{\varnothing} \dashv \bigcirc_*$, which map objects of the ∞-topos to the initial and terminal object.*

[Negation of the nothing modality?]

The next level is given by the localization by the double negation.

11.3 Cohesion

Cohesion is the first level of the hierarchy for which the higher morphisms start to be of importance.

Is the double negation subtopos of an ∞-topos also boolean?

As in the 1-categorical case, we consider the category of (∞-)sheaves over the terminal object as our base object, the ∞-category of ∞-groupoids,

$$\mathrm{Sh}_{\infty}(\mathbf{1}) \cong \infty\mathbf{Grpd} \tag{11.21}$$

where we pick as our forgetful functor the global section functor mapping between every two objects their groupoid, the hom-groupoid of the terminal groupoid.

$$\Gamma = \mathrm{Hom}_{\mathbf{H}}(1, -) \tag{11.22}$$

Example 11.3.1 *Consider a topological space (in some topological topos like* **Smooth***), with higher morphisms the homotopy transformations between lower morphisms given by the interval object of the real line (the A^1 homotopy).*

The global section functor of such a space is the set of its points, with the endomorphisms on those points being the homotopy equivalence classes between those points, and all higher morphisms likewise being higher homotopy equivalences.

For instance, given the circle S^1, we have

$$\tau_0 \Gamma(S^1) = |S^1| \tag{11.23}$$

For a point $p \in |S^1|$, its core is the equivalence class of paths from the interval, which for a circle is classified by the winding number :

$$\mathrm{core}(p) = \mathbb{Z} \tag{11.24}$$

As the higher homotopies of the circle are all trivial, any higher k-morphism is simply given by the identity.

As with before, we take the left adjoint functor of locally constant ∞-stacks LConst, which given a groupoid $\mathcal{G} \in \infty\mathbf{Grpd}$ associates the ∞-stack of

$$\mathrm{LConst}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{LConst}(?)) \tag{11.25}$$

This adjunction always exists for any ∞-topos (?), giving us the flat ∞-modality $\flat$:

$$\flat = \mathrm{LConst} \circ \Gamma \tag{11.26}$$

which first maps the object to its bare ∞-groupoid and then back to our ∞-topos, but importantly, unlike the 1-categorical case, while the (local) spatial information is lost, its *homotopical information*, global spatial information, is preserved, at least for each point. While we would not be able to reconstruct the underlying space(?), we still have all the equivalence classes of loop spaces up to homotopy preserved.

Example 11.3.2 *For our circle in the topos of smooth ∞-groupoids, its flat modality will be given by some object which is the coproduct of a single point with the core $\mathbb{Z}$ on each point.*

Definition 11.3.1 *If Γ admits a right adjoint functor, we will call it the codiscrete functor* CoDisc. *A topos with this right adjoint is ∞-locally local.*

Definition 11.3.2 *If* Disc *admits a left adjoint functor, we will call it the path ∞-groupoid functor,* Π. *A topos with this left adjoint is called ∞-locally connected.*

Definition 11.3.3 *If it exists, the left adjoint to* Disc $\cong$ LConst *is the* path ∞-groupoid *functor,*

$$\Pi : \mathbf{H} \to \infty\mathbf{Grpd} \tag{11.27}$$

Properties :

$$\mathrm{Hom}_{\mathbf{H}}(X, \mathrm{Disc}(G)) \cong \mathrm{Hom}_{\infty\mathbf{Grpd}}(\Pi(X), G) \tag{11.28}$$

The groupoid morphisms between the path ∞-groupoid $\Pi(X)$ and any other groupoid G is isomorphic to the hom-groupoid of that space with the discrete groupoid G

The modality Π maps the space given to its homotopy ∞-group. This is an expression of the homotopy groups at a base point being all isomorphic in a path-connected component, ie if $\int X = 1$ [?].

If we look at the 0-truncation, this is simply the same as the connected component functor of the 1-categorical case.

1-morphisms on a connected component : Given a morphism on the stack, ie on some U we have some non-trivial morphisms on an element of $X(U)$,

For cohesion, our subtopos will be some boolean ∞-category, typically the ∞**Grpd** category. As in the 1-categorical case, any ∞-stack ∞-topos admits a canonical geometric morphism and a left adjoint for it,

$$(\mathrm{LConst} \dashv \Gamma) : \mathbf{H} \underset{\mathrm{LConst}}{\overset{\Gamma}{\rightleftarrows}} \infty\mathbf{Grpd} \tag{11.29}$$

As Π_0 refers specifically to the fundamental group, ie the first homotopy group of the space, the higher order case will be Π, the full fundamental groupoid of the space.

Example 11.3.3 *On the circle S^1, defined by some smooth 0-type sheaf $S^1(\mathbb{R}^1) = 3$ with the usual good cover on a circle,*

[152]

11.3.1 Real cohesion

An extra condition to emulate the most common of spaces is that of real cohesion, saying that the shape modality is exhibited by the localization of the affine line object. This is a property which is specific to the homotopical case, as the notion of localization by an object is mostly relevant there.

In many cases, we will ask that the shape modality correspond to a retraction to a point of the (path)-connected components of the space. Using the usual notion of path connectedness, two points are path connected if there is a path $[0, 1] \to X$ between them, in which case the space is contractible if for any two points, there is a path between them.

The categorical equivalent is to consider the path space object I, and localization by I, typically $\mathbb{R}$.

$$\int \cong \mathrm{loc}_I \tag{11.30}$$

11.3.2 Negations

In the ∞-categorical case, the negation

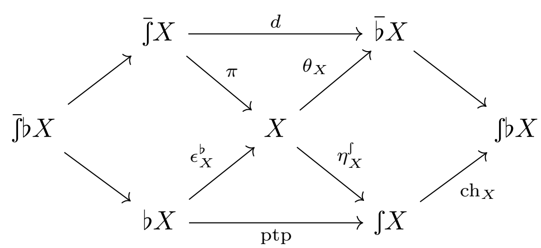
Theorem 11.3.1 *The negation of the shape modality $\overline{\int}$ is the ∞-universal cover.*

Proof 11.3.1 *From the negation of the shape modality, the object obtained by*

Definition 11.3.4 $\overline{\flat}$ *is the infinitesimal remainder[?]*

11.3.3 Cohesive hexagon

As the unity of opposites $(\int \dashv \flat)$ is an *sp*-unity, we can look at its modal hexagon



ch_X is the *Chern character* of the object X

$$\mathrm{ch}_X = \Pi(\theta_X) \tag{11.31}$$

de Rham modalities

Poincaré lemma : $\flat\mathbb{R} \cong \Omega^\bullet$??? Apply to fracture square

Example 11.3.4 *For the line object R, being a group object, its*

Example 11.3.5 *For the circle S^1, also known as the group object* U(1), *its loop space is homotopy equivalent to the group $\mathbb{Z}$, and its delooping is $\mathbb{Z}$.*

11.4 Singular cohesion

11.5 Elasticity

11.6 Solidity

12

Nature

12.1 Mechanics topos

The general topos typically used by nlab to respond to the requirements of all those modalities is the *super formal smooth ∞-groupoid*. This is the ∞-sheaf on a special site composed from the category of Cartesian spaces (to give the cohesion), the category of infinitesimally thickened points (to give the elasticity), and the category of superpoints (to give the solidity).

We have already seen the category of Cartesian spaces in detail, so let's now look into infinitesimally thickened points.

Infinitesimally thickened points are a geometrical realization of the formal notion of infinitesimals as provided by Weil algebras.

$$(x, \epsilon) \mapsto f(x, \epsilon) = f(x) \tag{12.1}$$

$$\mathbf{SuperFormalSmooth} = \mathrm{Sh}() \tag{12.2}$$

[...]

Lawvere's topos : nlab

[153]

12.2 Gauge theory

In a real cohesive topos, $\int$ will correspond to the path ∞-groupoid of an object. As we've seen, the path groupoid of an object lacks too much informations to give us the full informations of a connection, but as it identifies any two homotopic paths, it is however good enough for the space of flat connections.

$$\mathrm{Hom}(\int X, A) \tag{12.3}$$

Assignment to every point of X a fiber in A, to every path an equivalence of those fibers, ...

12.3 Geometry

The confluence of infinitesimal geometry, higher order morphisms and cohesion makes this system a good setting in which to work in the general notion of a geometrical theory of physics, such as general relativity or other such theories.

Infinitesimal disk :

$$D^X_p = 1 \times_{\Im(X)} X \tag{12.4}$$

Order k

$$D^X_{p,k} = 1 \times_{\Im(X)} \Im_{k)} X \tag{12.5}$$

[154]

Theorem 12.3.1 *In* **FormalSmooth**$_\infty$, *The automorphism group of the formal k-dimensional disk D_k is isomorphic to the jet group* $\mathrm{GL}_k(n)$

Proof 12.3.1 *Given the Weil algebra of the formal disk,*

$$D^n_k = \{x \in \mathbb{R}^n \mid x^k = 0\} \tag{12.6}$$

Any automorphism $\phi : D^n_k \to D^n_k$ will have the action, by microlinearity

$$\phi(x) = \phi(0) + cx \tag{12.7}$$

Automorphism in a category of rings : maps 0 *to* 0*? Linear? idk Inverse operation :*

$$\phi^{-1}(cx) = cc^{-1}x = x \tag{12.8}$$

Is the automorphism in the dual category isomorphic to the automorphism in the original category [137]

In particular, the first order jet group is simply the general linear group,

$$\mathrm{GL}(n) = \mathrm{Aut}(D^n) \tag{12.9}$$

We will define the jet group $\mathrm{GL}_k(V)$ in general to be the automorphism group

Definition 12.3.1 *The* jet group *of the typical infinitesimal disk of an object X is*

$$\mathrm{GL}(X) = \mathrm{Aut}(D_k^X) \tag{12.10}$$

Definition 12.3.2 *The ∞-jet bundle of a smooth groupoid is given by the action of the jet comonad*

$$J^\infty X = \mathrm{Jet}(X) \tag{12.11}$$

Definition 12.3.3 *The k-th jet bundle is the object given by the action of the k-th order jet comonad, where*

In particular, given the line object $\mathbb{R} \in \mathbf{FormalSmooth}_\infty$, the tangent bundle is the first jet bundle of the bundle given by the product projection. ie for the product $\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$, the tangent bundle of M is given by

$$T\mathbb{R} \times TM = \mathrm{Jet}(\mathrm{pr}_1) \tag{12.12}$$

Alternative viewpoint : the line $\mathbb{R}$ is reduced to the linelet D_1^1, $\pi : M \to$

Frame bundle : For

12.4 Quantization

Definition 12.4.1 *On a Cartesian monoidal ∞-category* **H**

$$\mathrm{Mod}(X) \tag{12.13}$$

12.5 Formal ontology

While the examples from physics can be useful, perhaps a more basic description of ontology would be more in line with the original spirit of the objective logic.

there exists a few attempts at formal descriptions of general ontologies

[...]

Topos for objects simply being assemblies of properties : products of discrete sites, or at least finite sites?

Consider a finite site S of elements $\{P_A, P_B, P_C, \ldots\}$. The only covering families on S are simply either the empty coverage or $\{A \hookrightarrow A\}$, giving a total of $2^{|S|}$ possible coverages.

A presheaf $\mathrm{PSh}(S)$

$$F(P_i) = \{\bullet, \bullet, \ldots\} \tag{12.14}$$

In other words, the sheaf associates to each property P_i some cardinality.

As the category is discrete, no need to concern about

13

The Ausdehnungslehre

One early attempt at mathematization of similar philosophical notions was the Ausdehnungslehre of Grassmann[5]

[155, 156, 157]

"Each particular existent brought to be by thought can come about in one of two ways, either through a simple act of generation or through a twofold act of placement and conjunction. That arising in the first way is the continuous form, or magnitude in the narrow sense, while that arising in the second way is the discrete or conjunctive form.

The simple act of becoming yields the continuous form. For the discrete form, that posited for conjunction is of course also produced by thought, but for the act of conjunction it appears as given; and the structure produced from the givens as the discrete form is a mere correlative thought. The concept of continuous becoming is more easily grasped if one first treats it by analogy with the more familiar discrete mode of emergence. Thus since in continuous generation what has already become is always retained in that correlative thought together with the newly emerging at the moment of its emergence, so by analogy one discerns in the concept of the continuous form a twofold act of placement and conjunction, but in this case the two are united in a single act, and thus proceed together as an indivisible unit. Thus, of the two parts of the conjunction (temporarily retaining this expression for the sake of the analogy), the one has already become, but the other newly emerges at the moment of conjunction itself, and thus is not already complete prior to conjunction. Both acts, placement and conjunction, are thus merged together so that conjunction cannot precede placement, nor is placement possible before conjunction. Or again, speaking in the sense appropriate for the continuous, that which newly emerges does so precisely upon that which has already become, and thus, in that moment of becoming itself, appears in its further course as growing there.

The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can also be regarded as continuous, and the continuous as discrete. The discrete may be regarded as continuous if that conjoined is itself again regarded as given, and the act of conjunction as a moment of becoming. And the continuous can be regarded as discrete if every moment of becoming is regarded as a mere conjunctive act, and that so conjoined as a given for the conjunction."

14
Kant's categories

[158]

Quantity : unity, plurality, totality

Quality : reality, negation, limitation

Relation : inherence and subsistence, causality and dependence, community

Modality : possibility, actuality, necessity

15

Lauter einsen

Cantor's original attempt at set theory[159] involved the notion of *aggregates* (*Mengen*), which is what we would call a sequence today, some ordered association of various objects. If we have objects $a, b, c, \ldots$, their aggregate M is denoted by

$$M = \{a, b, c, \ldots\} \tag{15.1}$$

Despite the notation reminiscent of sets, the order matters here. The notion closer to that of a set is given by an abstraction process $\overline{M}$, in which the order of elements is abstracted away (this would be something akin to an equivalence class under permutation nowadays).

The *cardinality* of the aggregate is given by a further abstraction, $\overline{\overline{M}}$, given by removing the nature of all of its element, leaving only "units" behind :

$$\overline{\overline{M}} = \{\bullet, \bullet, \bullet, \ldots\} \tag{15.2}$$

$\bullet$ here is an object for which all characteristics have been removed, and all instances of $\bullet$ are identical. In some sense this is the application of the being modality on its objects : we only have as its property that the object exists, similarly to *das eins*.

Comment from Zermelo [160, p. 351]:

“The attempt to explain the abstraction process leading to the ”cardinal number” by conceiving the cardinal number as a ”set made up of nothing but ones” was not a successful one. For if the ”ones” are all different from one another, as they must be, then they are nothing more than the elements of a newly introduced set that is equivalent to the first one, and we have not made any progress in the abstraction that is now required.”

Relation to the discrete/continuous modality

From Lawvere : The maps between the *Menge* and the *Kardinal* is the adjunction

$$(\text{discrete} \dashv \text{points}) : M \underset{\text{discrete}}{\overset{\text{points}}{\rightleftarrows}} K \tag{15.3}$$

The functor points maps all elements of an object in M to the ”bag of points” of the cardinal in K, the functor discrete sends back

[161]

From Iamblichus :

these, we say, are three fingers, the smallest, the next, and the middle one – consider me as intending them as seen from nearby. But please consider this about them: each of them appears to be equally a finger, I take it, and they are in that way in no way different, whether seen as being in the middle or at the end, whether pale or dark, fat or thin, and so on, for all similar categories.

16

Spaces and quantities

From Lawvere[162]

Distributive v. other categories

Intensive / extensive

"The role of space as an arena for quantitative "becoming" underlies the qualitative transformation of a spatial category into a homotopy category, on which extensive and intensive quantities reappear as homology and cohomology."

Definition 16.0.1 *A* distributive category $\mathbf{C}$ *is a category with finite products and coproducts such that the canonical distributive morphism*

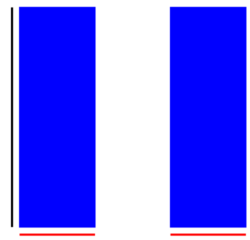
$$(X \times Y) + (X \times Z) \to X \times (Y + Z) \tag{16.1}$$

is an isomorphism, ie there exists a morphism

$$X \times (Y + Z) \to (X \times Y) + (X \times Z) \tag{16.2}$$

that is its inverse.

Distributive categories are typically categories that are "like a space" in some sense, in the context that concerns us. In terms of physical space, we can visualize it like this :



Whether we compose this figure by first spanning each red line along the black line and then summing them, or first summing the two red lines and spanning them along the black line does not matter and will give the same figure.

Example 16.0.1 *If a category is Cartesian closed and has finite coproducts, it is distributive.*

Proof 16.0.1 *In a Cartesian category, the Cartesian product functor $X \times -$ is a left adjoint to the internal hom functor $[X, -]$. As colimits are preserved by left adjoint tmp, we have*

$$(X + Y) \tag{16.3}$$

This means in particular that any topos is distributive, such as **Set** and **Smooth**.

Example 16.0.2 *The category of topological spaces* **Top** *is distributive.*

Proof 16.0.2 *In* **Top***, the product is given by spaces with the product topology,*

$$(X_1, \tau_1) \prod (X_2, \tau_2) = (X_1 \times X_2, \tau_1 \prod \tau_2) \tag{16.4}$$

where the product of two topologies is the topology generated by products of open sets in X_1, X_2 :

$$\tau_1 \prod \tau_2 = \{U \subset\} \tag{16.5}$$

[...] and the coproduct is the disjoint union topology :

[Category of frames?]

As can be seen, those are some of the most archetypal categories of spaces.

Definition 16.0.2 *A* linear category *is a category for which the product and coproduct coincide, called the* biproduct*. For any two objects $X, Y \in \mathbf{C}$, the biproduct is*

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\leftrightarrows}} Y \tag{16.6}$$

Proposition 16.0.1 *A linear category has a zero object* 0*, which is both the initial and terminal object.*

Proof 16.0.3 *This stems from the equivalence of initial and terminal objects as the product and coproduct over the empty diagram.*

Proposition 16.0.2 *In a linear category, there exists a zero morphism* $0 : X \to Y$ *between any two objects* X, Y*, which is the morphism given by the terminal object map* $0 \to Y$ *and the initial object map* $X \to 0$ *:*

$$0_{X,Y} : X \to 0 \to Y \tag{16.7}$$

A linear category is so called due to its natural enriched structure over commutative monoids.

Proposition 16.0.3 *For any two morphisms* $f, g : X \to Y$ *in a linear category, there exists a morphism* $f \oplus g$ *defined by the sequence*

$$X \to X \times X \cong X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \cong Y + Y \to Y \tag{16.8}$$

Proposition 16.0.4 *The sum of two morphisms is associative and commutative*

Proposition 16.0.5 *The zero morphism is the unit element of the sum.*

"in any linear category there is a unique commutative and associative addition operation on the maps with given domain and given codomain, and the composition operation distributes over this addition; thus linear categories are the general contexts in which the basic formalism of linear algebra can be interpreted."

Definition 16.0.3 *If in a linear category every morphism* $f : X \to Y$ *has an inverse denoted* $-f : X \to Y$*, such that* $f \oplus -f = 0$*, then it is enriched over the category of Abelian groups Ab, and is called an* additive category*.*

17

Parmenides

Arguments from Zeno & Parmenides :

“All objects are similar to each other and all objects are different from each other”

Parmenides proceeded: If one is, he said, the one cannot be many? Impossible. Then the one cannot have parts, and cannot be a whole? Why not? Because every part is part of a whole; is it not? Yes. And what is a whole? would not that of which no part is wanting be a whole?

Certainly. Then, in either case, the one would be made up of parts; both as being a whole, and also as having parts?

To be sure. And in either case, the one would be many, and not one? True. But, surely, it ought to be one and not many? It ought. Then, if the one is to remain one, it will not be a whole, and will not have parts?

No. But if it has no parts, it will have neither beginning, middle, nor end; for these would of course be parts of it.

Right. But then, again, a beginning and an end are the limits of everything?

Certainly. Then the one, having neither beginning nor end, is unlimited? Yes, unlimited. And therefore formless; for it cannot partake either of round or straight.

But why? Why, because the round is that of which all the extreme points are equidistant from the centre?

Yes. And the straight is that of which the centre intercepts the view of the extremes?

True. Then the one would have parts and would be many, if it partook either of a straight or of a circular form?

Being is indivisible, since it is equal as a whole; nor is it at any place more, which could keep it from being kept together, nor is it less, but as a whole it is full of Being. Therefore it is as a whole continuous; for Being borders on Being.

[...]

In categorical terms, this argument would obviously make the whole structure collapse immediately, as the ousting of non-being [negation?] would simply make the external logic incapable of producing an argument, but let's see what happens if we merely look at the internal logic.

If we try to look at this in terms of subobjects, the existence of a division, ie some subobject $S \hookrightarrow X$, relates to the existence of a negation, and therefore fundamentally to the existence of the initial object, ie "nothingness".

Given our poset of subobjects $\mathrm{Sub}(X)$, the existence of a division is that of some subobject $(S \hookrightarrow X) \in \mathrm{Sub}(X)$, and if we are considering it as fundamentally divisible, we have to consider at least another subobject S' which is not itself part of S. In other words, there exists no further subobject S'' of both S and S' which is part of both.

We could phrase this more elegantly with the use of the coproduct, as this is a more adapted notion to handle the existence of disjoint objects, but the coproduct will already imply the existence of an initial object, making it a non-starter.

Given this, if we assume that this poset of subobjects is a join semilattice, corresponding to the category being finitely complete (in other words, we just have the notion of logical conjunction in our category), this leads to a contradiction, as this subobject always exists, and this is the initial object. Our two subsets S, S' have the join

$$S \wedge S' = 0 \tag{17.1}$$

[equivalent method with negation from the Heyting implication]

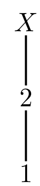
In terms of logic, the subobject relation $S \hookrightarrow X$ defines a proposition p, as well as $S' \hookrightarrow X$ defines p', and their lack of overlap is defined by their

$$p \wedge p' \leftrightarrow \bot \tag{17.2}$$

Theorem 17.0.1 *In a finitely complete category, the existence of disjoint objects requires the existence of an initial object.*

Any notion of division therefore requires some notion of nothingness.

We could of course conceive of posets of subobjects which can circumvent this, such as some entirely linear mereology



but this is not a particularly model of space. In mereological terms, if our space has at least two regions which are not related by parthood, $X \not\subseteq Y$, $Y \not\subseteq X$, and they do not overlap, $\neg(X \cap Y)$, then the

The only object that would avoid such an issue being of course the terminal object, simply having itself as a subobject and none other

18

Bridgman and identity

Another related philosophical issue to this is the notion of *identity*, ie what we can consider to be an individual object, both as a division (what specific parts of the world do we include in the object) as well as equivalence (what can we define to be the same object despite being differing from what we previously defined it as).

cf. Bridgman :

“We must, for example, be able to look continuously at the object, and state that while we look at it, it remains the same. This involves the possession by the object of certain characteristics — it must be a discrete thing, separated from its surroundings by physical discontinuities which persist.”

For instance, if we look at an object in physics, like a ball, to study its motion, this is somewhat arbitrary in a number of ways. If we look at it from an atomic perspective, it is nothing but some assemblage of particles, which we have chosen to consider together as it appears to form an individual object to the human mind. If we were to take the quantum field theory perspective, this would be even worse as due to the non-local nature of all relativistic quantum fields, there would be no obvious way to split

[12, 163]

Definition of a system?

19
Ludwig Gunther

Relation of moments to Ludwig's structuralism?

Axiomatization from Ludwig [164]

[165]

20

Dialectics of nature

When we look at dialectical logic in practice, ie [14], the examples given are much more concrete. We are considering the identity of some *entity*, such as an object, a group, etc, and considering what it means for at entity to be identical to itself. All concrete entities are never identical to themselves, either in time, context, etc.

From Engels :

The law of the transformation of quantity into quality and vice versa; The law of the interpenetration of opposites; The law of the negation of the negation.

Example : an object moving in space, an organization changing, ship of Theseus, etc

To keep things concrete, let's try to consider a simple example of both category theory and dialectical logic, which is an object in motion. We will simply consider the kinematics here and not the dynamics as this is unnecessary.

The simplest case is the $(1+1)$-dimensional case, of a point particle moving along a line $x : L_t \to L_s$.

First notion : We are considering the "identity" of an object under a certain *lens* (ie with respect to its relations with a number of other objects). Its identity is only assured under the full spectrum of those relationships, ie we say that two objects A, B are identical if

$$\forall X \in \mathbf{H},\ \mathrm{Hom}_{\mathbf{H}}(A,X) \cong \mathrm{Hom}_{\mathbf{H}}(B,X),\ \mathrm{Hom}_{\mathbf{H}}(X,A) \cong \mathrm{Hom}_{\mathbf{H}}(X,B) \quad (20.1)$$

Relation to Yoneda? Relation to Hegel's "only the whole is true" thing?

This corresponds to the hom covariant and contravariant functor

$$h^A \cong h^B,\ h_A \cong h_B \tag{20.2}$$

Two objects are identical if their hom functors are naturally isomorphic, ie if there exists a natural transformation

$$\eta : h^A \Rightarrow h^B,\ \epsilon : h_A \Rightarrow h_B \tag{20.3}$$

with two-sided inverses each.

Yoneda embedding :

$$\begin{aligned} \mathrm{Nat}(h_A, h_B) &\cong \mathrm{Hom}_{\mathrm{H}}(B, A) && (20.4) \\ \mathrm{Nat}(h^A, h^B) &\cong \mathrm{Hom}_{\mathrm{H}}(A, B) && (20.5) \\ & && (20.6) \end{aligned}$$

those isomorphisms are elements of this, therefore isomorphic to morphisms from A to B. Therefore A and B are identical in term of their relationships to every other object if they are identical in the more typical sense of the existence of an isomorphism.

As those are functors, this analysis also applies to elements of the objects X, or subobjects. For any two monomorphisms

$$x, y : S \to X \tag{20.7}$$

We say that those elements are identical if the hom functors applied to them [...]

The breakage of the law of identity comes by considering different perspectives. If given an object X [or element $x : \bullet \to X$?], we attempt to use different relations with other objects to "probe" it, its moments and the assessment of its identity will change.

[Abstract example ?]

Expression via the subobject classifier of the topos

Example : case of motion. What does it means for a moving object to *be* in different positions.

Ex : consider the position of an object wrt two different time intervals $[t_a, t_b]$, $[t'_a, t'_b]$, rather than all possible intervals.

Characterization by observables at those different times

Observables : $O_i : \mathrm{Conf} \to \mathbb{R}$ [Isbell duality thing idk]

21
misc

[166]

A
The Hegel dictionary

As a rather esoteric writer, it is not uncommon for commentaries related to Hegel to include a little lexicon explaining the terms involved, and this work will not detract from it.

As different translations of Hegel can show up, the original German word is also included.

Glossary

concept (*Begriff*) Test. 391

determinate (*bestimmt*) Test. 391

essence Test. 391

immediacy Test. 391

—*Birth of Isis, the heavens rained*
Rhind papyrus

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