Commentary on nLab's Science of Logic and other matters

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August 2024

Couples are things whole and things not whole, what is drawn together and what is drawn asunder, the harmonious and the discordant. The one is made up of all things, and all things issue from the one.

Heraclitus

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Chapter 1

Introduction

While not a new phenomenon by any mean, there is a certain recent trend of trying to mathematize certain philosophical theories, in particular ideas relating to dialectics.

An important focus of dialectics is to consider oppositions between concepts. In the case of metaphysics, those oppositions can be for instance

- One / Many
- Sameness / Difference
- Being / Nothing
- Space / Quantity
- General / Particular
- Objective / Subjective
- Qualitative / Quantitative

The oldest of such thoughts goes back to ancient greek traditions, the opposition between the ideas of Parmenides[1] and Heraclitus.

Medieval example : Nicholas of Cusa, De Docta Ignorantia (on learned ignorance) [2].

"Now, I give the name "Maximum" to that than which there cannot be anything greater. But fullness befits what is one. Thus, oneness—which is also being—coincides with Maximality. But if such oneness is altogether free from all relation and contraction, obviously nothing is opposed to it, since it is Absolute Maximality. Thus, the Maximum is the Absolute One which is all things. And all things are in the Maximum (for it is the Maximum); and since nothing is opposed to it, the Minimum likewise coincides with it, and hence the Maximum is also in all things. And because it is absolute, it is, actually, every possible being; it contracts nothing from things, all of which [derive] from it."

The first major author for the exact field that we will broach here is Kant and his transcendental logic [3]

The main author for these recent trends target is Hegel and his Science of Logic [4], where he describes his *objective logic.* The "classical" logic originally described by Aristotle, the logic of propositions and such, is described under the term of *subjective logic*

Heidegger?

The original attempt at the formalization of those ideas (or at least ideas similar to it) was by Grassmann[5], giving his theory of extensive quantities [vector spaces], which while it was commented on and inspired some things, mostly did not go much further.

In modern time, this programme was originally started by Lawvere[6]

"It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy."

nlab[7]

[8, 9, 10, 11, 12, 13]

As is traditional for such types of philosophy, the writings are typically fairly abstract and lacking example. For a more pedagogical exposition, I have tried here to include more examples and demonstrations to such ideas. I am not an expert in algebraic topology by any mean and have tried my best to explain those notions without using notions from this field.

Note that while some of these ideas could be argued to be fairly faithful translations of the philosophical ideas, others seem more to be generally inspired by them, the original notion of unity of opposites being more of a collection of different ideas on that theme than a rigorous construction. While such notions as quantity from the abstraction to pure being seem to have some parallels, I do not believe that Hegel had particularly in mind the concept of a graded algebra when he spoke of the opposition between das Licht and die Körper des Gegensatzes (in particular, this opposition is true regardless of dynamics, and therefore should not be particularly relevant to the solidity of a body), but if the opposition can somehow mirror the adjunction of bosonic and fermionic modalities, why not look into it. The notion of being-for-itself and being-for-one are [according to X] more related to consciousness than etc.

The notions described here are furthermore not firmly rooted in the formalism but merely described by it, as many of these notions are already needed to *define* the formalism. In particular, it is difficult to define any theory without the concept of discrete objects (in our case, by rooting the formalism of categories in the notion of sets, and in fact in general in any thought process requiring to have different ideas as discrete entities). The rooting of the notion of oneness in terminal objects are somewhat superficial, as the very notion of a terminal object already requires the notion of oneness, ie in the unique morphism demanded by universal properties.

Actual applications of dialectical logic [14, 15] also seem fairly disconnected from the formalism developed here, so that we will have to look into it separately.

It is therefore best to keep in mind that

[12, 16, 17, 18, 19, 20, 21, 22]

Transitions Into, With, and From Hegel‘s Science of Logic

Before going into those various formalizations, we will first have a rather in depth look at the formalisms on which these are based, which are type theory and category theory.

Chapter 2

Types

One basic element of the theory discussed is that of *types*[23], which will relate to the notion of categories and logic later on [trinitarity]

Relation with whatever Kant idk

A *type* is, as the name implies, a sort of object that some mathematical object can be. We denote that an object c is of type C by

$$c : C \tag{2.1}$$

and say that c is an *instance* of type C.

The typical simple example, as used in mathematics and computation theory, is that of integers. An integer n is an instance of the integer type, $n : \mathbb{N}$.

Types being themselves a mathematical object, we also have some type for types Type, although to avoid some easily foreseeable Russell style paradox (called the Girard paradox[24]), we will instead use some hierarchy of such types, called type universes :

$$C : \mathrm{Type}_0, \mathrm{Type}_0 : \mathrm{Type}_1, \mathrm{Type}_1 : \mathrm{Type}_2, \dots \tag{2.2}$$

Although as we will not really require much forray into the hierarchy of type universes, we will simply refer Type_0 as Type from now on.

Now given two types $A, B \in \mathrm{Type}$, we can define another type called the *function type* of A to B :

$$f : A \to B \tag{2.3}$$

2.1 Sequent calculus

The transformation rules of statements about types are given by the sequent calculus. In a type theory, the basic entities that we manipulate are the *formulas* and the *judgements*. A formula is simply some statement we have on our type theory, the basic one being the typing of a term. For instance, the statement "a is of type A" is a formula :

$$a : A \tag{2.4}$$

A judgement of a formula is the notion that the formula that we have can be proven in our system. For instance, if we consider the formula that 0 is an integer, $0 : \mathbb{N}$, our judgement is that this is indeed true in our system. This is denoted by the turnstyle $\vdash$:

$$\vdash 0 : \mathbb{N} \tag{2.5}$$

Judgement with assumption :

$$\Gamma \vdash P \tag{2.6}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P}$$

Definition 1 *An* equality type*, denoted by $a = b$, is a formula indicating the equality between two terms.*

Example 2 *A useful example of a type theory in our context is that of intuitionist logic, where the basic type is that of propositions,* $\mathrm{Prop} : \mathrm{Type}$*, with two terms* $\top, \bot : \mathrm{Prop}$

2.2 Dependent types

We speak of *dependent types* for a type that depends on a value, ie a "type" that is actually a function from one type to the universe of types,

$$a : A \vdash B(a) : \mathrm{Type} \tag{2.7}$$

B as a whole is the dependent type of A, with each instance $B(a)$

An example of this is the dependent type of vector spaces, where for some integer type $n : \mathbb{N}$, we associate a type of n-dimensional vector space, $\mathrm{Vect}(n)$, where we have the series of types

$$\mathrm{Vect}(0) : \mathrm{Type}, \mathrm{Vect}(1) : \mathrm{Type}, \mathrm{Vect}(2) : \mathrm{Type}, \ldots \tag{2.8}$$

2.3 Martin-Löf type theory

1

The basis for our type theory will usually be some Martin-Löf type theory, which corresponds to intuitionistic logic in the trinitarianism view, and is generally a rather universal sort of approach to logic (also corresponds to topos). [25]

There are three basic types for it, called the *finite types* :

- The zero type $\mathbf{0}$, or empty type $\varnothing$ or $\bot$, which contains no terms.
- The one type $\mathbf{1}$, or unit type $*$, which contains one canonical term.
- The two type $\mathbf{2}$, which contains two canonical term.

Empty type for nothingness, something that doesn't exist

Unit type for existence

Two type for a choice between two values, such as boolean values.

As with any type theory, those types also give us function types

$\nvdash \to \nvdash$: the empty function (no term or 1 term?) $\nVdash \to \nVdash$: $\nVdash \to \nvDash$: Two functions (can be interpreted as some boolean?)

In addition to these types, we have a variety of *type constructors*, which allow us to construct additional types from those basic types.

First are the constructions which just combine different types together. These are given by the sum type and the product type.

The sum type constructor takes two types A, B : Type and combines it in a single type $A + B$: Type, which can be understood as a type containing the terms of both A and B. This means that there is some map from each of those type to the sum type, denoted

$$\iota_A : A \quad \to \quad A + B \tag{2.9}$$
$$\iota_B : B \quad \to \quad A + B \tag{2.10}$$

: $\sum$

The sum of two types gives us a pair of the individual types, ie

$$a : A, b : B \vdash (a, b) : A \times B \tag{2.11}$$

Dependent sum : the type of the second element might depend on the value of the first.

$$\sum_{n:\mathbb{N}} \mathrm{Vect}(\mathbb{R}, n) \tag{2.12}$$

Dually to the sum type is the product type, where given two types A, B, we have the product type $A \times B$.

Indexed sets

Equality type

Inductive type

2.4 Homotopy types

A further refinement of type theory is the notion of *homotopy type*, where in addition to identity types, we also include the more general notion of equivalence types.

Definition 3

Definition 4 *Two types A, B :* Type *are said to be* equivalent*, denoted $A \cong B$, if there exists an equivalence between them.*

$$(A = B) \to (A \cong B) \tag{2.13}$$

Univalence axiom :

$$(A = B) \cong (A \cong B) \tag{2.14}$$

Correspondence between type theory and category theory :

- A universe of types is a category
- Types are objects in the category $T \in \mathrm{Obj}(C)$
- A term $a : A$ of A is a generalized element of A
- The unit type $*$ if present is the terminal object

- The empty type $\varnothing$ if present is the initial object
- A dependent type $x : A \dashv B(x) : \text{Type}$ is a display morphism $p : B \to A$, the fibers $p^{-1}(a)$ being the dependent type at $a : A$.

2.5 Modalities

Definition 5 *A* modality $\Box$ *on a type theory is a unary operator between two types*

$$\Box : \text{Type} \to \text{Type} \tag{2.15}$$

along with a modal unit Induction principle Computation rule Equivalence

The classic example of a modal theory is that of the necessity monad, or S4 modal logic.

Chapter 3

Categories

As is often the case in foundational issues in mathematics, the foundations used to define mathematics can easily become circular. In our case, although category theory can be used to define set theory and classical logic, as well as your other typical foundational field like model theory, type theory, computational theory, etc, we still need those concepts to define category theory itself. In our case we will simply use implicitly classical logic (what we will call the *external logic*, as opposed to the internal logic of a category we will see later on) and some appropriate set theory like ETCS[26] (as ZFC set theory will typically be too small to talk of important categories). We will not get too deeply into this, but it can be an important issue.

Definition 6 *A* category $\mathbf{C}$ *is a structure composed of a class of objects* $\mathrm{Obj}(\mathbf{C})$ *and a class of morphisms* $\mathrm{Mor}(\mathbf{C})$ *such that*

- *There exists two functions* $s, t : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{C})$*, the* source *and* target *of a morphism. If* $s(f) = X$ *and* $t(f) = Y$*, we write the morphism as* $f : X \to Y$*.*
- *For every object* $X \in \mathrm{Obj}(\mathbf{C})$*, there exists a morphism* $\mathrm{Id}_X : X \to X$*, such that for every morphism* g_1 *with* $s(g_1) = X$ *and every morphism* g_2 *with* $t(g_2) = X$*, we have* $g_1 \circ \mathrm{Id}_X = g_1$ *and* $\mathrm{Id}_X \circ g_2 = g_2$*.*
- *For any two morphisms* $f, g \in \mathrm{Mor}(\mathbf{C})$ *with s*

To simplify notation, if there is no confusion possible, we will write the set of objects and the set of morphisms as the category itself, ie :

$$X \in \mathrm{Obj}(\mathbf{C}) \quad := \quad X \in \mathbf{C} \tag{3.1}$$
$$f \in \mathrm{Mor}(\mathbf{C}) \quad := \quad f \in \mathbf{C} \tag{3.2}$$

Categories are often represented, in totality or in part, by *diagrams*, a directed graph in which objects form the nodes and morphism the edges, such that the direction of the edge goes from source to target. For instance, if we consider some category of two objects A, B with some morphism f with $s(f) = A$ and $t(f) = B$, we can write it as

[Diagram]

Throughout this section we will use a variety of common categories for examples. Some of them will be seen in more details later on as well in the section on the categories we will investigate for objective logic 5.

Before we go on detailing examples of categories, first a quick note on skeletonized categories. It is common in category theory to more or less assume the identity of objects that are isomorphic (we will see the exact definition of isomorphism later on but we can assume the usual definition here). This is not necessarily the case formally speaking (the category of sets can be seen as having multiple isomorphic sets in it, like $\{0, 1\}$, $\{1, 2\}$, etc), but for some purposes (such as trying to get a broad view of that category) it will be useful to consider the category where the set of objects is given as the equivalence class up to isomorphism.

Definition 7 *A category* $\mathbf{C}$ *is* skeletal *if all isomorphisms are identities, and the* skeleton *of a category* $\mathbf{C}$*, written* $\mathrm{sk}(\mathbf{C})$*, is given by the equivalence class of objects up to their isomorphisms, ie*

$$\mathrm{Obj}(\mathrm{sk}(\mathbf{C})) = \{[X] | \forall X' \in [X],\ \exists f \in \mathrm{iso}(\mathbf{C}),\ f : X \to X'\} \tag{3.3}$$

It will be pretty typical as we go on to implicitely consider the skeletal version of whatever category we talk about, as we will talk about *the* set of one element, *the* vector space of n dimensions, etc. If not specified, just assume that it is implicitly "up to isomorphism".

3.1 Examples

A few categories can easily be defined in categorical terms alone, such as the *empty category* $\varnothing$, which is the category with no objects and no morphisms (with the empty diagram as its diagram). We also have the *discrete categories* $\mathbf{n}$ for $n \in \mathbb{N}$, which consist of all the categories of exactly n objects and n morphisms (the identities of each object)

$$\begin{tikzcd} \cdots & \bullet \arrow[loop above, "\mathrm{Id}"] & \bullet \arrow[loop above, "\mathrm{Id}"] & \bullet \arrow[loop above, "\mathrm{Id}"] & \cdots \end{tikzcd}$$

The empty category is in fact itself the discrete category $\mathbf{0}$.

Many categories can also be defined using typical mathematical structures built on set theory, using sets as objects and functions as morphisms.

Example 8 *The category of sets* **Set** *has as its objects all sets (*$\mathrm{Obj}(\mathbf{Set})$ *is the* class *of all sets), and as its morphisms the functions between those sets.*

If we consider the skeletonized version of **Set**, where we only consider unique sets of a given cardinality, its diagram will look something like this for the first three elements classified by cardinality :

$$\begin{tikzcd}
\varnothing \arrow[r, "\varnothing"] \arrow[rr, bend right, "\varnothing"'] & \{\bullet_1\} \arrow[r, shift left, "X_1"] \arrow[r, shift right, "X_2"'] & \{\bullet_1, \bullet_2\} \arrow[l, bend right, "\Delta_\bullet"'] \dots
\end{tikzcd}$$

Example 9 *The category of topological spaces* **Top** *has as its objects the topological spaces* (X, τ_X)*, and as morphisms* $f : (X, \tau_X) \to (Y, \tau_Y)$ *the continuous functions between two such spaces.*

Example 10 *The category of vector spaces* $\mathbf{Vect}_k$ *over a field* k *has as its objects the vector spaces over* k*, and as its morphisms the linear maps between them. The hom-set between* V_1 *and* V_2 *is therefore the set of linear maps* $L(V_1, V_2)$ *(see later for enriched category)*

Diagram : finite dimensional case classified by dimension

$$k^0 \xrightarrow{x \in k} k \xrightarrow{L} k^2 \dots$$

Example 11 *The category of rings* **Ring** *has as its objects rings, and as morphisms ring homomorphisms.*

Example 12 *The category of groups* **Grp** *has as its objects groups* G*, and as its morphisms group homomorphisms.*

Example 13 *The category of smooth Cartesian spaces* **CartSp** *has as its objects subsets of* $\mathbb{R}^n$*, and as morphisms smooth maps between them.*

Besides concrete categories, another common type of categories is *partial orders*, which are defined as usual in terms of sets, ie a partial order $(X, \leq)$ is a set X with a relation $\leq$ on $X \times X$, obeying

- Reflexivity : $\forall x \in X,\ x \leq x$

- Antisymmetry : $\forall x, y \in X,\ x \leq y \wedge y \leq x \rightarrow x = y$
- Transitivity : $\forall x, y, z \in X,\ x \leq y \wedge y \leq z \rightarrow x \leq z$

As a category, a partial order is simply defined by $\mathrm{Obj}(\mathcal{C}) = X$. Its morphisms are defined by the relation $\leq$: if $x \leq y$, there is a morphism between x and y, which we will call $\leq_{x,y}$ formally, or simply $\leq$ if there is no risk of confusion. This obeys the categorical axioms for morphisms as the identity morphism Id_x is simply given by reflexivity, $\leq_{x,x}$, and obeys the triangular identities by transitivity :

$$\leq_{x,x} \circ \leq_{x,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq x \wedge x \leq y \rightarrow x \leq y \tag{3.4}$$

$$\leq_{x,y} \circ \leq_{y,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq y \wedge y \leq y \rightarrow x \leq y \tag{3.5}$$

Morphisms : $f : X \rightarrow Y$ means $X \leq Y$. Between any two objects, there are exactly zero or one morphisms. The identity is $X \leq X$, composition is transitivity

Example 14 *Given a topological space* (X, τ)*, its* category of open $\mathbf{Op}(X)$ *is given by its set of open sets* τ *with the partial order of inclusion* $\subseteq$*.*

Example 15 *The integers* $\mathbb{Z}$ *as a totally ordered set* $(\mathbb{Z}, \leq)$ *forms a category.*

$$\ldots \xrightarrow{\leq} -2 \xrightarrow{\leq} -1 \xrightarrow{\leq} 0 \xrightarrow{\leq} 1 \xrightarrow{\leq} 2 \xrightarrow{\leq} \ldots$$

The real line is such a category

The category of (locally small) categories **Cat**

Another type of such categories is the *simplicial category* Δ. This can be considered as a category of categories, where each object of Δ is the finite total order $\vec{\mathbf{n}}$, and the morphisms are order-preserving functions,

$$\forall f : \vec{m} \rightarrow \vec{n},\ f(m_i \rightarrow m_j) = f(m_i) \rightarrow f(m_j) \tag{3.6}$$

Example 16 *The* interval category I *is composed of two elements* $\{0, 1\}$ *and a morphism* $0 \rightarrow 1$.

$$I = \{0 \rightarrow 1\} \tag{3.7}$$

3.2 Morphisms

As our categories are fundamentally built from sets and classes, we can look at specific subsets of our set of morphisms, $\mathrm{Mor}(\mathbf{C})$, with specific properties.

A simple example is simply the set of all morphisms between two objects :

Definition 17 *The* hom-set *between two objects $X, Y \in \mathbf{C}$ is the set of all morphisms between X and Y :*

$$\mathrm{Hom}_{\mathbf{C}}(X,Y) = \{f \in \mathrm{Mor}(\mathbf{C}) | s(f) = X,\ t(f) = Y\} \tag{3.8}$$

This notion is more rigorously only valid for what are called *locally small categories*, where the cardinality of $\mathrm{Hom}_{\mathbf{C}}(X,Y)$ is small enough to be a set. If the cardinality is too large, and could only fit in say a class, it is more properly described as a *hom-object*. This will not affect our discussion much as most categories of interest are locally small.

Example 18 *The hom-set on the category of sets* $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$ *is the set of all functions between X and Y*

Example 19 *The hom-set on the category of vector spaces* $\mathrm{Hom}_{\mathbf{Vec}}(X,Y)$ *is the set of all linear functions between X and Y, also known as $L(X,Y)$.*

Example 20 *The hom-set on the category of topological spaces* $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$ *is the set of all continuous functions between X and Y, also known as $C(X,Y)$.*

A specific type of hom-set is the set of endomorphisms of an object.

Definition 21 *The endomorphisms of an object X are the hom-set*

$$\mathrm{End}(X) = \mathrm{Hom}_{\mathbf{C}}(X,X) \tag{3.9}$$

We can also define an induced function given an object between hom-sets thusly. For a morphism $f : X \to Y$, the induced function f_* on the hom-sets

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z,X) \to \mathrm{Hom}_{\mathbf{C}}(Z,Y) \tag{3.10}$$

is defined by composition. In other words, if we have a function $g : Z \to X$, we can map it to a function $g' : Z \to Y$ by post-composition with f :

$$g' = f_*(g) = f \circ g \tag{3.11}$$

Likewise, we have a similar induced function by pre-composition, defined by

$$f^* : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \rightarrow \mathrm{Hom}_{\mathbf{C}}(X, Z) \quad (3.12)$$
$$g \mapsto f^*(g) = g \circ f \quad (3.13)$$

(These will be transformations induced by the hom-functors later on)

3.2.1 Monomorphisms

Definition 22 *A* monomorphism $f : X \to Y$ *is a morphism such that, for every object* Z *and every pair of morphisms* $g_1, g_2 : Z \to X$*,*

$$f \circ g_1 = f \circ g_2 \to g_1 = g_2 \quad (3.14)$$

We say that a monomorphism is left-cancellable.

Example 23 *on* **Set***, a monomorphism is an injective function.*

Proof 1 *If* $f : X \to Y$ *is a monomorphism, then given two elements* $x_1, x_2 \in X$*, represented by morphisms* $x_1, x_2 : \{\bullet\} \to X$*, then we have*

$$f \circ x_1 = f \circ x_2 \to x_1 = x_2 \quad (3.15)$$

So that we can only have the same value if the arguments are the same, making it injective. Conversely, if f *is injective, take two functions* $g_1, g_2 : Z \to X$*. As the functions in sets are defined by their values, we need to show that*

$$\forall z \in Z,\ f(g_1(z)) = f(g_2(z)) \to g_1(z) = g_2(z) \quad (3.16)$$

By injectivity, the only way to have $f(g_1(z)) = f(g_2(z))$ *is that* $g_1(z) = g_2(z)$*. As this is true for every value of* z*,* $g_1 = g_2$*.*

Theorem 24 *A monomorphism is equivalent to the fact that the induced function* f_* *is injective on hom-sets :*

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \hookrightarrow \mathrm{Hom}_{\mathbf{C}}(Z, Y) \quad (3.17)$$

Proof 2 *If* $f : X \to Y$ *is a monomorphism, then for some hom-set* $\mathrm{Hom}_{\mathbf{C}}(Z, X)$*, we need to show that given two functions* g_1, g_2 *in there, the elements of* $\mathrm{Hom}_{\mathbf{C}}(Z, Y)$ *obtained by the induced map* f_* *obey*

$$f_*(g_1) = f_*(g_2) \to g_1 = g_2 \quad (3.18)$$

This is true by the definition of f_* *and left cancellability. Conversely, if* f_* *is an injective function on* $\mathrm{Hom}_{\mathbf{C}}(Z, X)$*, we have that*

It is tempting to try to view monomorphisms as generalizing injections, but categories may lack both elements on which to define such notions and other properties that

[counterexample?]

3.2.2 Epimorphisms

Definition 25 *An* epimorphism $f : X \to Y$ *is a morphism such that, for every object Z and every pair of morphisms g_1, g_2*

Mono and epi on sets, etc

3.2.3 Isomorphisms

Definition 26 *An* isomorphism $f : X \to Y$ *is a morphism with a two-sided inverse, ie there exists a morphism $f^{-1} : Y \to X$ such that*

$$f \circ f^{-1} = \mathrm{Id}_Y \tag{3.19}$$
$$f^{-1} \circ f = \mathrm{Id}_X \tag{3.20}$$

Example 27 *In the category of sets, isomorphisms are bijections.*

Despite the most typical examples, it is not in general true that a morphism that is both a monomorphism and an epimorphism is an isomorphism.

A specific type of isomorphisms are the isomorphic endomorphisms. Together, they are called the core of an object.

Definition 28 *For a given object X, the subset of all its endomorphisms which are isomorphisms are called its* core *:*

$$\mathrm{core}(X) = \{f \in \mathrm{Hom}_{\mathbf{C}}(X, X) \mid f \textit{ isomorphism}\} \tag{3.21}$$

Theorem 29 *The core of an object has a group structure.*

Proof 3 *If we pick as the set of the group* $\mathrm{Core}(X)$*, as its group product the composition of morphisms $\circ$, as unit the identity morphism and as inverse the function mapping a morphism to its inverse, we have that this fulfills all the group axioms.*

Theorem 30 *The composition of two monomorphisms is a monomorphism.*

Proof 4

Theorem 31 *The composition of two epimorphisms is an epimorphism.*

Proof 5

Theorem 32 *For f, g two morphisms that can be composed as $g \circ f$, if $g \circ f$ and g are monomorphisms, then so is f.*

Proof 6

As monomorphisms represent embeddings and inclusions, this means in particular that if we have the inclusion $S \hookrightarrow X$ and

3.2.4 Split morphisms

As we've seen, the notion of epis and monos do not generally directly correspond to injective and surjective functions, even in the case where objects are actual sets, but we do have a more accurate categorical notion for these, which are split epis and monos.

Definition 33 *A* split monomorphism *$f : X \to Y$ is a monomorphism possessing a left inverse $r : Y \to X$, called its* retraction*, so that*

$$r \circ f = \mathrm{Id}_X \tag{3.22}$$

Conversely, we say also that r is a *section* of f.

Idempotent

Example 34 *In the category of vector spaces* **Vec***, any monomorphism is split. The monomorphism is the inclusion map of a subspace $\iota : W \hookrightarrow V$, and its retract is the projection*

Definition 35 *A* split epimorphism

Theorem 36 *A morphism that is both split epi and split mono is an isomorphism.*

3.2.5 Lifts and extensions

Definition 37 *A* lift *of a morphism $f : X \to Y$ through some morphism $p : \overline{Y} \to Y$ is a morphism $\overline{f} : X \to \overline{Y}$ such that $f = p \circ \overline{f}$, ie it is the commutative triangle*

Extension

3.2.6 Subobjects

As monomorphisms are similar to injective functions, it is tempting to see a way to define

Definition 38 *A subobject is an equivalence class of monomorphisms to the same object up to isomorphism,*

$$[S] \in \mathrm{Sub}_{\mathbf{C}}(X) \leftrightarrow [S] = \{S' \in \mathbf{C} \wedge \exists\} \tag{3.23}$$

3.3 Functors

Functors are roughly speaking functions on categories that preserve their structures. In more details,

Definition 39 *A* functor $F : \mathbf{C} \to \mathbf{D}$ *is a function between two categories* $\mathbf{C}, \mathbf{D}$*, mapping every object of* $\mathbf{C}$ *to objects of* $\mathbf{D}$ *and every morphism of* $\mathbf{D}$ *to morphisms of* $\mathbf{D}$*, such that categorical properties are conserved :*

$$s(F(f)) = F(s(f)) \tag{3.24}$$
$$t(F(f)) = F(t(f)) \tag{3.25}$$
$$F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)} \tag{3.26}$$
$$F(g \circ f) = F(g) \circ F(f) \tag{3.27}$$

Example 40 *The* Identity functor $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ *maps every object and morphism to themselves.*

Example 41 *The* constant functor $\Delta_X : \mathbf{C} \to \mathbf{D}$ *for some object* $X \in \mathbf{D}$ *is the functor mapping every object in* $\mathbf{C}$ *to* X *and every morphism to* Id_X*.*

Example 42 *For a category where the objects can be built from sets (such as* **Top** *or* $\mathbf{Vect}_k$*), the* forgetful functor $U_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set}$ *is the functor sending every object to their underlying set, and every morphism to their underlying function on sets.*

A common functor is the *forgetful functor*, which maps a category that is composed of a set with extra structure its underlying set. For instance, the category **Top** has a forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$, which maps every topological space to its set, and every continuous function to the corresponding function on sets. If we have a set X and on it are all the different topologies (X, τ_i), then the forgetful functor maps

$$U((X, \tau_i)) = X \tag{3.28}$$

The forgetful functor on **Top** is obviously not injective, as two topological spaces with the same underlying set (such as any set of cardinality ≥ 1 with the discrete or trivial topology) will map to the same set.

Functors on partial orders : Order isomorphism

Example : negation

Definition 43 *A* contravariant functor *is a functor from the opposite category, ie $F : C \to D$ is a contravariant functor equivalently to a functor $F : C^{\mathrm{op}} \to D$ is a functor. In particular, this changes the rules as*

$$\begin{aligned} s(F(f)) &= F(t(f)) && (3.29) \\ t(F(f)) &= F(s(f)) && (3.30) \\ F(g \circ f) &= F(f) \circ F(g) && (3.31) \end{aligned}$$

Theorem 44 *The composition of contravariant and covariant functors works as follow :*

$$C \tag{3.32}$$

A generalization of functors is the notion of multifunctors (we mean here specifically the *jointly functorial* multifunctor)

Definition 45 *A multifunctor $F : \mathbf{C} \to \mathbf{D}$ is a function from a product of category to another category.*

"Functor categories serve as the hom-categories in the strict 2-category Cat."

3.3.1 The hom-functor

The hom-bifunctor $\mathrm{Hom}_{\mathbb{C}}(-,-)$ is the map

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(-,-) : \mathbf{C} \times \mathbf{C} &\to \mathbf{Set} && (3.33) \\ (X, Y) &\mapsto \mathrm{Hom}_{\mathbf{C}}(X, Y) && (3.34) \end{aligned}$$

mapping objects of $\mathbf{C}$ to their hom-sets. Given a specific object X, we can furthermore define two types of hom functors : the covariant functor $\mathrm{Hom}(X, -)$, also denoted by h^X, and the contravariant functor $\mathrm{Hom}(-, X)$, denoted by h_X.

Example :

Subobject functor :

$$\mathrm{Sub}_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set} \tag{3.35}$$

3.3.2 Full and faithful functor

Full, faithful functor

Definition 46 *A functor $F : \mathbf{C} \to \mathbf{D}$ induces the function*

$$F_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(F(X), F(Y)) \tag{3.36}$$

F is said to be

- faithful *if $F_{X,Y}$ is injective*
- full *if $F_{X,Y}$ is surjective*
- fully faithful *if $F_{X,Y}$ is bijective*

Despite the analogy of full and faithful functors with surjective and injective functions,

Functors that are fully faithful but not injective/surjective on objects or morphisms

3.3.3 Subcategory inclusion

An important type of functor is the inclusion of a subcategory. If we take a category $\mathbf{C}$, and then create a new category $\mathbf{S}$ for which $\mathrm{Obj}(\mathbf{S}) \subseteq \mathrm{Obj}(\mathbf{C})$

Definition 47 *An inclusion of a subcategory $\mathbf{S}$ in a category $\mathbf{C}$ is a functor $\iota : \mathbf{S} \hookrightarrow \mathbf{C}$, such that $\iota(\mathrm{Obj}(\mathbf{S})) \subseteq \mathrm{Obj}(\mathbf{C})$, $\iota(\mathrm{Mor}(\mathbf{S})) \subseteq \mathrm{Mor}(\mathbf{C})$, and*

- *If $X \in \mathbf{S}$, then $\mathrm{Id}_X \in \mathbf{S}$*
- *For any morphism $f : X \to Y$ in $\mathbf{S}$, then $X, Y \in \mathbf{S}$.*
-

For discrete categories, $\mathbf{n}$ in $\mathbf{m}$ if $n \leq m$

Full subcategories

Example 48 *The linear order of the integers $\mathbb{Z}$ has inclusion functors to $\mathbb{R}$*

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{R} \tag{3.37}$$

If treated as Canonical inclusion :

$$\begin{aligned} \iota_h : \mathbb{Z} &\to \mathbb{R} && (3.38) \\ k &\mapsto h + k && (3.39) \end{aligned}$$

Example 49 *The category of finite sets* **FSet** *is a subcategory of* **Set***, via the identity functor restricted to finite groups.*

Example 50 *The category of Abelian group* **Ab** *in* **Grp**

3.4 Natural transformations

Natural transformations are a type of transformations on functors (2-morphisms in **Cat**)

Definition 51 *For two functors $F, G : \mathbf{C} \to \mathbf{D}$, a* natural transformation η *between them is a map $\eta : F \to G$ which induces for any object $X \in \mathbf{C}$ a morphism on* **D**

$$\eta_X : F(X) \to G(X) \tag{3.40}$$

and for every morphism $f : X \to Y$ the identity

$$\eta_Y \circ F(f) = G(f) \circ \eta_X \tag{3.41}$$

[Commutative diagram]

Example 52 *The* identity transformation $\mathrm{Id}_F : F \to F$ *on the functor $F : \mathbf{C} \to \mathbf{D}$ is the natural tranformation for which every component $\mathrm{Id}_{F,X} : F(X) \to F(X)$ for $X \in \mathbf{D}$ is the identity map. This obeys the identity as*

$$\begin{aligned} \eta_Y \circ F(f) &= \mathrm{Id}_Y \circ F(f) && (3.42)\\ &= F(f) && (3.43)\\ &= F(f) \circ \mathrm{Id}_X && (3.44) \end{aligned}$$

Example 53 *The category of groups* **Grp** *has a functor to the category of Abelian groups* **AbGrp***, the Abelianization functor*

$$\begin{aligned} \mathrm{Ab} : \mathbf{Grp} &\to \mathbf{AbGrp} && (3.45)\\ G &\mapsto G/[G, G] && (3.46) \end{aligned}$$

[show functoriality] There is a natural transformation from the identity functor on groups to the abelianization endofunctor

$$\eta : \mathrm{Id}_{\mathbf{Grp}} \to \mathrm{Ab} \tag{3.47}$$

Example 54 *Given the category $\mathbf{Vect}_k$, for any vector space V we have the dual space V^* [see later in the internal hom section for why] of linear maps $V \to k$, and its double dual V^** of linear maps $V^* \to k$. We would like to show that there is an equivalence between V and V^{**}.*

Example 55 *The opposite group functor is simply given by the opposite category functor on* Grp. *Groups to opposite group*

For constant F and G : cone and cocone

3.5 Yoneda lemma

One of the common philosophical idea underlying category theory is that of the specific objects involved in a category are not as meaningful as the equivalence of all such objects under isomorphisms. That is, if we have two objects X, X' in a category, such that those two objects have identical behaviour in the category (all morphisms to other objects are mirrored on the other), then they are in essence the same object. In philosophical terms, this

"every individual substance expresses the whole universe in its own manner and that in its full concept is included all its experiences together with all the attendent circumstances and the whole sequence of exterior events. There follow from these considerations several noticeable paradoxes; among others that it is not true that two substances may be exactly alike and differ only numerically, solo numero."

[Leibniz discourse on metaphysics]

This is true in some sense in category theory as expressed by the Yoneda lemma. If we consider all the relationship of an object X to all other objects, this is given by the hom-sets of X to every other objects, that is,

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{3.48}$$

What we mean by two objects having the same relationships to every other objects is that if we consider their respective hom-sets to every other objects, they are isomorphic :

$$\forall Y \in \mathbf{C},\ \mathrm{Hom}_{\mathbf{C}}(X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{3.49}$$

Meaning that for any object Y and any morphism $f : X \to Y$, we will have some corresponding function $f' : X' \to Y'$, and the relationships between all those morphisms reflect each other. If we have $f : X \to Y$ and $g : Y \to Z$, then we have

$$\exists g',\ f' \circ g' : X \tag{3.50}$$

Equivalently (converse)

This equivalence is in fact an equivalence of the hom functors,

$$h_X \cong h_{X'} \tag{3.51}$$
$$h^X \cong h^{X'} \tag{3.52}$$

As an equivalence of functors, this means that we have a pair of natural transformations between them which are two-sided inverses of each other.

For a functor $F : \mathbf{C} \to \mathbf{Set}$, for any object $X \in \mathbf{C}$

$$よ : \mathbf{C} \to [\mathbf{C}, \mathbf{Set}] \tag{3.53}$$

Functor lives in the functor space $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

$$\mathrm{Nat}(h_A, F) \cong F(A) \tag{3.54}$$

Example 56 *Consider the category of a single group ($G = \mathrm{Aut}(*)$). A functor $F : \mathrm{G} \to \mathbf{Set}$ is a set X and a group homomorphism to its permutation group $G \to \mathrm{Perm}(X)$ (A G-set). Natural transformation is an equivariant map Cayley's theorem*

Example 57

3.6 Enriched categories

By default, we consider the hom sets of a category $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ to be sets, but many categories may have additional structure on the hom set. For instance, if we consider the category $\mathbf{Vect}_k$ of vector spaces over the field k, its morphisms are k-linear maps, and its hom-sets are

$$\mathrm{Hom}_{\mathbf{Vect}_k}(V, W) = L_k(V, W) \tag{3.55}$$

However, in addition to being a set, $L_k(V, W)$, the k-linear maps, also form themselves a vector space, ie we can define the sum $f + g$ of two linear maps, and the scaling αf, $\alpha \in k$, of a linear map.

Definition 58 *An* enriched category $\mathbf{C}$ *over* $\mathbf{V}$ *a monoidal category* $(\mathbf{V}, \otimes, I)$ *is a category such that each hom-set* $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ *is associated to a* hom-object $C(X, Y) \in \mathbf{V}$*, such that every hom-object in* $\mathbf{V}$ *obeys the same rules regarding composition and identity, which are*

$$\circ_{X,Y,Z} : C(Y, Z) \otimes C(X, Y) \to C(X, Z) \tag{3.56}$$

$$j_X : I \to C(X, X) \tag{3.57}$$

with the following commutation diagrams :

[composition is associative]

[Composition is unital]

Example 59 *A category enriched in* **Set** *is a locally small category.*

Example 60 *A* k-linear category *is enriched over* Vect_k.

3.7 Comma categories

The notion of a comma category can be used to describe categories whose objects are the morphisms of another category.

Definition 61 *The* comma category $(f \downarrow g)$ *of two functors* $f : C \to E$ *and* $g : D \to E$ *is the category composed of triples* (c, d, α) *such that* $\alpha : f(c) \to g(d)$ *is a morphism in* E*, and whose morphisms are pairs* (β, γ)

$$\beta \quad : c_1 \quad \to c_2 \tag{3.58}$$
$$\gamma \quad : d_1 \quad \to d_2 \tag{3.59}$$

that are morphisms in C *and* D*, such that* $\alpha_2 \circ f(\beta) = g(\gamma) \circ \alpha_1$ *[Commutative diagram]*

Composition

Def via pullback

Comma categories are rarely used directly, but are more typically used to define more specific operations. The three important one we will see are arrow categories, slice categories and coslice categories.

3.7.1 Arrow categories

The simplest kind of comma category is the arrow category, where we just consider the arrows of a category as a category.

Definition 62 *An arrow category is the comma category for the case where the two functors are the identity functors,* $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$

$$\mathrm{Arr}(\mathbf{C}) = (\mathrm{Id}_{\mathbf{C}} \downarrow \mathrm{Id}_{\mathbf{C}}) \tag{3.60}$$

Theorem 63 *The arrow category* $\mathrm{Arr}(\mathbf{C})$ *is equivalent to the category whose objects are the morphisms of* $\mathbf{C}$ *:*

$$\mathrm{Obj}(\mathrm{Arr}(\mathbf{C})) = \mathrm{Mor}(\mathbf{C}) \tag{3.61}$$

and its morphisms are given by pairs of morphisms (f, g) *in* $\mathbf{C}$ *obeying the commutative square*

3.7.2 Slice categories

Given an object $X \in \mathbf{C}$, we can define the *over category* (or *slice category*) $\mathbf{C}_{/X}$ by taking all morphisms emanating from X as objects :

$$\mathrm{Obj}(\mathbf{C}_{/X}) = \{f | s(f) = X\} \tag{3.62}$$

As a comma category, this is the comma category of the two functors $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ and $\Delta_X : \mathbf{1} \to \mathbf{C}$, of the identity functor on $\mathbf{C}$ and the inclusion of the object X, in which case $\mathbf{C}_{/X} = (\mathrm{Id}_{\mathbf{C}} \downarrow \Delta_X)$ is defined by the triples $(c, *, \alpha)$ of objects $c \in \mathbf{C}$, the unique object $* \in \mathbf{1}$, and morphisms in $\mathbf{C}$ $\alpha : c \to X$. As there is only one object in the terminal category, we can drop it as it is isomorphic to simply (c, α), and furthermore, c is implied by α as simply being the source term. Our slice category is therefore indeed just defined by the set of morphisms from objects of the category to our selected object.

Slice categories are useful to consider objects in a category as a category in themselves, where the objects are simply all the relations they have with all other objects in the category.

Example 64 *In* $\mathbf{Set}$*, given a set* X*, the slice category* $\mathbf{Set}_{/X}$ *has as its objects all functions with codomain* X*,* $f : Y \to X$*, and as morphisms all functions between sets* $g : Y \to Y'$ *for which*

$$f'(g(y)) = f(y) \tag{3.63}$$

Category of X-indexed collections of sets, object $f : Y \to X$ is the X-indexed collection of fibers $\{Y_x = f^{-1}(\{x\})| \in X\}$, morphisms are maps $Y_x \to Y'_x$

[fiber product $Y \times_X Y'$ is the product in the slice category]

If we look for instance at $\mathbb{N}$ as a set (the natural number object of sets), its slice category $\mathbf{Set}_{/\mathbb{N}}$ is the category of all functions to numbers

Example 65 *For a poset* P*, the slice category* $\mathbf{P}/p$ *is isomorphic to the downset of* p*, ie the subcategory of every element* $\{q | q \leq p\}$*.*

Objects : Id_p*, and every map* $\leq_{q,p}$ *(corresponding to* p *and object inferior to* p*), morphisms are*

Example 66 **Top**$/X$ *is the category of covering spaces over* X*.*

Definition 67 *Given a functor* $F : \mathbf{C} \to \mathbf{D}$*, we can defined a sliced functor for* $X \in \mathbf{C}$ *via :*

$$F_{/X} : \mathbf{C}_{/X} \to \mathbf{D}_{/F(X)} \tag{3.64}$$

Theorem 68 *If a category* $\mathbf{C}$ *has a limit for a given functor* $F : I \to$

Theorem 69 *If* $\mathbb{C}$ *has an initial object* 0*,* $\mathbf{C}_{/X}$ *has the initial object* $\varnothing_X : 0 \to X$*.*

Proof 7

3.7.3 Coslice categories

Coslice categories

comma categories, under categories?

Dependent sum, dependent product, indexed sets

3.7.4 Base change

Base change functor

3.8 Limits and colimits

In category theory, a limit or a colimit are roughly speaking a construction on a category. For some given set of objects $A, B, \ldots$ in our category C, and some morphisms between them, a limit or colimit of those objects will be some construction performed using those. Those constructions can be quite different, but overall, a limit will often be like a "subset", while a colimit is more of an "assemblage" of those.

Definition 70 *For a functor* $F : \mathbf{C} \to \mathbf{D}$*, a* universal property *from an object* $X \in \mathbf{D}$ *to* F

Universal property, kan extension?

A (co)limit is done using an indexing category, which is roughly the "shape" that our construction will take. An indexing category is a small category, ie it has a countable number of objects and morphisms small enough that you could fit them into sets. Typically they are fairly simple ones. As we are only interested in their shape, it's common to denote the objects by simple dots. Examples include the discrete categories of n elements $\mathbf{n}$,

$$\bullet \qquad \bullet \qquad \bullet \qquad \dots$$

The span category :

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

and the cospan :

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

and the parallel pair :

$$\bullet \rightrightarrows \bullet$$

Directed and codirected set, sequential and cosequential limit

To do our constructions, we need to send this diagram's shape into our category C, which we do by using some functor $F : I \to C$, producing a diagram :

Definition 71 *A* diagram *of shape I into C is a functor $F : I \to C$.*

The image of this functor in C will then be some subset of C that "looks like" I, although we are not guaranteed that the objects and morphisms of I will be mapped injectively into C (this simply corresponds to cases where our construction will use the same object or morphisms several times).

Let's now consider the constant functor $\Delta_X : I \to C$, which for $X \in C$ sends every object of I to X. If we can find a natural transformation between Δ_X and our diagram $F : I \to C$, we will have either a cone over F $\eta : \Delta_X \Rightarrow F$ or a cone under F $\eta : F \Rightarrow \Delta_X$.

Definition 72 *The* limit *of a diagram $F : I \to M$ is an object $\lim F \in \mathrm{Obj}(C)$ and a natural transformation $\eta : \Delta_{\lim F} \to F$, such that for any $X \in \mathrm{Obj}(C)$ and any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f : X \to \lim F$ such that $\alpha = \eta \circ F$. The cone of $\Delta_{\lim F}$ over F is the* universal cone *over F.*

In other words, if we pick any object X in our category C and define some collection of morphisms from X to other objects

Let's consider for instance the case of the trivial category **1**. Any functor $F : \mathbf{1} \to C$ is simply a choice of an object in C, mapping $\bullet$ to $F(\bullet) = A$, ie it is just the constant functor Δ_A for some A. A natural transformation $\eta : \Delta_X \to F$ is them simply $\eta : \Delta_X \to \Delta_A$, and conversely, $\eta : F \to \Delta_X$ is $\eta : \Delta_A \to \Delta_X$. The components of this natural transformations are simply a morphism from X to A (and a morphism from A to X).

Limit and colimit

Types of indexing category I and their limit and colimit :

3.8.1 Initial and teminal objects

The simplest kind of limit is the one done on the empty indexing category, **0**, as this leads to a unique diagram in the category **C**, the empty functor.

Definition 73 *Given the empty category* **0***, the limit* Lim$F$ *of a diagram* $F : \mathbf{0} \to C$ *is its* terminal object*, and the colimit is its* initial object*.*

Generally we will denote the terminal object as 1 and the initial object as 0, since they often correspond to objects of one and zero elements in concrete categories, except for a few cases where we will use a more specific notation for the given category, such as $\{\bullet\}$ and $\varnothing$ for **Set** (the singleton and empty set), or **1** and **0** for the category of categories **Cat**.

As there exists only one functor from the empty category to any other category (the empty functor $F_\varnothing$), the initial and terminal objects do not depend on specific objects and are simply special objects of the category. Every "constant functor" Δ_X sending objects of I to $X \in \mathrm{Obj}(C)$ is also the empty functor, sending them trivially to X by simply not having any objects to send. This is therefore also true of the constant functor to the limit $\lim F_\varnothing$, meaning that the natural transformation $\eta : \varnothing \to \varnothing$ is simply the identity transformation $\mathrm{Id}_{\varnothing^\varnothing}$. This means that the limit of the empty diagram in a category C is the object (defined by no other objects in the category) $\lim \varnothing$ such that for any natural transformation $\alpha : \Delta_X \to F$ (as we've seen, only possibly the identity transformation), there exists a unique morphism $f : X \to \lim \varnothing$

[...]

The terminal object $t = \lim \varnothing$ of a category, if it exists, is therefore an object for which there exists only one morphism from any object $X \in$ C to t

Dually, the *initial object* $i = \mathrm{colim}\varnothing$ of a category, if it exists, is an object for which there exists only one morphism from i to any object $X \in \mathrm{Obj}(C)$.

Theorem 74 *Initial and terminal objects are unique in a category up to isomorphisms.*

Proof 8

Initial and terminal objects occur in quite a lot of important categories, and tend to be somewhat similar objects. In **Set**, the

3.8.2 Products and coproducts

The product and coproduct are the limits where the diagrams are the discrete categories of n elements $\mathbf{n}$. This means obviously that the trivial case $\mathbf{0}$ of the diagram of zero object is the initial and terminal object, and this will correspond to the trivial product and coproduct as we will see later :

$$\sum_{\varnothing} = 0,\ \prod_{\varnothing} = 1 \tag{3.65}$$

Any diagram of shape $\mathbf{n}$ simply selects n objects in the category (and their identity functions),

$$\forall F : \mathbf{n} \to \mathbf{C},\ \exists X_1, \dots X_n \in \mathbf{C},\ \mathrm{Im}(F) = (X_1, \dots, X_n) \tag{3.66}$$

The constant functor Δ_X is the functor sending each of those points to X, $\bullet_i \to X$, and the identity of those points to the identity on X. We will denote the limit of a diagram F on the discrete category, selecting the objects $\{X_i\}$, by $\prod_i X_i$. This product is therefore such that for any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f = \Delta_X \to \prod X_i$ such that $\alpha = \eta \circ F$.

What this property means for the product is that given any object X picked in $\mathbf{C}$,

Definition 75 *for any object X in a category with products, the diagonal morphism $\Delta : X \to X \times X$ is the one given by the universal property of the product for the case where the two objects are X : [diagram]*

Codiagonal

3.8.3 Pullbacks and pushouts

The pullback and pushout are the limits of dual diagrams, the span and cospan, where the span is given by two morphisms to the same object,

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

or $\bullet \to \bullet \leftarrow \bullet$ for short, and two morphisms from the same object for the cospan :

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

or $\bullet \leftarrow \bullet \to \bullet$. Those are opposite diagrams, in the sense that **Span** = **Cospan**$^{\mathrm{op}}$

The pushout is the limit of the span, while the pullback is the limit of the cospan. From their opposition, we can also say that the pushout is the colimit of the cospan and the pullback the colimit of the span.

The span diagram can be mapped to any three objects connected thusly. For $A, B, C \in \mathbf{C}$, and $f : A \to C$, $g : B \to C$, the diagram of shape I will be

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

If we now look at the constant functor $\Delta_X : I \to \mathbf{C}$, this will map our three objects A, B, C to X, and f and g to Id_X. To find the limit of F, we therefore need to find $\eta : \Delta_{\lim F} \to F$

For any $X \in \mathbf{C}$, and any $\alpha : \Delta_X \to F$, there is a unique morphism $h : X \to \lim F$ such that $\alpha = \eta \circ F$.

Components of α : for every $Y \in I$, a morphism $\alpha_Y : \Delta_X(Y) \to F(Y)$, so

$$\alpha_Y : X \to F(Y) \tag{3.67}$$

$F(Y)$ can only be A, B, C, so we have three components for objects, and for $f : A \to C$ and $g : B \to C$, then using what we know of Δ_X we have the following commuting diagrams :

$$\begin{tikzcd}
X \arrow[r, "\alpha_A"] \arrow[dr, "\alpha_C"'] & A \arrow[d, "f"] \\
& C
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\alpha_B"] \arrow[dr, "\alpha_C"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

$$F(f) \circ \eta \tag{3.68}$$

The resulting cone is

$$\begin{tikzcd}
\lim F \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "p_A"] \arrow[ddr, bend right, "p_B"'] & & \\
& X \arrow[r, "\alpha_A"] \arrow[d, "\alpha_B"] \arrow[dr, "\alpha_C"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

The limit is the pullback, denoted as $A \times_C B$, along with the two and the universal cone that we have constructed gives us the commutative square

$$\begin{tikzcd}
A \times_C B \arrow[r, "p_A"] \arrow[d, "p_B"] \arrow[dr, "\eta_C"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

This means that our pullback diagram is given by this object and the two projectors, obeying

$$f \circ p_A = g \circ p_B \tag{3.69}$$

The interpretation of this is the *dependent sum* of the equality

$$\sum_{a:A} \sum_{b:B} (f(a) = g(b)) \tag{3.70}$$

Example 76 *In* **Set**, *the pullback by $f : A \to C$, $g : B \to C$ is the set*

$$A \times_C B = \{(a, b) \in A \times B | f(a) = g(b)\} \tag{3.71}$$

$$\bigcup_{c \in f(A) \cap g(B)} f^{-1} \tag{3.72}$$

Example 77 *A particular type of pullback in* **Set** *is the pullback of $f : A \to B$ by the identity function on B, $\mathrm{Id}_B : B \to B$*

Definition 78 *For two subobjects $\iota_1 : U_1 \hookrightarrow X$, $\iota_2 : U_2 \hookrightarrow X$, the pullback of $U_1 \to X \leftarrow U_2$ is called the intersection of U_1 and U_2*

$$U_1 \cap U_2 = U_1 \times_X U_2 \tag{3.73}$$

This is easy enough to see in terms of **Set**, where this pullback gives us

$$U_1 \times_X U_2 = \{(x_1, x_2) \in U_1 \times U_2 | \iota_1(x_1) = \iota_2(x_2)\} \tag{3.74}$$

which is the set of all points that are in both U_1 and U_2 in X [diagonal morphism?]

Likewise, we can define the intersection of two subobjects thusly

Definition 79 *The intersection of two subobjects $\iota_1 : U_1 \hookrightarrow X$, $\iota_2 : U_2 \hookrightarrow X$ is the pushout of the inclusion maps of their intersection, that is*

$$\iota_{1,U_1 \cap U_2} : U_1 \cap U_2 \to U_1 \tag{3.75}$$
$$\iota_{2,U_1 \cap U_2} : U_1 \cap U_2 \to U_2 \tag{3.76}$$

and so

$$U_1 \cup U_2 = U_1 \times_{U_1 \cap U_2} U_2 \tag{3.77}$$

Example 80 *A typical example of the pullback is given in fiber bundles, for instance in the category of manifolds, where given a bundle morphism $\pi : E \to B$ and some morphism $f : B' \to B$, the pullback $B' \times_B E$ is the* pullback bundle, *which is a fiber bundle with the same typical fiber as E but done over B', given by the equation*

$$f^* E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \tag{3.78}$$

Semantics of an equation

Properties :

Theorem 81 *The pullback of a function $f : X \to Y$ by the identity $\mathrm{Id}_X : X \to X$, $X \times_X Y$, is an isomorphism.*

Theorem 82 *The pullback to the terminal object, $X \to 1 \leftarrow Y$, is the product $X \times Y$.*

Dependent product/sum, indexed objects

[27]

A specific type of pullback that we will commonly use is the fiber of a morphism.

Definition 83 *The* fiber *of a morphism $f : A \to B$ by a given point in B, $x : 1 \to B$, is the pullback of the cospan with the terminal object :*

$$A \xrightarrow{f} B \xleftarrow{x} 1 \tag{3.79}$$

Ex : Fiber of a bundle $p : E \to B$ and $x : 1 \to B$ is the fiber at x, E_x.

Example 84 *If f is an epimorphism[?], such as in the case of a bundle space $\pi : E \to B$, the fiber $E \times_B 1$ at $x : 1 \to B$ is the subobject $E_x \hookrightarrow E$ which projects down to the point x :*

$$\pi(\iota(E_x)) = x(1) \tag{3.80}$$

"In an additive category fibers over the zero object are called kernels."

Dually to the fiber is also the cofiber, the equivalent for the pushout.

Definition 85 *A* cofiber *of a morphism $f : A \to B$ is the pushout of the span with the terminal object*

$$* \longleftarrow A \xrightarrow{f} B \tag{3.81}$$

$$\mathrm{cofib}(f : A \to B) = \tag{3.82}$$

Unlike the fiber, the cofiber does not depend on any base point as there is a unique morphism $X \to 1$.

Cokernels in additive categories

Pushout :

$$* \longleftarrow \square_{\varnothing} X \longrightarrow X \tag{3.83}$$

Pushout $A \sqcup_0 B \cong A \sqcup B$, therefore

$$\square_{\varnothing} X = X \sqcup 1 \tag{3.84}$$

Dependent product and sum

Definition 86 *The dependent product $\prod_f$ for a morphism $f : X \to 1$ is defined as the pullback of*

3.8.4 Equalizer and coequalizer

The equalizer and coequalizer are the limits and colimits corresponding to the diagram of a pair of parallel morphisms, ie :

$$\bullet \rightrightarrows \bullet \tag{3.85}$$

This diagram maps to some pair of morphisms in our category between two objects,

$$X \underset{f}{\overset{g}{\rightrightarrows}} Y \tag{3.86}$$

$$\mathrm{Hom}_{\mathbf{C}}(X, \lim F) \cong \mathrm{Nat}_{\mathbf{C}}(X, F) \tag{3.87}$$

Constant functor : $\Delta_X : I \to \mathbf{C}$ maps A, B to X, f, g to Id_X

Natural transformation $\eta : \lim F \to F$

Equalizer corresponds roughly to a solution of an equation, coequalizer to a quotient?

Example 87 *The equalizer of two morphisms $f, g : X \to Y$ in* **Set** *is the subset of X on which those morphisms agree, ie*

$$\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\} \tag{3.88}$$

Example 88 *The equalizer of two group homomorphisms $f, g : G \to H$*

Coequalizer

Example 89 *The coequalizer of two morphisms $f, g : X \to Y$ in* **Set** *is the quotient set for the equivalence defined by those morphisms :*

$$\mathrm{coeq}(f, g) = \{f(x) = g(x)\} \tag{3.89}$$

3.8.5 Directed limits

Given some directed set J forming our diagram, a directed limit is a limit of a functor $F : J \to \mathbf{C}$

Definition 90 *If the directed set J is furthermore a well-ordered set,*

$$\ldots \to 2 \to 1 \to 0 \tag{3.90}$$

the limit of this diagram is a sequential limit*.*

Example 91 *Given a sequence of inclusions*

$$\ldots \to X_2 \to X_1 \to X_0 \tag{3.91}$$

its sequential limit is the union over all indexes

$$\lim_J F = \bigcup_{i \in I} X_i \tag{3.92}$$

3.8.6 Properties of limits and colimits

Theorem 92 *For two diagrams I, J,*

$$f : \operatorname{colim} \lim D \tag{3.93}$$

Commutation

Effective morphisms

$$\bigsqcup_i U_i \to U \tag{3.94}$$

3.8.7 Limits and functors

A common tool in category theory to use is the behaviour of limits under the action of a functor.

Definition 93 *For a functor $F : \mathbf{C} \to \mathbf{D}$ and a diagram $J : \mathbf{I} \to \mathbf{C}$, a functor is said to preserve the limit $\lim_J$ if*

$$F \circ \lim_J \cong \lim_{F \circ J} \tag{3.95}$$

Preservation of limits and colimits

Left and right exact functors :

Definition 94 *A functor is* left exact *(resp.* right exact*)*

Maps inital objects to initial objects, products to products, and equalizers to equalizers

Even if we do not know the explicit limits and colimits of a category, we can verify that a functor preserves them, using the notion of a flat functor.

Definition 95 *A functor is* flat

[Flat functors preserve any limit and colimit]

Example 96 *The covariant hom-functor preserve limits :*

$$h^X(\lim F) = \lim(h^X F) \tag{3.96}$$

and the contravariant hom-functor preserves limits in the category $\mathbf{C}^{\mathrm{op}}$*, ie colimits in* $\mathbf{C}$ *:*

$$h_X() \tag{3.97}$$

Proof 9

Example : hom-sets of vector spaces with limits?

Example 97 *In the category of vector spaces* **Vec***, the covariant hom functor* h^V *gives us the set of linear transformations* $L(V, -)$*. If we take a look at various cases, we have for the terminal object* k^0 *:*

$$\begin{aligned} h^V(k^0) &= L(V, k^0) && (3.98)\\ &= \{0\} && (3.99)\\ &\cong \{\bullet\} && (3.100) \end{aligned}$$

The product of two vector spaces is the direct sum

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \tag{3.101}$$

The kernel of a linear map f *can be described as the equalizer of this map with the* 0 *map, Equalizer : for* $f, 0 : X \to Y$*, the equalizer is* $\ker(f)$*. The two functions* f, g *map to*

$$h^V(f) = \{a \in L(X, Y)\} \tag{3.102}$$

$$h^V(0) = \{0\} \tag{3.103}$$

$$\begin{aligned} h^V(\ker(f)) &= L(V, \ker(f)) && (3.104)\\ &= \mathrm{eq}(h^V(f), h^V(0)) && (3.105) \end{aligned}$$

$$\begin{aligned} h^V(f) &= L(V, \ker(f)) && (3.106)\\ &= && (3.107) \end{aligned}$$

The contravariant one : initial object is also k^0, therefore terminal object in op

$$\begin{aligned} h_V(k^0) &= L(k^0, V) && (3.108)\\ &= \{0\} && (3.109)\\ &\cong \{\bullet\} && (3.110) \end{aligned}$$

Same deal with the coproduct in op, which is also $\oplus$

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \tag{3.111}$$

but $\lim(h^V \circ *) = \lim \varnothing = *$

Reflected limits

Created limits

3.9 Monoidal categories

It is common in categories to have some need of defining a binary operation, ie some function of the type

$$A \cdot B = C \tag{3.112}$$

It is common in categories as well to have this notion applied to objects. Given two objects $A, B \in \mathbf{C}$, we want to find a third object C, such that there exists a bifunctor $(-) \cdot (-) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$

$$A \cdot B = C \tag{3.113}$$

While the notion of bifunctor covers this well enough, we often need to have additional conditions. A common case is that of a *monoid*, usually denoted as $\otimes$, where we ask that the operation be associative and unital, ie

$$\begin{aligned} (A \otimes B) \otimes C &= A \otimes (B \otimes C) && (3.114)\\ \exists I \in \mathbf{C},\ I \otimes A &= A \otimes I = A && (3.115) \end{aligned}$$

To categorify this notion, we define monoidal categories

Definition 98 *A* monoidal category $(\mathbf{C}, \otimes, I)$ *is a category* $\mathbf{C}$*, a bifunctor* $\otimes$*, and a specific object* $I \in \mathbf{C}$*, along with three natural transformations :*

$$a : ((-) \otimes (-)) \otimes (-) \overset{\cong}{\to} (-) \otimes ((-) \otimes (-)) \tag{3.116}$$
$$(X \otimes Y) \otimes Z \mapsto X \otimes (Y \otimes Z) \tag{3.117}$$
$$\lambda : (I \otimes (-)) \overset{\cong}{\to} (-) \tag{3.118}$$
$$I \otimes X \mapsto X \tag{3.119}$$
$$\rho : ((-) \otimes I) \overset{\cong}{\to} (-) \tag{3.120}$$
$$(X \otimes I) \mapsto X \tag{3.121}$$

which obey the following rules :

$$\begin{tikzcd} W \otimes (X \otimes (Y \otimes Z)) \arrow[r, "k"] \arrow[d, "I_W \otimes a_{X,Y,Z}"] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "k"] & ((W \otimes X) \otimes Y) \otimes Z)) \arrow[d, "f_i"] \\ W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "a_{W, X \otimes Y, Z}"] & & W \otimes (X \otimes Y)) \otimes Z \end{tikzcd}$$

We do not ask the equality for the associator and unitors, as equivalence is typically what we ask in general for a category. If in addition those equivalences are equality, we say that this is a *strict monoidal category*.

Example 99 *The tensor product of two vector spaces is a monoid in* $\mathbf{Vect}_k$*.*

Proof 10 *First we need to show that the tensor product is functorial. Given two morphisms* $f : X \to X'$*,* $g : Y \to Y'$*, the product* $f \otimes g : X \otimes Y \to X' \otimes Y'$

Proof that it is functorial, unit k*, associator, unitor, obeys the identities*

A basic example of monoidal structures on a category is the product and coproduct. If a category $\mathbf{C}$ has all finite products, it is automatically a monoidal category, called a *Cartesian monoidal category*, given by $(\mathbf{C}, \times, 1)$.

Likewise, a category with all finite coproducts is called a *co-Cartesian monoidal category*, given by $(\mathbf{C}, +, 0)$

Example 100 *The product and coproduct are both monoidal in* $\mathbf{Set}$*, leading to the monoidal category* $(\mathbf{Set}, \times, \{\bullet\})$*.*

A weaker notion to this is that of a *semi-Cartesian* monoidal category, which is when the unit of the tensor product is the terminal object. This is a useful notion to keep in mind due to this example :

Example 101 *The category of Poisson spaces,* **Poiss***, is a semi-Cartesian category.*

This will have consequences on the differences between classical and quantum mechanics later on.

Definition 102 *A* bimonoidal category

3.9.1 Braided monoidal category

Definition 103 *A* braided monoidal category $(\mathbf{C}, \otimes, I, B)$ *is a monoidal category equipped with a natural isomorphism B between the functors*

$$B : X \otimes (-) \to (-) \otimes X \tag{3.122}$$

Definition 104 *A monoidal category is* symmetric *if it is a braided monoidal category obeying*

3.10 Internalization

If a category admits set-like properties, typically properties such as finite limits, monoidal structures or Cartesian closedness, it is possible to recreate many types of mathematical structures inside the category itself.

3.10.1 Monoid in a monoidal category

The most generic form of internalization is that of an internal monoid (an even more general case would be that of a magma but we will not look at it here). Given some object M in the category, we would like to define what it means for M to be a monoid.

Definition 105 *In a monoidal category $(\mathbf{C}, \otimes, I)$, a monoid in $\mathbf{C}$ is given by a triplet (M, m, e), of an object $M \in \mathbf{C}$, a morphism $m : M \otimes M$, and a morphism $e : I \to M$, such that they make the following diagrams commute :*

All internal objects we will see as we go on will be some variation of a monoid in a monoidal category, most of them being more specifically Cartesian monoids, where the monoid will be that of the product, $(\mathbf{C}, \times, 1)$.

3.10.2 Internal group

Definition 106 *In a category* $\mathbf{C}$ *with finite products, a group object* G *is an object* $G \in \mathbf{C}$ *equipped with the morphisms*

- *The unique map to the terminal object* $p : G \to 1$
- *A neutral element morphism from the terminal object :* $e : 1 \to G$
- *An inverse endomorphism :* $(-)^1 : G \to G$
- *A binary morphism on the product :* $m : G \times G \to G$

such that all the following diagrams commute

If we want to write it out explicitly, the group object is not merely the object G itself but the quintuple $(G, p, e, (-)^{-1}, m)$.

Example 107 *Every category with finite product has the trivial group object* $\{e\}$ *which is the terminal object and the unique map to itself, so that*

$$\{e\} \cong (1, \mathrm{Id}_1, \mathrm{Id}_1, \mathrm{Id}_1, \mathrm{pr}_1) \tag{3.123}$$

Example 108 *The set of two elements* 2*, along with a given morphism* $e : 1 \to 2$*, the automorphism* $f : 2 \to 2$ *that is not the identity (the one that exchanges the two elements) and the morphism* $2 \times 2 \to 2$

form the internalized group $\mathbb{Z}_2$ *in* **Set** *:*

$$\mathbb{Z}_2 \cong (2, e, f, f) \tag{3.124}$$

Example 109 *As groups can be defined using sets, the category of sets contains every group as group objects using the traditional definition of groups.*

Example 110 *The group objects in the category* Top *are the topological groups, where all group operations are continuous functions. This can be done for instance by considering the set of all such quintuples that are mapped to a group object by the forgetful functor to* **Set***.*

Example 111 *The group objects in the category of smooth manifolds are the Lie groups, where the group operations are smooth maps.*

As the notation $(G, p, e, (-)^{-1}, m)$ is fairly cumbersome, from now on internal groups will be simply defined by their object G, with all morphisms left implicit.

Theorem 112 *For a category* $\mathbf{C}$ *with an internal group* G*, the hom-set* $\mathrm{Hom}_{\mathbf{C}}(X, G)$ *for any object* X *has a group structure.*

Proof 11 *If we consider two morphisms* $f, g \in \mathrm{Hom}_{\mathbf{C}}(X, G)$ *in* **Set** *(in other words an element in the product of two hom-sets), there*

Definition 113 *In a category with internal groups, the left action of an internal group* G *on an object* X *is defined by a morphism*

$$\rho : G \times X \to X \tag{3.125}$$

such that the following diagrams commute :

As a generalization of internal groups, we also have the notion of an internal groupoid.

Definition 114 *An internal groupoid* $\mathcal{G}$ *in a category with products is given by three objects in a category* (E, G, X)*, with* X *the underlying space of the groupoid,* G *the and the following morphisms :*

- *The morphism* $e : X \to G$ *associating the unit element to the underlying object* X
- *The inverse morphism* $(-)^{-1} : G \to G$
- *The*

3.10.3 Internal ring

Definition 115 *In a category* $\mathbf{C}$ *with finite products, a* ring object R *is an object* $R \in \mathbf{C}$ *along with the following morphisms :*

- *The addition morphism* $a : R \times R$
- *The multiplication morphism* $m : R \times R \to R$
- *The addition identity morphism* $0 : 1 \to R$
- *The multiplication identity morphism* $1 : 1 \to R$
- *The additive inverse morphism :* $-(-) : R \to R$

such that all those morphisms make the following diagrams commute :

Example 116 *The real line object* $\mathbb{R}$ *in many categories is an internal ring, with the elements* $0, 1 : 1 \to \mathbb{R}$*, the morphisms Those morphisms are for instance smooth functions and continuous functions, so that both* **SmoothMan** *and* **Top** *have the internal ring of the real line.*

It is common for many categories to have some variant of the real line, as

3.11 Groupoids

In many contexts, the objects of a category can themselves be categories, such as the category of categories **Cat**, the category of groupoids **Grpd** (where the monoid structure on the elements of the groupoids is built from morphisms) and so on.

If a category of that sort is equipped with a terminal object 1 which represents a category of a single object **1**, such as is the case for groupoids for instance, and we have that the object set of some other object category **C** is given by the hom-set

$$\mathrm{Obj}(\mathbf{C}) \cong \mathrm{Hom}_{?}(\mathbf{1}, \mathbf{C}) \tag{3.126}$$

In those circumstances, we can consider the morphisms of those objects by functors preserving 1 and C, as a functor will map those objects $X, Y : 1 \to C$ to each other (associativity?)

From this, we also have that the natural transformations η between the functors (or here morphisms) $f, g \in \mathrm{Mor}(\mathbf{C})$

3.12 Subobjects

Given an object X in a category $\mathbf{C}$, a *subobject* S of X is an isomorphism class of monomorphisms $\{\iota_i\}$

$$\iota_i : S_i \hookrightarrow X \tag{3.127}$$

so that the equivalence classes of $\{\iota_i\}$ is given by any two such monomorphisms if there exists an isomorphism between the two subobjects S_i;

$$S = [S_i]/(S_i \cong S_j \leftrightarrow \exists f : S_i \to S_j,\ \exists f^{-1} S_j \to S_j,\ f \circ f^{-1} = \mathrm{Id}) \tag{3.128}$$

Isomorphisms given by the automorphism group Aut()

This is meant to define the common mathematical notion of an object being *part* of another object in some sense.

Examples :

Example 117 *On* **Set***, subobjects are subsets (defined by injections up to the symmetric group)*

Example 118 *On* **Vect**$_k$*, subobjects are subspaces (defined by injections up to the general linear group?)*

On **Top**, subobjects are subspaces with the subspace topology

On **Ring**,

On **Grp**, subobjects are subgroups

Example 119 *On the category of smooth manifolds* **SmoothMan***, subobjects are* submanifolds*,* $\iota : S \hookrightarrow M$*, where the set of all submanifolds with the same image up to diffeomorphism of the base* $\mathrm{Diff}(S)$ *are equivalent.*

"Let C_c be the full subcategory of the over category C/c on monomorphisms. Then C_c is the poset of subobjects of c and the set of isomorphism classes of C_c is the set of subobjects of c. "

3.13 Simplicial categories

Simplicial category Δ is made of simplicial objects, ie

$$\left[\vec{\mathbf{0}}\right] = \{\bullet\} \tag{3.129}$$

$$\left[\vec{\mathbf{1}}\right] = \{\bullet \to \bullet\} \tag{3.130}$$

$$\left[\vec{\mathbf{2}}\right] = \{\bullet \to \bullet \to \bullet\} \tag{3.131}$$

$$\left[\vec{\mathbf{3}}\right] = \{\bullet \to \bullet \to \bullet \to \bullet\} \tag{3.132}$$

Simplicial morphisms

Despite the name, the simplicial categories (and its overarching simplicial 2-category) is not made of simplices, but they will be of use later on to *construct* simplices in the category of simplicial sheaves.

3.14 Equivalences and adjunctions

Like many objects in mathematics, it is possible to try to define some kind of equivalence between two categories. Like most such things, we try to consider two mappings between our categories. Let's consider two categories $\mathbf{C}, \mathbf{D}$, and two functors F, G between them,

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D} \tag{3.133}$$

The usual process of finding equivalent objects in such cases is to have those two maps be inverses of each other, ie $FG = \mathrm{Id}_{\mathbf{D}}$ and $GF = \mathrm{Id}_{\mathbf{C}}$. The composition functor FG maps every object and morphism of D to itself and likewise for GF on C, so that in some sense, the objects X and $F(X)$ are the same objects, and likewise, $f : X \to Y$ and $F(f) : F(X) \to F(Y)$ represent the same morphism.

If two categories admit such a pair of functors, we say that they are *isomorphic*. While this is the most obvious definition of equivalence, it is in practice not commonly used, as very few categories of interest are actually isomorphic, and this is generally considered too strict a definition in the philosophy of category theory, as we are generally more interested in the relationships between objects rather than the objects themselves. A good example of this overly strict definition is that the product of two objects done in a different order will not be isomorphic.

If we weaken the notion of equivalence, we can look at the case where our two functors are merely isomorphic to the identity, $FG \cong \mathrm{Id}_{\mathbf{D}}$ and $GF \cong \mathrm{Id}_{\mathbf{C}}$, where there exists a natural transformation η taking FG to $\mathrm{Id}_{\mathbf{D}}$.

$$\eta : FG \quad \to \quad \mathrm{Id}_{\mathbf{D}} \tag{3.134}$$

$$\epsilon : \mathrm{Id} \tag{3.135}$$

$$\mathbf{C} \qquad \mathbf{D}$$

Triangle identities

For every objects $X \in C, Y \in D$, the components of the relevant natural transformations are

$$\mathrm{Id}_{F(Y)} = \epsilon_{F(Y)} \circ F(\eta_Y) \tag{3.136}$$

$$\mathrm{Id}_{G(X)} = G(\epsilon_X) \circ \eta_{G(X)} \tag{3.137}$$

$F \dashv G$

Adjunction between two categories

Example 120 *A basic non-trivial example of adjoint functors is the even and odd functors. If we consider $\mathbb{Z}$ as a linear order category, with $\leq$ as its morphisms, functors are its order-preserving functions. Two specific functors that we have are the even and odd functors, give by*

$$\forall k \in \mathbb{Z},\ \mathrm{Even}(k) = 2k,\ \mathrm{Odd}(k) = 2k + 1 \tag{3.138}$$

These do indeed preserve the order so that for the unique morphism $k_1 \leq k_2$, it is mapped to the unique morphism $2k_1 \leq 2k_2$ and similarly for the odd functor. The corresponding "inverse" functor is the functor mapping any integer to the floor of its division by 2 :

$$\lfloor -/2 \rfloor : \mathbb{Z} \to \mathbb{Z} \tag{3.139}$$

Even *is then the left adjoint and* Odd *the right adjoint of $\lfloor -/2 \rfloor$. The unit and counit of* Even *are :*

$$\varepsilon_l = f \circ \lfloor -/2 \rfloor \tag{3.140}$$

[...]

This is however not an equivalence, as the floor function is not strictly an inverse of the even and odd functor (and not being a faithful functor to begin with), as $\lfloor (2n)/2 \rfloor = \lfloor (2n+1)/2 \rfloor$, and we have

Example 121 *Take the two linear order categories of $\mathbb{Z}$ and $\mathbb{R}$, with their elements being the objects and their order relations are the morphisms. The inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$, mapping $n \in \mathbb{Z}$ to its equivalent real number, is a functor*

We can try to define left and right adjoints for it, by finding two functions $f, g : \mathbb{R} \to \mathbb{Z}$ for which there exists

- *A left and right counit $\epsilon_l : f\iota \to \mathrm{Id}_{\mathbb{Z}}$ and $\epsilon_r : \iota g \to \mathrm{Id}_{\mathbb{R}}$*
- *A left and right unit $\eta_l : \mathrm{Id}_{\mathbb{R}} \to \iota f$ and $\eta_r : \mathrm{Id}_{\mathbb{Z}} \to g\iota$*

And all these must obey the triangle identities

If we take for instance the left adjoint, we need that our function f be such that there exists a natural transformation between the identity on $\mathbb{R}$ ($\mathrm{Id}_{\mathbb{R}}(x) = x$) and our function f reinjected into $\mathbb{R}$: $\iota(f(x))$. For every $x, y \in \mathbb{R}$, there's a morphism

$$\begin{tikzcd}
x \arrow[d, "\leq"] & x \arrow[r, "\eta_{l,x}"] \arrow[d, "\leq"] & \iota(f(x)) \arrow[d, "\iota(f(\leq))"] \\
y \arrow[r, phantom, "\overset{\eta}{\Longrightarrow}"] & y \arrow[r, "\eta_{l,y}"] & \iota(f(y))
\end{tikzcd}$$

In the context of our linear order, this means that for any two numbers x, y such that $x \leq y$, we have $x \leq \iota(f(x))$, $y \leq \iota(f(y))$ and $\iota(f(x)) \leq \iota(f(y))$

$$x \leq y \leq \iota(f(y)) \tag{3.141}$$

and for the counit, we need a natural transformation between the mapping of an integer into $\mathbb{R}$ and then back into $\mathbb{Z}$ via f with $f(\iota(n))$, and the identity on $\mathbb{Z}$, $\mathrm{Id}_{\mathbb{Z}}(n) = n$. For every n, m, $n \leq m$,

$$\begin{tikzcd} n \arrow[d, "\leq"] & & f(\iota(n)) \arrow[r, "\epsilon_{r,n}"] \arrow[d, "f(\iota(\leq))"] & n \arrow[d, "\leq"] \\ m \arrow[r, phantom, "\overset{\epsilon}{\Longrightarrow}"] & & f(\iota(m)) \arrow[r, "\epsilon_{r,m}"] & m \end{tikzcd}$$

We have the condition that if we inject n into $\mathbb{R}$, its left adjoint will be such that $f(\iota(n)) \leq n$, $f(\iota(m)) \leq m$ and $f(\iota(n)) \leq f(\iota(m))$. If we take the case $m = n+1$ and ignoring the injection ι for now, this means that $f(n) \leq f(n+1) \leq n+1$ and $f(n) \leq n$. As $f(n) \leq n+1$ cannot be equal to $n+1$, $f(n)$ can only be equal to n or smaller. If we pick the case $n-1 \leq n$ instead, the natural transformation implies $f(n-1) \leq f(n) \leq n$ and $f(n-1) \leq n-1 \leq n$, so that $f(n-1) < n$ and

Triangle identities : for any $n \in \mathbb{Z}$,

$$\mathrm{Id}_{\iota(n)} = \iota(\epsilon_{l,n}) \circ \eta_{l,\iota(n)} \tag{3.142}$$

The components of this natural transformations give us that, if we transform our integer n to a real and back,

$$\begin{tikzcd} x \arrow[d, "\leq"] & & x \arrow[r, "\eta_{l,x}"] \arrow[d, "\leq"] & \iota(f(x)) \arrow[d, "\iota(f(\leq))"] & \overset{\epsilon}{\Longrightarrow} \\ y \arrow[rr, phantom, "\overset{\eta}{\Longrightarrow}"] & & y \arrow[r, "\eta_{l,y}"] & \iota(f(y)) & \end{tikzcd}$$

Different definitions of adjunction

Adjoint functors : For two functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$, the functors form an *adjoint pair* $F \dashv G$, F the left adjoint of G and G the right adjoint of F, if there exists two natural transformations, η and ϵ

$$\eta : \mathrm{Id}_{\mathbf{C}} \to G \circ F \tag{3.143}$$
$$\epsilon : F \circ G \to \mathrm{Id}_{\mathbf{D}} \tag{3.144}$$

obeying the triangle equalities

$$\begin{tikzcd} F \arrow[rr, "\mathrm{Id}_F"] \arrow[dr, "F\eta"'] & & F \\ & FGF \arrow[ur, "\epsilon F"'] & \end{tikzcd}$$

Adjunct : for an adjunction of functors $(L \dashv R) : \mathbf{C} \leftrightarrow \mathbf{D}$, there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(LX, Y) \cong \mathrm{Hom}_{\mathbf{D}}(X, RY) \tag{3.145}$$

Two morphisms $f : LX \to Y$ and $g : X \to RY$ identified in this isomorphism are *adjunct*. g is the right adjunct of f, f is the left adjunct of g.

$$g = f^{\sharp},\ f = g^{\flat} \tag{3.146}$$

Adjunction for vector spaces

Galois connection

Theorem 122 *Given an adjunction $(L \dashv R)$, L preserves any colimits, while R preserves any limits.*

Proof 12 *As the hom-functor preserves limits, we have that*

$$\mathrm{Hom} \tag{3.147}$$

3.14.1 Dual adjunctions

A dual adjunction between two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ is given by a pair of natural transformations

$$\eta : \mathrm{Id}_{\mathbf{C}} \to GF \tag{3.148}$$
$$\theta : \mathrm{Id}_{\mathbf{D}} \to FG \tag{3.149}$$

which obey the triangle equalities $F\eta \circ \theta F = \mathrm{Id}_F$ and $G\theta \circ \eta G = \mathrm{Id}_G$:

[diagrams]

this is a notion that is simply the adjunction

[...]

This notion will be mostly useful for the notion of duality of two categories.

3.15 Grothendieck construction

3.16 Reflexive subcategories

Given two categories $\mathbf{C}$, $\mathbf{D}$, $\mathbf{C}$ is a reflexive subcategory of $\mathbf{D}$ if it is a full subcategory, $\mathbf{C} \hookrightarrow \mathbf{D}$, ie every morphism of $\mathbf{C}$ is a morphism of $\mathbf{D}$, and such that every object $d \in \mathrm{Obj}(\mathbf{D})$ and morphism $(f : d \to d') \in \mathrm{Mor}(\mathbf{D})$ have a reflection in $\mathbf{C}$.

Def : The inclusion functor $\iota : \mathbf{C} \hookrightarrow \mathbf{D}$ has a left adjoint T, the *reflector* :

$$(T \dashv \iota) : \mathbf{C} \overset{\iota}{\hookrightarrow} \mathbf{D} \tag{3.150}$$

T is the *reflector*

[...]

Examples : $\mathbf{Ab} \hookrightarrow \mathbf{Grp}$, reflection is the abelianization

Example 123 *The category of metric spaces with isometries as morphisms has as a full subcategory the category of complete metric spaces. The reflector associates the completion of the metric space to any metric space.*

3.17 Monads

Definition 124 *A* monad *is a category* $\mathbf{C}$ *with*

- *An object* $A \in \mathbf{C}$
- *An endomorphism* $t : A \to A$
- *A natural transformation* $\eta : \mathrm{Id}_A \to t$
- *A natural transformation* $\mu : t \circ t \to t$

Theorem 125 x

Monads from adjunctions

3.17.1 Algebra of a monad

Monads naturally form an algebra over each of the objects that they act upon.

Definition 126 *For a monad (T, η, μ) on a category $\mathbf{C}$, a T-algebra is a pair (X, f) of an object $X \in \mathbf{C}$ and a morphism $f : TX \to X$ making the following diagrams commute :*

The precise category in which this algebra is defined

Definition 127 *The* Eilenberg-Moore category *of a monad*

Example 128 *The basic adjoint modality example is the* even/odd modality *pair,*

$$\mathrm{Even} \dashv \mathrm{Odd} \tag{3.151}$$

This is done on the category of integers as an ordered set, $(\mathbb{Z}, \leq)$, for which the morphisms are the order relations, and endofunctors are order-preserving functions.

The functor we consider here is the largest integer which is smaller to $n/2$:

$$\begin{aligned} \lfloor -/2 \rfloor : (\mathbb{Z}, \leq) &\to (\mathbb{Z}, \leq) && (3.152) \\ n &\mapsto \lfloor n/2 \rfloor && (3.153) \end{aligned}$$

This functor has a left and right adjoint functor,

$$\begin{aligned} \mathrm{even} : (\mathbb{Z}, \leq) &\hookrightarrow (\mathbb{Z}, \leq) && (3.154) \\ n &\mapsto 2n && (3.155) \\ \mathrm{odd} : (\mathbb{Z}, \leq) &\hookrightarrow (\mathbb{Z}, \leq) && (3.156) \\ n &\mapsto 2n + 1 && (3.157) \\ & && (3.158) \end{aligned}$$

Proof :

$\lfloor -/2 \rfloor$ has as a domain the whole category

For a total order, the hom-set $\mathrm{Hom}(X, Y)$ is simply empty if $X > Y$ and has a single element otherwise. For $\lfloor -/2 \rfloor$, the hom-set

The left adjoint of $\lfloor -/2 \rfloor$ is a functor such that

$$\mathrm{Hom}_{\mathbb{Z}}(L(-), -) \cong \mathrm{Hom}_{\mathbb{Z}}(-, \lfloor -/2 \rfloor) \tag{3.159}$$

In the case of a total order, the isomorphism simply means that both sets have the same cardinality, ie they either have no elements (the two objects are not ordered) or one (the two objects are ordered). So

$$\mathrm{Hom}_{\mathbb{Z}}(L(n), m) \cong \mathrm{Hom}_{\mathbb{Z}}(n, \lfloor m/2 \rfloor) b \Leftrightarrow L(n) \leq m \leftrightarrow n \leq \lfloor m/2 \rfloor \tag{3.160}$$

If we have $L(n) = 2n$, we need to show this equivalence both ways.

If $2n \leq m$, then dividing by 2, we have $n \leq m/2$, which we can then apply the floor to both sides (it is monotonous), so $\lfloor n \rfloor \leq \lfloor m/2 \rfloor$. As n is an integer, $n \leq \lfloor m/2 \rfloor$

Converse : If $n \leq \lfloor m/2 \rfloor$: From properties of floor :

$$n \leq \lfloor \frac{m}{2} \rfloor \leftrightarrow 2 \leq \frac{\lceil m \rceil}{n} \tag{3.161}$$

As m is an integer, $2n \leq m$.

So the even function is indeed left adjoint.

Odd function is right adjoint :

From these three functions, we can define adjoint monads :

$$(\mathrm{Even} \vdash \mathrm{Odd}) \tag{3.162}$$

which send numbers to their half floor and then to their corresponding even and odd number :

$$\mathrm{Even}(n) = 2\lfloor n/2 \rfloor \tag{3.163}$$

$$\mathrm{Odd}(n) = 2\lfloor n/2 \rfloor + 1 \tag{3.164}$$

n	Even(n)	Odd(n)
-2	-2	-1
-1	-2	-1
0	0	1
1	0	1
2	2	3
3	2	3

Table 3.1: Caption

Monad and comonad, unit and counit, multiplication

Example 129 *Integrality modality : Given the two total order categories* $(\mathbb{Z}, \leq)$ *and* $(\mathbb{R}, \leq)$*, the inclusion functor*

$$\begin{aligned} \iota : (\mathbb{Z}, \leq) &\hookrightarrow (\mathbb{R}, \leq) && (3.165) \\ n &\mapsto n \textit{ (as a real number)} && (3.166) \end{aligned}$$

Left and right adjoints :

An important class of monads are the ones which are associated (in the internal logic of the category, cf. [X]) to the classical *modalities*, ie necessity and possibility.

Kripke semantics

Example 130 *Necessity/Possibility modalities*

3.18 Linear and distributive categories

Chapter 4

Spaces

One of the main type of category we will use for objective logic are categories which are *spaces* or relate to spaces, in a broad sense, such as frames, sheaves and toposes.

[28, 29]

4.1 General notions of a space

Before looking into how spaces work in category theory, let's first look at how spaces are treated both intuitively, in philosophical analysis, and the most common ways to treat spaces in mathematics.

4.1.1 Mereology

One important aspect of a space in philosophical terms is that of *mereology*. The mereology of a space is the study of its parts, where we can decompose a space into regions with some specific properties. A space X is composed of a collection of regions $\{U_i\}$, which are ordered by a relation of inclusion $(\{U_i\}, \subseteq)$, called *parthood*, which obeys the usual partial order relations :

- Reflexion :
$$U \subseteq U$$
- Symmetry :
$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_1 \to U_1 = U_2$$
- Transitivity :
$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_3 \to U_1 \subseteq U_3$$

Those are the typical notion of a partial order : reflexivity (a region is part of itself), antisymmetry (if a region is part of another, and the other region is part of the first, they are the same region) and transitivity (if a region is part of another region, itself part of a third region, the first is part of the third). The antisymmetry allows us to define equality in terms of parthood, simply as $U_1 = U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_2 \subseteq U_1$.

We also have the *proper parthood* relation $\subset$, which is defined simply as parthood excluding equality :

$$U_1 \subset U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_1 \neq U_2 \tag{4.1}$$

Other relations we can define are proper extension, underlap, disjointness, indiscernibility ,

Definition 131 *The* overlap *of two regions is the existence of a third region which is a part of both :*

$$U_1 \circ U_2 \leftrightarrow \exists U_3,\ [U_3 \subseteq U_1 \wedge U_3 \subseteq U_1] \tag{4.2}$$

Unless a mereological nihilist, we also typically define an operation to turn several regions into one, the *fusion* :

Definition 132 *Given a set of regions $\{U_i\}_{i \in I}$, we say that U is the* fusion *of those region, $\sum(U, \{U_i\})$,*

Mereologies can vary quite a lot depending on what you wish to model or your own philosophical bent. *Mereological nihilism* will assume for instance that there are no objects with proper parts (so $U_1 \subseteq U_2$ implies $U_1 = U_2$), and we can only consider a collection of atomic points with no greater structure (in particular, there is no *space* itself which is the collection of all its regions), while on the other end of the spectrum is *monism* (such as espoused by Parmenides[1]), where the only region is the whole space itself, with no subregion.

Typically however, we tend to consider some specific base axioms for a mereology. Beyond the partial ordering axioms (M1 to M3), we also have

- M4 - Weak supplementation : if U_1 is a proper part of U_2, there's a third region U_3 which is part of U_2 but does not overlap with U_1 : $U_1 \subset U_2 \to \exists U_3,\ [U_3 \subseteq U_2 \wedge \neg U_3 \circ U_1]$

Systems[30, 31] : mereology **M**, minimal mereology **MM**, extensional mereology **EM**, classical extensional mereology **CEM**, general mereology **GM**, general extensional mereology **GEM**, atomic general extensional mereology **AGEM**

In terms of categories, the various formalizations of mereology are expressed by different types of algebraic structures on posets. **M** is simply a poset with no extra structure.

The TOP axiom corresponds to the existence of a greatest element in this partial order (if we consider this applying to spaces, this is the object X of the space itself), BOTTOM to a least element (the empty set).

Most axiomatizations of mereology do not include the bottom element, but we will keep it for a better analogy with spaces in terms of a category, as they typically include one.

4.1.2 Topology

A common approach for space in mathematics is the notion of *topology*. We have already briefly defined the category **Top** of topological spaces, but as they form the basis for most of the common understanding of spaces in math, we should look into them more deeply : what their motivations are, what they are for, and how they may relate to other objects.

If we look at one of the common structuration of mathematics, popularized by Bourbaki[ref on structuralism], spaces are built first as sets, then as topological spaces, and they may afterward get further specified into other structures, such as metric spaces, etc.

This is only a convention, as there are many other structures one may choose, that can be easier, more general, more specific to a given property, etc. The point of topological spaces is that they are a good compromise between those constraints, being fairly easy to define and allowing to talk about quite a lot of properties.

First, let's look at the basic structure of sets. From the perspective of mereology, sets are a rather specific choice of structure, corresponding to an atomic unbounded relatively complemented distributive lattice 1, [see definition of points in philosophy too etc]

Historically, the notion that spaces are made of point is quite ancient [cf. Sextus Empiricus], but it has not had the modern popularity it now has until the works of [Riemann?] Cantor, Hausdorff, Poincaré, etc, and the notion was put into the modern mathematical canon with such works as Bourbaki, etc.

If we consider spaces only as sets however, the informations we can derive from them is rather limited. This is an observation from antiquity [Sextus again]

In modern terms, we can talk about subsets, cardinalities, overlap and unions, but we would be missing on quite a lot of intuitively important properties of a space. If we consider physical space as our example, as we've seen from mereology, some points seem to "belong together" more than other points, some may be "next to" a give subset even if they do not belong to it, two subsets may "touch" without any overlap, and so on.

To illustrate those notions, we can consider some subsets of the plane. If we compare let's say some kind of continuous shape, a disk and an uncountable set of points sprinkled in an area (for instance the two-dimensional Cantor dust),

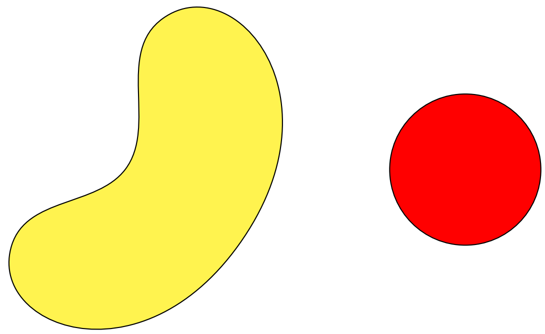


Figure 4.1: Three subsets of the plane of identical cardinality

We would expect the first two objects to have more in common than with the third, having what we will later call the same *shape*, but as sets, they are all isomorphic, simply by the virtue of containing the same amount of points. The same goes for comparing one connected shape and one composed of two disconnected shapes

[...]

If we consider the set D of all points that are at a distance strictly inferior to a given value r from a central point o, we would like to say that a point at exactly a distance of r is somehow closer to D than other points, but as a set, this point is simply equivalent to every point outside of D, as $D \cap \{p\} = \varnothing$.

Convergence

quasitopological spaces, approach spaces, convergence spaces, uniformity spaces, nearness spaces, filter spaces, epitopological spaces, Kelley spaces, compact Hausdorff spaces, δ-generated spaces Cohesion, pretopology, proximity spaces, convergence spaces, cauchy spaces, frames, locales

Approaches via open/closed sets, closure operators, interior operators, exterior operators, boundary operators, derived sets

Definition 133 *Given a set X, a* net *on X is a directed set $(A, \leq)$ and a map $\nu : A \to X$*

Example 134 *A* sequence *is a net with directed set $(\mathbb{N}, \leq)$*

Example 135 *Net of neighbourhoods*

Definition 136 *A net is said to* converge *to an element $x \in X$*

Definition 137 *A* filter

4.2 Frames and locales

[Hasse diagram]

All those notions of mereology and topology can be formalized within the context of category theory using the notion of frames and locales.

As we've seen, any formalization of a space can be at least formalized as a poset ordered by inclusion, already a category. All further notions relating to spaces will therefore be extra structures on posets, typically relating to their limits.

First we need to define the notion of semilattices for joins and meets.

Definition 138 *A meet-semilattice is a poset $(S, \leq)$ with a meet operation $\wedge$ corresponding to the greatest lower bound of two elements (which is assumed to always exist in a meet-semilattice) :*

$$m = a \wedge b \leftrightarrow m \leq a \wedge m \leq b \wedge (\forall w \in S,\ w \leq a \wedge w \leq b \rightarrow w \leq m) \tag{4.3}$$

Example 139 *In $\mathbb{Z}$ and $\mathbb{R}$ (in fact for any total order), the meet of two numbers is the min function :*

$$k_1 \wedge k_2 = \in (k_1, k_2) \tag{4.4}$$

Theorem 140 *In a poset category, the meet is the coproduct.*

Example 141 *In the partial order defined by the power set of a set, the meet is the intersection of two sets.*

Definition 142 *A join-semilattice is a poset $(S, \leq)$ with a join operation $\vee$ corresponding to the least upper bound of two elements (which is assumed to always exist in a join-semilattice) :*

$$m = a \vee b \leftrightarrow a \leq m \wedge b \leq m \wedge (\forall w \in S,\ a \leq w \wedge b \leq w \rightarrow m \leq w) \tag{4.5}$$

Theorem 143 *In a poset category, the join is the product.*

Proof 13 *Product :* $\prod_i a_i$

Natural transformation (by components) :

$$\eta_X : \prod_i a_i \to a_i \tag{4.6}$$

There is one morphism from the join to each element, therefore $\prod_i a_i \leq a_i$

Upper bound is least : for any other b *such that* $b \leq a_i$ *(ie the natural transformation* $\alpha_b : b \to a_i$ *for some* a_i*), then the unique morphism* $f : b \to \prod a_i$ *(b smaller than* a_i*)*

Properties :

Proposition 144 *The meet is commutative :* $a \wedge b = b \wedge a$.

Proof 14 *As the roles of* a *and* b *in the definition of the meet are entirely symmetrical, due to the commutativity of the logical conjunction, this is true.*

Proposition 145 *The meet is associative :* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof 15 *If* $m = b \wedge c$*, then* $a \wedge (b \wedge c) = a \wedge m$*, meaning that the meet can be defined by some element* m' *such that*

$$(m' \leq a) \wedge (m' \leq m) \wedge (m \leq b) \wedge (m \leq c) \tag{4.7}$$

$$\wedge (\forall w \in S,\ w \leq b \wedge w \leq c \to w \leq m) \tag{4.8}$$

$$\wedge (\forall w' \in S,\ w' \leq a \wedge w \leq m \to w \leq m') \tag{4.9}$$

as $(m' \leq m) \wedge (m \leq b)$

Definition 146 *A* lattice *is a poset that is both a meet and join semilattice, such that* $\wedge$ *and* $\vee$ *obey the* absorption law

$$a \tag{4.10}$$

Definition 147 *A* Heyting algebra *is a bounded lattice*

A common poset structure that we will use is the algebra generated by a family of subsets. If we have a set X, and a family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, $\mathcal{B}$ forms a poset by the inclusion relation ordering, $(\mathcal{B}, \subseteq)$. Some common families of subsets of interest are the power set $\mathcal{P}(X)$, and more generally the set of opens $\mathrm{Op}(X)$ for a given topology.

It is a directed set with X the top element

In this context, the meet is the intersection

Theorem 148 *The intersection of two sets is their meet.*

Proof 16 *As the intersection of A and B is defined via*

$$A \cap B = \{x | x \in A \wedge x \in B\} \tag{4.11}$$

We can

We already know that $A \cap B \subseteq A, B$. If we assume a set $C \neq A \cap B$ such that $A \cap B \subseteq C$ and $C \subseteq A, B$, this means that C contains all the same elements as $A \cap B$ with some additional elements (since $A \cap B$ is a subset, we have $C = (A \cap B) \cup (A \cap B)^C$, and as they are different, $(A \cap B)^C \neq \varnothing$). However, as C is a subset of A and B, that complement can only contains elements of A and B

Theorem 149 *The union of two sets is their join.*

Proof 17 *As the union of A and B is defined via*

$$A \cup B = \{x | x \in A \vee x \in B\} \tag{4.12}$$

A family of sets is therefore a meet semilattice if it is closed under intersection, and a join semilattice if it is closed under union. If it is both, it is automatically a lattice as the absorption laws are obeyed by union and intersection.

[proof]

If the empty set is furthermore included, it is a bounded lattice, with $1 = X$, $0 = \varnothing$

Semi-lattice, lattice, Heyting algebra, frame (complete Heyting algebra)

Definition 150 *A* Heyting algebra *H is a bounded lattice for which any pair of elements $a, b \in H$ has a greatest element x, denoted $a \to b$, such that*

$$a \wedge x \leq b \tag{4.13}$$

Definition 151 *The* pseudo-complement *of an element a of a Heyting algebra is*

$$\neg a = (a \to 0) \tag{4.14}$$

Example 152 *A bounded total order $0 \to 1 \to \ldots \to n$ is a Heyting algebra given by*

$$a \to b = \begin{cases} n & a \leq b \\ b & a > b \end{cases} \tag{4.15}$$

The pseudo-complement is therefore just $\neg a = 0$.

Example 153 *For the power set $\mathcal{P}(X)$ poset, the relative pseudo-complement of two sets A, B is*

$$C = (X \setminus A) \cup B \tag{4.16}$$

This follows the property as $A \cap C = A \cap (B \setminus A)^C$

(eq. to the discrete topology)

Definition 154 *A Heyting algebra is* complete *if*

Definition 155 *A* frame *$\mathcal{O}$ is a poset that has all small coproducts (called joints $\vee$) and all finite limits (called meets $\wedge$), and satisfied the distribution law*

$$x \wedge (\bigvee_i y_i) \leq \bigvee (x \wedge y_i) \tag{4.17}$$

Frames define a mereology by considering its objects as regions, its poset structure by the parthood relation, and joins and meets by

Mereological axiom for distribution law?

Category of frames : **Frm**

Dual category : the locales $\mathbf{Frm}^{\mathrm{op}} = \mathbf{Locale}$

Definition 156 *A* boolean algebra $a \wedge \neg a = 0$

Example 157 *A power set is a boolean algebra*

Proof 18 *The power set $\mathcal{P}(X)$*

The basic example of a frame in math is that of the frame of opens for a topological space (X, τ).

Example 158 *The category of open sets of a topological space X,* $\mathrm{Op}(X)$*, is a frame.*

Proof 19 *If we consider the poset of opens, as a union and intersection of open sets is itself an open set, we have a lattice, which is bounded by X itself and the empty set $\varnothing$.*

The frame of open is not boolean typically, as the negation $\neg$ can be defined as $\neg a \to 0$, and the implication

$$\begin{aligned} U \to V &= \bigcup \{W \in \mathrm{Op}(X) | U \cap W \subseteq V\} && (4.18) \\ &= (U^c \cup V)^\circ && (4.19) \end{aligned}$$

$$\begin{aligned}
\neg U &= (U^c \cup \varnothing)^\circ && (4.20)\\
&= (U^c)^\circ && (4.21)\\
&= X \setminus \mathrm{cl}(U \cap X) && (4.22)\\
&= X \setminus \mathrm{cl}(U) && (4.23)
\end{aligned}$$

The interior of the complement

$$\begin{aligned}
U \cup (X \setminus U)^\circ &= (X \cup U) \setminus (\mathrm{cl}(U) \setminus U) && (4.24)\\
&= X \setminus \partial U && (4.25)\\
& && (4.26)
\end{aligned}$$

Therefore a frame of open is boolean if open sets never have a boundary, which is that every open set is a clopen set.

Stone theorem

Theorem 159 *The category* **Sob** *of sober topological spaces with continuous functions and the category* **SFrm** *of spatial frames are dual to each other.*

Examples :

Example 160 *For a given set X, the partial order defined by inclusion of the power set $\mathcal{P}(X)$, is a complete atomic Boolean algebra.*

Definition 161 *A sober topological space*

Theorem 162 *Stone duality : The category of sober topological spaces* **Sob** *is dual to the category of spatial frames* **SFrm**

4.2.1 Sublocales

Moore closure

double negation sublocale

Consider the map

$$\begin{aligned}
\neg\neg : L &\to L && (4.27)\\
U &\mapsto \neg\neg U && (4.28)
\end{aligned}$$

A nucleus on L (a frame) is a function $j : L \to L$ which is monotone ($j(a \wedge b) = j(a) \wedge j(b)$), inflationary ($a \leq j(a)$) and $j(j(a)) \leq j(a)$

A meet-preserving monad.

Properties :

- $j(\top) = \top$
- $j(a) \leq j(b)$ if $a \leq b$
- $j(j(a)) = j(a)$

Quotient frames : L/j is the subset of L of j-closed elements of L (such that $j(a) = a$).

Double negation sublocale :

4.2.2 Lattice of subobjects

If we are to consider some category or object of a category as representing a space in some sense, a useful method to model it is to consider the structure given by its subobjects, as this is the best analogue that we have to a subregion of a space.

Given a category $\mathbf{C}$, we can look at the *poset of subobjects* $\mathrm{Sub}(X)$ for a given object X, with the following definition :

Definition 163 *The poset of subobjects of X is the skeletal subcategory of the slice category $\mathbf{C}_{/X}$ from which we take every object of $\mathbf{C}_{/X}$ which is a monomorphism in $\mathbf{C}$, and identify every isomorphism to the identity.*

Theorem 164 *Every morphism in $\mathrm{Sub}(X)$ is a monomorphism in $\mathbf{C}$*

Proof 20 *As our objects are only monomorphisms, and any morhpism in $\mathrm{Sub}(X)$ will be a slice morphism g between two monomorphisms f, f', so that $f' \circ g = f$, we can just use the property that of $f \circ g$ is a monomorphism, then so is g. Therefore all morphisms in $\mathrm{Sub}(X)$ are monomorphisms in $\mathbf{C}$.*

They are in fact monomorphisms between the subobjects of X themselves.

Theorem 165 $\mathrm{Sub}(X)$ *is a poset category*

Proof 21 *As we are assuming the equivalence class for everything here, if we have two morphisms f, g between two objects $\alpha, \beta \in \mathrm{Sub}(X)$, this means that we have two subobjects A, B of X with inclusion $\alpha : A \hookrightarrow X$ and $\beta : B \hookrightarrow X$*

It is best to try to keep in mind when a statement about a category regards the category itself versus when it is about the category of subobjects, as those are typically categories of spaces versus a poset category, in which the different terms have vastly different meanings, and it is common for textbooks to not be terribly clear on this point.

As a poset category, we can say a few things about the limits of $\mathrm{Sub}(X)$, if they exist.

Initial, terminal object : bottom and top element product, coproduct : join, meet

If our subobject poset is a join semilattice, this means that it is equipped with a product and a terminal object (X itself), meaning that we have in fact a monoidal category.

In this case, we can define the functor $(-) \times S$, which is simply a function on the poset mapping every object to their meet with S, which is a map from the subobject poset $\mathrm{Sub}(X)$ to the subobject poset $\mathrm{Sub}(S)$ by intersection.

If the class of functors $(-) \times S$ admits a right-adjoint for every S, we will also have an internal hom. The adjunction gives us

$$\mathrm{Hom}_{\mathrm{Sub}(X)}(S, [X, Y]) \cong \mathrm{Hom}_{\mathrm{Sub}(X)}(S \times X, Y) \tag{4.29}$$

Meaning that

$$S \leq [X, Y] \leftrightarrow S \wedge X \leq Y \tag{4.30}$$

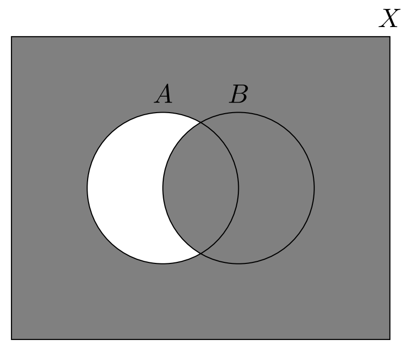
The internal hom $[X, Y]$ is therefore some subobject for which every subobjects is such that their meet with X is a subobject of Y. As we will see, this is what is defined as an implication in a Heyting algebra, denoted by

$$Y^X = X \to Y \tag{4.31}$$

Example 166 *In* **Set***, the implication between subsets $A \to B$ is the union of B and the complement $X \setminus A$,*

$$A \to B = A^c \cup B \tag{4.32}$$

with the Venn diagram



Properties :

Proposition 167 *X belongs to* $\mathrm{Sub}(X)$ *and is the top element.*

Proposition 168 *If the category has an initial object* 0*, it is in* $\mathrm{Sub}(X)$ *and is the bottom element.*

Theorem 169 *If* $\mathbf{C}$ *admits all finite limits,* $\mathrm{Sub}(X)$ *is a meet-semilattice.*

Proof 22 *As we've seen before, the intersection of two subobjects is defined by the pullback of their inclusion map. If a category has all finite limits, this is in particular true of the pullback, so that any two subobjects of X also admit an intersection, itself a subobject, such that $A \cap B$ is indeed a lower bound of both A and B. This lower bound is indeed the greatest by the universal property of the pullback : if we have any other lower bound C of A and B (monomorphisms to A and B, then there is a unique morphism $\iota_{C,A\cap B} : C \to A \cap B$ which is such that*

$$\iota_{A\cap B,A} \circ \iota_{C,A\cap B} = \iota_{C,A} \tag{4.33}$$

which must be a monomorhpism itself since its composition here is also a monomorphism, so that we indeed have $C \subseteq A \cap B$ for any other lower bound.

By the same reasoning, admitting all colimits will also make it a join-semilattice, with the join the union [?]

Definition 170 *If the poset of subobjects* $\mathrm{Sub}(X)$ *admits*

Theorem 171 *The implication $A \to B$ in a poset of subobject is given by the exponential object*

Proof 23 *As an implication, we have that the subobject $A \to B$ is a subobject of X for which all subobjects C of $A \to B$ are such that*

$$(C \wedge A) \leq B \tag{4.34}$$

Definition 172 *If the poset of subobjects has a bottom element* 0 *(the inclusion* $0 \hookrightarrow X$ *in* $\mathbf{C}$*) and an implication, the negation* $\neg_X A$ *of a subobject* A *is the operation*

$$\neg_X A = A \to 0 \tag{4.35}$$

This negation (what is called the pseudo-complement) means that for any subobject of the pseudocomplement,

$$S \subseteq \neg_X A \tag{4.36}$$

we have that

$$S \wedge A \subseteq 0 \tag{4.37}$$

As the bottom element, this simply means that any subobject of $\neg_X A$ is disjoint from A, hence its name of pseudocomplement. The pseudo being here due to the fact that we are however not guaranteed that this pseudocomplement contains *all* subobjects not in A, which is only true for the boolean case

Definition 173 *If the poset of subobjects is a Heyting algebra and has the boolean property,*

$$a \to b = \neg a \vee b \tag{4.38}$$

then it is a boolean category idk

In a boolean category we have indeed that $\neg_X A = A \to 0 = \not{A} \vee 0$

4.3 Coverage and sieves

To define a space in categorical terms, we need to have some formalization of an equivalent notion to mereology, open sets, frames or such that we saw earlier. The notion of *coverage* that we will see will be more general than that (in particular not necessarily be about subregions) but contain those as a special case.

[define cover/covering family first?]

Definition 174 *Given an object* X *in a category* $\mathbf{C}$*, a* coverage J *is a collection of morphisms to that object indexed by some indexing set* I*,*

$$J = \{U_i \to X\}_{i \in I} \tag{4.39}$$

such that morphisms between two objects of $\mathbf{C}$ *induce a coverage. For* $g : Y \to X$*, there exists a covering family* $\{h_j : V_j \to Y\}_{j \in J}$ *such that* gh_j *factors through* f_i *for some* i *:*

$$\begin{array}{ccc} V_j & \xrightarrow{k} & U_i \\ \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow{g} & X \end{array}$$

If we take the case of topology that we've seen as an example, we define the standard coverage of a space X to be the collection of all families of open subsets that cover it, ie

$$J(X) = \{\{U_i \to X\} \mid U_i \subseteq X, \bigcup_i U_i = X\} \tag{4.40}$$

Its stability under pullback corresponds to the fact that for any continuous function $f : Y \to X$, as the pre-image of any open set is itself an open set, we can define a family

$$\{f^{-1}(U_i) \to Y\} \tag{4.41}$$

and as any point in X is covered by some U_i, any point in Y will similarly be covered by $f^{-1}(U_i)$, obeying the properties of a coverage.

"Another perspective on a coverage is that the covering families are "postulated well-behaved quotients." That is, saying that $\{f_i : U_i \to U\}_{i \in I}$ is a covering family means that we want to think of U as a well-behaved quotient (i.e. colimit) of the U_i. Here "well-behaved" means primarily "stable under pullback." In general, U may or may not actually be a colimit of the U_i; if it always is we call the site subcanonical. " To define spaces in the mathematical sense of the word, we need to have some sort of equivalent definition of a *topology*.

If $\mathbf{C}$ has pullback : the family of pullbacks $\{g^*(f_i) : g^* U_i \to V\}$ is a covering family of V.

Grothendieck topology :

An important class of coverage is the *Grothendieck topology*

Cech nerve

Sieve

Definition 175 *For a covering family $\{f_i : U_i \to U\}$ in a coverage J, its* sieve *is the coequalizer*

$$\coprod_{j,k} j(U_j) \times_{j(U)} j(U_k) \rightrightarrows \coprod_i j(U_i) \to S(\{U_i\}) \tag{4.42}$$

with j the Yoneda embedding $j : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C})$

Other definition : A sieve $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ on $X \in \mathbf{C}$ is a subfunctor of $\mathrm{Hom}_{\mathbf{C}}(-, X)$

Objects $S(Y)$ are a collection of morphisms $Y \to X$, and for any morphism $f : Y \to Z$, $S(f)$

Pullback by a sieve :

Ordering : $S \subseteq S'$ if $\forall X$, $S(X) \subseteq S'(X)$

Category of sieves is a partial order, with intersection and union, it is a complete lattice

Grothendieck topology : covering sieves

4.3.1 Cech nerves

Given a covering sieve $\{U_i \to X\}$ with respect to a coverage,

$$C(U) = \left(\cdots U \times_X U \times_X U \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} U \times_X U \rightrightarrows U \right) \tag{4.43}$$

4.4 Subobject classifier

In a category $\mathbf{C}$ with finite limits, a subobject classfier is an object Ω (the object of truth values) and a monomorphism

$$\top : * \to \Omega \tag{4.44}$$

from the terminal object $*$, such that for every monomorphism [inclusion map] $\iota : U \hookrightarrow X$, there is a unique morphism $\chi_U : X \to \Omega$ such that U is the pullback of $* \to \Omega \leftarrow X$

$$\begin{array}{ccc} U & \xrightarrow{!_U} & * \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow{\chi_U} & \Omega \end{array}$$

so that $U \cong X \times_\Omega 1$, or, if we look at it via the equalizer of a product,

$$U \cong X \times_\Omega 1 \to X \times 1 \underset{\chi_U \circ \mathrm{pr}_1}{\overset{\top \circ \mathrm{pr}_2}{\rightrightarrows}} \Omega$$

From our rough intuition of the pullback, we can see imagine that this means something like the object for which χ_U is equal to true, which is the usual sense of what a characteristic function is.

This diagram is furthermore universal, in the sense that for any other subobject V of X, with $\iota_V : V \to X$, the following diagram only commutes if V is itself a subobject of U :

$$\begin{tikzcd}
V \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "!_V"] \arrow[ddr, bend right, "\iota_V"'] & & \\
& U \arrow[r, "!_U"] \arrow[d, "\iota_U"] & 1 \arrow[d, "\top"] \\
& X \arrow[r, "\chi_V"'] & \Omega
\end{tikzcd}$$

ie that V has the same type of valuation in Ω as U through the characteristic function χ_U.

Theorem 176 *In a locally finite category* **C** *with a terminal object,* **C** *has a subobject classifier if and only if there is some object* Ω *for which the hom functor is naturally isomorphic to the subobject functor.*

A particularly important case of the pullback is the subobject defined by the monomorphism $0 \hookrightarrow 1$ (if the category admits an initial object), in which case we get the following diagram

$$\begin{tikzcd}
0 \arrow[r, "!_0"] \arrow[d, "\iota"] & 1 \arrow[d, "\top"] \\
1 \arrow[r, "\chi_0"] & \Omega
\end{tikzcd}$$

The morphism $\chi_0 : 1 \to \Omega$ is another truth value of Ω, which is the *false* truth value, denoted as

$$\perp : 1 \to \Omega \tag{4.45}$$

As this defines a subobject for Ω itself, we also have the pullback

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{Id}_1"] \arrow[d, "\perp"] & 1 \arrow[d, "\top"] \\
\Omega \arrow[r, "\chi_\perp"] & \Omega
\end{tikzcd}$$

$\chi_\perp$ is then some endomorphism on Ω, which we call the *negation*, $\neg$. From this diagram, we have the first equality on Ω,

$$\neg \circ \perp = \top \tag{4.46}$$

the negation of falsity is truth. Furthermore, if we

Theorem 177 *The composition of the negation with any characteristic morphism $\chi_U : X \to \Omega$ causes the failure of U or any of its subobject to be subobjects of X.*

Proof 24 *If we have the span*

$$X \xrightarrow{\neg\chi_U} \Omega \longleftarrow 1 \tag{4.47}$$

$$\begin{tikzcd}
U \arrow[r, "!_U"] \arrow[d, "\iota_U"] & 1 \arrow[d, "\top"] \arrow[dr, "\top"] & \\
X \arrow[r, "\chi_U"] & \Omega \arrow[r, "\neg"] & \Omega
\end{tikzcd}$$

For U to be a subobject of X, we need the lower composition to form a pullback with

However, the triangle here does not commute

$$\begin{tikzcd}
1 \arrow[d, "\top"] \arrow[dr, "\top"] & \\
\Omega \arrow[r, "\neg"] & \Omega
\end{tikzcd}$$

[proof?]

Furthermore, if we call the pullback of this span $\neg U$, and we try to look at any subobject of U itself, none of those subobjects are subobjects of $\neg U$.

$$\begin{tikzcd}
V \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "!_V"] \arrow[ddr, bend right, "\iota_V"'] & & & \\
& U \arrow[r, "!_U"] \arrow[d, "\iota_U"] & 1 \arrow[dr, "\top"] & \\
& X \arrow[r, "\chi_V"'] & \Omega \arrow[r, "\neg"] & \Omega
\end{tikzcd}$$

So that the negation of a characteristic function contains none of the original subobjects. We will call this the *pseudo-complement* of a subobject.

Definition 178 *For a characteristic function $\chi_U : X \to \Omega$, its pseudo-complement is the negation*

$$\neg\chi_U = \neg \circ \chi_U \tag{4.48}$$

and its pullback defines the pseudo-complement

$$\neg U = X \times_{\Omega, \neg\chi_U} 1 \tag{4.49}$$

The exact nature of this pseudo-complement will depend on the exact category that we are working on. As we will see, it is not necessarily true that the pseudo-complement is to be understood as "everything in X not in U" (that is, $U \cup \neg U = X$), but it does have a few of the characteristics we would expect from the complement.

Theorem 179 *A subobject and its pseudo-complement are disjoint :*

$$U \cap \neg U = 0 \tag{4.50}$$

but one thing that is true is that it does split the points of X into either. To show this, we need to consider a few things.

Proof that set is boolean

Proof that the hom set commutes with negation?

Theorem 180 *In a category with a strict initial object and a two-valued subobject classifier, given the hom-functor h^X, the negation morphism on $\mathbf{C}$ is mapped to a negation morphism on* **Set***(???)*

Proof 25 *First we have to show that the falsity morphism is preserved. By the strictness of the initial object, we have that* $\mathrm{Hom}_{\mathbf{C}}(1, 0) = 0$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(X, 0) \arrow[r, "!_0"] \arrow[d, "\iota"] & 1 \arrow[d, "\top"] \\
1 \arrow[r, "\chi_0"] & \Omega
\end{tikzcd}$$

By preservation of limits, we simply map the pullback

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{Id}_1"] \arrow[d, "\perp"] & 1 \arrow[d, "\top"] \\
\Omega \arrow[r, "\neg"] & \Omega
\end{tikzcd}$$

to

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{Id}_1"] \arrow[d, "\perp"] & 1 \arrow[d, "\top"] \\ \Omega \arrow[r, "\chi_\perp"] & \Omega \end{tikzcd}$$

(using $\mathrm{Hom}_{\mathbf{C}}(X, 1) \cong 1$*)*

Theorem 181 *Any point of X, $x : 1 \to X$ in $\mathrm{Hom}_{\mathbf{C}}(1, X)$, is either in U or $\neg U$.*

Proof 26 *As the pullback preserves limits, if we take the hom functor h^1, this leads to the following pullback in* $^{\mathbf{Set}}$ *:*

$$\begin{tikzcd} \mathrm{Hom}(1, U) \arrow[r, "\mathrm{Id}_1"] \arrow[d, "\iota"] & 1 \arrow[d, "\top"] \\ \mathrm{Hom}(1, X) \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

As **Set** *is a boolean category*

Theorem 182 *Conjunction*

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{Id}_1"] \arrow[d, "(\top,\top)"'] & 1 \arrow[d, "\top"] \\ \Omega \times \Omega \arrow[r, "\cap"] & \Omega \end{tikzcd}$$

test

Theorem 183 *For any object X, the initial object 0 is always a subobject.*

Proof 27

This is best exemplified by the simple case for sets :

Example 184 *In* **Set***, Ω is the set containing the initial object, $\Omega = \{\varnothing, \{\bullet\}\}$, also noted as $2 = \{0, 1\}$.*

For a subset $S \subseteq X$ with an inclusion map $\iota : S \hookrightarrow X$, the characteristic function $\chi_S : X \to 2$ is the function defined by $\chi_S(x) = 1$ for $x \in S$ and $\chi_S(x) = 0$ otherwise. The truth function simply maps $*$ to 1 in Ω.

$$\forall x \in U,\ \chi_U(\iota_U(x)) = 1 \tag{4.51}$$

And conversely, if we look at another subobject $V \subseteq X$, the pullback works out

$$\forall x \in V,\ \chi_U(\iota_U(x)) = 1 \tag{4.52}$$

only if $V \subseteq U$, ie there exists a monomorphism from $V \to U$

Subobject classifiers can be more complex than the simple boolean domain true/false. A good illustration of this is the subobject classifier in the category of graphs[32]. A graph is composed by two sets, those of nodes N and of arrows A, with two functions s, t for the source and target of each arrow.

$$\begin{tikzcd} 0_N \arrow[loop left, "0_A"] \arrow[r, "t"] & N \arrow[loop above, "A"] \end{tikzcd}$$

For a subgraph $\iota : S \hookrightarrow G$, the classifying map χ_S has the following behaviour :

- If a node in G is not in S, it is mapped to 0_N.
- If a node in G is in S, it is mapped to N.

Subobject classifier for a topological space

Negation complement :

Definition 185 *Given the classifying arrow $\perp : 1 \to \Omega$ of the initial object $!_0 : 0 \hookrightarrow 1$, with the associated negation morphism $\neg : \Omega \to \Omega$ the classifying arrow of $\perp$, the complement $\neg_X A$ of a subobject $A \hookrightarrow X$ is the pullback*

4.4.1 In a sheaf topos

Given the sheaf topos $\mathbf{H} = \mathrm{Sh}(\mathbf{C}, J)$, there is a natural subobject classifier

4.5 Elements

One of the important difference between set theory and category theory is that while sets are typically composed of elements, as defined by the $\in$ relation, categories (for which the objects are often somewhat similar to sets themselves) do not seem to have a naturally equivalent notion.

If we wish to define elements of a set in terms of the morphisms of sets (functions), this is best done via the use of functions from the singleton set $\{\bullet\}$, as those functions are in bijection with the elements of a set

$$\mathrm{Fun}(\{\bullet\}, X) \cong X \tag{4.53}$$

As functions from the singleton are all of the form $\{(\bullet, x)\}$ for every $x \in X$ (From the properties of the Cartesian product)

Generalized elements : Given the yoneda embedding $Y : C \hookrightarrow [C^{\mathrm{op}}, \mathbf{Set}]$, the representable functor

$$\mathrm{GenEl}(X) : C^{\mathrm{op}} \to \mathbf{Set} \tag{4.54}$$

Sends each object U of C to the set of generalized elements of X at stage U.

For an object $X \in \mathrm{Obj}(\mathbf{C})$, its *global elements* are morphisms $x : 1 \to X$. It's a generalized element at stage of definition 1.

Definition 186 *An object $S \in \mathbf{C}$ is a* separator *if for every pair of morphisms $f : X \to Y$, and every morphism $e : S \to X$, then $f \circ e = g \circ e$ implies $f = g$.*

In other words, the global elements generated by the separator are enough to entirely define the morphisms. For instance, in the case of **Set**, $\{\bullet\}$ is a separator, essentially saying that the elements of a set entirely define its functions : a function $f : X \to Y$ is defined by its value $f(x)$ for every $x \in X$.

If we have a topos E such that its terminal object 1 is a separator, and $1 \neq 0$, we say that the topos is *well-pointed*, meaning that

Other definitions : global section functor is faithful

Prop : well-pointed topos are boolean, its subobject classifier is two-valued,

Definition 187 *A* concrete category $\mathbf{C}$ *is such that there exists a faithful functor F*

$$F : \mathbf{C} \to \mathbf{Set} \tag{4.55}$$

What the faithful functor implies

In the general case of a concrete category, we can define its set of points internally to the category by considering its left adjoint to the forgetful functor, the free functor

Theorem 188 *For a concrete category with a forgetful functor $U : \mathbf{C} \to \mathbf{Set}$, and a left adjoint free functor $F : \mathbf{Set} \to \mathbf{C}$, the set of all points of a given object X taken as a set $U(X)$ is equivalent to the hom-set of its free object on a single element $F_1 = F(\{\bullet\})$.*

Proof 28

Example 189 *For the category of topological spaces, the free functor maps sets to their equivalent discrete topologies, and the free object of a single element F_1 is the terminal object in the category. Therefore*

$$U(X) \cong \mathrm{Hom}_{\mathbf{Top}}(F_1, X) \tag{4.56}$$

Example 190 *For the category of vector spaces, the free functor maps sets to the vector space generated by a basis of those elements (a set of n elements generates an n-dimensional vector space). The free object of a single element is the vector space isomorphic to the underlying field, $F_1 = k$, so that*

$$U(V) \cong \mathrm{Hom}_{\mathbf{Vec}}(k, V) \tag{4.57}$$

which is just the statement that a vector space is isomorphic to the vector space of linear maps from the field to itself.

Example 191 *For the category of groups, the free functor maps sets to their free group (the group of all strings generated by those elements by concatenation). The free object of one element is the group of integers $\mathbb{Z}$*

$$U(G) \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \tag{4.58}$$

This is commonly the case in monoidal categories, such as **Hilb**

[Concrete categories and well-pointed ones do not imply each other in any direction, depends on if elements are global elements?]

Given a set-valued functor $F : \mathbf{C} \to \mathbf{Set}$, its *category of elements*

Category of elements

[...]

Global elements also interact with the subobject classifier.

What is the relation of global elements wrt the subobject classifier?

4.5.1 Points

As we have seen, we can represent a general notion of space as that of a frame, but a more contentious issue is how to define *points* in a space. This is an issue that goes back all the way to the foundation of geometry [33], and to this day is not an uncontentious one, as the assumption of point-like structures in space is still a thorny issue.

Categories for which we have fairly simple notions of discrete elements such as finite sets do not have too much trouble defining what a point could be, corresponding in some sense to the notion of discrete objects that were used uncontroversially in antiquity, but given a frame like that of continuous physical space, this becomes a more complex notion to define, as there is no internal notion of what a point is in the context of the frame of opens $\mathcal{O}(X)$.

In the abstract, we can define a point just as we would define an element for another category, simply as morphisms from some terminal object to the regions of space, but we are neither guaranteed the existence of such an object nor that the space is in some sense *composed* by those points rather than just those merely inhabiting it.

The intuitive notion, going back at least as far as [33], would be to consider a point as the limit of a shrinking family of open sets, but we could have for instance a family of regions $\{U_i\}$ which converge to another (non-point like) region, such as a family of disks of radius $r_n = 1 + 2^{-n}$. Furthermore, two different such sequences can converge to the same point so that we also need to be able to define the equivalence of such sequences.

To represent the notion of several sequences of regions converging to the same result, we need to use the notion of *filter*

A subset F of a poset L is called a filter if it is upward-closed and downward-directed; that is:

If $A \leq B$ in L and $A \in F$, then $B \in F$; for some A in L, $A \in F$; if $A \in F$ and $B \in F$, then for some $C \in F$, $C \leq A$ and $C \leq B$.

Points given a locale?

Given a locale X, a concrete point of X a completely prime filter on $O(X)$. [Show equivalence with a continuous map $f : 1 \to X$: treat $f^* : O(X) \to O(1)$ as a characteristic function]

Completely prime filter :

A filter F is prime if $\perp \notin F$ and if $x \vee y \in F$, then $x \in F$ and $y \in F$. For every finite index set I, $x_k \in F$ for some k whenever $\bigvee_{i \in I} x_i \in F$.

[Some descent of open sets for a topological space?]

Example 192 *Take the frame defined by the poset* $0 \leq A \leq 1$*, the simplest frame which is not boolean. Its joins and meets can all be deduced from the properties of the join and meet with respect to the top and bottom element and idempotency. Let's attempt to find a frame homomorphism* ϕ *to the terminal(?) frame* $\{0 \leq 1\}$*. As it must preserve joins and meets and bounds, we should have* $\phi(0) = 0$*,* $\phi(1) = 1$*, and therefore* $\phi(A)$ *must be mapped to either* 1 *or* 0*. We should therefore have that*

$$\phi(A \to 0) = \phi(0) \tag{4.59}$$
$$\phi(A) \to 0 = 0 \tag{4.60}$$

and

$$\phi(A \to 1) = \phi(1) \tag{4.61}$$
$$\phi(A) \to 1 = 1 \tag{4.62}$$

The first implies that $\phi(A) = 0$, while the second implies that $\phi(A) = 1$. There is therefore no such map from this frame to the terminal frame, and therefore no points.

Example 193 *The classic example of a pointless locale is the locale of surjections from the discrete space $\mathbb{N}$ to the continuous space $\mathbb{R}$ with its standard topology[34]. This locale is defined by the*

As $|\mathbb{N}| < |\mathbb{R}|$, there is no such surjection.

Stone theorem

4.6 Internal hom

One component of the definition of a topos regards the behaviour of its *internal homs*, which are a way with which to include the hom-set of the category in its objects. In other words, every space of morphisms between two objects of the topos is itself an object of the topos.

Definition 194 *In a symmetric monoidal category $(\mathbf{C}, \otimes, I)$, an* internal hom *is a bifunctor*

$$[-,-] : \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{C} \tag{4.63}$$

such that for any object $X \in \mathbf{C}$, the functor $[X, -]$ is right adjoint to the functor $(-) \otimes X$:

$$((-) \otimes X \dashv [X, -]) : \mathbf{C} \to \mathbf{C} \tag{4.64}$$

The simplest way to see the content of internal homs from their definition is via the adjunction of hom sets :

$$\mathrm{Hom}_{\mathbf{C}}(Z, [X, Y]) \cong \mathrm{Hom}_{\mathbf{C}}(Z \otimes X, Y) \tag{4.65}$$

If we consider the case where $Z = I$, we have the equivalence

$$\mathrm{Hom}_{\mathbf{C}}(I, [X, Y]) \cong \mathrm{Hom}_{\mathbf{C}}(I \otimes X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{4.66}$$

In other words, if we look at $\mathrm{Hom}_{\mathbf{C}}(I, [X, Y])$, the set of generalized elements of the monoidal object, then it is isomorphic to the actual hom-set of the category. This internal hom is isomorphic to the hom-set as a set, but also carry the structure of $\mathbf{C}$ objects. If we pick the case of the Cartesian monoidal category $(\mathbf{Set}, \times, \{\bullet\})$ in particular, we recover an exact definition of the hom-set.

As an adjunction of two functors, we have two natural transformations

$$\begin{aligned} \eta : \mathrm{Id}_{\mathbf{C}} &\to [X, ((-) \otimes X)] && (4.67) \\ \epsilon : ([X, -] \otimes X) &\to \mathrm{Id}_{\mathbf{C}} && (4.68) \end{aligned}$$

The natural transformation ϵ is the notion of an *evaluation map*. If we look at its components at Y, we can define the map on objects

$$\mathrm{eval}_{X,Y} = \epsilon_X = ([X, Y] \otimes X) \to Y \tag{4.69}$$

The meaning here is that given a function from X to Y and a value in X, we can obtain a value in Y, hence its name of evaluation map.

If we have a morphism $Y \otimes X \to Z$, there is equivalently some morphism $Z \to [X, Y]$

Example : take $Z = [X, Y]$, take the morphism $\mathrm{Id}_{[X,Y]} : [X, Y] \to [X, Y]$. Its adjunct is

$$Y \otimes X \to [X, Y] \tag{4.70}$$

Evaluation map :

The counit of the adjunction for $[X, -]$ is called (evaluated at a component Y) the *evaluation map*

$$\mathrm{eval}_{X,Y} : [X, Y] \otimes X \to Y \tag{4.71}$$

Internal hom bifunctor

Example 195 *In the category of vector spaces* **Vec***, where the monoidal structure is the tensor product, the internal hom $[V, W]$ is the space of linear maps from V to W, which is indeed itself a vector space. The space of dual vectors is for instance given by $[V, I]$, leading to the equivalence A map between two vector spaces can be considered as the tensor product of the target space and the dual space of the domain.* $[X \otimes Y, Z] \cong [X, [Y, Z]]$

Example 196 *There is a category of smooth spaces which is the generalization of the category of smooth manifolds, as we will see later [x]. Partly this a way to include the internal hom for smooth manifolds, so that maps between manifolds are themselves a type of generalized manifold, which can be thought of as diffeological spaces, infinite dimensional manifolds like Hilbert manifoldd, inverse limit manifolds, pro-manifolds, etc.*

4.6.1 Cartesian closed categories

As the product is an example of a monoidal structure, it is quite common to define the internal hom for it.

Definition 197 *A category is* Cartesian closed *if it is a closed monoidal category wrt the product*

The internal hom of the Cartesian product is generally called more specifically the *internal object* and denoted by

$$[X, Y] = Y^X \tag{4.72}$$

By analogy with the notation for the set of functions between two sets, which is the prototypical example of a Cartesian closed category.

Adjunction $- \times A \dashv (-)^A$

Theorem 198 *As an adjunction, the monoidal product functor preserves colimits and the internal hom functor preserves limits.*

Example : the set of all function from X to some object $[X, -]$ preserves the terminal object :

$$[X, 1] \cong 1 \tag{4.73}$$

which is just the basic property of the hom-set for terminal object. Similarly, for two morphisms $f : Y \to X_1$ and $g : Y \to X_2$, we have

$$[Y, X_1 \times X_2] \cong [Y, X_1] \times [Y, X_2] \tag{4.74}$$

where the functions to a product are the product of those functions to each, ie morphism of the form (f, g)

Pullback : $[X, Y_1 \times_Z Y_2] \cong [X, Y_1] \times_{[X,Z]} [X, Y_2]$

Equalizer : $[X, \mathrm{eq}(f, g)] = \mathrm{eq}([X, g], [X, g])$.

Similarly for $(-) \otimes X$, $0 \otimes X = 0$. This is true for the Cartesian product with the empty set in **Set**

Theorem 199 *Isomorphism*

$$[X \otimes Y, Z] \cong [X, [Y, Z]] \tag{4.75}$$

4.7 Presheaves

Definition 200 *A* presheaf *on a small category* **C** *is a functor* F

$$F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set} \tag{4.76}$$

This definition also generalizes to any category. If we replace **Set** with any category **S**, we speak of an S-valued presheaf, defined as

$$F : \mathbf{C}^{\mathrm{op}} \to \mathbf{S} \tag{4.77}$$

In a similar manner, we have the dual of presheaves, called *copresheaves*, and defined as sheaves on the opposite category :

$$F : \mathbf{C} \to \mathbf{Set} \tag{4.78}$$

And similarly, for an S-valued copresheaf,

$$F : \mathbf{C} \to \mathbf{S} \tag{4.79}$$

Fundamentally, any functor can be described as a (co)presheaf, as any functor from a category **C** (or its opposite) fits the definition, but a presheaf is typically gonna be studied with more specific goals in mind, ie to turn them into sheaves or topos.

An example for the motivation of (co)presheaves is to consider a topological space (X, τ). The category of interest here is the frame of opens $\mathrm{Op}(X)$. A sheaf on the frame of open is some functor associating a set to every open set :

$$\forall U \in \mathrm{Op}(X),\ F(U) = A \in \mathbf{Set} \tag{4.80}$$

In a way that preserves the functions in some sense. In particular, if we have an inclusion $\iota : U \hookrightarrow U'$, its opposite is $\iota^{\mathrm{op}} : U' \to U$, and the functor maps it to

$$F(\iota^{\mathrm{op}}) : F(U') \to F(U) \tag{4.81}$$

We will see in the section on sheaves the meaning of this construction.

Example 201 *An S-valued presheaf on* **C** *is a* constant presheaf *if it is a constant functor, ie for some element $X \in \mathbf{S}$, the presheaf is just*

$$\Delta_X : \mathbf{C}^{\mathrm{op}} \to \mathbf{S} \tag{4.82}$$

interpretation of presheaves $X(U)$ as a function U to X via Yoneda

Definition 202 *A* representable presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$ *is a presheaf that has a natural isomorphism to the hom-functor h_X fr some object $X \in \mathbf{C}$:*

$$\eta : F \xrightarrow{\cong} h_X = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{4.83}$$

In other words a representable presheaf sends every object $Y \in \mathbf{C}$ to the set $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ of all morphisms from Y to X, and every morphism $f : Y \to Z$ to the function sending any morphism $g : Y \to X$ to its composite $f \circ g$.

As we will see later on, representable presheaves are often use to carry the notion that the objects of a category $\mathbf{C}$ correspond in some sense to some of the objects of a presheaf category. This is used for instance in the notion of constructing spaces from simpler spaces as presheaves, in which case the basic spaces are also included. For instance, we can consider manifolds as constructed by atlases to be presheaves on the category of open sets of $\mathbb{R}^n$, in which case the representable presheaves in this category will correspond to the manifolds which are simply raw open sets of $\mathbb{R}^n$.

Theorem 203 *A representable presheaf is determined by a unique object $X \in \mathbf{C}$.*

Proof 29 *By the Yoneda lemma,*

Theorem 204 *Representable functors preserve all limits.*

Proof 30

4.7.1 Simplexes

A basic example of presheaves is given by simplicial sets, which are presheaves over the simplex category $\boldsymbol{\Delta}$:

$$X : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Set} \tag{4.84}$$

By the Yoneda embedding [representable presheaves etc], any object in the simplex category is a simplicial set

Furthermore, we can consider simplexes which are constructed from the combination of different simplexes

Example 205 *Take the basic simplex of the triangle. Let's call this sheaf S_3.*

Take the two simplexes $\vec{\mathbf{2}}$, $\vec{\mathbf{1}}$. We want to identify the edges of each to form a triangle.

$$\begin{aligned} S_3(\vec{\mathbf{0}}) &= \varnothing && (4.85)\\ S_3(\vec{\mathbf{1}}) &= \{\bullet\} && (4.86)\\ S_3(\vec{\mathbf{2}}) &= \{\bullet\} && (4.87) \end{aligned}$$

and this maps the morphism (functor) which maps each extremity of 1 *and* 2 *together :*

$$S_3(f : \vec{\mathbf{1}} \to \vec{\mathbf{2}}) = f'(S_3(\vec{\mathbf{1}}) \to S_3(\vec{\mathbf{2}})) \tag{4.88}$$

As an equalizer? $1 + 2/f$

4.8 Sheaves

The more important construction based on (co)presheaves is that of (co)sheaves, which are (co)presheaves with some additional conditions, meant to signify the spatial nature of the construction : the category corresponds in some sense to the piecing together of regions.

[...]

Consider the Yoneda embedding of $\mathbf{C}$:

$$j : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C}) \tag{4.89}$$

Definition 206 *Given a presheaf* $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$*, and given a coverage* J *of* $\mathbf{C}$*,* F *is a sheaf with respect to* J *if*

- *for every covering family* $\{p_i : U_i \to U\}_{i \in I}$ *in* J
- *for every compatible family of elements* $(s_i \in F(U_i))_{i \in I}$*,*

there is a unique element $s \in F(U)$ *such that* $F(p_i)(s) = s_i$ *for all* $i \in I$*.*

If we consider the case where a covering family is composed of monomorphisms with subobjects (assuming the equivalence something something), then $p_i : U_i \hookrightarrow U$ can be considered [something], and the morphism generated by the sheaf is understood to be a *restriction* : $F(p_i)(s) = s_i$, we are restricting the section s on U to the subobject on U_i.

[The section of a sheaf is defined by its local elements]

The easiest example of this is to pick once again a presheaf on the frame of open $F \in \mathrm{Psh}(\mathrm{Op}(X))$, and as the coverage, pick the subcanonical coverage. For any open set $U \subseteq X$, a subcanonical coverage is a family of open sets $\{U_i\}$ such that

$$\bigcup_i U_i = U \tag{4.90}$$

[...]

A nice class of such sheaves are the sheaves given by function spaces over the appropriate sets, *sheaves of functions*.

Example 207 *For an S-valued sheaf on (C, J), the* constant sheaf

Examples with sheaves on frames of opens

For $\mathrm{Op}(X)$ with the canonical coverage, a presheaf F is a sheaf if for every complete subcategory $\mathcal{U} \hookrightarrow \mathrm{Op}(X)$,

$$F(\lim_{\rightarrow} \mathcal{U}) \cong \lim_{\leftarrow} F(\mathcal{U}) \tag{4.91}$$

Proof 31 *Complete full subcategory is a collection $\{\iota_i : U_i \hookrightarrow X\}$ closed under intersection. The colimit*

$$\lim_{\rightarrow}(\mathcal{U} \hookrightarrow \mathrm{Op}(X)) \cong \bigcup_i U_i \tag{4.92}$$

is the union of these open subsets. By construction,

Example 208 *The presheaf mapping all objects to the empty set $\Delta_\varnothing$ is a sheaf, called the* empty sheaf *(or* initial sheaf *as it is the initial object in the category of sheaves). As every covering family is mapped to the compatible family of elements of the empty set,*

Example 209 *The presheaf mapping all objects to the singleton $\{\bullet\}$ is a sheaf, called the* terminal sheaf.

Definition 210 *A* sheaf morphism *$\phi : F \to G$ for two sheaves on a site $\mathbf{C}$ is a natural transformation on the functors F, G, in particular preserving restriction maps via the commutative diagram*

$$\begin{tikzcd}
F(V) \arrow[r, "\rho_{V,U}"] \arrow[d, "\phi_V"] & F(U) \arrow[d, "\phi_U"] \\
G(V) \arrow[r, "\rho'_{V,U}"] & G(U)
\end{tikzcd}$$

4.8.1 Concrete sheaves

4.9 Descent

An important notion that we will need for various notions relating to spaces in category theory is that of descent. This is the process by which we will consider our various spaces as a collection of subspaces and how the various structures on them are meant to work together. For instance we can describe some topological space X by the quotient of the coproduct of its open sets, ie

$$X \cong \bigsqcup_i U_i / \cong \tag{4.93}$$

where we define the equivalence relation to be that if two open sets have a non-empty intersection, then the points in the intersection are identified. If we have for instance two open sets U_1, U_2 and their intersection $U_{12} = U_1 \cap U_2$,

$$\forall x, y \in U_1 \sqcup U_2,\ x \cong y \leftrightarrow x = y \vee x \in \iota(U_1) \wedge y \in \iota(U_2) \wedge \exists U_{12} \tag{4.94}$$

This notion can be used to derive a variety of global structures from local ones

Before this we need to talk about the notion of fibered categories, and first of Cartesian morphisms.

Definition 211 *Given a functor $P : \mathbf{C} \to \mathbf{D}$, we say that a morphism $f : X \to Y$ in $\mathbf{C}$is P-*Cartesian *if for every $Z \in \mathbf{C}$ and every morphism $g : Z \to Y \in \mathbf{C}$, and for every morphism $\alpha : P(Z) \to P(X)$ in $\mathbf{D}$ such that $P(g) = P(f) \circ \alpha$, there is a unique morphism $\nu : Z \to X$ in $\mathbf{C}$ such that*

$$g = f \circ \nu \tag{4.95}$$

and

$$\alpha = P(\nu) \tag{4.96}$$

ie we have the following commuting diagram

$$\begin{tikzcd} Z \arrow[d, "P"] \\ Y \end{tikzcd}$$

Definition 212 *For a functor $P : \mathbf{C} \to \mathbf{D}$, this is a* fibration *if for all $X \in \mathbf{C}$ and $f_0 : Y_0 \to P(X) \in \mathbf{D}$, there exists a Cartesian morphism $f : Y \to X \in \mathbf{C}$ such that $P(f) = f_0$.*

Example 213 *Category of arrows*

4.10 Topos

[35, 36, 37]

One important type of categories that will be the main focus of study here is that of a *topos*. There are many possible definitions and intuitions of what a topos is, many of them listed in [34], but for our purpose, a topos will mostly be about

- A universe of types in which to do mathematics
- A category of spaces
- A categorification of some types of logics

There are a few different nuances to what a topos can be, but the most general case we will look at for now (disregarding things such as higher topoi) is that of an elementary topos.

Definition 214 *An* elementary topos $\mathbf{H}$ *is a category which has all finite limits, is Cartesian closed, and has a subobject classifier.*

An elementary topos' definition fits best in the first sense of the definitions, in that it is a universe in which to do mathematics. These properties are overall modeled over **Set**, and in some sense it is the generalization of a set. As we will see in details in 5.1, **Set** itself is a topos.

Its use as a mathematical universe is given by its closure under limits (and as we will show, colimits) and exponentiation. We can easily talk about any (finite) construction of objects in a topos, as well as any function between two elements of a topos, without leaving the topos itself, and the subobject classifier will give us some notion of subobjects in the category.

Theorem 215 *An elementary topos has all finite colimits.*

Proof 32 *Contravariant power set functor :*

$$\Omega^{(-)} : \mathbf{H}^{op} \to \mathbf{H} \tag{4.97}$$

Properties : locally Cartesian closed, finitely cocomplete, Heyting category,

Giraud-Rezk-Lurie axioms

4.10.1 Lattice of subobjects

From the existence of all limits, we can deduce that the poset of subobjects

Theorem 216 *In a topos, the subobjects* $\mathrm{Hom}_{\mathbf{H}}(X, \Omega)$ *of an object* X *form a Heyting algebra.*

Proof 33

Heyting algebra of subobject : given two subobjects $A, B \in \mathbf{H}$

$$A \wedge \tag{4.98}$$

Subobject classifier : the subobject presheaf of X is isomorphic to the hom-set

$$\mathrm{Sub}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \Omega) \tag{4.99}$$

4.10.2 Grothendieck topos

The most common type of topos, and the one we will typically use, is the Grothendieck topos, based around the category of sheaves on a given site.

As most of the proofs related to this will be easier going first through presheaves as a topos, let's first look at the presheaf topos.

Definition 217 *A* presheaf topos *is the functor category of all presheaves on a category,* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Theorem 218 *A presheaf topos is an elementary topos.*

Proof 34 *By the property of limits on presheaves, and as* **Set** *has all limits, this means that* $\mathrm{Psh}(\mathbf{C})$ *has all limits. The*

Definition 219 *A* Grothendieck topos E *on a site* $\mathbf{C}$ *with coverage* J *is the category of sheaves over the site* $\mathbf{C}$

$$\mathcal{E} \cong \mathrm{Sh}(\mathbf{C}, \mathcal{J}) \tag{4.100}$$

Proposition 220 *A Grothendieck topos is an elementary topos*

Proof 35

Example 221 *A trivial example of a Grothendieck topos is the initial topos, which is the sheaf topos over the empty category with the empty topology (which is the maximal topology on this category),* $\mathrm{Sh}(\mathbf{0})$*. The only element of this topos is the empty functor, with the identity natural transformation on it (as there is no possible components to differentiate them on the empty category, this is the only one). We therefore have*

$$\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1} \tag{4.101}$$

Its only object $*$ *is both the initial and terminal object (therefore a zero object), its product and coproduct are simply* $* + * \cong *$ *and* $* \times * \cong *$

One important nuance in topos theory is that a topos can be considered alternatively as a space in itself, or as a category in which every object is a space. The former is referred to as a *petit topos*, while the latter is a *gros topos*. A typical example of this would be for instance the topos of smooth spaces **Smooth**, which contains (among other things) all manifolds, as a gros topos, while the topos of the site of opens on a topological space $\mathrm{Sh}(\mathrm{Op}(X))$ would be an example of a petit topos.

Theorem 222 *For any topos* $\mathbf{H}$*, the slice category given by one of its object* $\mathbf{H}_{/X}$ *is itself a topos.*

This construction allows us to bridge the gros and petit topos, in that given a space X in a gros topos $\mathbf{H}$, its corresponding petit topos will be $\mathbf{H}_{/X}$.

"Also in 1973 Grothendieck says the objects in any topos should be seen as espaces etales over the terminal object of the topos, in a generalized sense that includes saying any orbit of a group action lies "etale" over a fixed point. "

As a topos, a Grothendieck topos has a subobject classifier Ω, which, as an object of a sheaf category, will be itself a sheaf. From the properties of a sheaf, we can also look at its various properties.

Any sheaf topos has an object with the property of a subobject classifier, given by the sheaf of principal sieves. That is, for any object U of the site,

$$\Omega(U) = \{S | S \text{ is a sieve on } U\} \tag{4.102}$$

$$\Omega(f) : \Omega(U) \to \Omega(V),\ \Omega(g)(S) = S\big|_g = \{h | g \circ h \in S\} \tag{4.103}$$

To show the subobject classifier, we first need to show the existence of a sheaf morphism from the terminal sheaf to this sheaf, $\top : 1 \to \Omega$. This is simply given by the components

$$\begin{aligned} \top_X : 1(X) &\to \Omega(X) && (4.104)\\ * &\mapsto t(X) && (4.105) \end{aligned}$$

Example 223 *In the sheaf topos* **Set** $\cong \mathrm{Sh}(\mathbf{1})$

Sheaf topos is Cartesian closed, internal hom from this monoidal structure

Theorem 224 *In a Grothendieck topos, all coproducts are disjoint, in the sense that for any two subobjects of the same object, $\iota_1, \iota_2 : X_1, X_2 \hookrightarrow Y$, their intersection (pullback) is empty (the initial object) if their union (pushout) is isomorphic to the coproduct. For the diagram $F = X_1 \hookrightarrow Y \hookleftarrow X_2$,*

$$\lim_I F \cong 0 \to \mathrm{colim}_I(F) \cong X_1 + X_2 \tag{4.106}$$

4.10.3 Lawvere-Tierney topology

On a topos $\mathbf{H}$, a *Lawvere-Tierney topology* is given by a morphism j on the subobject classifier, ie

$$j : \Omega \to \Omega \tag{4.107}$$

Analog of Grothendieck topology for a topos? [38]

First to define a sheaf on a topological space (X, τ) (the category $\mathrm{Op}(X)$ with the canonical coverage). Take a collection $C = \{U_i\}_{i \in I}$.

The locality operator j maps C to the open sets covered by C.

$$j(C) = \{U \in \mathrm{Op}(X) | U \subseteq \bigcup_{i \in I} U_i\} \tag{4.108}$$

Properties :

If $U \in C$, $U \in j(C)$

$j(\mathrm{Op}(X)) = \mathrm{Op}(X)$

$j(j(C)) = j(C)$

$j(C_1 \cap C_2) \subseteq j(C_1) \cap j(C_2)$

If C_1, C_2 are sieves, $j(C_1 \cap C_2) = j(C_1) \cap j(C_2)$

If $C = S(U)$, $j(C)$ is also a sieve.

If C is a sieve, it is an element of $\Omega(U)$, the subobject classifier on the topos of presehaves on X.

Generalization : j is a map $j : \Omega \to \Omega$ on the subobject classifier of a topos, the *Lawvere-Tierney topology*, with properties

- $j \circ \top = \top$

- $j \circ j = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

j is a modal operator on the truth values Ω.

Example 225 *In the Grothendieck topos $E = \mathrm{Set}^{\mathrm{Op}(X)^{\mathrm{op}}}$ of presheaves on X a topological space, Classifier object $U \mapsto \Omega(U)$, the set of all sieves S on U, a set of open subsets $V \subseteq U$ such that $W \subseteq V \in S$ implies $W \in S$.*

Each open subset $V \subseteq U$ determines a principal sieve $\hat{V}$ consisting of all opens $W \subseteq V$

The map $\top_U : 1 \to \Omega(U)$ is the map that picks the maximal sieve $\hat{U}$ on U.

$$J(U) = \{S |\ S \text{ is a sieve on } U \text{ and } S \text{ covers } U\ \} \tag{4.109}$$

S covers U means

4.11 Site

A site is roughly speaking the elements from which a (Grothendieck) topos is stitched together.

Definition 226 *A site (C, J) is a category C equipped with a coverage J.*

Sieve

Example :

Example 227 *The terminal category* 1 *is a site. The covering family is simply the only function, $\{\mathrm{Id}_* : * \to *\}$. As there are no other objects in the category, we only need to check the induced coverage on itself. Given the morphism $\mathrm{Id}_* : * \to *$, the diagram commutes trivially by using the identity function everywhere.*

$$\begin{tikzcd} * \arrow[r, "\mathrm{Id}_*"] \arrow[d, "\mathrm{Id}_*"] & * \arrow[d, "\mathrm{Id}_*"] \\ * \arrow[r, "\mathrm{Id}_*"] & * \end{tikzcd}$$

Example 228 *The category of opens of a topological space is a site*

In particular, Cartesian spaces?

"Every frame is canonically a site, where U is covered by $\{U_i\}$ precisely if it is their join."

Is there some kind of relationship between the sheaves of a Grothendieck topos, and the elements of the site taken as (representable) sheaves + coproduct and equalizer

4.11.1 Site morphisms

Site morphism : for C, D sites, a functor $f : C \to D$ is a morphism of sites if it is covering-flat and preserves covering families : for every covering $\{p_i : U_i \to U\}$ of $U \in C$, $\{f(p_i) : f(U_i) \to f(U)\}$ is a covering of $f(U) \in D$.

Covering-flat :

For a set-valued functor $F : C \to \mathbf{Set}$,

Filtered category : A filtered category is a category in which every diagrgam has a cocone.

For any finite category D and functor $F : D \to C$, there exists an object $X \in C$ and a nat. trans. $F \to \Delta_X$.

Simpler version :

- There exists an object of C (non-empty category)
- For any two objects $X, Y \in C$, there is an object Z and morphisms $X \to Z$, $Y \to Z$
- For any two parallel morphism, $f, g : X \to Y$, there exists a morphism $h : Y \to Z$ such that $hf = hg$.

Every category with a terminal object is filtered.

Every category which has finite colimits is filtered.

Interpretation : for any limit that the site has, they are preserved.

4.12 Stalks and étale space

[38]

4.12.1 In a topological context

In **Top**, consider an object B (the base space), and take the slice category $\mathbf{Top_{/B}}$, the category of bundles $\pi : E \to B$ over B.

If π is a local homeomorphism, ie for every $e \in E$, there is an open neighbourhood U_e such that $\pi(U_e)$ is open in B, and the restriction $\pi|_{U_e} : U_e \to \pi(U_e)$ is a homeomorphism, then we say that $\pi : E \to B$ is an *etale space*, with $E_x = \pi^{-1}(x)$ the *stalk* of π over x.

For $\mathrm{Sh}(\mathbf{C}, J)$ a topos, if F is a sheaf on $(\mathbf{C}, J)$, the slice topos $\mathrm{Sh}(\mathbf{C}, J)/F$ has a canonical étale projection $\pi : \mathrm{Sh}(\mathbf{C}, J)/F \to \mathrm{Sh}(\mathbf{C}, J)$, a local homeomorphism of topoi, the étale space of F.

For any object $X \in \mathbf{C}$, $y(X)$ the Yoneda embedded object,

$$U(X) = \mathrm{Sh}(\mathbf{C}, J)/y(X) \tag{4.110}$$

Sections of π_F over $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ are in bijection with elements of $F(X)$.

If $(\mathbf{C}, J)$ is the canonical site of a topological space, each slice $\mathrm{Sh}(\mathbf{C}, J)/F$ is equivalent to sheaves on the etale space of that sheaf. In particular, $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ corresponds to the inclusion of an open subset.

4.13 Topological spaces

While the category of topological spaces **Top** is *not* a topos, I feel like I should bring up some comments on this.

The common notion of a "space" in mathematics (common as in defined on set theory) is usually done using that of a topological space, for various reasons such as history, ease of use, broadness, etc. The common grounding as with the rest in general is done using set theory as

Definition 229 *A topological space (X, τ) is a set X and a set of subsets $\tau \subset \mathcal{P}(X)$, called* open sets*, such that for any collection of open sets $\{U_i\} \subseteq \tau$, we have*

- *The entire space X and the empty set $\varnothing$ are both open sets : $X, \varnothing \in \tau$*
-

In terms of category, we have the category of topological spaces **Top**, with objects the class of all topological spaces, and morphisms the class of all continuous functions. From this definition, we can tell that **Top** is a concrete category, where the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is simply the functor associating the underlying set X of a topological space (X, τ), and it is faithful as we have

$$\mathrm{Hom}_{\mathbf{Top}}(X,Y) = C(X,Y) \tag{4.111}$$

The injectivity to $\mathrm{Hom}_{\mathbf{Set}}(U(X), U(Y))$ implies that two different continuous functions lead to two different functions, which is trivially true since we defined our continuous functions to merely be a subset of the functions here.

Adjoints

Theorem 230 *The initial and terminal objects of* **Top** *are the empty topological space* $\varnothing$ *and the singleton topological space* $\{\bullet\}$*, both of which have a unique topology :*

$$\begin{aligned} \tau_{\varnothing} &= \mathcal{P}(\varnothing) = \{\varnothing\} && (4.112) \\ \tau_{\{\bullet\}} &= \mathcal{P}(\{\bullet\}) = \{\varnothing, \{\bullet\}\} && (4.113) \end{aligned}$$

Proof 36 *As the initial object is given by the empty set, for which there is only one possible function to any other function (the empty function), we only need to prove that this is always a continuous function, which is vacuously true since an empty function has an empty preimage.*

The same goes for the terminal object, as any set has a single morphism to the singleton, so that we only need to show it to be a continuous function. This is

Limits and colimits

Subobject classifier

Theorem 231 *There is no exponential object in* **Top**.

Proof 37

Problem with exponential object

There are several possible ways to try to find a compromise to have a topos of topological spaces. One is to take some subcategory that does form a topos

category of compact Hausdorff spaces **CHaus**

Convenient category of topological spaces : subcategory of Top such that

Every CW complex is an object C is Cartesian closed C is complete and cocomplete Optional : C is closed under closed subspaces in Top : if X in C and $A \subseteq X$ is a closed subspace, then A belongs to C.

Another way is to find some other category that has a broad intersection with **Top** (ex smooth spaces)

4.14 Geometry

The broad notion of "geometry" in a topos involved the use of so-called *geometric morphisms* (although in terms of the topos itself, those are actually functors)

For two toposes E, F, a geometric morphism $f : E \to F$ is a pair of adjoint functors (f^*, f_*)

$$f_* : E \to F \tag{4.114}$$
$$f^* : F \to E \tag{4.115}$$

such that the left adjoint f^* preserves finite limits. f_* is the *direct image functor*, while f^* is the *inverse image functor*.

To get a better idea of what a geometric morphism is, let's look at the more concrete case of a Grothendieck topos. If the sites are $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$, with a morphism of site $f : X \to Y$, inducing a functor by precomposition :

$$(-) \circ f : \mathrm{PSh}(Y) \to \mathrm{PSh}(X) \tag{4.116}$$
$$(F : Y^{\mathrm{op}} \to \mathbf{Set}) \mapsto (G : X^{\mathrm{op}} \to \mathbf{Set}) \tag{4.117}$$

ie for some element $y \in Y$, and a presheaf F, we have the map

$$F \circ f : Y \to \mathrm{Set} \tag{4.118}$$

Upon restriction to act on sheaves, this is our inverse image functor f^*, with the right adjoint to this being the direct image functor.

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y) \tag{4.119}$$
$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X) \tag{4.120}$$

Example 232 *The basic example which gives the morphisms their names is the case of a sheaf over a topological space X, where the site is the category of opens $\mathrm{Op}(X)$, and a common type of sheaf is the sheaf of functions to some set A : $\mathrm{Sh}(\mathrm{Op}(X)) = C(X, A)$.*

$$[X] \tag{4.121}$$

A morphism of site in this case is a continuous function $f : X \to Y$, which induces a functor on $\mathrm{Op}(X)$ by restriction :

$$f(U \in \mathrm{Op}(X)) = \tag{4.122}$$

$$f_*F(U) = F(f^{-1}(U)) \tag{4.123}$$

Example 233 *Another similar example to look at the basic functioning of the geometric morphisms is to consider two slice topos from the category of sets. Taking two sets X and Y, consider the slice topoi $\mathbf{Set}_{/X}$ and $\mathbf{Set}_{/Y}$.*

Example : One of the most common type of geometric morphism on a (Grothendieck) topos is the case of global sections. The site morphism involved is from whichever site we decide on our topos X to the trivial site $*$, so that our geometric morphism is between our topos and the topos of sets, $\mathbf{Set} = \mathrm{Sh}(*)$. The only site morphism available here is the constant functor

$$p : X \to * \tag{4.124}$$

which is a site morphism as any covering family of X is sent to $\mathrm{Id}_* : * \to *$, which is the only covering of 1. As the terminal category does not have much in the way of limits, we will have to show that this functor is filtered.

The induced functor is therefore some functor from the category of sets to our topos, so that for any object $F : X^{\mathrm{op}} \to \mathbf{Set}$ in our topos, and any object $x \in X$ in the site, the precomposition becomes

$$\begin{aligned} (-) \circ p : \mathrm{Set} &\to \mathrm{Sh}(X) && (4.125)\\ (A : * \to \mathrm{Set}) &\mapsto (A \circ p : X^{\mathrm{op}} \to * \to \mathbf{Set}) && (4.126) \end{aligned}$$

ie for any "sheaf" $* \to$ A (a set), we obtain a sheaf on X simply giving us back this set.

there is only one morphism between two sites, $\mathrm{Id}_* : * \to *$. The induced functor on the sheaf is

$$F \circ \mathrm{Id}_* : * \tag{4.127}$$

direct image functor is

$$x \tag{4.128}$$

If f^* has a left adjoint $f_! : E \to F$, f is an essential geometric morphism.

Direct image functor :

$$f_*F(U) = F(f^{-1}(U)) \tag{4.129}$$

Global section : if $p : X \to *$, $*$ the terminal object of the site

Inverse image functor :

$$f^{-1}G(U) = G(f(U)) \tag{4.130}$$

Local geometric morphism

4.15 Subtopos

Definition 234 *Given a topos* $\mathbf{H}$*, a subtopos* $\mathbf{S}$ *is a topos for which there exists a geometric morphism* $\iota : \mathbf{S} \hookrightarrow \mathbf{H}$

Example 235 *The initial topos* $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$ *is always a subtopos of any topos.*

Theorem 236 *The slice category* $\mathbf{H}_{/X}$ *of a topos* $\mathbf{H}$ *is itself a topos and a subtopos of* $\mathbf{H}$.

Proof 38

over topos, comma topos?

Definition 237 *A subtopos* $\iota : \mathbf{H}_j \hookrightarrow \mathbf{H}$ *with Lawvere-Tierney topology* j *is a* dense subtopos

Lawvere-Tierney topology j

Level of a topos : an essential subtopos $H_l \hookrightarrow H$ is a *level* of H.

"the essential subtoposes of a topos, or more generally the essential localizations of a suitably complete category, form a complete lattice"

"If for two levels $H_1 \hookrightarrow H_2$ the second one includes the modal types of the idempotent comonad of the first one, and if it is minimal with this property, then Lawvere speaks of “Aufhebung” (see there for details) of the unity of opposites exhibited by the first one."

4.16 Motivic yoga

[39, 40] Six functor formalism

[41]

4.17 Localization

In some cases we wish to

[42] For a category C and a collection of morphisms $S \subseteq \mathrm{Mor}(C)$, an object $c \in C$ is S-local if the hom-functor

$$C(-, c) : C^{\mathrm{op}} \to \mathrm{Set} \tag{4.131}$$

sends morphisms in S to isomorphisms in Set, so that for every $(s : a \to b) \in S$,

$$C(s, c) : C(b, c) \to C(a, c) \tag{4.132}$$

is a bijection

"localization of a category consists of adding to a category inverse morphisms for some collection of morphisms, constraining them to become isomorphisms"

"In homotopy theory, for example, there are many examples of mappings that are invertible up to homotopy; and so large classes of homotopy equivalent spaces"

Example 238 *The basic example of a localization is that of a commutative ring. localizing with prime* 2 *:* $\mathbb{Z}[1/2]$*, localization away from all primes :* $\mathbb{Q}$

Example 239 *Localization of* $\mathbb{R}[x]$ *away from* a *: rational functions defined everywhere except at* a

Localization at a class of morphisms W : reflective subcategory of W-local objects (reflective localization).

Localization of an internal hom : localization of the morphisms defined by $\prod_{X,Y}[X, Y]$?

Localization of a topos corresponds to a choice of Lawvere topology, localization of a Grothendieck topos to a Grothendieck topology.

Duality of a localization?

4.18 Number objects

One benefit of topoi as a category is the guaranteed existence of a natural number object[43], which are some internalization of the notion of positive integers.

Definition 240 *A* natural number object *for a topos is an object denoted* $\mathbb{N}$ *such that there exists the morphisms*

- *The morphism* $z : 1 \to \mathbb{N}$
- *The* successor morphism $s : \mathbb{N} \to \mathbb{N}$

such that for any diagram $q : 1 \to X$*,* $f : A \to A$*, there is a unique morphism* u

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "q"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbb{N} \arrow[d, "u"] \\ & A \arrow[r, "f"] & A \end{tikzcd}$$

The map $z : 1 \to N$ is the zero element of N, while the $s : N \to N$ is the successor morphism, analogous to the operation $s(n) = n + 1$.

This number object is in fact (isomorphic to) the algebra generated by the maybe monad $=$ Maybe(n)

f defines a sequence, such that $a_0 = q$ and $a_{n+1} = f(a_n)$

Relation to maybe monad

Show that a morphism $\mathbb{N} \to A$ induces a diagram $A \to A \to A \to \ldots$, which induces a limit

$$\lim_f A \tag{4.133}$$

Topos also induce a *real number object*

4.19 Duality

In our analysis of topos, a very common way to define them is via duality.

Definition 241 *A duality is an equivalence between a category* $\mathbf{C}$ *and the dual of a category* $\mathbf{D}$*.*

$$S : \mathbf{C} \to \mathbf{D} \tag{4.134}$$
$$T : \mathbf{D} \to \mathbf{C} \tag{4.135}$$

$$\eta : \mathrm{Id}_{\mathbf{C}} \to TS \tag{4.136}$$
$$\epsilon : \mathrm{Id}_{\mathbf{D}} \to ST \tag{4.137}$$

obeying that for every $X \in \mathbf{C}$ *and* $Y \in \mathbf{D}$

$$T\epsilon_X \circ \eta_{TX} = \mathrm{Id}_{TX} \tag{4.138}$$
$$S\eta_Y \circ \epsilon_{SY} = \mathrm{Id}_{SY} \tag{4.139}$$

Definition 242 *If two dual categories* $\mathbf{C}$, $\mathbf{D}$ *are concrete, ie with*

$$
\begin{aligned}
U : \mathbf{C} &\rightarrow \mathbf{Set} && (4.140)\\
V : \mathbf{D} &\rightarrow \mathbf{Set} && (4.141)
\end{aligned}
$$

that are faithful and representable, $U \cong \mathrm{Hom}_{\mathbf{C}}(E_{\mathbf{C}}, -)$, $V \cong \mathrm{Hom}_{\mathbf{D}}(E_{\mathbf{D}}, -)$

[is concreteness and representability needed]

Definition 243 *A* concrete duality

[44]

Dualizing object :

Example 244 *The standard example of duality is that of dual vectors, with dualizable object* $I \cong k \in \mathbf{Vec}_k$. *The two functors are simply* $U = V = \mathrm{Hom}_{\mathrm{Vec}_k}(I, -)$, *mapping vector spaces to their set of points. Then a dualizable object is a vector space* X

$$
\begin{aligned}
T : \mathrm{Vec}_k^{op} &\xrightarrow{\mathrm{Hom}_{\mathrm{Vec_k}}(-,\mathrm{X})} \mathrm{Vec}_k && (4.142)\\
S : \mathrm{Vec}_k^{op} &\xrightarrow{\mathrm{Hom}_{\mathrm{Vec_k}}(-,\mathrm{X})} \mathrm{Vec}_k && (4.143)
\end{aligned}
$$

$$
\begin{aligned}
US : \mathbf{Vec}_k &\rightarrow \mathbf{Set} && (4.144)\\
VT : \mathbf{Vec}_k &\rightarrow \mathbf{Set} && (4.145)
\end{aligned}
$$

representable : $\exists X, Y \in \mathbf{Vec}_k$ *such that*

$$
\begin{aligned}
US &= \mathrm{Hom}_{\mathbf{Vec}_k}(-, X) && (4.146)\\
VT &= \mathrm{Hom}_{\mathbf{Vec}_k}(-, Y) && (4.147)
\end{aligned}
$$

Representing elements : $\phi = \mathrm{Hom}_{\mathbf{Vec}_k}(X, X)$, $\psi = \mathrm{Hom}_{\mathbf{Vec}_k}(Y, Y)$, *space of square matrices*

nat :

$$
\begin{aligned}
\eta : \mathrm{Id}_{\mathbf{Vec}_k} &\rightarrow TS && (4.148)\\
\epsilon : \mathrm{Id}_{\mathbf{Vec}_k} &\rightarrow ST && (4.149)
\end{aligned}
$$

Those are respectively the resolution of the identity and [trace?]

Contravariant representation?

Canonical isomorphism $\omega : |X| \rightarrow |Y|$ *via*

$$
|X| \xrightarrow{X} USTX \qquad (4.150)
$$

Adjunction

$$(-) \otimes A \dashv (-) \otimes A^* \cong [A, -] \tag{4.151}$$

$$X^* \cong k \otimes X^* \cong [A, k] \tag{4.152}$$

[...]

In the context of presheaves, this situation is simply the case where

As a presheaf can be considered as the generalization of a space, a copresheaf can be considered as the generalization of an algebra.

The basic example for this is to consider the algebra of smooth functions on a Cartesian space, via the functor associating R-algebras to Cartesian spaces :

$$C^\infty : \mathbf{SmoothMan} \to R{-}\mathbf{Alg} \tag{4.153}$$

Associating the algebra $C^\infty(M)$ to any manifold M. Adding the forgetful functor $U : R{-}\mathbf{Alg} \to \mathbf{Set}$, this gives us some copresheaf on the category of Cartesian spaces.

Theorem 245 *Milnor duality :*

$$\mathrm{Hom}_{\mathrm{SmoothMan}}(X, Y) \cong \mathrm{Hom}_{\mathbf{CAlg}_{\mathbb{R}}}(C^\infty(Y), C^\infty(X)) \tag{4.154}$$

This copresheaf is furthermore a cosheaf. If we consider the restriction of our manifold to some submanifold $\iota : S \hookrightarrow M$,

Copresheaf : functor $F : \mathbf{C} \to \mathbf{Set}$

Cosheaf : copresheaf has for a covering family $\{U_i \to U\}$

$$F(U) \cong \varinjlim \left(\coprod_{ij} F(U_i \times_U U_j) \rightrightarrows \coprod_i F(U_i) \right) \tag{4.155}$$

Equivalence via Yoneda :

$$\mathrm{CoSh}(\mathbf{C}) \cong \tag{4.156}$$

The copresheaf assigns to each test space $U \in \mathbf{C}$ the set of allowed maps from A to U - U-valued functions on A

Example 246 *There is a duality between the algebra* $\mathbb{R}$ *and the single point space* $\{0 < 1\}$

Proof 39 *As* $\mathbb{R}$ *has only a single ideal,* 0*,*

$$\mathrm{Spec}(\mathbb{R}) \tag{4.157}$$

4.20 Ringed topos

As with sheaves in general, we do not have to consider our topos to be exclusively set-valued, and we can give it a variety of other types. One of the most commonly used is that of a ringed topos.

Definition 247 *A* ringed topos $(X, \mathcal{O}_X)$ *is*

Chapter 5

Example categories

For the consideration of the methods to be studied, we need to look at a few good examples of appropriate categories. We will mostly look at topos (the main focus of this), more specifically Grothendieck topos, as well as a few categories of some physical interest relating to quantum theory, which may differ from the topos structure, hopefully illuminating the differences with the more classical logic associated with classical mechanics.

Quantales? Topos of the sheaves of commutative algebras on Hilbert space? Effectus categories?

Why **Top** isn't a topos : Not balanced, not Cartesian closed or localy Cartesian closed[45]

Simplicial category? Sierpinski topos?

5.1 Category of sets

The most basic topos (outside of the initial topos $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$) is the category of sets **Set**, with objects made from the class of all sets and morphisms the class of all functions.

In terms of a sheaf topos, sets can be defined as the sheaf on the terminal category : $\mathbf{Set} = \mathrm{Sh}(\mathbf{1})$, which is the functor from $*^{\mathrm{op}} = *$ to Set. As the set of all functions from $\{\bullet\}$ to any set X is isomorphic to X itself, this is easily seen to be isomorphic to **Set**. We only need to consider sieves on its unique object $*$, and as there is only one possible morphism there, there are only two possible sieves : the empty sieve $S_\varnothing$ which maps *ast* to the empty set, and the maximal sieve S_* which maps it to the singleton containing the identity map.

This allows us two possible topologies, the *chaotic topology*

As the site only has a trivial coverage, there is only a fairly limited amount of assembly that we can do from it.

A variation of **Set** is **FinSet**, the category of finite sets. Trivially a full subcategory of **Set** via the obvious inclusion functor.

Theorem 248 *The category* FinSet *is a subtopos of* **Set**.

Proof 40 *As we are only concerned with finite limits here, all basic arguments for* **Set** *being a topos apply here. All finite limits of finite sets in* **Set** *are also finite sets, as are all internal homs and the subobject classifier.*

5.1.1 Limits and colimits

As the prototype for the very notion of limits and colimits, **Set** has all the finite limits.

Theorem 249 *The empty set* $\varnothing$ *is the initial object of* **Set**.

Proof 41 *We need to show that for any set* $X \in \mathrm{Obj}(\mathbf{Set})$*, there is a unique function* $f : \varnothing \to X$*. A function* $f : A \to B$ *is a subset of* $A \times B$ *obeying some properties, therefore we need to look at the set of subsets of* $\varnothing \times A$*. By properties of the Cartesian product,*

$$\varnothing \times A = \{\varnothing\} \tag{5.1}$$

There is therefore only one element to choose from, $\varnothing$*, which is indeed a function since it obeys (vacuously) the constraints on functions.*

Theorem 250 *Any singleton set* $\{\bullet\}$ *is a terminal object of* **Set***, all isomorphic.*

Proof 42 *We need to show that for any set* $X \in \mathrm{Obj}(\mathbf{Set})$*, there is a unique function* $f : X \to \{\bullet\}$.

$$X \times \{\bullet\} \tag{5.2}$$

The terminal object is also the unique representable presheaf on $\mathbf{1}$, simply given by $\mathrm{Hom}_{\mathbf{1}}(*, *)$, mapping the unique element of $\mathbf{1}$ to the set of the unique identity morphism of $*$, as we would expect.

As a category of sets, which are fundamentally defined by $\in$, **Set** has global elements $x : I \to X$. Those global elements are separators

Well-pointed topos

Products and coproducts :

Theorem 251 *The product on* **Set** *is isomorphic to the Cartesian product.*

Proof 43 *The Cartesian product has by construction*

Theorem 252 *The coproduct on* **Set** *is isomorphic to the disjoint union.*

Proof 44

Theorem 253 *Given two functions $f, g : A \to B$, the equalizer in* **Set** *is the subset $C \subseteq A$ on which those functions coincide,*

$$\mathrm{eq}(f, g) = \{c \in A | f(c) = g(c)\} \tag{5.3}$$

Theorem 254 *The equalizer of two functions $f, g : A \rightrightarrows B$ is the set of elements of A whose image agree :*

$$\mathrm{eq}(f, g) = \{x \in A | f(x) = g(x)\} \tag{5.4}$$

Theorem 255 *The coequalizer of two functions $f, g : A \rightrightarrows B$ is the quotient set on A by the equivalence relation*

$$x \sim y \leftrightarrow f(x) = g(y) \tag{5.5}$$

Proof 45

$$A \to B \to C \tag{5.6}$$

Definition 256 *The pullback of the (co?)span $A \to C \leftarrow B$ is the indexed set*

Theorem 257 *The pushout of the (co?)span $A \leftarrow C \to B$ is the*

Theorem 258 *Directed limit*

Given these, we can see that **Set** has all small limits and colimits.

5.1.2 Elements

Fairly obviously, given its status as the model for it, **Set** has global elements : $x : \{\bullet\} \to X$, corresponding to the functions

$$\forall x \in X,\ x(\bullet) = x \tag{5.7}$$

so that explicitly, $x = \{(\bullet, x)\}$ (this set can be shown to exist with the axiom of pairing).

5.1.3 Subobject classifier

For **Set**, the subobject classifier is the set $\{\varnothing, \{\bullet\}\}$, also denoted by $\{0,1\}$ or $\{\perp, \top\}$, corresponding to the two boolean valuations of a subobject : either being a subset or not being a subset. In terms of set theory, a subobject is a subset (up to isomorphism), that is

$$A \subseteq X \leftrightarrow \exists \chi_A : X \to \Omega \tag{5.8}$$

For a subset $\iota : S \hookrightarrow X$

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, "\iota"] & * \arrow[d, "\mathrm{true}"] \\ X \arrow[r, "\chi_S"] & \Omega \end{tikzcd}$$

The function χ_S is more typically called the characteristic function and uses the notation $\chi_S : U \to \mathbb{B}$

$$\chi_U(x) = \begin{cases} 0 \\ 1 \end{cases} \tag{5.9}$$

Internal hom : The set of functions

$$\begin{aligned} [X,Y] &= \{f | f : X \to Y\} && (5.10) \\ &= \{f \subseteq X \times Y | \forall x \in X,\ \exists y \in Y,\ (x,y) \in f \wedge ((x,y) \in f \wedge (x,z) \in f \to y && (5.11) \end{aligned}$$

Natural number object : the natural number construction.

The maybe monad is simply

$$\mathrm{Maybe}(X) = X \sqcup \{\bullet\} \tag{5.12}$$

Coproduct for natural number :

$$0 \sqcup 0 = \{\} \tag{5.13}$$

Lawvere-Tierney topology : some morphism $j : \mathbf{2} \to \mathbf{2}$

Properties : $j(\{\bullet\}) = \{\bullet\}$, $j(j(x)) = j(x)$, $j(a \wedge b) =$

$j(\{\bullet\}) = \{\bullet\}$ reduces the choice to $j = \mathrm{Id}_\Omega$ and $j = \{\bullet\}$, the constant map.

For the identity map : Given a subset $\iota : S \hookrightarrow X$, with classifier $\chi_S : A \to \Omega$, the composition $j \circ \chi_S$ defines another subobject $\bar{\iota} : \overline{S} \hookrightarrow A$ such that s is a subobject of $\bar{\iota}$, $\bar{s}$ is the j-closure of s

Identity map closure : every object is its own closure. This is the *discrete topology.*

Constant map $j(x) = \{\bullet\}$: the composition $j \circ \chi_S$ is the "always true" characteristic function, which is just χ_A. The closure of a set S in A is the entire set A. This is the *trivial* or *codiscrete* topology.

Those are the only two allowed topologies in **Set**.

Relation to $\mathrm{loc}_{\neg\neg}$: $\neg : \Omega \to \Omega$ is

$$\neg(\{\bullet\}) = \varnothing \tag{5.14}$$

$$\neg(\varnothing) = \{\bullet\} \tag{5.15}$$

$\neg\neg$ is simply the identity on **Set**. The j-closure associated to it is the identity map.

Localization?

Power object

5.1.4 Closed Cartesian

That the category of sets is closed Cartesian simply stems from the usual construction of functions on sets in basic set theory. As functions from A to B are defined as a subset of the Cartesian product $A \times B$, this is guaranteed by the existence of the Cartesian product of sets and the axiom schema of specification.

Definition 259 *For any two sets A, B, the hom-set $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ is itself a set, by the traditional set definition of functions*

$$f : A \to B \leftrightarrow f = \{(a, b) \subseteq A \times B | f(a) = b\} \tag{5.16}$$

so that []

The evaluation map of a function in **Set** is given by the traditional formulation of function evaluations in set theory. For a function $f : A \to B$, its evaluation by an element $x \in A$ is the unique element y in Y for which $(x, y) \in R_f$. In set theoretical terms, this can be defined Russell's iota operator,

$$\mathrm{ev}(x, f) = \iota y,\ xR_f y \tag{5.17}$$

$$= \bigcup \{z \mid \{y \mid xR_f y\} = \{z\}\} \tag{5.18}$$

Counit of the tensor product/internal hom adjunction?

$$S \times (-) \dashv [S, -] \tag{5.19}$$

currying

5.2 Topos on a set

As any set forms a site with the power set as coverage, we can consider the Grothendieck topos

$$\mathrm{Sh}(X, \mathcal{P}(X)) \tag{5.20}$$

for some set X.

5.3 Topos of a topological space

Another common example of a topos is the sheaf of a topological space $\mathrm{Sh}(\mathrm{Op}(X))$ with the subcanonical coverage, simply written as $\mathrm{Sh}(X)$ for short. This is the type of topos we originally saw in the definition of a sheaf as an ur-exemple, with such common examples as the structure sheaf of continuous functions to some topological space Y. If we call our topos $E = \mathrm{Sh}(X)$, then we have the interpretation that we saw earlier.

This is the classic example of a topos, where the restriction maps $\rho_{U,V}$ are simply given by the domain restriction

Restriction maps, gluing, locality

$\mathrm{Sh}(\mathrm{Op}(X))$

"The problem with excluded middle in topological models is that it may not hold continuously: e.g. a subspace $U \subseteq X$ and its complement $X \setminus U$ will together contain all the points of X, but the map from their coproduct $U + (X \setminus U)$ to X does not have a continuous section because the topology on the domain is different. Thus, we cannot expect to have the full law of excluded middle in a topological model"

5.4 Category of smooth spaces

[46, 47]

A more geometric category for a topos is the category of smooth spaces **Smooth**, which is defined as the sheaf over the category of smooth Cartesian spaces,

$$\mathbf{Smooth} = \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{Smooth}}) \tag{5.21}$$

The category of smooth Cartesian spaces can be given a variety of sites equivalently, such as the site of smooth manifolds SmoothMan, the site of open subsets of $\mathbb{R}^n$, or the site of $\mathbb{R}^n$, all with morphisms being the smooth maps between such objects. As we will see, all those possible sites end up producing the same topos, and we will typically just use the category of Cartesian spaces **CartSp** where the only objects are $\mathbb{R}^n$, as this will make for the simpler site. This stems from the fact that the other sites can be constructed by limits and colimits of each other that we do not have to worry too much. While the objects are $\mathbb{R}^n$, this is of course only up to diffeomorphism so that this will include all manners of contractible domains like open balls.

The coverage of this site is slightly tricky. The most obvious cover is simply the coverage by open sets (diffeomorphic to $\mathbb{R}^n$ in our case), where we consider the lattice generated by all open sets of **CartSp**. While we can construct a sheaf over this coverage (and it will in fact lead to an equivalent topos), there are coverages with better properties.

Definition 260 *A* good open cover *is an open cover for which any finite intersection of open sets is contractible, ie a good open cover $\{f_i : U_i \to X\}$ of X*

$$\int \prod_{i\in I,\ X} U_i \cong \star \tag{5.22}$$

This is taken typically taken as the cover as, from algebraic topology, we know that such covers have better properties, as the Cech cohomology of the cover will be identical to that of the space itself.

Theorem 261 *Any smooth manifold admits a good open cover.*

This implies a homeomorphism to the open ball

Definition 262 *A good open cover $\{f_i : U_i \to X\}$ is a* differentially good open cover *if finite intersections of the cover are all diffeomorphic to the open ball.*

$$\prod_{i\in I,\ X} U_i \cong B^k \tag{5.23}$$

Theorem 263 *All three coverage of* $\mathbf{CartSp}_{\mathrm{smooth}}$ *lead to isomorphic sheaves :*

$$\mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{open}) \cong \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{good}) \cong \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{diff})$$

Therefore for our purpose we can pick the best behaved coverage.

Sheaves on the category of Cartesian spaces is best understood, in the context of geometry, as being *plots*, the more general version of what would be an atlas in the case of manifolds.

Definition 264 *A* plot *is a map between an open set of a Cartesian space $\mathcal{O} \subseteq \mathbb{R}^n$ and a topological space X*

From the Yoneda lemma, we have that for any sheaf $X \in \mathbf{Smooth}$ and Cartesian space U, we have the isomorphism

$$X(U) \cong \mathrm{Hom}_{[\mathbf{CartSp},\mathbf{Set}]}(y(U), X) \tag{5.24}$$

$$\begin{aligned} y(U) : \mathbf{CartSp}^{\mathrm{op}} &\to \mathbf{Set} && (5.25) \\ O &\mapsto && (5.26) \end{aligned}$$

While this is a good intuitive way to understand the spaces probed by plots, it can be useful to know that in fact the topological space X itself is not necessary as a data to define a space cattaneo.

Theorem 265 *Given the set of transition functions on a manifold, the topological space can be reconstructed as*

$$M = \bigsqcup O_i / \sim \tag{5.27}$$

where two points in $O_i \sqcup O_j$ are equivalent if $\tau_{ij}(x_i) = x_j$

This is what we do with the smooth sets topos, as we are only considering the existence of those maps (as the set $F(\mathbf{Cartsp})$), and the behaviour of those plots over overlapping regions.

Example 266 *Take the circle S^1, which we will defined a bit simplistically as a plot over $\mathbb{R}$. To avoid any issue of bad covers, let's take three different overlapping plots so that all overlaps are pairwise contractible. Just considering those plots, we get*

$$S^1(\mathbb{R}) = \{\varphi_1, \varphi_2, \varphi_3\} \tag{5.28}$$

In terms of an atlas, if we considered our circle as the interval $[0, 1]$ with ends identified, we could cover it via the three intervals $(0, 3/4)$, $(1/2, 1)$ and $(3/4, 1/2)$ (simply pick any map $\mathbb{R} \to (0, 1)$ to appropriately rescale everything such as the arctan *map)*

[diagram]

with overlaps

$$\begin{aligned} 1;2 &\quad:\quad (1/2,3/4) &(5.29)\\ 1;3 &\quad:\quad (0,1/2) &(5.30)\\ 2;3 &\quad:\quad (3/4,1) &(5.31)\end{aligned}$$

with the transition maps

$$\begin{aligned} \tau_{12} &= &(5.32)\\ \tau_{13} &= &(5.33)\\ \tau_{23} &= &(5.34)\end{aligned}$$

In terms of overlap, we have $U^+ \cap U^- \cong I \sqcup I$, so that we need to consider additionally the plot of that Cartesian space (slightly complicated by the non-connected aspect of it, but we can consider the open set of the line $(-1,0)\cup(0,1)$. While we can do it this is partly why we generally consider good open covers)

$$S^1(I \sqcup I) = \{\varphi^{\pm}\} \tag{5.35}$$

which maps this overlap region onto S^1. The inclusion of this overlap area is done as

$$\begin{aligned} \iota_+(x \in I \sqcup I) &= x &(5.36)\\ \iota_-(x \in I \sqcup I) &= 2-x &(5.37)\end{aligned}$$

The overlap in terms of the plot is that we map $(-1,0)\cup(0,1)$ to the interval I as

Those morphisms on **CartSp** *are mapped onto opposite mappings on* **Set** *:*

$$S^1(\iota_+) : \{\varphi^+, \varphi^-\} \to \{\varphi^{\pm}\} \tag{5.38}$$

If we take the less abstract case of a concrete sheaf to look at smooth spaces, considering CartSp is a concrete site, the concrete presheaf of Sh(CartSp) is the category of *diffeological spaces* DiffeoSp, where each global element $X : 1 \to$ DiffeoSp is a diffeological space.

[Diff is a quasitopos]

An important subcategory is also the category of smooth manifolds **SmoothMan**.

$$\mathbf{SmoothMan} \subseteq \mathbf{DiffeoSp} \subseteq \mathbf{Smooth} \tag{5.39}$$

SmoothMan is not itself a topos, as it lacks an exponential object (Hom sets between manifolds are not themselves manifolds, although they are close to it [48]), and the quotients or equalizers of manifolds are not themselves manifolds [examples]

Smooth manifolds are locally representable objects of **Smooth**. If $X : 1 \to \mathbf{Smooth}$ is a concrete smooth space (diffeological space), it is locally representable if there xists $\{U_i \hookrightarrow X\}$, $U_i \in \mathbf{Smooth}$ such that the canonical morphism out of the coproduct

$$\bigsqcup_i U_i \to X \tag{5.40}$$

Is an effective epimorphism in **Smooth**.

$$\bigsqcup_i U_i \times_X \bigsqcup_j U_j \rightrightarrows \bigsqcup_i U_i \to X \tag{5.41}$$

By commutativity of coproduct and pullback [prove it]

$$\bigsqcup_{i,j} (U_i \times_X U_j) \rightrightarrows \bigsqcup_i U_i \to X \tag{5.42}$$

Theorem 267

$$\mathbf{Smooth} \cong \mathrm{Sh}(\mathbf{SmoothMan}) \tag{5.43}$$

An important property of **Smooth** is that it contains a large proportion of **Top**, more specifically the category of

Status wrt top, delta generated top, etc

Due to this wide variety of physically important objects in **Smooth**, it will typically be (or at least some wider categories that we will define later) the topos serving as the setting for physics in general.

Subobject classifier : for any $U \in \mathbf{CartSp}$, $\Omega(U)$ is the set of subsheaves of h_U.

5.4.1 Limits and colimits

Theorem 268 *The initial object of* **Smooth** *is the constant functor*

$$\Delta_\varnothing : \mathbf{CartSp}_{\text{Smooth}} \to \mathbf{Set} \tag{5.44}$$

which maps every cartesian space to the empty set $\varnothing$*.*

Proof 46

The interpretation of this is that the initial object can be seen as the empty manifold, with the empty atlas.

Theorem 269 *The terminal object of* **Smooth** *is the constant functor*

$$\Delta_{\{\bullet\}} : \mathbf{CartSp}_{\text{Smooth}} \to \mathbf{Set} \tag{5.45}$$

which maps every cartesian space to the singleton $\{\bullet\}$*.*

The interpretation of this is that the terminal object of **Smooth** is a *point*, as in particular the plot of points $p : \mathbb{R}^0 \to \{\bullet\}$ is the only one which is an isomorphism.

Theorem 270

Important functors :

Forgetful functor to **Set** $U_{\mathbf{Set}} : \mathbf{Smooth} \to \mathbf{Set}$

Logic?

5.4.2 Subcategories of smooth sets

As a sheaf topos, we have that all objects of the site correspond themselves to objects of the topos via the representable presheaves. Any Cartesian space $\mathbb{R}^k$ is a smooth set via

$$X_{\mathbb{R}^k} = \mathrm{Hom}_{\mathbf{CartSp}}(-, \mathbb{R}^k) \tag{5.46}$$

such that their plots are given by

$$\mathrm{Plot}_{X_{\mathbb{R}^n}}(\mathbb{R}^l) = \mathrm{Hom}_{\mathbf{CartSp}}(\mathbb{R}^l, \mathbb{R}^k) = C^\infty(\mathbb{R}^l, \mathbb{R}^k) \tag{5.47}$$

and their transition functions are

Concrete sheaves (diffeological space)

Example 271

5.4.3 Stalks

Stalks via family of open balls?

5.4.4 Non-concrete objects

As we've seen, the concrete sheaves in **Smooth** do not form the entire topos, leaving non-concrete sheaves.

The archetypical example of this is the smooth set of differential k-forms,

$$\begin{aligned} \Omega^k : \mathbf{CartSp} &\to \mathbf{Set} && (5.48)\\ U &\mapsto \Omega^k(U) && (5.49) \end{aligned}$$

which associates to every Cartesian space the set of k-forms over that space.

If we attempt to look at the set of "points" of this space, if that term can be applied here, that would be the plot of the terminal object in the site, $\mathbb{R}^0$. But of course, in the sense of the sheaf as described here, this will just be the set of all k-forms over the point $\Omega(\mathbb{R}^0)$, which will just include the zero section, so that if we try to consider this plot as the "point content" of the space, there is but a single point :

$$\Omega(\mathbb{R}^0) = \{0\} \tag{5.50}$$

As we would not really consider the elements of this space to be that single section, it is therefore important to be mindful of what the plots of the sheaf represent.

Global sections :

$$\Gamma(\Omega^k) = \mathrm{Hom}_{\mathbf{Smooth}}(1, \Omega) \tag{5.51}$$

5.4.5 Important objects

The category of smooth spaces contains most of the objects of importance in physics and other fields, so that it is useful to look at the various types of objects within it.

First, as a topos, it has a terminal object as we've seen (the constant sheaf 1 which maps all probes to a single element, the constant plot). From this and the coproduct, we can construct objects similar to sets as we wish (this is in fact what the discrete functor will be later on), and as with any topos, a natural number object in particular.

As the coverage is subcanonical, the Yoneda embedding makes any Cartesian space a smooth space via its representable presheaf,

$$U \mapsto \mathrm{Hom}_{\mathbf{CartSp}}(-, U) \tag{5.52}$$

As we have seen, any diffeological space is a smooth space, in fact every concrete smooth space is a diffeological space.

Manifolds

By the Cartesian closed character of the topos, for any pair of manifolds, the set of all smooth maps between them is itself a smooth space, ie

$$C^\infty(M, N) \in \mathbf{Smooth} \tag{5.53}$$

Important classes of non-concrete sheaves are the *moduli spaces*, which are sheaves giving back appropriate function spaces on a Cartesian space. For instance the moduli space of Riemannian metrics Met is a sheaf

$$\begin{aligned} \mathrm{Met} : \mathbf{CartSp}^{\mathrm{op}} &\to \mathbf{Set} && (5.54) \\ U \subseteq \mathbb{R}^n &\mapsto && (5.55) \end{aligned}$$

where $\mathrm{Met}(U)$ is the set of all Riemannian metrics on U. For instance, as there is only one metric on a point (since the tangent bundle there is zero dimensional), we have

$$\mathrm{Met}(\mathbb{R}^0) = \{0\} \tag{5.56}$$

And there is only one component to the metric on the line which must also be positive, so its set of metric is that of the positive definite smooth functions.

Moduli space of differential forms

Moduli space of symplectic forms

True for any section?

Theorem 272 *The moduli spaces of sections is a smooth space*

5.5 Category of classical mechanics

The exact category to give to classical mechanics is somewhat controversial, due to the difficulties of finding an appropriate notion of morphisms, but a common pick is the *category of Poisson manifolds*

Definition 273 *A* Poisson manifold (P, π) *is a manifold* P *equipped with a Poisson bivector* $\pi \in \Gamma(\bigwedge^2 P)$

Poisson bracket :

$$\{f, g\} = \langle df \otimes dg, P\rangle \tag{5.57}$$

Definition 274 *An* ichtyomorphism *is a smooth map preserving the Poisson bivector :* $f^*\pi = \pi$

From this, the category of Poisson manifolds **Poiss** is the category with Poisson manifolds as objects and ichtyomorphisms as morphisms. In terms of the smooth topos, we have to look at the moduli space of Poisson bivectors :

$$\Pi(-) : \mathbf{CartSp} \rightarrow \mathbf{Set} \tag{5.58}$$
$$\mathbb{R}^n \mapsto \tag{5.59}$$

which is the set of all Poisson bivectors over $\mathbb{R}^n$. A choice of a Poisson manifold is therefore given by the internal hom of our underlying manifold and the moduli space of Poisson bivectors :

$$P(X) = [X, P] \tag{5.60}$$

Slice topos $\mathrm{Smooth}_{/\Omega^2}$?

Poisson manifold : locally representable concrete object?

5.5.1 Logic

The logic of classical mechanics is tied to the logic of measurement of observables. If we have some classical theory, with a Poisson manifold

Example : phase space of a point particle in n dimensions $\mathbb{R}^{2n}$, with the Poisson bracket

If we have some observable

$$f_o : \mathbb{R}^{2n} \rightarrow \mathbb{R} \tag{5.61}$$

Inversely, f_o selects a subset of the Poisson manifold. The statement that the measurement m_o is in the Borel subset $\Delta_o \subseteq \mathbb{R}$ is equivalent to a subobject of **Poiss**

$$S_o = f_o^{-1}(\Delta_o) \tag{5.62}$$

Limits and colimits :

$$S_{o_1} \sqcup S_{o_2} \tag{5.63}$$

What is the topos

Logic and presheaf etc

[49, 50, 51]

5.6 Categories for quantum theories

To contrast with the topos we have seen for classical mechanics, we need to also consider an appropriate category to look at the behaviour of quantum mechanics, to see if specific categorical properties have some important things to tell us about quantum systems.

There are for this quite a lot of different categories we can look at. These are :

- The category of Hilbert spaces
- The slice category of a Hilbert space
- The spectral presheaf of a Hilbert space
- The Bohr topos

Those categories can all be used for quantum mechanics with various results [etc]

[52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66]

5.6.1 Quantum mechanics as a symmetric monoidal category

The basic formulation of quantum mechanics in terms of category theory is to simply look at the categories of its main objects, which are Hilbert spaces and C^*-algebras.

Definition 275 *The category* **Hilb** *of Hilbert spaces has as its objects Hilbert spaces and as morphisms bounded linear maps between two Hilbert spaces.*

The condition of bounded linear maps is here to guarantee the existence of a dual on every operator, which will be of use for us, as otherwise linear maps are not guaranteed a dual in the infinite dimensional case, for instance by picking the linear map on $L^2(\mathbb{R}^n)$

$$f(\psi) = \psi(0) \tag{5.64}$$

which is indeed linear etc, but corresponds to a "dual" vector $\chi = \delta_0$, which is not an $L^2(\mathbb{R}^n)$ Hilbert space etc

[proof]

Hilb can be entirely defined in categorical terms[67], so that we will look at this in a bit more detail to get some feel of the categorical structures involved here.

If we were to say them all at once, then **Hilb**, the category of (complex) Hilbert spaces with bounded linear operators, is defined as :

- A dagger compact symmetric monoidal category with duals, $(\mathbf{Hilb}, \otimes, I, \dagger, ^*)$
- The monoidal unit I is a separator

[...]

Interpretations :

The monoidal unit I can be shown to be equivalent to the complex numbers. First we take the hom-set

$$\mathrm{End}(I) = \mathrm{Hom}_{\mathbf{Hilb}}(I, I) \tag{5.65}$$

and show that it has some scalar structure.

In terms of quantum mechanics, we have that for a given object $\mathcal{H}$, the Hilbert space vectors are given by morphisms $\psi : \mathbb{C} \to \mathcal{H}$, as the underlying field is the free Hilbert space here and therefore the appropriate object for generalized elements. Dually, maps of the form $\omega : \mathcal{H} \to \mathbb{C}$ form dual vectors on the Hilbert space

Dagger

In addition to all we've seen, we can also consider the enrichment of the category **Hilb** over Banach spaces, turning it into a *C^*-category.* This enrichment means that for any three Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$,

[...]

Zero map, sum of maps, scalar product of maps, involution

Those properties are in fact those of a C^*-algebra. As the notion of a C^*-algebra will be quite important later on for our other quantum categories, it is good to keep in mind.

Theorem 276 *The C^* algebra of an object $\mathcal{H} \in$ **Hilb** is its hom-set*

In addition to C^*-algebras, we will also need to look at the von Neumann algebras of our Hilbert spaces

Definition 277 *A von Neumann algebra (or W^*-algebra) is a C^*-algebra A that admits a predual, a complex Banach space A_* with an isomorphism of complex Banach spaces*

$$* : A \to (A_*)^* \tag{5.66}$$

5.6.2 Slice category of Hilbert spaces

For a short detour in terms of interpreting quantum theory in a categorical manner, we will need to look at slice categories of **Hilb**. If we pick some Hilbert space $\mathcal{H}$, the slice category $\mathbf{Hilb}_{/\mathcal{H}}$ will be the appropriate setting to talk about this specific Hilbert space.

The objects of $\mathbf{Hilb}_{/\mathcal{H}}$ are the (bounded) linear maps from any Hilbert space to $\mathcal{H}$. For our interest later on, this contains in particular all the subspace maps from all subspaces of $\mathcal{H}$. As the slice category of an enriched category, it is furthermore an enriched slice category,

5.6.3 Daseinisation

While monoidal categories are a perfectly serviceable setting for dealing with quantum mechanics, it has a few issues making it unsuitable for this analysis. In some sense it corresponds to the construction of an actual "quantum object" with an existence independent of measurement, giving it fairly problematic properties from a logical perspective (this is the content of the Kochen-Specker theorem). Due to this it also famously fails to be a topos, which is the main object we are concerned with here.

To deal with those problems, we have to deal with the *Daseinisation*[61] of the category, where rather than deal with some quantum object directly, we only consider its measurements in some context.

The simplest way to consider a measurement in quantum mechanics is to look at the projectors P of the theory. If we ignore the wider case of positive operator-valued measure and only look at projection-valued measure (we will assume no additional source of uncertainty beyond quantum theory), every measurement in a quantum theory can be modelled by this. If a measurement is associated with an observable A with spectrum $\sigma(A)$, and of projection-valued measure

$$E : \Sigma(\sigma(A)) \to \mathrm{Proj}(\mathcal{H}) \tag{5.67}$$

$$\Delta \mapsto E(\Delta) \tag{5.68}$$

The Born rules is that the probability of the measurement lying in some measurable subset of the spectrum Δ is

$$P(X \in \Delta|\psi) = \langle \psi, E(\Delta)\psi \rangle \tag{5.69}$$

After said measurement the system will collapse to the state $E(\Delta)\psi$. Our logic is that a system is indeed such that $X \in \Delta$ if it was last measured to be so. The creation of a *context* from there is to consider the set of all measurements composed from commutative operators so that they can be said to be both true at the same time in a manner consistent with classical logic. If we have another measurement derived from an observable A' with a projection-valued measure E', the two PVM commute, in the sense that for any two measurable subsets of their spectra, $\Delta \subset \sigma(A), \Delta' \subset \sigma(A')$, we have

$$E(\Delta)E(\Delta') = E(\Delta')E(\Delta) \tag{5.70}$$

Meaning that if we have done a first measure $E(\Delta)$ (meaning $x \in \Delta$), and a second measure $E'(\Delta')$ ($x' \in \Delta'$), a third measure of the original quantity will yield the same result :

First measurement : Collapse

$$\psi \to \frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \tag{5.71}$$

Second measurement : Collapse $E(\Delta)\psi$ to $E'(\Delta')E(\Delta)\psi$

$$\frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \to \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|} \tag{5.72}$$

Third measurement :

$$\begin{aligned} P(X \in \Delta|\frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}) &= \langle \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}, E(\Delta)\frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}\rangle \\ &= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2}\langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)E(\Delta)\psi\rangle \\ &= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2}\langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)\psi\rangle \\ &= 1 \end{aligned} \tag{5.73}$$

Therefore in a context, we can say that the measured values are "real" in that they do not depend on the measurement.

As the identity and the zero projector both commute with every operator, they are a part of every context.

We will furthermore need the notion of ordering of projectors, which corresponds to the ordering of the lattice in quantum logic, ie we say that two projectors P_1, P_2 are ordered if

$$P_1 \leq P_2 \leftrightarrow \mathrm{im}(P_1) \subseteq \mathrm{im}(P_2) \tag{5.74}$$

or equivalently, $P_1 P_2 = P_2 P_1 P = P_1$. This means

Example : Given a projection-valued measure P and a measurable set of its spectrum Δ, with some subset $\Delta' \subseteq \Delta$, by the rules

$$E(\Delta') = E(\Delta' \cap \Delta) = E(\Delta')E(\Delta) \tag{5.75}$$

We therefore have $E(\Delta') \leq E(\Delta)$.

In terms of interpretation, this means that for $P \leq P'$, P' is *weaker* : we only know that our state is in some subspace larger than for P. This can be seen in the case of projection-valued measures on some interval, where the weaker statement is $x \in [a - \varepsilon_1, b + \varepsilon_2]$ compared to the more precise statement $x \in [a, b]$. The best one could find is in fact the 1-dimensional projector, as no projector is smaller than that (except for the zero projector which cannot provide any information), and corresponds to the measurement of the exact state. Due to this, two 1-dimensional projectors are never ordered, unless they are the same

$$\forall P, P',\ \dim(\mathrm{im}(P)) = \dim(\mathrm{im}(P')) = 1 \to (P \leq P' \leftrightarrow P = P') \tag{5.76}$$

Properties :

$$\forall P,\ 0 \leq P \tag{5.77}$$

$$\forall P,\ P \leq \mathrm{Id} \tag{5.78}$$

The point of daseinisation is to consider measurements in general not as projectors in the category of Hilbert spaces, but spread onto all possible contexts that a system may have by considering the closest approximation of that measurement in a given context. This approximation is given by the narrowest projection that is superior to our projector, ie for all the projectors P' in the context, we wish to find the one such that $P \leq P'$, and for any other projector P'' which also obeys $P \leq P''$, $P' \leq P''$. This projector is denoted by, for a context V, $\delta(P)_V$, the V-support of P. In terms of lattice notation, this is given by

$$\delta(P)_V = \bigwedge \{P' \in \mathrm{proj}(V) \mid P \leq P'\} \tag{5.79}$$

As Id is always part of every context and the supremum of any context, we are always guaranteed to have such a projector more precise or equal to the identity projector, which merely informs us that the state is in the Hilbert space at all and nothing more. If $P \in \mathrm{proj}(V)$, $\delta(P)_V = P$.

Example of a subset again

We will need to consider the approximation of $E(\Delta)$ in every possible contexts

$$P \to \{\delta(E(\Delta))_V | V \in \mathcal{V}(\mathcal{H})\} \tag{5.80}$$

Why is this a sheaf? Contexts are ordered

von Neumann algebras

To formalize this idea, we will need to use the notion of von Neumann algebra. While we could merely use C^* -algebras, there will be a difficulty if we do so : the projectors of C^*-algebras do not form a complete lattice, ie there may be subsets $S \subseteq \mathrm{proj}(A)$ which lack a lower or upper bound.

An example for this would be the algebra of compact operators $K(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$

Every projection in $K(\mathcal{H})$ has finite rank :

$$\dim(\mathrm{im}(P)) < \infty \tag{5.81}$$

If we consider an infinite dimensional Hilbert space, like $L^2(\mathbb{R})$, consider this subset : $\{P_i\}_{i\in\mathbb{N}}$, such that P_i maps to an i-dimensional subspace, and we have

$$\mathrm{im}(P_i) \subset \mathrm{im}(P_{i+1}) \tag{5.82}$$

Union of these is dense in $\mathcal{H}$?

$$\overline{\bigcup_{i\in\mathbb{N}} \mathrm{im}(P_i)} = \mathcal{H} \tag{5.83}$$

Supremum : $\sup(\{P_i\}) = I$, but I is not a compact operator.

[Why it happens? Relation to topology]

This possible lack of supremum and infimum would lead to the absence of disjunctions and conjunctions in our category [cf logic chapter]. While not tragic (this would only affect infinite conjunctions of propositions), we will try to keep things complete.

To insure the completeness of the lattice, we will use instead von Neumann algebras

Definition 278 *A* von Neumann algebra *A is a C^*-algebra with a predual A_*, a Banach space dual to A.*

Weak operator topology : The basis of neighbourgoods of 0 given by sets of the form

$$U(x, f) = \{A \in L(V, W) | f(A(x)) < 1\} \tag{5.84}$$

for $x \in V$, $f \in W^* = \mathrm{Hom}_{\mathrm{TVS}}(W, k)$. A sequence of operators (A_n) converges to A iff $(A_n(x))$ in the weak topology on W.

von Neumann algebras are closed in weak operator topology : any limit of net converges.

[...]

Definition 279 *An Abelian von Neumann algebra*

Definition 280 *For any Abelian von Neumann algebra over $\mathcal{B}(\mathcal{H})$, there exists a self-adjoint operator generating it as [...]*

5.6.4 The Bohr topos

A broader notion of categorical quantum theory is given by the *Bohr topos*

(in the usual category of compact symmetric monoidal objects etc of quantum logic) is transformed to a (clopen) sub-object $\delta(P)$ of the spectral presheaf in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$

Kochen-Specker theorem : equivalent to the presheaf on the category of self-adjoint operator has no global element

Take a C^*-algebra (von Neumann?) A.

Subcategory of commutative subalgebras $\mathrm{ComSub}(A)$ is the poset wrt inclusion maps

for any operator (self-adjoint?) A, let W_A be the spectral algebra. W_A is the boolean algebra of projectors $E(A \in \Delta)$ that projects onto the eigenspaces associated with the Borel subset Δ of the spectrum $\sigma(A)$. $E[A \in \Delta]$ represents the proposition $A \in \Delta$

Spectral theorem : for all Borel subsets J of the spectrum of $f(A)$, the spectral projector $E[f(A) \in J]$ for $f(A)$ is equal to the spectral projector $E[A \in f^{-1}(J)]$ for A. In particular, if $f(\Delta)$ is a Borel subset of $\sigma(f(A))$, since $\Delta \subseteq f^{-1}(f(\Delta))$,

$$E[A \in \Delta] \leq E[A \in f^{-1}(f(\Delta))] \tag{5.85}$$

$$E[A \in \Delta] \leq E[f(A) \in f(\Delta)] \tag{5.86}$$

This means $f(A) \in f(\Delta)$ is weaker than $A \in \Delta$. $f(A) \in f(\Delta)$ is a *coarse graining* of $A \in \Delta$.

If $A \in \Delta$ has no truth value defined, $f(A) \in f(\Delta)$ may have for some f

Relations between two logical systems here :

First, any proposition corresponding to the zero element of the Heyting algebra should be valued as false, $\nu(0_L) = 0_{T(L)}$.

If $\alpha, \beta \in L$, $\alpha \leq \beta$, then α implies β. Ex : $A \in \Delta_1$, $A \in \Delta_2$, $\Delta_1 \subseteq \Delta_2$. Valuation should be $\nu(\alpha) \leq \nu(\beta)$ (monotonicity).

If $\alpha \leq \alpha \vee \beta$, $\beta \leq \alpha \vee \beta$, then $\nu(\alpha) \leq \nu(\alpha \vee \beta)$ and $\nu(\beta) \leq \nu(\alpha \vee \beta)$, and therefore

$$\nu(\alpha) \vee \nu(\beta) \leq \nu(\alpha \vee \beta) \tag{5.87}$$

Not as strong as $\nu(\alpha) \vee \nu(\beta) = \nu(\alpha \vee \beta)$. For instance for $A = a_1$, $A = a_2$, the projection operator for both of these proposition projects on the 2D span of the eigenvectors, not their union.

Similarly,

$$\nu(\alpha \wedge \beta) \leq \nu(\alpha) \wedge \nu(\beta) \tag{5.88}$$

Exclusivity : a condition and its complementation cannot both be totally true :

$$\alpha \wedge \beta = 0_L \wedge \nu(\alpha) = 1_{T(L)} \to \nu(\beta) \leq 1_{T(L)} \tag{5.89}$$

Unity condition : $\nu(1_L) = 1_{T(L)}$

Take the boolean subalgebra W of the lattice $P(H)$ of projection operators. Forms a poset under subalgebra inclusion. W is a poset category.

Take the set $\mathcal{O}$ of all bounded, self-adjoint operators on $\mathcal{H}$. Spectral representation :

$$A = \int_{\sigma(A)} \lambda dE^A_\lambda \tag{5.90}$$

$\sigma(A) \subseteq \mathbb{R}$ the spectrum of A, $\{E^A_\lambda | \lambda \in \sigma(A)\}$ a spectral family of A.

$$E[A \in \Delta] = \int_\Delta dE^A_\lambda \tag{5.91}$$

for Δ a borel subset of $\sigma(A)$. If a belongs to the discrete spectrum of A, the projector ontop the eigenspace with eigenvalue a is

$$E[A = a] := E[A \in \{a\}] \tag{5.92}$$

for $f : \mathbb{R} \to \mathbb{R}$ any bounded Borel function,

$$f(A) = \int_{\sigma(A)} f(\lambda) dE_\lambda^A \tag{5.93}$$

Categorification of $\mathcal{O}$: Objects are elements of $\mathcal{O}$, morphisms from B top A if an equivalence class of Borel functions $f : \sigma(A) \to \mathbb{R}$ exists such that $B = f(A)$, ie

$$B = \int_{\sigma(A)} f(\lambda) dE_\lambda^A \tag{5.94}$$

Definition 281 *The spectral algebra functor $W : \mathcal{O} \to W$ is*

- *Objects mapped $W(A) = W_A$, W_A is the spectral algebra of A*
- *Morphisms : if $f : B \to A$, then $W(f) : W_B \to W_A$ is the subset inclusion of algebras $i_{W_B W_A} : W_B \to W_A$.*

Spectral algebra for $B = f(A)$ is naturally embedded in the spectral algebra for A since $E[f(A) \in J] = E[A \in f^{-1}(J)]$ for all Borel subsets $J \subseteq \sigma(B)$

$$i_{W_{f(A)} W}(E[f(A) \in J]) = E[A \in f^{-1}(J)] \tag{5.95}$$

[...]

Category $\mathcal{O}_d$ of discrete spectra self-adjoint operators

Definition 282 *The spectral presheaf on $\mathcal{O}_d$ is the contravariant functor $\Sigma : \mathcal{O}_d \to$* **Set**

- *$\Sigma(A) = \sigma(A)$ (spectrum of A)*
- *if $f_{\mathcal{O}_d} : B \to A$, so that $B = f(A)$, then $\Sigma(f_{O_d}) : \sigma(A) \to \sigma(B)$ is defined by $\Sigma(f_{\mathcal{O}_d})(\lambda) = f(\lambda)$ for all $\lambda \in \sigma(A)$*

Works because on discrete spectrum $\sigma(f(A)) = f(\sigma(A))$.

$$\Sigma(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d}) = \Sigma(f_{\mathcal{O}_d}) \circ \Sigma(g_{\mathcal{O}_d}) \tag{5.96}$$

global section : function γ that assigns for every object of the site an element γ_A of the topos, such that if $f : B \to A$, then $H(f)(\gamma_A) = \gamma_b$.

For the spectral functor, a global section / element is a function that assigns to each self-adjoint operator A with a discrete spectrum a real number $\gamma_A \in \sigma(A)$, such that if $B = f(A)$, then $f(\gamma_A) = \gamma_B$.

Kochen-Specker theorem : if $\mathrm{Dim}(\mathcal{H}) > 2$, there are no global sections of the spectral presheaf.

Continuous case

By Gelfand duality, the presheaf topos PSh(ComSub(A)) contains a canonical object, the presheaf

$$\Sigma : C \mapsto \Sigma_C \tag{5.97}$$

which maps a commutative C^*-algebra $C \hookrightarrow A$ to (the point set underlying) its Gelfand spectrum Σ_C.

[68]

projection

5.6.5 The finite dimensional case

Take the case $\mathbb{C}^n$ of the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$, which is for instance used in quantum computing. The C^*-algebra is just

$$C^*(\mathcal{H}) = L(\mathcal{H}, \mathcal{H}) \tag{5.98}$$

(denoted $L(\mathcal{H})$ for short), with operator composition as its algebraic operation and complex conjugate as involution, as all finite-dimensional linear maps are bounded. If we pick a specific basis, this is the algebra of $n \times n$ matrices on $\mathbb{C}^n$ with matrix multiplication.

The algebra $L(\mathcal{H})$ is also a von Neumann algebra [proof]

any subalgebra is a von Neumann subalgebra

The projections of this algebra are formed by the orthogonal projections, as any oblique projection would not be self-adjoint, classified by the Grassmannians of the space,

$$\bigoplus_{i=0}^{n} \mathrm{Grass}(i, \mathcal{H}) \tag{5.99}$$

$$P = I_r \oplus 0_{d-r} \tag{5.100}$$

An operator A will simply be one of the linear map $A \in L(\mathcal{H})$

A context here is an Abelian von Neumann subalgebra of $L(\mathcal{H})$. The category of contexts $\mathcal{V}(L(\mathcal{H}))$, equivalently a set of commuting matrices

"An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator."

Theorem 283 *Any abelian von Neumann algebra on a separable Hilbert space is $*$-isomorphic to either*

- $\ell^\infty(\{1, 2, \ldots, n\})$
- $\ell^\infty(\mathbb{N})$
- $L^\infty([0, 1])$
- $L^\infty([0, 1] \cup \{1, 2, \ldots, n\})$
- $L^\infty([0, 1] \cup \mathbb{N})$

There is therefore some surjection from the self-adjoint operators to commutative von Neumann algebras :

$$f : \mathcal{B}_{\mathrm{sa}}(\mathcal{H}) \to \mathcal{V}(W^*(\mathcal{H})) \tag{5.101}$$

Spectral theorem :

Theorem 284 *For a bounded self-adjoint operator, there is a measure space (X, Σ, μ) and a real-valued essentially bounded measurable function f on X and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that*

$$U^\dagger T U = A \tag{5.102}$$

$$[T\varphi](x) = f(x)\varphi(x) \tag{5.103}$$

and $\|T\| = \|f\|_\infty$

Finite dimensional :

Theorem 285 *There exists eigenvalues $\{\lambda_i\}$ (ordered by value by i) of A and eigen subspaces $V_i = \{\psi \in \mathcal{H} | A\psi = \lambda_i \psi\}$ such that*

$$\mathcal{H} = \bigoplus_{i=1}^{n} V_j \tag{5.104}$$

Theorem 286 *For self-adjoint A, there exists an orthonormal basis of eigenvectors of A.*

Theorem 287 *For a self-adjoint operator A with respect to an orthogonal matrix, there exists an orthogonal matrix T such that $T^{-1}AT$ is diagonal.*

Theorem 288 *For a self-adjoint operator A, there exists different eigenvalues $\{\lambda_i\}$, $i \leq j \to \lambda_i \leq \lambda_j$, and eigen subspaces,*

$$W_i = \{\psi \in \mathcal{H} \mid A\psi = \lambda_i \psi\} \tag{5.105}$$

Let P_i be the orthogonal projection of $\mathcal{H}$ onto W_i, then

- *$\mathcal{H}$ is an orthogonal direct sum of W_i : $\mathcal{H} = \bigoplus_{i=1}^n W_i$, and $W_i \perp W_j$ for $i \neq j$*
- *$P_i P_j = \delta_{ij} P_i$ and $\mathrm{Id}_{\mathcal{H}} = \sum_i P_i$*
- *$A = \sum_i \lambda_i P_i$*

Theorem 289 *For a normal operator A (ie, commutes with its adjoint), there is a spectral resolution of A.*

Spectrum in finite dimension :

$$\sigma(A) = \tag{5.106}$$

For our observable A,

$$A = \sum_i \lambda_i^m P_i \tag{5.107}$$

The commutative algebra generated is that which is spanned by those projective operators, ie

$$\forall B \in \mathrm{ComSub}(A),\ \exists \{c_i\} \in \mathbb{C}^k,\ B = \sum_i^m c_i P_i \tag{5.108}$$

All those operators are commutative, simply by the commutativity of the projectors between themselves.

Example of two operators with the same commutative subalgebra : any two operators with the same projectors but different eigenvalues

Alternatively : define them entirely by sets of projectors (up to a scale?), ie some subset of commutative projector (between 0 and n)

The Gelfand spectrum of a von Neumann algebra is the unique measurable space we define

"The predual of the von Neumann algebra B(H) of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm ——A——= Tr(—A—). The Banach space of trace class operators is itself the dual of the C*-algebra of compact operators (which is not a von Neumann algebra)."

Self-duality in finite dimension due to every operator being trace-class

Spectral measure [69] (1) :

For (X, Ω) a Borel space, a spectral measure is

$$\Phi : \Omega \to \mathcal{B}(\mathcal{H}) \tag{5.109}$$

- $\Phi(U)$ is an orthogonal projection for all U, $\Phi(U)^2 = \Phi(U) = \Phi(U)^*$
- $\Phi(\varnothing) = 0$ and $\Phi(X) = \mathrm{Id}$
- $\Phi(U \cap V) = \Phi(U)\Phi(V)$
- For a sequence (U_i) of pairwise disjoint Borel subsets,

$$\Phi(\bigcup_i U_i) = \sum_i \Phi(U_i)$$

(convergence wrt strong operator topology)

[...]

Spectral measure for finite dimensional case : Given the Abelian von Neumann algebra generated by

$$A = \sum_i \lambda_i P_i \tag{5.110}$$

with functions

$$f(A) = \sum f(\lambda_i) P_i \tag{5.111}$$

$$\begin{aligned}
\int f d\mu_\psi &= \langle \psi, f(A)\psi \rangle && (5.112)\\
&= \sum_i f(\lambda_i)\langle \psi, P_i \psi \rangle && (5.113)\\
&= \sum_i f(\lambda_i)\|P_i\psi\|^2 && (5.114)
\end{aligned}$$

measure is the counting measure

$$\mu_\psi = \sum_i \|P_i\psi\|\delta_{\lambda_i} \tag{5.115}$$

Gelfand dual :

- The space is the discrete space $\sigma(A)$
- The sigma-algebra is the discrete sigma algebra given by $\mathcal{P}(\sigma(A))$
- The measure is the counting measure

The spectral presheaf is then the presheaf

$$\underline{\Sigma} : \mathcal{V}(\mathrm{VNA}(\mathcal{H}))^{\mathrm{op}} \to \mathbf{Set} \tag{5.116}$$

which maps

Decomposition of operators : Given a set of n 1-dimensional orthogonal projectors, $\{P_i\}$, $P_i P_j = 0$,

The two-dimensional case

The simplest case we can use is the one-dimensional case, $\mathbb{C}$, but having only a single state in its projective Hilbert space, is a bit too trivial, so let's look at $\mathbb{C}^2$.

To classify its orthogonal projectors, let's look at the Grassmannians of various dimensions for $\mathbb{C}^2$:

- $\mathrm{Gr}_0(\mathbb{C}^2) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^2) \cong \mathbb{C}P^1$
- $\mathrm{Gr}_2(\mathbb{C}^2) = \{\mathbb{C}^2\}$

The zero and two dimensional cases are simple enough, the zero-dimensional projection operator being the zero operator 0, with Abelian von Neumann algebra the trivial algebra $\{0\}$, and the two-dimensional projection operator is the identity map $\mathrm{Id}_{\mathbb{C}^2}$, with Abelian von Neumann algebra the scaling matrices, $c\mathrm{Id}_{\mathbb{C}^2}$

The one-dimensional case will contain most of the cases of interest. for some point $p \in \mathbb{C}P^1$, ie a point on the Riemann sphere $p \in S^2$, $p = (\theta, \phi)$, there is a projector to that line in the complex plane.

Given any self-adjoint operator $\mathcal{B}_{\mathrm{sa}}(\mathbb{C}^2)$, the finite-dimensional spectral theorem tells us that the Hilbert space can be decomposed into orthogonal subspaces $\{W_i\}$ which each contain one or more of the eigenvectors of the operator. As there can only be as many orthogonal spaces as the sum of their dimension being inferior or equal to the total dimension, this will only allow the trivial case (Just the 0-dimensional subspace), a single 1-dimensional subspace, two 1-dimensional subspace, or a single 2-dimensional subspace. The first case is simply the projector 0, corresponding only to the 0 operator. The second case is, for the choice of a point (θ, ϕ) on the Riemann sphere,

$$A = \lambda P_{(\theta,\phi)} \tag{5.117}$$

The third case is

$$A = \lambda_1 P_{(\theta_1,\phi_1)} + \lambda_2 P_{(\theta_1,\phi_2)} \tag{5.118}$$

And the last case is a diagonal operator,

$$A = \lambda \mathrm{Id}_{\mathbb{C}^2} \tag{5.119}$$

The Abelian von Neumann algebras are therefore classified by those two points on the Riemann sphere,

$$((\theta_1, \phi_1), (\theta_2, \phi_2)) \to \mathrm{VNA}(\mathbb{C}^2) \tag{5.120}$$

The category of contexts is therefore such that

- The trivial von Neumann algebra is included in all algebras
- The scaling von Neumann algebra is not included in any other algebra?
- The von Neumann algebra constructed from a single one dimensional projection $P_{(\theta,\phi)}$ is included in any von Neumann algebra constructed from two one-dimensional projections, as long as they share that projection.

Diagram of the category

$\mathrm{VNA}(P_{(\theta,\phi)}) \leq \mathrm{VNA}(P_{(\theta,\phi)}, P_{(\theta',\phi')})$

Approximation of a projection : For any projection P, there is only two possible cases :

- The projection is 0, and the V-

Kochen-Specker : $\mathbb{C}^2$ is not concerned by this.

The three-dimensional case

To have a case that is actually covered by the big quantum theorems properly, we will have to consider the case of the Hilbert space $\mathbb{C}^3$. This is for instance the case given by massive spin 1 particles.

The classification of projectors is much the same as previously, thanks to the duality of Grassmannians,

- $\mathrm{Gr}_0(\mathbb{C}^3) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_2(\mathbb{C}^3) = \mathrm{Gr}_{3-2}(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_3(\mathbb{C}^3) = \{\mathbb{C}^3\}$

The orthogonal subspaces of an operator will be

1. The empty subspace 0
2. One 1-dimensional subspace
3. Two 1-dimensional subspace
4. Three 1-dimensional subspace
5. One 2-dimensional subspace
6. One 1-dimensional subspace and one 2-dimensional subspace
7. One 3-dimensional subspace

As before, the first and last case are trivial, consisting of the trivial subspace and the whole subspace.

5.6.6 The infinite-dimensional case

For a Bohr topos with a more interesting structure, such as a differential cohesive structure, let's consider instead a simple infinite dimensional case, of the theory of a quantum particle on the unit interval,

$$\mathcal{H} = L^2([0,1], \ell) \tag{5.121}$$

with the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_0^1 \psi_1^\dagger \psi_2 \mu_\ell \tag{5.122}$$

where two functions ψ, ψ' are identified if they have the same inner product with all other functions, ie up to differences on a set of measure zero.

This is the Hilbert space used for the particle in a box problem. The boundedness of the underlying space allows us to freely use the position operator

$$\hat{x}\psi(x) = x\psi(x) \tag{5.123}$$

as it is a bounded operator in this case, using the inequality $x\|\psi(x)\| \leq \|\psi(x)\|$ for all $x \in [0,1]$

$$\|\hat{x}\| \leq 1 \tag{5.124}$$

proof self-adjoint

1-dimensional classification : every projector is part of the set of all 1-dimensional subspaces of $\mathcal{H}$, is it the projective limit $\mathcal{C}P^\infty$? The Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, classifier of U(1) bundles

Kuiper's theorem?

Due to its much more complex nature, the full classification of projection operators, and therefore contexts, is not gonna be attempted here, so that only a few representative examples will be look at here.

Examples of operators [projectors?] with continuous spectrum

As a continuous operator, $\hat{x}$ does not have an eigenbasis (outside of the more general case of the Gelf'and triple rigged Hilbert space), but we can instead compute its projection-valued measure.

One hierarchy of projections we can do for this is to consider the projections for the localization of the particle (we are assuming the non-relativistic case here where the particle *can* be localized). If the particle ψ is entirely localized in some interval $[a, b]$, that is,

$$\int \psi^\dagger(x)x\psi(x)dx = 1 \tag{5.125}$$

then we will denote this by the projector operator $P_{[a,b]}$. This selects wavefunctions of support $[a, b]$ (almost everywhere)

Those projectors obey the following properties :

$$P_{[a,a]} = 0 \tag{5.126}$$

$$[a, b] \subset [a', b'] \,,\ P_{[a,b]} \tag{5.127}$$

Heyting algebra

If we consider our position operator $\hat{x}$ as the generator of an Abelian von Neumann algebra,

Chapter 6

Logic

One element of interest of topoi as a good foundation for math is that there is a connection between topos and logical theories. Given some category $\mathbf{C}$, there exists some equivalence with a type theory $L(\mathbf{C})$, which, given some appropriate constraints on the category, can be interpreted as a logic on this category.

To look at this mapping, first let's look at how logics, in the broad sense, are composed.

Definition 290 *In a logical theory, a (first-order) signature is composed of*

- *A set* Σ_0 *of* sorts
- *A set* Σ_1 *of* function

Example 291 *For classical logic, its signature is given by*

Definition 292 *For a category* $\mathbf{C}$ *with finite products, and a signature* Σ*, a* Σ-structure M *in* $\mathbf{C}$ *defines :*

- *A function between sorts in* Σ_0 *and objects in* $\mathbf{C}$
- *A function between functions in* Σ_1 *and morphisms in* $\mathbf{C}$
- *A function between relations and subobjects*

[internal v. external logic]

Support object, h-propositions

$$\text{isProp}(A) = \prod_{x:A}\prod_{y:A}(x = y) \tag{6.1}$$

difference between h-propositions and propositions as types?

[70]

Logic from types, logic from topos, Heyting algebra [71] The subobjects of objects X in a topos H form a Heyting algebra, with operations $\cap, \cup, \rightarrow$ the partial ordering $\subseteq$ and the greatest and smallest elements 1_A, 0_A.

The language $L(H)$ of a topos is a many-sorted first-order language having the objects $X \in H$ as types for the terms of $L(H)$, there is a type operator τ which assigns to any term of $L(H)$ an object $\tau(p)$ of H called the type of p.

- 0_H is a constant term of type 1.
- For any object A of E, there is a countable number of variables of type A
- For any map $f : A \rightarrow B$, there is an "evaluation operator" $f(-)$ for terms of type A to terms of type B : p of type $A \Rightarrow f(p)$ of type B
- For any ordered pair (A, B) of H, there is an ordered pair operator $\langle -, - \rangle$
- For any subobject $M : A \rightarrow \Omega$, there is a unary "membership-predicate $(-) \in M$ for elements of A. $x \in M$ is an atomic formula provided $x \in A$.
- The propositional connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$ are allowed for new formulas
- For any object A and variable $x \in A$, the quantifier $\exists x \in A$ and $\forall x \in A$ are allowed

Definition 293 *Two objects $x, y \in A$ are equal if*

$$x = y \leftrightarrow \langle x, y \rangle \in \Delta_A \tag{6.2}$$

$\Delta_A : A \times A \rightarrow \Omega$ *the diagonal operator*

Unique equality :

$$(\exists! x \in A)\phi(x) \leftrightarrow \exists x \in A, \forall y \in A, (\phi(y) \leftrightarrow x = y) \tag{6.3}$$

Membership :

$$x \in y \leftrightarrow \langle y, x \rangle \in (\mathrm{ev} : PA \times A \rightarrow \Omega) \tag{6.4}$$

For $x \in A$ and $F \in B^A$,

$$F(x) = (\mathrm{ev} : B^A \times A \rightarrow B)\langle F, x \rangle \tag{6.5}$$

For any map $f : A \rightarrow B$ with exponential adjoint $\overline{f} : 1 \rightarrow B^A$, we define an element $f_e = \overline{f}(0_e) \in B^A$ which represents f internally.

Example 294 *Some of the most barebones "logics" one can have for a category are the initial category* $\mathbf{0}$ *and the terminal category* $\mathbf{1}$*. The initial category has no types and therefore no terms or propositions, corresponding to the empty logic. The terminal category has a single type (the unit type), and only one term given by the identity, and a single proposition* $* \to *$*, which corresponds to the truth statement. As all the limits in the terminal category are its single object, this transfers to its category of subobjects, meaning that every logical construction there is also* $1 \to 1$*, so that this is the trivial logic, for which any proposition is true (this can be seen as stemming from the initial and terminal object being the same here, and therefore not having a notion of falsehood different from truth).*

Example 295 *A standard example of an internal logic is given by a group category, in the sense of a groupoid of one element with endomorphisms isomorphic to* G*. Having only one object, there is only one associated type, the group type* G*, and only one subobject relation, which is* $\mathrm{Id}_G : G \hookrightarrow G$*.*

6.1 The internal logic of Set

The internal language $L(\mathbf{Set})$ roughly corresponds to classical logic (as applied to sets). Many-sorted first order language having the objects of the topos as types

Boolean algebra of subsets : for $X \in \mathbf{Set}$, we consider the boolean algebra of subobjects $\mathrm{Sub}(X)$ with the correspondences

Boolean algebra	Set operator
$a, b, c, \ldots$	$A, B, C \subseteq X$
$\wedge$	$\cap$
$\vee$	$\cup$
$\leq$	$\subseteq$
0	$\varnothing$
1	X
$\neg A$	$X \setminus A = A^c$
$A \to B$	$(X \setminus A) \cup B = A^c \cup B$

Table 6.1: Caption

$\to$: every element except the elements of A that aren't also elements of B.

Boolean algebra identities :

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{6.6}$$

$$A \cup B = B \cup A \tag{6.7}$$

$$A \tag{6.8}$$

[...]

A basic example of statement in our topos is given by the truth morphism $\top : 1 \to \Omega$, which trivially factors through itself,

$$1 \xrightarrow{\mathrm{Id}_1} 1 \xrightarrow{\top} \Omega \tag{6.9}$$

which corresponds to the trivial statement

$$\vdash \top \tag{6.10}$$

Simply stating that truth is always internally valid. Conversely, the falsity morphism $\perp : 1 \to \Omega$, representing falsehood, should not have any such factoring, ie

$$1 \xrightarrow{f} 1 \xrightarrow{\top} \Omega \tag{6.11}$$

such that $\top \circ f = \perp$. As there is only one endomorphism on 1, this can only be the identity map, and therefore is only true if $\top = \perp$. This would however mean that, as $\top$ is the classifying map of 1 and $\perp$ that of 0 as subobjects of 1, by pullback this would mean that both 0 and 1 are the same object, which is not the case in **Set**.

Therefore, we can write that falsity is indeed false,

$$\nvdash \perp \tag{6.12}$$

but on the other hand, the negation of falsity, $\neg\perp$, is true, as

[...]

More generally, we can derive this

Theorem 296 *If the negation of a morphism is true, ie*

$$A \xrightarrow{f} \Omega \xrightarrow{\neg} \Omega \tag{6.13}$$

factors through 1 *:*

$$A \xrightarrow{f'} 1 \xrightarrow{\top} \Omega \tag{6.14}$$

such that $\neg \circ f = \top \circ f'$, *then this morphism is false, ie there is no such factoring of* f *through* $\top$.

Proof 47

Axioms :

Negation : for $p : A \hookrightarrow X$, ie

$$A \overset{p}{\hookrightarrow} X \xrightarrow{\chi_A} \Omega \tag{6.15}$$

The negation of the set $\neg A$ is such that $\neg p : \neg A \hookrightarrow X$ factors through the negation and p,

$$\chi_{\neg A} = \chi_A \circ \neg \tag{6.16}$$

Statement with context : $p, p \to q \dashv q$: slice category $\mathbf{Set}_{/(p:A\to X)\times[]}$.

Localization modality : $\bigcirc_j$ for $j = \mathrm{Id}_\Omega$:

[...]

An example of internal logic we can derive in **Set** is by looking at the natural number object **Set** as an internal ring. If we just look at $\mathbb{N}$ and $\{\bullet\}$, we have the integer type $\mathbb{N}$ and the unit type 1, so that

The successor function $s : \mathbb{N} \to \mathbb{N}$ is an integer term with a free integer variable, ie we have

$$n : \mathbb{N} \dashv s(n) : \mathbb{N} \tag{6.17}$$

We also have the subobjects

Proof of induction?

6.2 The internal logic of a spatial topos

Logic of $\mathrm{Sh}(\mathbf{X})$, $\mathrm{Sh}(\mathbf{CartSp_{Smooth}})$

Locality modality j

6.3 The internal logic of smooth spaces

As a logical system, the topos of smooth sets has as types the various smooth sets

Proposition : subspaces $S \hookrightarrow X$

Subobject classifier : Ω is the sheaf associating to any $U \subseteq \mathbb{R}^n$ such that

$$\Omega(U) = \{S | S \text{ is a } J\text{-closed sieve on } U\} \tag{6.18}$$

$$\Omega(f) = f^* \tag{6.19}$$

$\top : 1 \to \Omega :$ maximal sieve on each object

Points in $\Omega :$ all the maps $1 \to \Omega$, all the J-closed sieves on $\mathbf{R}^0$.

6.4 The internal logic of classical mechanics

Internal logic for Poisson manifolds

Symmetric monoidal category with projection

A particular Poisson structure we can give is the trivial Poisson structure,

$$\{f, g\} = 0 \tag{6.20}$$

If we consider the map $\mathbb{R} \to P$ corresponding to this trivial structure on the real line, we will get our local example of a real line object. Statements about the measurements of a classical system can therefore be understood as morphisms from

Are the measurement given by $\mathbb{R}$ with the trivial Poisson structure

types given by morphisms to R?

For many practical circumstances, we will want to know more specifically propositions about values. If we have a phase space Phase and some point in that phase space $(x, p) : 1 \to \mathrm{Phase}$

6.5 The internal logic of quantum mechanics

There are three possible internal logics that we can consider for quantum mechanics here. If we consider it as a symmetric monoidal category, this is a form of *linear logic*. If we consider a given Hilbert space $\mathcal{H}$, the logic of the slice category $\mathbf{Hilb}_{\mathcal{H}}$ is *quantum logic*. And finally, we will look at the internal logic of the Bohr topos that we have constructed.

6.5.1 Linear logic

The category of Hilbert spaces and linear logic are not quite like the other ones that we have looked into so far, not forming a topos.

As we do not have a subobject classifier here, we will not be able to use a Mitchell-Benabou language. But we can still perform that translation using the basic translation as a type theory.

If we pick a given Hilbert space $\mathcal{H}$ as a reference, propositions are given by monomorphisms $\iota : W \hookrightarrow \mathcal{H}$ (up to isomorphisms). In the category of Hilbert spaces, all monomorphisms are split, meaning that there exists a retraction P_W

$$W \overset{\iota_W}{\hookrightarrow} \mathcal{H} \xrightarrow{P_W} W \tag{6.21}$$

such that $P_W \circ \iota_W = \mathrm{Id}_W$. This is the notion we've seen before for the state of a system depending on a projection of the Hilbert space via some measurement operator.

The category of Hilbert spaces, like the category of vector spaces, has an initial and terminal object that are the same, the *zero object* 0, corresponding to the Hilbert space $\mathbb{C}^0$. Being the subobject of any Hilbert space, the unique map $0 : 0 \to \mathcal{H}$ has an interpretation as a proposition for any Hilbert space,

Interpretation of daggers logically

Chapter 7

Higher categories

To generalize categories, we can introduce the concept of *higher categories*, in which in addition to objects and morphisms, we also introduce the possibility of transformations between any two objects of the same type, called a k-morphism.

Definition 297 *A k-morphism is defined inductively as an arrow for which the source and target is a $(k-1)$-morphism, and an object is a 0-morphism.*

[Identity of indistinguishables The notion of Haecceity by Scott Dunn

"Because there is among beings something indivisible into subjective parts—that is, such that it is formally incompatible for it to be divided into several parts each of which is it—the question is not what it is by which such a division is formally incompatible with it (because it is formally incompatible by incompatibility), but rather what it is by which, as by a proximate and intrinsic foundation, this incompatibility is in it. Therefore, the sense of the questions on this topic [viz. of individuation] is: What is it in [e.g.] this stone, by which as by a proximate foundation it is absolutely incompatible with the stone for it to be divided into several parts each of which is this stone, the kind of division that is proper to a universal whole as divided into its subjective parts?"

]

Therefore the morphisms we saw so far are 1-morphisms (arrows between 0-morphisms or objects), 2-morphisms are morphisms between two 1-morphisms, and so forth. In diagrammatical terms, we will express higher order morphisms by arrows between arrows, such as the 2-morphism $\alpha : f \to g$

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

A basic example of this is that just as functors can be considered as a morphism in the category of categories **Cat**, so can natural transformations be considered as 2-morphisms in the 2-category **Cat**.

"The objects in the hom-category C(x,y) are the 1-morphisms in C from x to y, while the morphisms in the hom-category C(x,y) are the 2-morphisms of C that are horizontally between x and y."

Arrow category? Over category?

Example 298 *In the category* **Top***, homotopies between two continuous functions are* 2*-morphisms.*

From this we can define the notion of n-category to be categories with morphisms up to level n, with the appropriate rules regarding composition and identity.

Definition 299 *An* n*-category* $\mathbf{C}$ *is given by a sequence of* n *classes of* k*-morphisms* $(\mathrm{Mor}_k(\mathbf{C}))_{0\leq k\leq n}$*, such that for every* $k > 0$*, we have two functions*

$$s_k, t_k : \mathrm{Mor}_k(\mathbf{C}) \to \mathrm{Mor}_{k-1}(\mathbf{C}) \tag{7.1}$$

and a function of composition of k*-morphisms,*

$$\circ_k : \mathrm{Mor}_k(\mathbf{C}) \times \mathrm{Mor}_k(\mathbf{C}) \to \mathrm{Mor}_k(\mathbf{C}) \tag{7.2}$$

which have to obey the condition that and such that for every $k > 0$*, for every* $k-1$*-morphism* α*, there exists a* k*-morphism* Id_α *for which* $s_k(\mathrm{Id}_\alpha) = t_k(\mathrm{Id}_\alpha) = \alpha$

We can technically extend this definition to include cases for $k = 0$

The concept of an ∞-category is for the case where the set of classes of k-morphism is of infinite cardinality (typically countable).

We will also define more specifically the concept of (n, k)-categories :

Definition 300 *An* (n, k)*-category for* $n, k \in \mathbb{N} \cup \{\infty\}$ *is an* n*-category for which every* l*-morphism for* $k < l \leq n$ *possesses an inverse, ie for any* k*-morphism* $\alpha \in \mathrm{Mor}_k(\mathbf{C})$*,* $k > 1$*, then there exists another* k*-morphism* α^{-1} *which obeys*

$$\alpha \circ \alpha^{-1} = \mathrm{Id}_{t(\alpha)} \tag{7.3}$$

$$\alpha^{-1} \circ \alpha = \mathrm{Id}_{s(\alpha)} \tag{7.4}$$

Example 301 *A group* G *interpreted as a category* $\mathbf{G}$ *(*$\mathrm{Mor}(\mathbf{G}) \cong G$*) is a* $(1, 0)$*-category.*

Example 302 *The category of topological spaces with continuous maps as morphisms and homotopie equivalences between any two continuous map is a* $(2,1)$*-category.*

Gauge example?

"An (n,r)-category is an r-directed homotopy n-type." Ex : a $(0,0)$-category is isomorphic to a set (the set of all objects), a $(1,0)$-category is a groupoid, a $(1,1)$-category is a category

$(\infty,0)$: ∞-groupoid (∞,∞) :

Descent to negative degrees : $(-1,0)$-category : truth values $(-2,0)$-category : Point

n-truncation : a category is n-truncated if it is an n-groupoid

Truncation

Definition 303 *The n-truncation functor τ_n is an endofunctor on an ∞-category mapping all morphisms of order $k \geq n$ to the identity k-morphism.*

An n-truncated category therefore acts as the equivalent n-category. For the case of a 0-truncation, this is simply a basic category.

7.1 Groupoids

[72, 73] The prototypical ∞-category is $\infty\mathbf{Grpd}$, the ∞-category of ∞-groupoids, along with its various truncations, the k-groupoids,

$$k\mathbf{Grpd} \cong \tau_k \infty\mathbf{Grpd} \tag{7.5}$$

where in particular, the 0-groupoids, simply being given by 0-morphisms, are just sets [size concerns?], and the 1-groupoids are the usual notion of groupoids.

There are a number of different ways that we can consider the various levels of groupoids

First, any ∞-category with all invertible k-morphisms can be embedded in $\infty\mathbf{Grpd}$. Therefore any such diagram can be considered as such, which we will write either as their pure diagram

$$\mathcal{G} = \{\} \tag{7.6}$$

Or as the traditional groupoid notation $\mathcal{G} : G_1 \rightrightarrows G_0$, where G_0 is the set of objects, G_1 the set of morphisms, and s, t are the source and target.

Delooping

7.1.1 Truncation

7.2 Homotopy category

There exists a weaker notion than that of a higher category which is that of a category with weak equivalences, where rather than any higher morphisms, we define classes of morphisms which are weak equivalences, a special class of morphisms which are not strictly speaking isomorphisms (they do not admit an inverse such that $f \circ f^{-1} = \mathrm{Id}$), but we consider them to have a weak inverse which we will consider to be something akin to an equivalence.

Definition 304 *A category* $\mathbf{C}$ *with weak equivalences is given by some subcategory* $\mathbf{W} \hookrightarrow \mathbf{C}$ *which contains all isomorphisms and which obeys the two out of three property : for any two composable morphisms* f, g *in* $\mathbf{C}$*, if two out of* $\{f, g, g \circ f\}$ *are in* $\mathbf{W}$*, then so is the third.*

In particular, if we have composable morphisms of the types $f : X \to Y$ and $g : Y \to X$ in $\mathbf{W}$, then we can say that X and Y are weakly equivalent

[...]

In terms of n-categories, weak equivalences are meant to represent the notion of 2-morphisms to the identity. That is

[...]

Definition 305 *A category* $\mathbf{C}$ *is* homotopical *if in addition to being a category with weak equivalences, the subcategory* $\mathbf{W}$ *also obeys the two out of six property : for any sequence of composable morphisms,*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \tag{7.7}$$

if $h \circ g$ *and* $g \circ f$ *are in* $\mathbf{W}$*, then so are* f, g, h *and* $h \circ g \circ f$*.*

The homotopy category $\mathrm{Ho}(\mathbf{C}, \mathbf{W})$ of a category with weak equivalences $(\mathbf{C}, \mathbf{W})$ is the localization of that category along the weak equivalences, ie for any morphism $f \in \mathbf{W}$, we adjoin an actual inverse f^{-1} to transform the weak equivalence into an equivalence.

Example 306 *For the category of topological spaces* **Top***, we take the subcategory of weak equivalences to be that of continuous maps which are bijective on the path components, ie for* $f : X \to Y$*,* $f \in \mathbf{W}$ *if*

$$f_* : \pi_0(X) \to \pi_0(Y) \tag{7.8}$$

[Higher homotopy groups?]

In addition to weak equivalences to reflect the notion of homotopy equivalence, we can also look at two additional notions from homotopy theory, which are fibrations and cofibrations.

As a reminder, those notions in topology are

Definition 307 *A fibration is a continuous map $p : E \to B$ such that every space X satisfies the homotopy lifting property.*

Definition 308 *A cofibration is a continuous map $i : A \to X$ such that for every space Y*

"good subspace embedding". Example of use : given a subspace $\iota : S \hookrightarrow X$, and a map $f : S \to R$, is there an extension of f to X?

Example 309 $0 \hookrightarrow X$ *is a cofibration*

Example 310 *A homeomorphism is a cofibration*

Definition 311 *A fibration in a category with weak equivalences is a*

Definition 312 *A* model category *is a category* $\mathbf{C}$ *with three classes of morphisms* $\mathbf{W}, \mathbf{Fib}, \mathbf{Cofib}$ *where* $(\mathbf{C}, \mathbf{W})$ *is a category with weak equivalences, and*

Similarly to how any category can be considered an ∞-category by simply considering all higher morphisms to be identities, we can also consider any category to be a category with weakly equivalences by simply picking W to be only isomorphisms, or a homotopy category by

in which case the homotopy projection is simply the skeletal category $\mathrm{Ho}(\mathbf{C}, \mathbf{W}) \cong \mathrm{Sk}(\mathbf{C})$

Definition 313 *Given a category with weak equivalences* $(\mathbf{C}, \mathbf{W})$ *and some functor* $F : \mathbf{C} \to \mathbf{D}$*, its* left derived functor LF *is the right Kan extension of* F *along the projection* $p : \mathbf{C} \to \mathrm{Ho}(\mathbf{C}, \mathbf{W})$

$$\begin{tikzcd}
 & \mathrm{Ho}(\mathbf{C}, \mathbf{W}) \arrow[dr, "LF"] \arrow[d, phantom, "\Downarrow f"] & \\
\mathbf{C} \arrow[ur, "\pi"] \arrow[rr, "F"'] & & \mathbf{D}
\end{tikzcd}$$

In a category with weak equivalences, the notions of limits and colimits do not typically preserve weak equivalences. ie if two diagram functors F, F' are weakly equivalent ($F(X) \cong_w F'(X)$ for all X), it is however not guaranteed that the limits from such a diagram are.

Example : pushout for $1 \leftarrow S^n \to D^{n+1}$ and $1 \leftarrow S^n \to 1$. The first colimit is S^{n+1} while the other is 1, which are not weakly equivalent in homotopy top

Definition 314 *A* homotopy limit *of a functor $F : I \to \mathbf{C}$ with diagram of shape I is the right derived functor of the limit functor* $\lim_I : [I, C] \to C$

$$\mathrm{holim}_{\mathbf{I}} F = (R \lim_I) F \tag{7.9}$$

Theorem 315 *Any discrete limit is homotopy invariant under [some condition] :*

$$\lim_{\mathbf{n}} F = \mathrm{holim}_{\mathbf{n}} F \tag{7.10}$$

Definition 316

Homotopy limit

7.3 Intervals, loops and paths

The interaction of categories of spaces like topoi and ∞-categories or homotopy categories is what gives rise to concepts of homotopy. To talk of a notion of homotopy in our spaces (at least the type of homotopy we typically associate to spaces), we need to be able to talk about some notion of a path. Obviously our main idea behind such notions is that of the standard path in geometry, a map from the unit interval to a space

$$\gamma : [0, 1] \to X \tag{7.11}$$

But we will need to keep things somewhat general.

Definition 317 *An interval object I is some object containing two points, the boundary points*

$$1 \sqcup 1 \underset{\iota_0}{\overset{\iota_1}{\rightrightarrows}} I \tag{7.12}$$

This is a pretty broad definition but we also typically depend that it be contractible.

Definition 318 *A* contractible interval object *is an interval object*

Typical examples of interval objects are the 1-simplex in the category of simplicial sets, where its boundary points are the maps $\vec{1} :\to \Delta[1]$, the unit interval $[0, 1]$ for a variety of spatial categories, or the standard interval object $0 \to 1$ in the category of categories.

A path space object is some

Definition 319

$$X \xrightarrow{S} \mathrm{Path}_X \xrightarrow{(d_0,d_1)} X \times X \tag{7.13}$$

Loop space object and suspension object

Theorem 320 *The localization of a category by its*

7.4 ∞-sheaves

Just as categories have sheaves as functors to the category of sets, ∞-categories have ∞-sheaves as functors to the category $\infty\mathbf{Grpd}$.

Before we get into ∞-sheaves, however, let's first look at the simpler case of stacks, or groupoid stacks more specifically (throughout we will simply refer to groupoid stacks as stacks, as we will not consider any other category for valuation)

Definition 321 *A* pre-stack *is a groupoid-valued presheaf, ie a functor*

$$S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Grpd} \tag{7.14}$$

Definition 322 *A* stack *is a prestack*

Example 323 *A traditional example of a stack is given by the stack of similar triangles[74, 75]. Take the set of all possible triangles, with sides (a, b, c). To only consider similar triangles, ie up to equivalence $(a, b, c) \cong (\alpha a; \alpha b, \alpha c)$, we will take triangles of a constant perimeter :*

$$a + b + c = 2 \tag{7.15}$$

The space of all such values is therefore some two-dimensional subspace of $[0, 2]^3 \subseteq \mathbb{R}^3$, and more specifically this is a simplex.

For each point of this simplex, we have an associated triangle, and we attach to each of those point the group of symmetries of those triangle. The structure thus formed is a groupoid where each point is a group. The groups involved are typically the symmetric groups S_2 (isoceles triangles), S_3 (equilateral triangles) as well as the reflection symmetry $\mathbb{Z}_2$

Example 324 *The simplest non-trivial example of a smooth stack is that of an* orbifold*, which are spaces modelled locally by quotients of open subsets of $\mathbb{R}^n$ by some finite group action. The simplest example of this being a line acted on by $\mathbb{Z}_2$, which resembles the half-line $[0, \infty)$.*

As in the context of a stack we are instead interested in mapping a Cartesian space to a groupoid, we will instead look at the action of the functor H (for half line) on $\mathbb{R}$.

We are

Definition 325 *A ∞-presheaf F is given by a functor*

$$F : \mathbf{C}^{\mathrm{op}} \to \infty\mathbf{Grpd} \tag{7.16}$$

Definition 326 *A ∞-sheaf F is an ∞-presheaf*

Example 327 *If we consider a set X as an ∞-sheaf on the terminal site of 0-type, and we consider its quotient under group action by some group object G, via the action*

$$\rho : G \times X \to X \tag{7.17}$$

7.5 ∞-topos

7.5.1 Principal bundles

Definition 328 *A G-*principal bundle *P over an object X is a pullback of a morphism $X \to \mathbf{B}G$ and (some point?) $1 \to \mathbf{B}G$.*

The only non-trivial morphism of the pullback is the projection $\pi : P \to X$.

7.6 Smooth groupoids

The higher categorical equivalent of the category of smooth spaces is the ∞-topos of smooth ∞-groupoids over the site of Cartesian spaces, which are smooth sets along with their homotopy equivalences. This will allow us to treat the notion of groups without internalization in the context of smooth spaces, such as what we would need for gauge theories.

If we look at those spaces once more in the context of their analogies with diffeological spaces, rather than mapping our probes to some sets of atlases, we are mapping them to some ∞-groupoid. All of our previous diffeological spaces are simply sheaves that are non-empty only for the discrete objects of the ∞-groupoid (in the sense of being 0-truncated, ie all their higher order morphisms are just the identities). Equivalently this can be done via the composition of the smooth set and the embedding of **Set** into ∞**Grpd**.

Definition 329 *A* pre-smooth groupoid *is a presheaf of groupoids on the site* **CartSp***,*

$$X : \mathbf{CartSp}^{op} \to \mathbf{Grpd} \tag{7.18}$$

The most basic kind of non-0-truncated smooth groupoid is a 1-truncated smooth groupoid of a single point, which simply corresponds to a group, which as a sheaf is given by

$$G(\mathbb{R}^0) = G \tag{7.19}$$

Another example is the *classifying space* of a group. This will be in the context of a smooth groupoid a diffeological space for which the k-homotopy groups will be equal to G.

Example 330 *The circle S^1 is the classifying space of the group $\mathbb{Z}$. As a smooth groupoid, this is given by considering the sheaves on $\mathbb{R}^1$ (for instance using the stereographic projection)*

A less trivial example for this is the notion of a G-manifold, ie a manifold M equipped with a smooth group action by the group G, such as the pair of the line manifold $L \cong \mathbb{R}$ and the one-dimensional translation group $T \cong \mathbb{R}$ (as a group).

$$\rho : M \times G \to M \tag{7.20}$$

Action groupoid

7.6.1 Principal bundle

In the case of a smooth groupoid, the principal bundle is the usual notion of a G-principal bundle.

The basic example is that of a manifold M, ie some locally representable smooth 0-type, with G some Lie group of the same type. If we consider some open cover $\{U_i \to M\}$, and its Cech groupoid $C(\{U_i\})_\bullet$,

7.6.2 Connections

An important element of higher category as it relates to physics is the description of connections, and more generally of transport operators.

7.7 Cohomology

Definition 331 *Given some ∞-topos $\mathbf{H}$, the cohomology of $X \in \mathbf{H}$ with values in $A \in \mathbf{H}$ is the set of connected components of the*

Chapter 8

Moments

The formalization of qualities of an instance of an object are given by the concept of *moments* of such an instance, given by some "projection operator" $\bigcirc : C \to C$, such that

$$\bigcirc \bigcirc X \cong \bigcirc X \tag{8.1}$$

If we reduce the object X to merely the qualities given by $\bigcirc$, there is nothing left to remove so that any subsequent projection will be isomorphic to it. In categorical term, we also demand that the projection given as $X \to \bigcirc X$ be, within the category of qualities $\bigcirc$, an equivalence :

$$\bigcirc(X \to \bigcirc X) \in \mathrm{core}(X) \tag{8.2}$$

In terms of types, this is an idempotent monad.

Dually, we can also define comonads $\Box$, $\Box X \to X$, with $\Box(\Box X \to X)$ is an equivalence

Definition 332 *A* moment *on a type system/topos* $\mathbf{H}$ *is either an idempotent monad or comonad.*

As idempotent monads, the Eilenberg-Moore category of the monad $\bigcirc$, $\mathbf{H}_{\bigcirc}$, is a full subcategory of the initial topos,

$$\iota : \mathbf{H}_{\bigcirc} \hookrightarrow \mathbf{H} \tag{8.3}$$

$$\mathbf{H}_{\bigcirc} \hookrightarrow \mathbf{H} \tag{8.4}$$

Semantics as similarity : two objects are $\bigcirc$-similar if their modality is identical

$$X \cong_{\bigcirc} Y \leftrightarrow \bigcirc X = \bigcirc Y \tag{8.5}$$

$\mathbf{H}_{\bigcirc}$ is the Eilenberg-Moore category of $\bigcirc$

"we may naturally make sense of "pure quality" also for (co-)monads that are not idempotent, the pure types should be taken to be the "algebras" over the monad."

A particular moment that we will see more in details later but that will be of great importance throughout is the one given by the smallest possible subtopos, which is just the terminal topos $\mathbf{1} \cong \mathrm{Sh}(\mathbf{0})$. Being the smallest possible subtopos, it corresponds in some sense to an object with no qualities left beyond being an object, what we will call its quality of *being*.

Accidence : A moment $\bigcirc$ is exhibited by a type J if $\bigcirc$ is a J-homotopy localization :

$$\bigcirc \cong \mathrm{loc}_J \tag{8.6}$$

$\bigcirc J \cong *$

Homotopy localization : for an object $A \in \mathbf{C}$, take the class of morphisms W_A

$$X \times (A \overset{\exists!}{\to} *) : X \times A \xrightarrow{p_1} X \tag{8.7}$$

what means

"The idea is that if A is, or is regarded as, an interval object, then "geometric" left homotopies between morphisms $X \to Y$ are, or would be, given by morphisms out of $X \times A$, and hence forcing the projections $X \times A \to X$ to be equivalences means forcing all morphisms to be homotopy invariant with respect to A."

Example using the real line for homotopy localization?

Notation : $\bigcirc$

Example 333 *The adjunction* Even $\dashv$ Odd *is an opposition of the form* $\Box \dashv \bigcirc$

As monads	As moments	Notation
Comonad	p-moment	$\Box$
Monad	s-moment	$\bigcirc$

Table 8.1: Caption

8.1 Unity of opposites

[76]

Given an adjoint pair of modal operators, ie an adjoint triple

$$F \dashv G \dashv H : C \underset{G}{\overset{F}{\rightleftarrows}} D \tag{8.8}$$

where $F, H : C \to D$ and $G : D \to C$

The two adjunctions imply that G preserves all limits and colimits in D

Gives rise to an adjoint pair of monads,

$$(GF \dashv GH) : C \underset{GF}{\overset{GH}{\rightleftarrows}} C \tag{8.9}$$

and the pair

$$(FG \dashv HG) : D \underset{FG}{\overset{HG}{\rightleftarrows}} D \tag{8.10}$$

Theorem 334 *For $F \dashv G \dashv H$, F is fully faithful iff H is.*

F being fully faithful is equivalent to $\eta : \mathrm{Id} \to GF$ being a natural isomorphism.

H being fully faithful is equivalent to $\varepsilon : GH \to \mathrm{Id}$ being a natural isomorphism.

GF is isomorphic to the identity if GH is

$F \dashv G \dashv H$ is a fully faithful adjoint triple in this case. "This is often the case when D is a category of “spaces” structured over C, where F and H construct “discrete” and “codiscrete” spaces respectively."

The opposite of a moment $\bigcirc$ is a moment $\Box$ such that they form either a left or right adjunction, ie :

$$(\Box \dashv \bigcirc) : \mathbf{H}_{\Box} \simeq \mathbf{H}_{\bigcirc} \underset{\hookrightarrow}{\overset{\hookrightarrow}{\longleftarrow}} \mathbf{H} \tag{8.11}$$

or

$$(\bigcirc \dashv \Box) : \mathbf{H}_{\Box} \simeq \mathbf{H}_{\bigcirc} \underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}} \mathbf{H} \tag{8.12}$$

We will denote the unity of opposites $\Box \dashv \bigcirc$ as a unity of a preceding to a successive moment, or *ps* unity, and $\bigcirc \dashv \Box$ as an *sp* unity.[8]

Theorem 335 *A ps unity defines an essential subtopos.*

(level of a topos)

Theorem 336 *As sp unity defines a bireflective subcategory*

”$\Box \dashv \bigcirc$ – Here are two different opposite “pure moments” .”

”$\bigcirc \dashv \Box$ – Here is only one pure moment, but two opposite ways of projecting onto it.”

8.2 Determinate negation

In addition to opposition, monads and comonads can also have *negations*, what are called more specifically determinate negations. The negation of a moment will be, if it exists, an operator which removes specifically the attributes of a given moment. In other words, if we have both the moment and its negation acting on an object,

$$\Box\overline{\Box}X = 1 \tag{8.13}$$

We are left with the terminal object with no specific properties. Equivalently, $\Box\overline{\Box} = \bigcirc_*$, the modality of being that we will see later on.

To do this, we need to find a map from the category to the subcategory containing only objects that the moment map to the terminal object. Given the counit $\Box X \to X$, this is the cofiber :

Definition 337 *The determinate negation of a comonadic moment $\Box$ is given object-wise by the cofiber of its counit :*

$$\overline{\Box}X = \mathrm{cofib}(\Box X \to X) \tag{8.14}$$

Or as a limit, it is the pullback of the cospan $1 \leftarrow \Box X \to X$:

$$\begin{tikzcd} \Box X \arrow[r, "\epsilon_X"] \arrow[d, "!_{\Box X}"] & X \arrow[d] \\ 1 \arrow[r] & \overline{\Box}X \end{tikzcd}$$

Dually we can also define the determinate negation of a monadic moment, but while a cofibration only involves the pushout $1 \leftarrow \Box X \to X$, where the morphism $\Box X \to 1$ is unique, the fibration is the pullback $1 \to X \leftarrow \bigcirc X$, which depends on a specific choice of a point $1 \to X$ in the object. This means that this negation will either be defined if the object contains a single point, if we are given a specific choice of a point, if the result is independent from the choice of basepoint, or if we allow more flexible negations such as the homotopy fiber of a connected object.

Definition 338 *The determinate negation of a monadic moment*

$$\overline{\bigcirc}X = \mathrm{fib}(X \to \bigcirc X) \tag{8.15}$$

$$\Box\overline{\Box} = * \tag{8.16}$$

Show that the intersection of subcategories is something

Example 339

Definition 340 Determinate negation *of a unity of opposite moments* $\bigcirc \dashv \Box$ *if* $\Box, \bigcirc$ *restrict to* 0*-types and*

- $\bigcirc * \cong *$
- $\Box \to \bigcirc$ *is an epimorphism.*

"For an opposition with determinate negation, def. 1.14, then on 0-types there is no $\bigcirc$-moment left in the negative of $\Box$-moment"

$$\bigcirc\overline{\Box} \cong * \tag{8.17}$$

Proof 48 *By left adjoints preserving colimits,*

$$\bigcirc\overline{\Box}X = \bigcirc\mathrm{cofib}(\Box X \to X) \cong \mathrm{cofib}(\Box X \to \bigcirc X) \tag{8.18}$$

Since $\Box X \to \bigcirc X$ *is epi, which is preserved by pushout, this is an epimorphism from the terminal object, therefore the terminal object itself.*

"For opposite moments of the form $\bigcirc \dashv \Box$, then for stable types X the hexagons"

$$\begin{tikzcd}
 & \overline{\bigcirc}X \arrow[rr] \arrow[dr] & & \overline{\Box}X \arrow[dr] & \\
\overline{\bigcirc}\Box X \arrow[ur] \arrow[rr] \arrow[dr] & & X \arrow[ur] \arrow[rr] \arrow[dr] & & \bigcirc\Box X \\
 & \Box X \arrow[rr] \arrow[ur] & & \bigcirc X \arrow[ur] &
\end{tikzcd}$$

are homotopy exact in that

- both squares are homotopy Cartesian, hence are fracture squares
- the boundary sequences are long homotopy fiber sequences

"In particular every stable type is the fibered direct sum of its pure $\bigcirc$-moment and its pure $\overline{\Box}$-moment:"

$$X \simeq (\bigcirc X) \underset{\bigcirc \overline{\Box} X}{\oplus} (\overline{\Box} X)\,. \tag{8.19}$$

"However, this fiber depends on a chosen basepoint, so it only makes sense on types which have only one constituent (but possibly this constituent has (higher) equalities), or, thought of homotopically, have only one connected component. In this case,$\bigcirc T$ contains that part of the structure of T that is trivialized by $T \to \bigcirc T$. Note that $\Box$ and $\overline{\Box}$ (or the dual notions) do not form a unity of oppositions. However, for each object T , a sequence $\Box T \to T \to \overline{\Box} T$ exists and this sequence decomposes each T , in the sense that T could be reconstructed from its aspects under a moment and its negative, as well as their relation. This is not generally true for unities of oppositions."

8.3 Sublation

Sublation (or Aufhebung in the original German), levels of a topos

§180 The resultant equilibrium of coming-to-be and ceasing-to-be is in the first place becoming itself. But this equally settles into a stable unity. Being and nothing are in this unity only as vanishing moments; yet becoming as such is only through their distinguishedness. Their vanishing, therefore, is the vanishing of becoming or the vanishing of the vanishing itself. Becoming is an unstable unrest which settles into a stable result.

§181 This could also be expressed thus: becoming is the vanishing of being in nothing and of nothing in being and the vanishing of being and nothing generally; but at the same time it rests on the distinction between them. It is therefore inherently self-contradictory, because the determinations it unites within itself are opposed to each other; but such a union destroys itself.

§182 This result is the vanishedness of becoming, but it is not nothing; as such it would only be a relapse into one of the already sublated determinations, not the resultant of nothing and being. It is the unity of being and nothing which has settled into a stable oneness. But this stable oneness is being, yet no longer as a determination on its own but as a determination of the whole.

§183 Becoming, as this transition into the unity of being and nothing, a unity which is in the form of being or has the form of the onesided immediate unity of these moments, is determinate being.

Chapter 9

Objective logic

Yoneda v.

"These many different things stand in essential reciprocal action via their properties; the property is this reciprocal relation itself and apart from it the thing is nothing"

As there will be many notations for very similar concepts of different types, we will require the following convention :

- Unless a more specific unambiguous symbol exists, monads will be denoted by $\bigcirc_{(-)}$, with its subscript to differentiate it
- Similarly, comonads will be denoted by $\Box_{(-)}$, with its subscript to differentiate it
- A generic topos $\mathbf{H}$
- The terminal object is 1
- The initial object is 0

This to avoid circumstances such as $*$ to represent both an object, functor, category and monad.

A trivial opposition we have in the objective logic is

$$\mathrm{Id} \dashv \mathrm{Id} \tag{9.1}$$

This is an opposition defined by the triple of endofunctors

$$(\mathrm{Id} \dashv \mathrm{Id}) : \mathbf{H} \overset{\leftarrow}{\hookrightarrow} \tag{9.2}$$

Representing three identity functors, composing into the two identity monad and comonad, with the subtopi being $\mathbf{H}$ itself. (Moment of identity?)

9.1 Being and nothingness

"Being, pure being, [...] it has no diversity within itself nor any with a reference outwards"

"Nothing, pure nothing: it is simply equality with itself, complete emptiness"

The most basic type of moments are the monads of *being* (*Sein*) and *nothingness* (*Nichts*). This can be seen easily enough by considering that the smallest subtopos is the initial topos $\mathrm{Sh}(\varnothing) \cong \mathbf{1}$. Let's consider the unique functor from our topos to it, the constant functor on this subtopos,

$$\Delta_* : \mathbf{C} \to 1 \tag{9.3}$$

which maps every object of $\mathbf{C}$ to the unique object $*$ of the terminal category $\mathbf{1}$. This corresponds to the *unit type* in type theory term, and this is what is referred to as *The One* (*Das Eins*) in objective logic. From this already we can see that the opposition will be a *ps*-unity.

We give this functor a left and right adjoint, which as we will see are the constant functors of the initial and terminal object, Δ_0 and Δ_1, forming the adjoint cylinder

$$(\Delta_0 \dashv \Delta_* \dashv \Delta_1) : \mathbf{C} \xrightarrow{\Delta_*} \mathbf{1} \underset{\Delta_0}{\overset{\Delta_1}{\rightrightarrows}} \mathbf{C} \tag{9.4}$$

An easy way to see this is via the adjunction of hom-sets :

$$\mathrm{Hom}_{\mathbf{C}}(\Delta_0(*), X) \cong \mathrm{Hom}_{\mathbf{1}}(*, \Delta_* X) \tag{9.5}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between $\Delta_0(*)$ and any object X, making it the initial object of the topos. Similarly,

$$\mathrm{Hom}_{\mathbf{1}}(\Delta_*(X), *) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Delta_1(*)) \tag{9.6}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between any object X and $\Delta_1(*)$, making it the terminal object of the topos, confirming our choice of those adjoints as constant functors.

We can also look at this in terms of the unit and counit of the adjunction. First if we look at the adjunction $\Delta_0 \dashv \Delta_*$, as adjoint functors, they are equipped with the unit and counit natural transformations,

$$\begin{aligned} \eta_0 &: \mathrm{Id}_{\mathbf{1}} \Rightarrow \Delta_* \circ \Delta_0 & (9.7)\\ \epsilon_0 &: \Delta_0 \circ \Delta_* \Rightarrow \mathrm{Id}_{\mathbf{C}} & (9.8)\end{aligned}$$

they have to obey the triangle identities

$$\begin{tikzcd} \Delta_0 \arrow[rr, "\mathrm{Id}_{\mathbf{1}}"] \arrow[dr, "\Delta_0\eta_0"'] & & \Delta_0 \\ & \Delta_0\Delta_*\Delta_0 \arrow[ur, "\epsilon_0\Delta_0"'] & \end{tikzcd}$$

In terms of components, this means that for any object $X \in \mathbf{C}$ (and the only object $*$ in $\mathbf{1}$),

$$\begin{aligned} \mathrm{Id}_{\Delta_0(*)} &= \epsilon_{\Delta_0(*)} \circ \Delta_0(\eta_*) & (9.9)\\ \mathrm{Id}_{\Delta_*(X)} &= \Delta_*(\epsilon_X) \circ \eta_{\Delta_*(X)} & (9.10)\end{aligned}$$

We have the identities $\Delta_*(X) = *$, and any component of the counit can only be the identity morphism on $*$, so that

$$\begin{aligned} \mathrm{Id}_{\Delta_0(*)} &= \epsilon_{\Delta_0(*)} \circ \mathrm{Id}_{\Delta_0(*)} & (9.11)\\ \mathrm{Id}_* &= \mathrm{Id}_* & (9.12)\end{aligned}$$

The second line is trivial, but the first line tells us that ...

for any object $X \in \mathbf{C}$, there exists an object $\Delta_*(X) \in \mathbf{1}$ and a morphism $\epsilon_X : \Delta_0 \circ \Delta_*(X) \to X$ such that for every object in $\mathbf{1}$ (so only for $*$), and every morphism $f : \Delta_0(*) \to X$, there exists a unique morphism $g : * \to \Delta_*(X) = *$ with $\epsilon_X \circ \Delta_0(g) = f$.

The unique morphism g is simply the identity function Id_1, and our natural transformation at c is simply $\epsilon_X : \Delta_0(*) \to X$. We therefore need the constraint

$$\epsilon_X \circ \Delta_0(\mathrm{Id}_1) = f \qquad (9.13)$$

However, as g can only be one function, we cannot have more than one such possible morphism f, and we need exactly one to map to Id_1. This means that the empty functor Δ_0 maps the single object of 1 to the initial object 0 in $\mathbf{C}$ (if the category contains one), as its name indicates, justifying our notation of the constant functor Δ_0.

[Do the other triangle?]

Conversely, the right adjoint Δ_1 has to obey

$$\begin{tikzcd}
\Delta_1 \arrow[rr, "\mathrm{Id}_{\mathbf{1}}"] \arrow[dr, "\eta_0 \Delta_1"'] & & \Delta_1 \\
& \Delta_1 \Delta_* \Delta_1 \arrow[ur, "\Delta_1 \epsilon_0"'] &
\end{tikzcd}$$

Δ_1 maps the unique object of the terminal category to the terminal object 0 of $\mathbf{C}$, if this object exists.

From the adjoint cylinder $(\Delta_0 \dashv \Delta_* \dashv \Delta_1)$, we can see that this is a *ps*-type unity of opposites, giving rise to an adjoint modality that we will write as $\Box_\varnothing \dashv \bigcirc_*$, with $\Box_\varnothing$ the modality of nothingness (or empty comonad) and $\bigcirc_*$ the modality of being (or unit monad), defined by

$$\Box_\varnothing = \Delta_0 \circ \Delta_* \tag{9.14}$$
$$\bigcirc_* = \Delta \circ \Delta \tag{9.15}$$

where the unit and counit of the adjunction are given by those that we have seen,

In terms of their modal action, the empty monad maps any object of the category to its initial element,

$$\Box_\varnothing(X) = 0 \tag{9.16}$$

while the unit monad maps any object of the category to its terminal element,

$$\bigcirc_*(X) = 1 \tag{9.17}$$

In the Hegelian sense : $\bigcirc_*$ maps every object of $\mathbf{C}$ to a single object (they all share the same characteristic of existence, "pure being", and there is nothing differentiating them in that respect, no further qualities). The image of any two objects under this modality are identical, as there are no other characteristics to differenciate them, and the only relation they can have is that of the identity. This is also true of the empty object []

"In its indeterminate immediacy it is equal only to itself. It is also not unequal relatively to an other; it has no diversity within itself nor any with a reference outwards"

Conversely, $\Box_\varnothing$ maps every object to nothing, the opposition of being.

The unit of the monad and counit of the comonad are given in terms of components by the typical morphisms of the terminal and initial object,

$$\epsilon_X : X \quad \to \quad \bigcirc_* X \cong 1 \tag{9.18}$$
$$\eta_X : 0 \cong \Box_\varnothing X \quad \to \quad X \tag{9.19}$$

Both of those adjoint functors roughly reflect the fact that each has to map elements to a single element and morphisms between that element and every other element to a single morphism.

The composition of the unit and counit give us the *unity of opposites* for being and nothingness

$$\varnothing \to X \to * \tag{9.20}$$

"there is nothing which is not an intermediate state between being and nothing."

An alternative interpretations of this modality is given by the opposition of the dependent sum and dependent product on the empty context

$$\sum_\varnothing (-) \vdash \prod_\varnothing (-) \tag{9.21}$$

Cartesian product v. internal home adjunction of the unit type

$$((-) \times \varnothing) \dashv (\varnothing \to (-)) \tag{9.22}$$

Negation in categories : internal hom to the initial object : $\neg = [-, \varnothing]$

Examples of those two modalities on a topos will not give us very different results overall, as they all roughly have the same behaviour. For **Set** for instance,

$$\forall A \in \mathrm{Obj}(\mathbf{Set}),\ \Box_\varnothing(A) \quad = \quad \varnothing \tag{9.23}$$
$$\forall f \in \mathrm{Mor}(\mathbf{Set}),\ \Box_\varnothing(f) \quad = \quad \mathrm{Id}_\varnothing \tag{9.24}$$

$$\forall A \in \mathrm{Obj}(\mathbf{Set}),\ \bigcirc_*(A) \quad = \quad \{\bullet\} \tag{9.25}$$
$$\forall f \in \mathrm{Mor}(\mathbf{Set}),\ \bigcirc_*(f) \quad = \quad \mathrm{Id}_{\{\bullet\}} \tag{9.26}$$

[...]

As the examples we have given thus far do not have particularly varied definitions for the initial and terminal object (being mostly concrete categories where $I = \varnothing$ and $T = \{\bullet\}$, the modality of being and nothingness do not offer particularly more insight in those topos. Smooth sets simply get mapped to the empty space and the one point space, etc

"In geometric language these are categories equipped with a notion of discrete objects and codiscrete objects."

As any further adjoint would have to be a new functor from $\mathbf{H}$ to $\mathbf{1}$, they would simply be just the functor Δ_* again, so that the further adjoints would simply be the same monads again. There is therefore no further moments at this level.

9.1.1 Negations

The modality of being admits a determinate negation

Theorem 341 *The negation of the monad of being is the identity monad :*

$$\overline{\bigcirc}_* = \mathrm{Id} \tag{9.27}$$

Proof 49 *The computation is simple enough, as*

$$\begin{aligned}
\overline{\bigcirc}_*(X) &= \mathrm{Fib}(X \to \bigcirc_* X) && (9.28)\\
&= \mathrm{Fib}(X \to 1) && (9.29)\\
&= \mathrm{Fib}(!_X) && (9.30)\\
&= X \times_* * && (9.31)\\
&= X && (9.32)
\end{aligned}$$

This is independent of the choice of basepoint, so that the negation of being is therefore the identity.

Interpretation? "contains that part of the structure that is trivialized by $X \to \bigcirc X$". As the unit monad removes all struture from the object, the trivialized part is all of it.

Theorem 342 *The left dual*

On the other hand, the modality of nothingness's cofibration does not give us exactly a determinate negation :

Theorem 343 *The negation of the comonad of nothingness is the maybe monad :*

$$\overline{\Box}_\varnothing = \mathrm{Maybe} \tag{9.33}$$

Proof 50

$$\begin{aligned}
\overline{\Box}_\varnothing &= \mathrm{cofib}(\Box_\varnothing X \to X) && (9.34)\\
&= X +_{\Box_\varnothing X} 1 && (9.35)\\
&= X +_\varnothing 1 && (9.36)\\
&= X + 1 && (9.37)
\end{aligned}$$

While the determinate negation is well-defined, it is not an idempotent monad, as it simply adds a new element to any object.

$$\mathrm{Maybe}^2 X = (\mathrm{Maybe} X) \sqcup \{\bullet\} = X \sqcup \{\bullet_1, \bullet_2\} \tag{9.38}$$

$$\overline{\Box}_\varnothing \Box_\varnothing X = 1 \tag{9.39}$$

$\Box_\varnothing \overline{\Box}_\varnothing = 0$ Interpretation?

Is the hexagonal diagram valid?

9.1.2 Algebra

As a monad, the unit monad has an associated algebra. For a given element X, and a morphism $x : \bigcirc_* X \cong 1 \to X$, ie a point of X,

Free algebra : algebra on 1 with morphism $\mathrm{Id}_1 : 1 \to 1$, trivial algebra

Coalgebra of the empty comonad $\Box_\varnothing$: given X and a morphism $f : X \to \Box_\varnothing X \cong 0$

If there is no such map : empty coalgebra? Except on $\varnothing$, the cofree coalgebra, which is the coalgebra with

9.1.3 Logic

In terms of logic, those two monads will correspond to modalities

For sets : given a proposition $X \to \Omega$, that factors through either 0 or 1, the corresponding modality is

$$\Box_\varnothing (X \to 1 \to \Omega) = \tag{9.40}$$

subset relation defined by $\chi_U : X \to \Omega$,

$$\bigcirc_* (\chi_U) = \tag{9.41}$$

Projection to the trivial logic?

9.2 Necessity and possibility

Before looking further into the first sublation of the ground opposition, let's briefly look at another direction to generalize.

The interpretation of being and nothingness as the duality between the dependent product and sum on the empty context gives us a possibility of generalization in this direction, in which we simply generalize to an arbitrary context.

As we've seen before, the context Γ of the internal logic corresponds to the slice category $\mathbf{C}_\Gamma$. If we wish to change our context, this is done via a display morphism $f : X \to Y$ which induces the functor

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{9.42}$$

The ground that we've seen is done on the empty context, which is given by the terminal object 0. The corresponding context is that of the display morphism $!_0 : 0 \to 1$, changing the context from 0, falsity, to 1, truth. The corresponding contexts are $\mathbf{C}_1 \cong \mathbf{C}$ and $\mathbf{C}_0 = \mathbf{1}$, giving the base change functor

$$f^* : \mathbf{C} \to \mathbf{1} \tag{9.43}$$

which is exactly the functor that we used as its basis.

The interpretation in this sense is therefore that $\bigcirc_*$ is adding the context

For a morphism $f : X \to 1$ (what context is that?) :

$$f^* : \mathbf{C} \to \mathbf{C}_{/X} \tag{9.44}$$

Adjoints : $(f_! \dashv f^* \dashv f_*)$

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}_X \overset{\Pi_f}{\underset{\Sigma_f}{\rightrightarrows}} \mathbf{H}_X \xrightarrow{f^*} \mathbf{H}_X \tag{9.45}$$

for $f : X \to 1$:

[...]

Theorem 344 *$\prod_f$ is a right adjoint and therefore preserves limits, in particular the terminal object :*

$$\prod_f 1 = 1 \tag{9.46}$$

Theorem 345 *$\sum_f$ is a left adjoint and therefore preserves colimits, in particular the initial object :*

$$\sum_f 0 = 0 \tag{9.47}$$

From this, we can see that each of those adjunctions is a sublation of the ground.

Base change f^* preserves limits and colimits because toposes are stable under pullbacks?

so $f_* f^* = \prod_f f^* = \Box_f$ is such that

$$\Box_f 0 = \prod_f 0 \tag{9.48}$$

Adjoint modality : $(f_! f^* \dashv f_* f^*)$

Writer comonad and reader monad

Possibility comonad and necessity monad $(\Box \dashv \Diamond)$

[...]

The interpretation of this modality in terms of standard modal logic (the modality of necessity and possibility) can be understood using the Kripke semantics of modal logic.

interpretation of the ground in this context : $\bigcirc_*$ sends every proposition to true and $\Box_\varnothing$ to false. Every proposition is *possibly* true and none are *necessarily* true.

9.3 Determinate being

Determinate being (dasein), being in a certain place (hacceity?)

Sublation of being/non-being into non-becoming/becoming? idk via localization of $\neg\neg$

The localization of $\neg\neg$ allows for a boolean topos : we only consider subobjects which have a complement. For any object X and subobject $\iota : S \hookrightarrow X$, there is a complementary object $\bar{\iota} : \overline{S} \hookrightarrow$. We can *separate* S from X.

9.4 Cohesion

There are other directions in which to sublate the initial opposition $(\Box_\varnothing \dashv \bigcirc_*)$ to find higher ones. To find some resolution of $(\Box_\varnothing \dashv \bigcirc_*)$ to some higher adjunction $(\Box'_\varnothing \dashv \bigcirc'_*)$, we need some adjunction obeying the property

$$\begin{aligned} \Box'_{\varnothing}\Box_{\varnothing} &= \Box_{\varnothing} && (9.49)\\ \bigcirc'_{*}\bigcirc_{*} &= \bigcirc_{*} && (9.50) \end{aligned}$$

In other words, each of these preserve the initial and terminal object

$$\begin{aligned} \Box'_{\varnothing}0 &= 0 && (9.51)\\ \bigcirc'_{*}1 &= 1 && (9.52) \end{aligned}$$

We will look here more specifically at a *right sublation*, with the additional property

$$\bigcirc'_{*}\Box_{\varnothing} \cong \Box_{\varnothing} \qquad (9.53)$$

From the properties of the empty comonad, this simply means that the sublated monad preserves the terminal product :

$$\bigcirc'_{*}0 \cong 0 \qquad (9.54)$$

which is the exact property of $\mathbf{H}_{\bigcirc'_{*}}$ being a dense subtopos. Fortunately there is a natural choice for this, given by this theorem

Theorem 346 *The smallest dense subtopos of a topos is that of local types with respect to double negation* $\sharp = \mathrm{loc}_{\neg\neg}$. *(Johnstone 02, corollary A4.5.20)*

From this, we have that the natural sublation of the ground opposition can be constructed from the localization by the double negation. This is called the *sharp modality.*

To understand the role of the sharp modality here, let's look at the internal logic, the most natural setting for the double negation. As a proposition in the internal logic corresponds to some subobject relation,

$$\begin{array}{ccc} U & \xrightarrow{!_U} & 1 \\ \downarrow{\scriptstyle \iota_U} & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[\chi_U]{} & \Omega \end{array}$$

[Prove the sharp law of excluded middle]

From Shulman : The proof of sharp LEM is based on LEM in the category of discrete objects and the introduction rule

$$\frac{\Delta|\Gamma, x : A \vdash b : B}{\Delta|\Gamma \vdash \lambda x.b : \prod_{x:A} B}$$

Equivalence to the eval map in a topos?

What is the Thing?

In a topos we always have $A \cap \neg_X A = 0$, therefore we need to show the relationship on $A \cup \neg_X A = X$

The union is the pushout of the inclusion of this intersection,

As a Heyting category, the subobjects of X are such that we have the monomorphism $A + \neg_X A \to X$

To have the category be boolean, we would need to show furthermore that this is an epimorphism.

if so,

$$\sharp(A + \neg_X A) \to \sharp(X) \tag{9.55}$$

Show the invertibility?

In cohesive types :

In logical term, the subobject U associated to a proposition p is the largest context in which p holds, and its negation $\neg_X U$ the largest context in which $\neg p$ holds, but as our categories are not required to be boolean, the proposition $p \vee \neg_X p$ is not required to be "true", in the sense that the object associated to it is not X itself. For instance in a Grothendieck topos over a topological space, given some open subset $U \subseteq X$,

$$\neg U = \tag{9.56}$$

The point structure is preserved however, as for any point $x : 1 \to X$, we have that either this point belongs to U

[diagram of $1 \to U \to X$]

or to its negation

[diagram of $1 \to \neg U \to X$]

[...]

"The double negation subtopos is Boolean topos." (Johnstone 02, lemma A4.5.21)

The subtopos $\mathbf{H}_\sharp$ is therefore a *Boolean topos*. It can be understood in terms of the existence of a complement for any subobject, where every object can be split exactly in two parts by a subobject, one part which contains every point of A, and another part which contains every other point. [etc etc]

Any adjoint modality $\Box \dashv \bigcirc$ that includes the modalities $\varnothing \dashv *$, ie $\varnothing \subset \Box$, $* \subset \bigcirc$, formalizes a more determinate being (Dasein)

Sublation by $\sharp$ always exists for any topos?

"A topos E is Boolean iff E has exactly one dense subtopos, namely E neg neg = E."

Left sublation?

$$\Box'_{\varnothing}\bigcirc_* \cong \bigcirc_* \tag{9.57}$$

Preserves the unit monad

$$\Box'_{\varnothing}1 \cong 1 \tag{9.58}$$

This is true of $\sharp$ but is it the first possible sublation?

[...] [77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88]

"Quantity is the unity of these moments of continuity and discreteness"

Qualität/Quality, Etwas/Something, Die Endlichket/Finitude, Etwas und ein Anderes/Something and another

As we've seen, the natural (right) sublation of the ground is given by the localization by the double negation, $\mathrm{loc}_{\neg\neg}$, the sharp modality $\sharp$. To get the full sublation of the adjunction, we will need also the existence of an adjoint modality, called the *flat modality* $\flat$. If this adjoint exists, the given adjoint cylinder

$$\mathbf{H}_\sharp \overset{\mathrm{Disc}}{\hookrightarrow} \mathbf{H} \xrightarrow{\Gamma} \mathbf{H}_\sharp \tag{9.59}$$

As a boolean topos, $\mathbf{H}_\sharp$ is typically either the topos of sets **Set** or some variant thereof, such as some ETCS variant of it or plus minus the axiom of choice. We will generally assume that we are using **Set** here, but most of the properties used here should generalize to any boolean topos.

Definition 347 *A topos that admits an adjoint* $\flat$ *to* $\sharp$ *is called a* local topos

[89, 90]

As $\sharp$ can be understood as the space generated from the set of its points with a trivial topology, $\flat$ is the space generated from the set of its points with a discrete topology, ie as a topos it is entirely determined by its points, and is simply the coproduct of every point.

$$\flat X \cong \bigsqcup_{s:\Gamma(X)} 1 \tag{9.60}$$

[...]

Properties of $\sharp$:

As a reflector preserving all limits (by being right adjoint to $\flat$), $\sharp$ defines a Lawvere-Tierney topology. That is, for any $X \in \mathbf{H}$, we have some closure operator

$$j_\sharp : \Omega \xrightarrow{\chi_{\sharp\top}\circ\sharp} \Omega \tag{9.61}$$

Proof that this is the codiscrete one

[...]

Negation : internal hom into the initial object, $\neg = [-, \varnothing]$. In a topos, $\neg A$ is the internal hom object 0^A with 0 the initial object

Double negation : 0^{0^A}

$$\neg\neg = [[-, \varnothing], \varnothing] \tag{9.62}$$

For an object $X \in \mathbf{H}$, $\neg X = [X, \varnothing]$, the internal hom of all morphisms $X \to \varnothing$

Topos localized by $\neg\neg$, the new topos is $H_\sharp$. The opposition $\sharp \dashv \flat$ is the ground topos of $H_\sharp$ Localization of a topos

If the flat modality admits a further adjunction, we will call its left adjoint the *shape modality*, $\int$. This will allow us to define the full notion of cohesion properly by adding some appropriate requirements to it. The shape modality roughly corresponds to the topological notion of connected objects, so that first we need to define what it means for an object to be connected in a topos. If we take a connected object to be the lack of decomposition into the disjoint sum of other objects, as they are for topological spaces, we will show that this definition gives out that result :

Definition 348 *An object X in a topos $\mathbf{H}$ is* connected *if* $\mathrm{Hom}_{\mathbf{H}}(X, -)$ *preserves finite coproducts.*

Theorem 349 *If an object is connected, it cannot be expressed as the coproduct of more than one non-empty subobject.*

Proof 51

Definition 350 *A topos is* locally connected *if every object is the coproduct of connected objects* $(X_i)_{i\in I}$ *:*

$$X \cong \coprod_i X_i \tag{9.63}$$

Theorem 351 *The index set* I *is unique up to isomorphism*

Definition 352 *The connected component functor maps objects of a locally connected topos to its index set of connected objects :*

$$\Pi_0(X) \cong \Pi_0(\coprod_{i\in I} X_i) \cong I \tag{9.64}$$

Theorem 353 *The connected component functor is left adjoint to the discrete object functor*

Definition 354 *A topos* $\mathbf{H}$ *is* cohesive *over a base topos* $\mathbf{B}$ *if it is equipped with the geometric morphisms*

$$(f^* \dashv f_*) : \mathbf{H} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbf{B} \tag{9.65}$$

and obeys the following properties :

- *It is a locally connected topos :*

As an adjoint triple of modalities, we therefore have two pairs of units and counits. First is the pair for $(\flat \dashv \sharp)$, which are components-wise

$$\begin{aligned} \epsilon^\flat X : \flat X &\to X &(9.66)\\ \eta^\sharp X : X &\to \sharp X &(9.67)\end{aligned}$$

which give us relations of objects and their discrete and codiscrete equivalents, and the pair for $(\int \dashv \flat)$

$$\begin{aligned} \epsilon^\flat X : \flat X &\to X &(9.68)\\ \eta^{\int} X : X &\to \textstyle\int X &(9.69)\end{aligned}$$

which deal with the relations of objects, their points and their connected components.

Those are called respectively the *pieces-to-points* transformations and the *points-to-pieces* transformation, which are meant to signify that the first one deals with how the counit maps connected components of the space to

For most of our cases here, the base topos will be consider will be the topos of sets, **Set**, so that it is best to understand cohesion in terms of functors to sets. The archetypical cohesion is done using the global section functor Γ :

Cohesion is a sublation of the ground opposition

Theorem 355 *If the terminal object* 1 *is connected,* $\mathrm{Hom}_{\mathbf{H}}(1,-)$ *preserves coproducts, then the point content of an object is given by any given subobject and its negation.*

$$\Gamma(A+\neg_X A)\cong\Gamma(X) \tag{9.70}$$

Proof 52 *From the preservation of coproducts, we have that*

$$\Gamma(A+\neg_X A)\cong\Gamma(A)+\Gamma(\neg_X A) \tag{9.71}$$

Preservation of pushout requires the geometric morphism to be surjective idk

Similarly $\flat(A+\neg_X A)\cong\flat(X)$?

First, we ask that $\int$ admits a definite negation. As this is an *sp*-unity, the two moments are meant to project on the same moment, so that the negation [...]

$$\int * \quad\cong\quad * \tag{9.72}$$

and $\int\to\int$ is an epimorphism.

Vague : $\flat$ maps to the points of X, and every point of $\int X$ is an image of one of those points. (point to pieces transform)

To give it its proper meaning of being about the connected components of

$$(\Pi_0\dashv\mathrm{Disc}):\mathbf{H}\xrightarrow{\Pi_0}\mathbf{Set}\xrightarrow{\mathrm{Disc}}\mathbf{H} \tag{9.73}$$

$$\mathrm{Hom}_{\mathbf{Set}}(\Pi_0(X),A)\cong\mathrm{Hom}_{\mathbf{H}}(X,\mathrm{Disc}(A)) \tag{9.74}$$

The hom-set of functions from our space to the discrete space from a set A is isomorphic to the set of functions from $\Pi_0(X)$ to that set.

Interpretation : Π_0 send each element - subobject in the same "connected component" to a different point.

Connected object : X is a connected object if the hom-set functor preserves coproducts

Shape preserves connected objects?

Shape modality : $\int = \mathrm{Disc} \circ \Pi_0$. *sp*-unity, so $\int$ and $\flat$ share the same space : $\int X$ is a discrete space : $\flat \int = \int$

In many cases, we will ask that the shape modality correspond to a retraction to a point of the (path)-connected components of the space. Using the usual notion of path connectedness, two points are path connected if there is a path $[0,1] \to X$ between them, in which case the space is contractible if for any two points, there is a path between them.

The categorical equivalent is to consider the path space object I, and localization by I, typically $\mathbb{R}$.

$$\int \cong \mathrm{loc}_I \tag{9.75}$$

Commutating diagram for adjoint quadruples :

for $(\Pi_0 \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc})$, $X \in \mathbf{H}$ and $S \in \mathbf{Set}$,

$$\begin{tikzcd}
\Gamma X \arrow[r, "\epsilon^{-1}_{\Gamma X}"] \arrow[d, "\Gamma(\eta_X)"'] \arrow[dr, "\theta_X"] & \Pi_0 \mathrm{Disc}\Gamma(X) \arrow[d, "\Pi_0(\epsilon_X)"] \\
\Gamma \mathrm{Disc}\Pi_0(X) \arrow[r, "\eta^{-1}_{\Pi_0(X)}"'] & \Pi_0(X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Disc} S \arrow[r, "\eta \mathrm{Disc} S"] \arrow[d, "\mathrm{Disc}(\epsilon^{-1}_S)"'] \arrow[dr, "\phi_S"] & \mathrm{CoDisc}\Gamma\mathrm{Disc}(S) \arrow[d, "\mathrm{CoDisc}(\eta^{-1}_S)"] \\
\mathrm{Disc}\Gamma\mathrm{CoDisc}(S) \arrow[r, "???"'] & \mathrm{CoDisc}(S)
\end{tikzcd}$$

Another intuitive property that we could ask of a topos is that its codiscrete objects are connected, that is

$$\int \sharp X \cong 1 \tag{9.76}$$

This is not generally true (**Set** is cohesive and does not obey it), not even for a sufficiently cohesive topos, but it is true in a category called codiscretely connected[83]

Definition 356 *A cohesive topos is* codiscretely connected *if for some set S, the unique map* $\Pi_0 \mathrm{CoDisc} S \to \{\bullet\}$ *is a monomorphism.*

This property is equivalent to it, as we can show that

Theorem 357 *A cohesive topos is codiscretely connected if and only if* $\int \mathrm{CoDisc}(2) \cong 1$

property

Theorem 358 *For any codiscrete object $\sharp X$, this object is connected and contractible :*

$$\int \sharp X \cong 1,\ \forall Y,\ \int (\sharp X)^Y \cong 1 \tag{9.77}$$

Proof 53 *Given a set S and its associated codiscrete object $\overline{S} = \mathrm{CoDisc}(S)$,*

[...]

Hierarchy :

Topos (every topos has a terminal geometric morphism with adjoint?), sheaf topos (geometric morphism?)

splitting : existence of a further left / right adjoint :

local topos (codisc adjoint), locally connected topos ()

essential topos? (???)

Theorem 359 *If a topos has a site with an initial object*

Theorem 360 *If a topos has a site with an initial and terminal object, it is both locally local and locally connected.*

Proof 54

Cohesive site

Definition 361 *Given a cospan composed of a point $x : 1 \to X$ and the counit of the $(\flat \dashv \sharp)$ adjunction, $\epsilon^\flat_X : \flat X \to X$, the* de Rham flat modality $\flat_{\mathrm{dR}}$ *is given by the pullback*

Interpretation : $\bar{\flat}\flat = \square_*$, ie the de Rham flat modality negates the flat modality. The de Rham flat modality of a discrete space is the terminal object.

For co-concrete objects, $\flat X \to X$ is an epimorphism

Example 362 *As we have that for the real line, $\flat\mathbb{R} \cong |\mathbb{R}|$, the discrete space with the cardinality of $\mathbb{R}$, given some point $x : 1 \to \mathbb{R}$ and the inclusion map $|\mathbb{R}| \to \mathbb{R}$, its de Rham flat modality is given by the dependent product*

$$\bar{\flat}\mathbb{R} = |\mathbb{R}| \times_{\mathbb{R}} 1 = 1 \tag{9.78}$$

which does give us $\flat\bar{\flat}\mathbb{R} = 1$

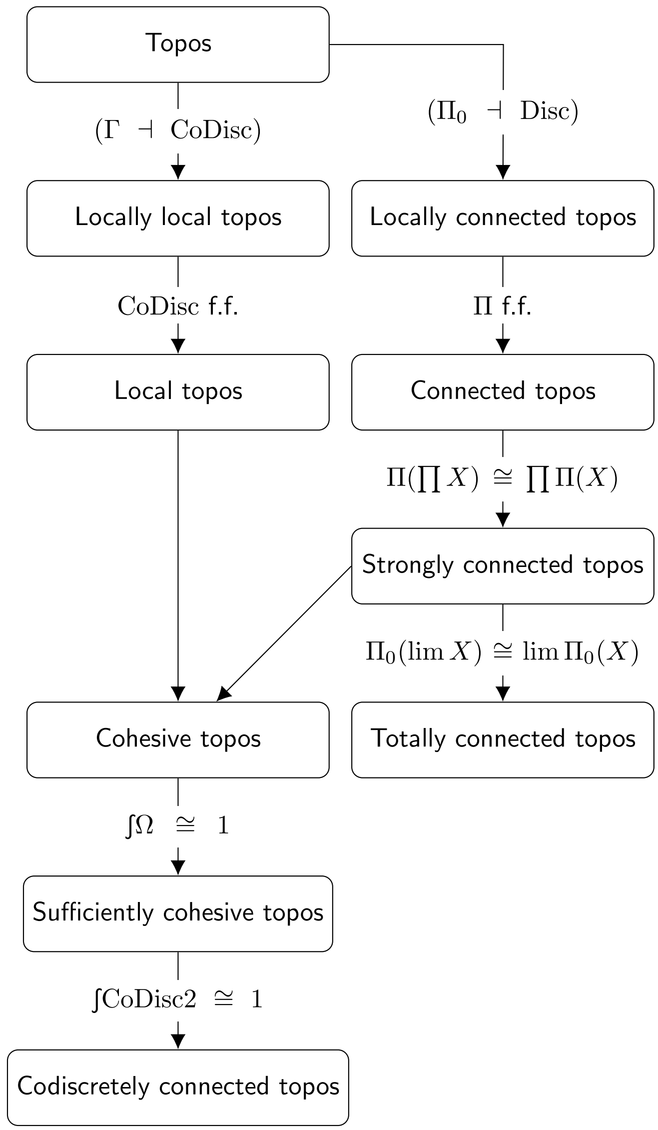


Figure 9.1: Hierarchy of cohesive topoi

The de Rham flat modality does not seem to be of much use, however it will reveal itself to be much more important later on for the case of higher category theory, and will earn its name of "de Rham" there.

9.4.1 Negation

The negation of the sharp modality :

$$\bar{\sharp}X = \mathrm{Cofib}(\sharp X \to X) \tag{9.79}$$

$$X \sqcup_{\sharp X} * \tag{9.80}$$

Universal property with this object : for any object Y such that there is a morphism $f : Y \to X$, there exists a unique function $\beta : Y \to \bar{\sharp}X$ such that

Negation of the flat modality?

$$\bar{\flat} = \mathrm{Fib}(X \to \flat X) \tag{9.81}$$

Negation of the shape modality

$$\bar{\int} X = \mathrm{Cofib}(\int X \to X) \tag{9.82}$$

9.4.2 Algebra

9.4.3 Concrete and co-concrete objects

Definition 363 *An object in a cohesive topos is* concrete *if the unit of the adjunction* $(\Gamma \dashv \mathrm{CoDisc})$ *is a monomorphism, so that*

$$X \hookrightarrow \sharp X \tag{9.83}$$

In terms of what we saw previously about concrete categories, the largest possible subcategory of $\mathbf{H}$ that is concrete is the category of all concrete objects. This is because for any functor $L : \mathbf{C} \to \mathbf{D}$ with a right adjoint R, the unit $X \to RLX$ is a monomorphism iff U is faithful on morphisms with target X, so Γ (forgetful functor) is faithful, therefore concrete

Proof 55 *By composing with the unit,*

$$\mathrm{Hom}_{\mathbf{C}}(X', X) \to \mathrm{Hom}_{\mathbf{D}}(LX', LX) \cong \mathrm{Hom}_{\mathbf{C}}(X', LRX) \tag{9.84}$$

function is injective etc

$$\mathbf{H} \underset{\iota_{\mathrm{conc}}}{\overset{\mathrm{conc}}{\rightleftarrows}} \mathbf{H}_{\mathrm{conc}} \rightleftarrows \mathbf{Set} \tag{9.85}$$

Definition 364 *If a morphism has a concrete codomain, ie $f : X \to Y$ is such that X is a concrete object, it is called* intensive*.*

This is the generalization of the notion used in thermodynamics, where the function is determined entirely by its value at points. We could see this as considering our function as associating values to points

For any subobject $S \hookrightarrow X$, the

Concrete objects and separated presheaves

Copresheaves of algebras?

Definition 365 *For an intensive morphism, we have [some isomorphism idk]*

Extensive objects : maximally non-concrete codomain, ie X is

9.4.4 Infinitesimal cohesive topos

One particular special type of topos regarding cohesion is the case where the points to pieces transform is an equivalence, ie

$$\flat \cong \int \tag{9.86}$$

In other words, for every connected component of a topos, there is a unique point in that piece. This leads to the collapse of the cohesion adjoint triple $(\int \dashv \flat \dashv \sharp)$ into a single modality $\natural$.

The notion of such spaces being infinitesimal comes from the general idea that infinitesimal spaces in the context of a topos are really just single points with a "halo" of pointless regions around them, such as the infinitesimal disks [...]

Beware of overextending this idea however as this is not always an appropriate interpretation. **Set** is such an "infinitesimal cohesive topos", but has no real extension around its points (although it could be understood as the topos over the infinitesimally thickened point of order 0, or the dual to the trivial Weil algebra that is just $\mathbb{R}$, ie just a point).

9.4.5 Logic

Definition 366 *A proposition $p : A \hookrightarrow X$ is* discretely true *if in the pullback*

$$\begin{tikzcd}
\sharp A\big|_X \arrow[r] \arrow[d] & \sharp A \arrow[d] \\
X \arrow[r, "\eta_X"] & \sharp X
\end{tikzcd}$$

$\sharp A\big|_X \to X$ is an isomorphism

Proposition that is true over discrete spaces.

Theorem 367 *If p*

9.4.6 Cohesion on sets

Set is trivially a cohesive topos, but it fails to have the stronger property of being *sufficiently cohesive*. If we consider the case where its base topos is itself, then we need to investigate the adjoint cylinder from its functor of global sections, ie

$$(\mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc}) : \mathbf{Set} \underset{\mathrm{Disc}}{\overset{\mathrm{CoDisc}}{\rightrightarrows}} \mathbf{Set} \xrightarrow{\Gamma} \mathbf{Set} \tag{9.87}$$

The global section functor, as the hom functor from the terminal object, is simply the identity on sets, as sets are entirely defined by their points. As the adjoints of the identity are always the identity, any adjunction string is simply going to be more identity functor. Our adjunction for cohesion is therefore

$$(\mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}}) \tag{9.88}$$

And its corresponding monads are just the identity monad.

$$(\mathrm{Id} \dashv \mathrm{Id} \dashv \mathrm{Id}) \tag{9.89}$$

From there, every property of cohesion is fulfilled simply by every limit and colimit being preserved by the identity functor and every unit and counit being appropriately mono or epi, as they are always isomorphisms.

The behaviour of Disc is about what we would expect for discrete objects, sending an object to the coproduct of terminal objects over its point content, in other words a set with the same cardinality. However, the codiscrete objects, being the same, does not seem to follow what we would expect of a codiscrete object, of being "one whole" in some sense, although they do obey the more decisive property of every function on its points lifting to a continuous function. Connected components are likewise about what we would expect since every point is its own component in the "topology" of a set. We have that every piece (a single point) has a point, and every point has its piece, and that this is a sublation of the ground by $\sharp 0 = 0$. That pieces of powers are powers of pieces is obviously true again due to the identity :

$$\Pi_0(X^{\mathrm{Disc}(S)}) = (\Pi_0 X)^S = X^S \tag{9.90}$$

The issue that prevents it from being a more intuitive cohesive topos is that of the connectedness of the subobject classifier, 2, as trivially, $\Pi_0 2 = 2$, and not 1 as we would need to. This stems from an obstruction found in [91], saying that a localic topos cannot both have a shape modality over sets that preserves the product and have a connected subobject classifier. [proof?]

This means that we cannot embed any object into some connected or contractible space, and in fact, there is no connected space outside of 1, meaning that its topology is fairly uninteresting as we would guess. In other words, there's no equivalence of our own "standard" Euclidian space, some simply connected space that can contain multiple disjoint objects.

As we've seen in the case of **Set**, there are two possible Lawvere-Tierney closure operators we could try over sets, and the $\mathrm{loc}_{\neg\neg}$ closure is the discrete topology, in the sense that every subset $S \subseteq X$ is its own closure, including singletons $\{\bullet\}$. No point is "in contact" with another, we can entirely separate a given element from the whole. The other closure operator is the trivial one $j(x) = 1$, giving us the *chaotic* topology on 1.

Equivalence between the lawvere-tierney topology and Grothendieck topology, chaotic grothendieck topology, collapse of sets into triviality?

As $\neg\neg = \mathrm{Id}_\Omega$, the localization does not do anything and the smallest dense subtopos is simply itself, there are no levels in between Set and the ground.

The global section functor $\Gamma(-) \cong \mathrm{Hom}_{\mathbf{Set}}(1, -)$ is simply the identity, as

$$\Gamma(S) \cong \mathrm{Hom}_{\mathbf{Set}}(1, S) \cong S \tag{9.91}$$

Since the hom-set of the point to a set is of the same cardinality as the set itself, and

$$\Gamma(f) \tag{9.92}$$

The left adjoint functor Disc here will work out as

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Disc}(-), -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, -) \tag{9.93}$$

while the right adjoint is

$$\mathrm{Hom}_{\mathbf{Set}}(-,-) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{9.94}$$

As far as the objects go, these are identities, so that the discrete and codiscrete objects of sets are the same objects (they have the same point content). However, the actions on morphisms does change, and most importantly, on the Lawvere-Tierney topology of the topos. For the morphism $j : \Omega \to \Omega$, $j \in \mathrm{Hom}_{\mathbf{Set}}(2, 2)$, we have

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(-),-) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{9.95}$$

The left adjoint modality $\flat \dashv \sharp$

"Note that in this example, the "global sections" functor $S \to Set$ is not the forgetful functor $Set/U \to Set$ (which doesn't even preserve the terminal object), but the exponential functor $\Pi U = \mathrm{Hom}(U, -)$. This is the direct image functor in the geometric morphism $Set/U \to Set$, whereas the obvious forgetful functor is the left adjoint to the inverse image functor that exhibits S as a locally connected topos."

Negation of sharp modality :

$$\begin{aligned} \bar{\sharp} X &= X \sqcup_{\sharp X} * && (9.96) \\ &= X \sqcup_{\sharp X} * && (9.97) \end{aligned}$$

Is it just $*$? The negation of the moment of continuity (maximally non-concrete object)

9.4.7 Cohesion of a spatial topos

Spatial topos always has disconnected truth values hence not a sufficiently cohesive topos cf Lawvere

9.4.8 Cohesion of smooth spaces

As a Grothendieck topos, the cohesiveness of smooth spaces relies on the cohesiveness of its site, the category of Cartesian spaces.

First, **CartSp** has the terminal object ($\mathbb{R}^0$), and every object $\mathbb{R}^n$ admits global sections (at least $0 : \mathbb{R}^0 \to \mathbb{R}^n$).**CartSp** is also cosifted in that it has finite products, ie for any $U_1, U_2 \subseteq \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$,

$$U_1 \times U_2 \subseteq \mathbb{R}^{n_1+n_2} \tag{9.98}$$

We therefore just have to prove that it is a locally connected site with its topology of differentiably good open covers, ie for any object $X \in \mathbf{CartSp}$, any covering sieve on X is a connected subcategory of $\mathbf{CartSp}_{/X}$.

Theorem 368 **Smooth** *is sufficiently connected.*

Proof 56

[...]

As a cohesive site, **CartSp** has the global section functor Γ which simply assigns to each open set of $\mathbb{R}^n$ its set of points

$$\Gamma(U_{\mathbf{CartSp}}) = U_{\mathbf{Set}} \tag{9.99}$$

and as its left adjoint the functor $\Gamma^* \cong \mathrm{LConst} \cong \mathrm{Disc}$ which associate to any set the smooth space composed by an equinumerous number of disconnected copies of $\mathbb{R}^0$,

$$\mathrm{Disc}(X) = \bigsqcup_{x \in X} \mathbb{R}^0 \tag{9.100}$$

This is equivalently the discrete (fine?) diffeology on a set, given by some

Its right adjoint is given by the coarse diffeology on a set

Right adjunction :

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(U), X) = \mathrm{Hom}_{\mathbf{CartSp}}(U, \mathrm{CoDisc}(X)) \tag{9.101}$$

Every possible function (as a set) between some Cartesian space U and our codiscrete space X correspond to a valid plot.

In terms of intuition, this means that every point is "next to" every other point in some sense. For instance, given the probe $[0,1]$, there is a smooth curve for every possible combination of points, ie the set of smooth curves is just $X^{[0,1]}$.

There is therefore no meaningful way to separate points (as we would expect from the smooth equivalent of the trivial topology).

Given these two functors, we can construct our two modalities. Our flat modality is

$$\flat = \mathrm{Disc} \circ \Gamma \tag{9.102}$$

which first maps a smooth space to its points and then to the coproduct of $\mathbb{R}^0$ over those points, giving us the discrete space $\flat X$, and the sharp modality $\sharp$ is

$$\flat = \mathrm{CoDisc} \circ \Gamma \tag{9.103}$$

which maps a smooth space to its points and then to the

Are all smooth spaces in the Eilenberg category diffeological spaces?

fine diffeology v. coarse diffeology

"Every topological space X is equipped with the continuous diffeology for which the plots are the continuous maps."

Negation of

Extensive and intensive quantities

9.4.9 Cohesion of classical mechanics

9.4.10 Cohesion of quantum mechanics

The terminal object in the Bohr topos is given by the constant sheaf which maps all commutative operators in a context to a spectrum of a single element, the singleton $\{\bullet\}$ in **Set**. This is the spectral presheaf which has a spectrum of a single value for all

Global section functor : hom-set of

$$\Gamma : \mathbb{C}^0 \to \tag{9.104}$$

$$\Gamma : \mathbb{C}^0 \to \tag{9.105}$$

9.4.11 Sublation

Sublation of $(\int \dashv \flat)$ to $(\Im \dashv \mathrm{Et})$

Sublation :

$$X \to \Im \to \int X \tag{9.106}$$

Partial contraction?

9.5 Elastic substance

A further sublation we can get is that of ($\int \dashv \flat$), the opposition given by

$$\mathbf{H}_\Gamma \xrightarrow{\mathrm{Disc}} \mathbf{H} \underset{\Pi_0}{\overset{\Gamma}{\rightrightarrows}} \mathbf{H}_\Gamma \tag{9.107}$$

As ($\int \dashv \flat$) is an *sp*-unity, any sublation of it will be automatically a left and right sublation. We will call this sublation ($\Im \dashv \&$), and from the properties described, we will have

$$\begin{aligned} \Im\int &= \int && (9.108)\\ \&\flat &= \flat && (9.109)\\ \&\int &= \int && (9.110)\\ \Im\flat &= \flat && (9.111) \end{aligned}$$

Those modalities will therefore not act on any space contracted to its connected components or reduced to a discrete space. They are therefore in some sense modalities about the continuous structure of a space.

[92, 93]

to , $\bigcirc \dashv \Box$ opposition

$$X \to \Im X \to \int X \tag{9.112}$$

Differential cohesion

None of the topos we have seen thus far are differentially cohesive, but it is simple enough to extend them to be. This is generally done concretely by changing the site to include an infinitesimal structure. The basic example for this is the site of *formal Cartesian spaces*, **FormalCartSp**, which is the site with objects being open sets of $\mathbb{R}^n$ composed with an *infinitesimally thickened point*, D. As we will see, D is a point if we forget the elastic structure, $\Re(D) \cong *$, but in basic mathematical terms, this relates to Weil algebras, the algebras of infinitesimal objects.

The basic example of a Weil algebra is the algebra of *dual numbers*, a vector space composed by a pair of real numbers, $\mathbb{R}[\varepsilon]/\varepsilon^2$

$$(x + \epsilon y)^2 = x^2 + xy\varepsilon \tag{9.113}$$

As a copresheaf?

Its formal dual is the spectrum

$$D = \mathrm{Spec}(\mathbb{R}[\varepsilon]/\varepsilon^2) \tag{9.114}$$

which is the first order infinitesimally thickened point in one dimension (also sometimes called a "linelet", an infinitesimally small line).

[Open subset $D(f) = \mathrm{Spec}(A) \setminus V(f)$, $V(f)$ all prime ideals containing f, then $D(f)$ is empty if f is nilpotent?]

As a topos?

The topos from that site is the *Cahier topos*,

$$\mathbf{Cahier} = \mathrm{Sh}(\mathbf{FormalCartSp}) \tag{9.115}$$

[94, 95, 96]

9.5.1 Synthetic infinitesimal geometry

Probably the most common model of differential cohesion in topos theory is the one given by synthetic differential geometry.

Definition 369 *On a topos T with a ring object R, if there exists an object D such that*

$$D = \{x \in R \mid x^2 = 0\} \tag{9.116}$$

for which the canonical map

$$\begin{aligned} R \times R &\to R^D && (9.117)\\ (x, d) &\mapsto (\varepsilon \mapsto x + \varepsilon d) && (9.118) \end{aligned}$$

is an isomorphism of objects.

Koch-Lawvere axioms

This axiom cannot work properly in the context of a boolean internal logic. Otherwise, if we picked two different "points" in D, given a function $g : D \to R$, we would have

$$x \tag{9.119}$$

9.5.2 Non-standard analysis

While a perfectly good model for differential structures in a topos, there are other alternative models to synthetic differential geometry outside of formal spaces.

As a space, infinitesimally thickened points are entirely structureless. Even beyond the single point, it also has a completely trivial mereology, so that its only differentiating aspect from a normal point is given by its sheaf structure. Unlike actual points, there are multiple possible maps $X \to D$, including the core $\mathrm{Aut}(D)$

[Due to the nilpotence of the algebra?]

An alternative to this is to have invertible infinitesimals

9.5.3 Differential cohesion of the Cahier topos

The various topos we have seen previously will not typically be models of differential cohesion,

Formal Cartesian space **FormalCartSp**

9.5.4 Crystalline cohomology

9.6 Solid substance

Given the opposition $\mathfrak{R}$ and $\mathfrak{I}$, a *ps*-unity, we seek another sublation, ie we want a further opposition $(\mathfrak{R}' \dashv \mathfrak{I}')$ obeying

$$\mathfrak{R}'\mathfrak{R} = \mathfrak{R} \tag{9.120}$$
$$\mathfrak{I}'\mathfrak{I} = \mathfrak{I} \tag{9.121}$$

We will look here specifically at a left sublation, so that furthermore

$$\mathfrak{I}'\mathfrak{R} = \mathfrak{R} \tag{9.122}$$

We will call the opposition of this sublation the opposition of *solidity*, given by the bosonic modality $\rightsquigarrow$ and the rheonomic modality Rh,

$$(\rightsquigarrow \dashv \mathrm{Rh}) \tag{9.123}$$

where the sublation gives us the identities

$$\rightsquigarrow \mathfrak{R} = \mathfrak{R} \tag{9.124}$$
$$\mathrm{Rh}\mathfrak{I} = \mathfrak{I} \tag{9.125}$$
$$\mathrm{Rh}\mathfrak{R} = \mathfrak{R} \tag{9.126}$$

Meaning that those modalities have no effect on a reduced space and that the rheonomic modality has no effect on an infinitesimal space.

$$\mathfrak{R}X \to \overset{\rightsquigarrow}{X} \to X \tag{9.127}$$

$$x \tag{9.128}$$

Bosonic v. fermionic spaces

Definition 370 *A k Grassmann algebra on a k-vector space is an algebra defined by the anticommuting product*

$$x \wedge y = -y \wedge x \tag{9.129}$$

As this is a non-commuting algebra, the interpretation of its dual as a space is much more complicated. There is no obvious way to define a sober space from a non-commutative algebra[97], nor a locale[98], nor a sheaf[99], so that the geometric interpretation of that dual should be taken cautiously.

This dual space is the superpoint :

Definition 371 *The* superpoint *is the formal dual to the Grassmann algebra [...]*

What is its lattice structure

The general naming scheme of the opposition of solidity is meant to reflect Hegel's talking points on light and matter, not from the Science of Logic but Hegel's Encyclopaedia of the Philosophical Sciences [100].

"Light behaves as a general identity, initially in this determination of diversity, or the determination by the understanding of the moment of totality, then to concrete matter as an external and other entity, as to darkening. This contact and external darkening of the one by the other is colour."

meant to mirror the parallel that light relates to the bosonic moment as it is a bosonic particle while matter is of a fermionic nature.

This is probably the least convincing parallel in the hierarchy of nlab's objective logic here, as while the rigidity of matter can be connected to its fermionic nature, this is not something that can be deduced simply from the raw properties of fermionic spaces and is very much dependent on the model of physics that we use, which is a much more empirical fact than purely metaphysical. It is probably just included as solidity for purely aesthetic reasons, much like the I Ching notation for the classification of cohesive toposx

9.7 Higher order objective logic

We have done everything thus far in the context of category theory, but a generalization of this can be done in the context of ∞-categories. This will be useful as the most interesting parts of physics work best in the context of category theory where we use the full ∞-category context to account for gauge groups in a more natural way.

In the case of being and nothingness, this will be done with the terminal ∞-topos, which is the topos defined by a single k-morphism for every level k connecting level $k-1$ morphisms to themselves. There is not a lot to say on this level, as most everything related to this unity is roughly similar to the 1-categorical case.

[Negation of the nothing modality?]

The next level is given by the localization by the double negation.

[...]

If we consider the subtopos $\mathrm{Sh}_\infty(\mathbf{1}) \cong \infty\mathbf{Grpd}$ here, the basic unity is given by the global section ∞-functor, which maps every

and the functor of locally constant ∞-stacks LConst, which given a groupoid $\mathcal{G} \in \infty\mathbf{Grpd}$ associates the

$$\mathrm{LConst}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{LConst}(?)) \tag{9.130}$$

For cohesion, our subtopos will be some boolean ∞-category, typically the $\infty\mathbf{Grpd}$ category. As in the 1-categorical case, any ∞-stack ∞-topos admits a canonical geometric morphism and a left adjoint for it,

$$(\mathrm{LConst} \dashv \Gamma) : \mathbf{H} \underset{\mathrm{LConst}}{\overset{\Gamma}{\rightleftarrows}} \infty\mathbf{Grpd} \tag{9.131}$$

As Π_0 refers specifically to the fundamental group, ie the first homotopy group of the space, the higher order case will be Π, the full fundamental groupoid of the space.

Example 372 *On the circle S^1, defined by some smooth 0-type sheaf $S^1(\mathbb{R}^1) = 3$ with the usual good cover on a circle,*

Chapter 10

Nature

10.1 Mechanics topos

The general topos typically used by nlab to respond to the requirements of all those modalities is the *super formal smooth infinity-groupoid*. This is the ∞-sheaf on a special site composed from the category of Cartesian spaces (to give the cohesion), the category of infinitesimally thickened points (to give the elasticity), and the category of superpoints (to give the solidity).

We have already seen the category of Cartesian spaces in detail, so let's now look into infinitesimally thickened points.

Infinitesimally thickened points are a geometrical realization of the formal notion of infinitesimals as provided by Weil algebras.

Example : dual numbers

$$(x, \epsilon) \mapsto f(x, \epsilon) = f(x) \tag{10.1}$$

10.2 Gauge theory

Chapter 11

The Ausdehnungslehre

One early attempt at mathematization of similar philosophical notions was the Ausdehnungslehre of Grassmann[5]

[101, 102, 103]

Chapter 12

Kant's categories

[104]

Quantity : unity, plurality, totality

Quality : reality, negation, limitation

Relation : inherence and subsistence, causality and dependence, community

Modality : possibility, actuality, necessity

Chapter 13

Lauter einsen

Cantor's original attempt at set theory[105] involved the notion of *aggregates* (*Mengen*), which is what we would call a sequence today, some ordered association of various objects. If we have objects $a, b, c, \ldots$, their aggregate M is denoted by

$$M = \{a, b, c, \ldots\} \tag{13.1}$$

Despite the notation reminiscent of sets, the order matters here. The notion closer to that of a set is given by an abstraction process $\overline{M}$, in which the order of elements is abstracted away (this would be something akin to an equivalence class under permutation nowadays).

The *cardinality* of the aggregate is given by a further abstraction, $\overline{\overline{M}}$, given by removing the nature of all of its element, leaving only "units" behind :

$$\overline{\overline{M}} = \{\bullet, \bullet, \bullet, \ldots\} \tag{13.2}$$

$\bullet$ here is an object for which all characteristics have been removed, and all instances of $\bullet$ are identical. In some sense this is the application of the being modality on its objects : we only have as its property that the object exists, similarly to *das eins*.

Comment from Zermelo [106, p. 351]:

"The attempt to explain the abstraction process leading to the "cardinal number" by conceiving the cardinal number as a "set made up of nothing but ones" was not a successful one. For if the "ones" are all different from one another, as they must be, then they are nothing more than the elements of a newly introduced set that is equivalent to the first one, and we have not made any progress in the abstraction that is now required."

Relation to the discrete/continuous modality

From Lawvere : The maps between the *Menge* and the *Kardinal* is the adjunction

$$(\text{discrete} \dashv \text{points}) : M \underset{\text{discrete}}{\overset{\text{points}}{\rightleftarrows}} K \tag{13.3}$$

The functor points maps all elements of an object in M to the "bag of points" of the cardinal in K, the functor discrete sends back

[107]

Chapter 14

Spaces and quantities

From Lawvere[108]

Distributive v. other categories

Intensive / extensive

"The role of space as an arena for quantitative "becoming" underlies the qualitative transformation of a spatial category into a homotopy category, on which extensive and intensive quantities reappear as homology and cohomology."

Definition 373 *A* distributive category **C** *is a category with finite products and coproducts such that the canonical distributive morphism*

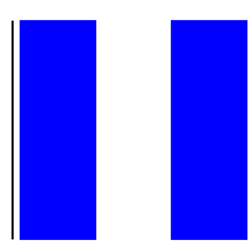
$$(X \times Y) + (X \times Z) \to X \times (Y + Z) \tag{14.1}$$

is an isomorphism, ie there exists a morphism

$$X \times (Y + Z) \to (X \times Y) + (X \times Z) \tag{14.2}$$

that is its inverse.

Distributive categories are typically categories that are "like a space" in some sense, in the context that concerns us. In terms of physical space, we can visualize it like this :



Whether we compose this figure by first spanning each red line along the black line and then summing them, or first summing the two red lines and spanning them along the black line does not matter and will give the same figure.

Example 374 *If a category is Cartesian closed and has finite coproducts, it is distributive.*

Proof 57 *In a Cartesian category, the Cartesian product functor $X \times -$ is a left adjoint to the internal hom functor $[X, -]$. As colimits are preserved by left adjoint tmp, we have*

$$(X + Y) \tag{14.3}$$

This means in particular that any topos is distributive, such as **Set** and **Smooth**.

Example 375 *The category of topological spaces* **Top** *is distributive.*

Proof 58 *In* **Top***, the product is given by spaces with the product topology,*

$$(X_1, \tau_1) \prod (X_2, \tau_2) = (X_1 \times X_2, \tau_1 \prod \tau_2) \tag{14.4}$$

where the product of two topologies is the topology generated by products of open sets in X_1, X_2 :

$$\tau_1 \prod \tau_2 = \{U \subset\} \tag{14.5}$$

[...] and the coproduct is the disjoint union topology :

[Category of frames?]

As can be seen, those are some of the most archetypical categories of spaces.

Definition 376 *A* linear category *is a category for which the product and coproduct coincide, called the* biproduct*. For any two objects $X, Y \in \mathbf{C}$, the biproduct is*

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y \tag{14.6}$$

Proposition 377 *A linear category has a zero object 0, which is both the initial and terminal object.*

Proof 59 *This stems from the equivalence of initial and terminal objects as the product and coproduct over the empty diagram.*

Proposition 378 *In a linear category, there exists a zero morphism $0 : X \to Y$ between any two objects X, Y, which is the morphism given by the terminal object map $0 \to Y$ and the initial object map $X \to 0$:*

$$0_{X,Y} : X \to 0 \to Y \tag{14.7}$$

A linear category is so called due to its natural enriched structure over commutative monoids.

Proposition 379 *For any two morphisms $f, g : X \to Y$ in a linear category, there exists a morphism $f \oplus g$ defined by the sequence*

$$X \to X \times X \cong X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \cong Y + Y \to Y \tag{14.8}$$

Proposition 380 *The sum of two morphisms is associative and commutative*

Proposition 381 *The zero morphism is the unit element of the sum.*

"in any linear category there is a unique commutative and associative addition operation on the maps with given domain and given codomain, and the composition operation distributes over this addition; thus linear categories are the general contexts in which the basic formalism of linear algebra can be interpreted."

Definition 382 *If in a linear category every morphism $f : X \to Y$ has an inverse denoted $-f : X \to Y$, such that $f \oplus -f = 0$, then it is enriched over the category of Abelian groups Ab, and is called an* additive category*.*

Chapter 15

Parmenides

Arguments from Zeno & Parmenides :

"All objects are similar to each other and all objects are different from each other"

Parmenides proceeded: If one is, he said, the one cannot be many? Impossible. Then the one cannot have parts, and cannot be a whole? Why not? Because every part is part of a whole; is it not? Yes. And what is a whole? would not that of which no part is wanting be a whole?

Certainly. Then, in either case, the one would be made up of parts; both as being a whole, and also as having parts?

To be sure. And in either case, the one would be many, and not one? True. But, surely, it ought to be one and not many? It ought. Then, if the one is to remain one, it will not be a whole, and will not have parts?

No. But if it has no parts, it will have neither beginning, middle, nor end; for these would of course be parts of it.

Right. But then, again, a beginning and an end are the limits of everything?

Certainly. Then the one, having neither beginning nor end, is unlimited? Yes, unlimited. And therefore formless; for it cannot partake either of round or straight.

But why? Why, because the round is that of which all the extreme points are equidistant from the centre?

Yes. And the straight is that of which the centre intercepts the view of the extremes?

True. Then the one would have parts and would be many, if it partook either of a straight or of a circular form?

Being is indivisible, since it is equal as a whole; nor is it at any place more, which could keep it from being kept together, nor is it less, but as a whole it is full of Being. Therefore it is as a whole continuous; for Being borders on Being.

Idea : the non-existence of non-being, no initial object, leading to a lack of subobjects due to the impossibility of negation

Chapter 16

Bridgman and identity

what the fuck is that about [12, 109]

”We must, for example, be able to look continuously at the object, and state that while we look at it, it remains the same. This involves the possession by the object of certain characteristics — it must be a discrete thing, separated from its surroundings by physical discontinuities which persist.”

Definition of a system?

Chapter 17

Ludwig Gunther

Relation of moments to Ludwig's structuralism?

Axiomatization from Ludwig [110]

[111]

Chapter 18

Dialectics of nature

When we look at dialectical logic in practice, ie [14], the examples given are much more concrete. We are considering the identity of some *entity*, such as an object, a group, etc, and considering what it means for at entity to be identical to itself. All concrete entities are never identical to themselves, either in time, context, etc.

From Engels :

The law of the transformation of quantity into quality and vice versa; The law of the interpenetration of opposites; The law of the negation of the negation.

Example : an object moving in space, an organization changing, ship of Theseus, etc

To keep things concrete, let's try to consider a simple example of both category theory and dialectical logic, which is an object in motion. We will simply consider the kinematics here and not the dynamics as this is unnecessary.

The simplest case is the $(1+1)$-dimensional case, of a point particle moving along a line $x : L_t \to L_s$.

First notion : We are considering the "identity" of an object under a certain *lens* (ie with respect to its relations with a number of other objects). Its identity is only assured under the full spectrum of those relationships, ie we say that two objects A, B are identical if

$$\forall X \in \mathbf{H},\ \mathrm{Hom}_{\mathbf{H}}(A, X) \cong \mathrm{Hom}_{\mathbf{H}}(B, X),\ \mathrm{Hom}_{\mathbf{H}}(X, A) \cong \mathrm{Hom}_{\mathbf{H}}(X, B) \quad (18.1)$$

Relation to Yoneda? Relation to Hegel's "only the whole is true" thing?

This corresponds to the hom covariant and contravariant functor

$$h^A \cong h^B,\ h_A \cong h_B \tag{18.2}$$

Two objects are identical if their hom functors are naturally isomorphic, ie if there exists a natural transformation

$$\eta : h^A \Rightarrow h^B,\ \epsilon : h_A \Rightarrow h_B \tag{18.3}$$

with two-sided inverses each.

Yoneda embedding :

$$\begin{aligned}
\mathrm{Nat}(h_A, h_B) &\cong \mathrm{Hom}_{\mathrm{H}}(B, A) && (18.4)\\
\mathrm{Nat}(h^A, h^B) &\cong \mathrm{Hom}_{\mathrm{H}}(A, B) && (18.5)\\
& && (18.6)
\end{aligned}$$

those isomorphisms are elements of this, therefore isomorphic to morphisms from A to B. Therefore A and B are identical in term of their relationships to every other object if they are identical in the more typical sense of the existence of an isomorphism.

As those are functors, this analysis also applies to elements of the objects X, or subobjects. For any two monomorphisms

$$x, y : S \to X \tag{18.7}$$

We say that those elements are identical if the hom functors applied to them [...]

The breakage of the law of identity comes by considering different perspectives. If given an object X [or element $x : \bullet \to X$?], we attempt to use different relations with other objects to "probe" it, its moments and the assessment of its identity will change.

[Abstract example ?]

Expression via the subobject classifier of the topos

Example : case of motion. What does it means for a moving object to *be* in different positions.

Ex : consider the position of an object wrt two different time intervals $[t_a, t_b]$, $[t'_a, t'_b]$, rather than all possible intervals.

Characterization by observables at those different times

Observables : $O_i : \mathrm{Conf} \to \mathbb{R}$ [Isbell duality thing idk]

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