Commentary on nLab's Science of Logic and other matters

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Couples are things whole and things not whole, what is drawn together and what is drawn asunder, the harmonious and the discordant. The one is made up of all things, and all things issue from the one.

Heraclitus

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Chapter 1

Introduction

While not a new phenomenon by any mean, there is a certain recent trend of trying to mathematize certain philosophical theories, in particular ideas relating to dialectics.

An important focus of dialectics is to consider oppositions between concepts. In the case of metaphysics, those oppositions can be for instance

- One / Many
- Sameness / Difference
- Being / Nothing
- Space / Quantity
- General / Particular
- Objective / Subjective
- Qualitative / Quantitative

The oldest of such thoughts goes back to ancient greek traditions, the opposition between the ideas of Parmenides[1] and Heraclitus.

Medieval example : Nicholas of Cusa, De Docta Ignorantia (on learned ignorance) [2].

"Now, I give the name "Maximum" to that than which there cannot be anything greater. But fullness befits what is one. Thus, oneness—which is also being—coincides with Maximality. But if such oneness is altogether free from all relation and contraction, obviously nothing is opposed to it, since it is Absolute Maximality. Thus, the Maximum is the Absolute One which is all things. And all things are in the Maximum (for it is the Maximum); and since nothing is opposed to it, the Minimum likewise coincides with it, and hence the Maximum is also in all things. And because it is absolute, it is, actually, every possible being; it contracts nothing from things, all of which [derive] from it."

The notion of Haecceity by Scott Dunn

"Because there is among beings something indivisible into subjective parts—that is, such that it is formally incompatible for it to be divided into several parts each of which is it—the question is not what it is by which such a division is formally incompatible with it (because it is formally incompatible by incompatibility), but rather what it is by which, as by a proximate and intrinsic foundation, this incompatibility is in it. Therefore, the sense of the questions on this topic [viz. of individuation] is: What is it in [e.g.] this stone, by which as by a proximate foundation it is absolutely incompatible with the stone for it to be divided into several parts each of which is this stone, the kind of division that is proper to a universal whole as divided into its subjective parts?"

The first major author for the exact field that we will broach here is Kant and his transcendental logic [3]

The main author for these recent trends target is Hegel and his Science of Logic [4], where he describes his *objective logic*. The "classical" logic originally described by Aristotle, the logic of propositions and such, is described under the term of *subjective logic*

Heidegger?

The original attempt at the formalization of those ideas (or at least ideas similar to it) was by Grassmann[5], giving his theory of extensive quantities [vector spaces], which while it was commented on and inspired some things, mostly did not go much further.

In modern time, this programme was originally started by Lawvere[6]

"It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy."

nlab[7]

[8, 9, 10, 11, 12, 13]

As is traditional for such types of philosophy, the writings are typically fairly abstract and lacking example. For a more pedagogical exposition, I have tried here to include more examples and demonstrations to such ideas. I am not an expert in algebraic topology by any mean and have tried my best to explain those notions without using notions from this field.

Note that while some of these ideas could be argued to be fairly faithful translations of the philosophical ideas, others seem more to be generally inspired by them, the original notion of unity of opposites being more of a collection of different ideas on that theme than a rigorous construction. While such notions as quantity from the abstraction to pure being seem to have some parallels, I do not believe that Hegel had particularly in mind the concept of a graded algebra when he spoke of the opposition between das Licht and die Körper des Gegensatzes (in particular, this opposition is true regardless of dynamics, and therefore should not be particularly relevant to the solidity of a body), but if the opposition can somehow mirror the adjunction of bosonic and fermionic modalities, why not look into it. The notion of being-for-itself and being-for-one are [according to X] more related to consciousness than etc.

The notions described here are furthermore not firmly rooted in the formalism but merely described by it, as many of these notions are already needed to *define* the formalism. In particular, it is difficult to define any theory without the concept of discrete objects (in our case, by rooting the formalism of categories in the notion of sets, and in fact in general in any thought process requiring to have different ideas as discrete entities). The rooting of the notion of oneness in terminal objects are somewhat superficial, as the very notion of a terminal object already requires the notion of oneness, ie in the unique morphism demanded by universal properties.

Actual applications of dialectical logic [14, 15] also seem fairly disconnected from the formalism developed here, so that we will have to look into it separately.

It is therefore best to keep in mind that

[12, 16, 17, 18, 19, 20, 21, 22]

Transitions Into, With, and From Hegel‘s Science of Logic

Before going into those various formalizations, we will first have a rather in depth look at the formalisms on which these are based, which are type theory and category theory.

Chapter 2

Types

One basic element of the theory discussed is that of *types*[23], which will relate to the notion of categories and logic later on [trinitarity]

Relation with whatever Kant idk

A *type* is, as the name implies, a sort of object that some mathematical object can be. We denote that an object c is of type C by

$$c : C \tag{2.1}$$

and say that c is an *instance* of type C.

The typical simple example, as used in mathematics and computation theory, is that of integers. An integer n is an instance of the integer type, $n : \mathbb{N}$.

Types being themselves a mathematical object, we also have some type for types Type, although to avoid some easily foreseeable Russell style paradox (called the Girard paradox[24]), we will instead use some hierarchy of such types, called type universes :

$$C : \mathrm{Type}_0, \mathrm{Type}_0 : \mathrm{Type}_1, \mathrm{Type}_1 : \mathrm{Type}_2, \ldots \tag{2.2}$$

Although as we will not really require much forray into the hierarchy of type universes, we will simply refer Type_0 as Type.

Now given two types $A, B \in \mathrm{Type}$, we can define another type called the *function type* of A to B :

$$f : A \to B \tag{2.3}$$

Sequent calculus with types? Logic?

2.1 Dependent types

We speak of *dependent types* for a type that depends on a value, ie a ”type” that is actually a function from one type to the universe of types,

$$a : A \vdash B(a) : \mathrm{Type} \tag{2.4}$$

B as a whole is the dependent type of A, with each instance $B(a)$

2.2 Martin-Löf type theory

The basis for our type theory will usually be some Martin-Löf type theory, which corresponds to intuitionistic logic in the trinitarianism view, and is generally a rather universal sort of approach to logic (also corresponds to topos). [25]

There are three basic types for it, called the *finite types* :

- The zero type **0**, or empty type $\varnothing$ or $\perp$, which contains no terms.
- The one type **1**, or unit type $*$, which contains one canonical term.
- The two type **2**, which contains two canonical term.

Empty type for nothingness, something that doesn't exist

Unit type for existence

Two type for a choice between two values, such as boolean values.

As with any type theory, those types also give us function types

$\nvdash \to \nvdash$: the empty function (no term or 1 term?) $\nVdash \to \nVdash$: $\nVdash \to \nvDash$:

In addition to these types, we have a variety of *type constructors*, which allow us to construct additional types from those basic types.

Sum type constructor : $\sum$

The sum of two types gives us a pair of the individual types, ie

$$a : A, b : B \vdash (a, b) : A \times B \tag{2.5}$$

Dependent sum : the type of the second element might depend on the value of the first.

$$\sum_{n:\mathbb{N}} \mathrm{Vect}(\mathbb{R}, n) \tag{2.6}$$

Indexed sets

Product type $\prod$

Equality type

Inductive type

2.3 Homotopy types

A further refinement of type theory is the notion of *homotopy type*, where in addition to identity types, we also include the more general notion of equivalence types.

Definition 1 *Two types* $A, B : \mathrm{Type}$ *are said to be* equivalent*, denoted* $A \cong B$*, if there exists an equivalence between them.*

$$(A = B) \to (A \cong B) \tag{2.7}$$

Univalence axiom :

$$(A = B) \cong (A \cong B) \tag{2.8}$$

Correspondence between type theory and category theory :

- A universe of types is a category
- Types are objects in the category $T \in \mathrm{Obj}(C)$
- A term $a : A$ of A is a generalized element of A
- The unit type $*$ if present is the terminal object
- The empty type $\varnothing$ if present is the initial object
- A dependent type $x : A \dashv B(x) : \mathrm{Type}$ is a display morphism $p : B \to A$, the fibers $p^{-1}(a)$ being the dependent type at $a : A$.

2.4 Modalities

Chapter 3

Categories

As is often the case in foundational issues in mathematics, the foundations used to define mathematics can easily become circular. In our case, although category theory can be used to define set theory and classical logic, as well as your other typical foundational field like model theory, type theory, computational theory, etc, we still need those concepts to define category theory itself. In our case we will simply use implicitly classical logic [the external logic?] and some appropriate set theory like ETCS[26] (as ZFC set theory will typically be too small to talk of important categories).

Definition 2 *A* category $\mathbf{C}$ *is a structure composed of a class of objects* $\mathrm{Obj}(\mathbf{C})$ *and a class of morphisms* $\mathrm{Mor}(\mathbf{C})$ *such that*

- *There exists two functions* $s, t : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{C})$*, the* source *and* target *of a morphism. If* $s(f) = X$ *and* $t(f) = Y$*, we write the morphism as* $f : X \to Y$*.*
- *For every object* $X \in \mathrm{Obj}(\mathbf{C})$*, there exists a morphism* $\mathrm{Id}_X : X \to X$*, such that for every morphism* g_1 *with* $s(g_1) = X$ *and every morphism* g_2 *with* $t(g_2) = X$*, we have* $g_1 \circ \mathrm{Id}_X = g_1$ *and* $\mathrm{Id}_X \circ g_2 = g_2$*.*
- *For any two morphisms* $f, g \in \mathrm{Mor}(\mathbf{C})$ *with* s

To simplify notation, if there is no confusion possible, we will write the set of objects and the set of morphisms as the category itself, ie :

$$X \in \mathrm{Obj}(\mathbf{C}) \quad := \quad X \in \mathbf{C} \tag{3.1}$$
$$f \in \mathrm{Mor}(\mathbf{C}) \quad := \quad f \in \mathbf{C} \tag{3.2}$$

Categories are often represented, in totality or in part, by *diagrams*, a directed graph in which objects form the nodes and morphism the edges, such that the direction of the edge goes from source to target.

Throughout this section we will use a variety of common categories for examples. Some of them will be seen in more details later on as well in the section on the categories we will investigate for objective logic 5.

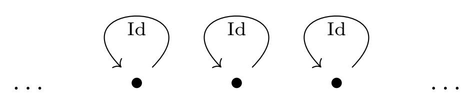
Before we go on detailing examples of categories, first a quick note on skeletonized categories. It is common in category theory to more or less assume the identity of objects that are isomorphic. This is not necessarily the case formally speaking (the category of sets can be seen as having multiple isomorphic sets in it, like $\{0, 1\}$, $\{1, 2\}$, etc), but for some purposes (such as trying to get a broad view of that category) it will be useful to consider the category where the set of objects is given as the equivalence class up to isomorphism.

Definition 3 *A category* $\mathbf{C}$ *is* skeletal *if all isomorphisms are identities, and the* skeleton *of a category* $\mathbf{C}$*, written* $\mathrm{sk}(\mathbf{C})$*, is given by the equivalence class of objects up to their isomorphisms, ie*

$$\mathrm{Obj}(\mathrm{sk}(\mathbf{C})) = \{[X] | \forall X' \in [X],\ \exists f \in \mathrm{iso}(\mathbf{C}),\ f : X \to X'\} \tag{3.3}$$

3.1 Examples

A few categories can easily be defined in categorical terms alone, such as the *empty category* $\varnothing$, which is the category with no objects and no morphisms (with the empty diagram as its diagram). We also have the *discrete categories* $\mathbf{n}$ for $n \in \mathbb{N}$, which consist of all the categories of exactly n objects and n morphisms (the identities of each object)

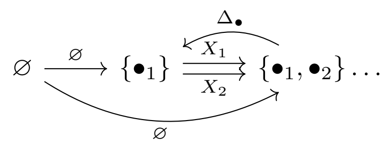


The empty category is in fact itself the discrete category $\mathbf{0}$.

Many categories can also be defined using typical mathematical structures built on set theory, using sets as objects and functions as morphisms.

Example 4 *The category of sets* **Set** *has as its objects all sets (*$\mathrm{Obj}(\mathbf{Set})$ *is the* class *of all sets), and as its morphisms the functions between those sets.*

If we consider the skeletonized version of **Set**, where we only consider unique sets of a given cardinality, its diagram will look something like this for the first three elements classified by cardinality :



Example 5 *The category of topological spaces* **Top** *has as its objects the topological spaces* (X, τ_X)*, and as morphisms* $f : (X, \tau_X) \to (Y, \tau_Y)$ *the continuous functions between two such spaces.*

Example 6 *The category of vector spaces* $\mathbf{Vect}_k$ *over a field* k *has as its objects the vector spaces over* k*, and as its morphisms the linear maps between them. The hom-set between* V_1 *and* V_2 *is therefore the set of linear maps* $L(V_1, V_2)$ *(see later for enriched category)*

Diagram : finite dimensional case classified by dimension

$$k^0 \xrightarrow{x \in k} k \xrightarrow{L} k^2 \dots$$

Example 7 *The category of rings* **Ring** *has as its objects rings, and as morphisms ring homomorphisms.*

Example 8 *The category of groups* **Grp** *has as its objects groups* G*, and as its morphisms group homomorphisms.*

Example 9 *The category of smooth Cartesian spaces* **CartSp** *has as its objects subsets of* $\mathbb{R}^n$*, and as morphisms smooth maps between them.*

Besides concrete categories, another common type of categories is *partial orders*, which are defined as usual in terms of sets, ie a partial order $(X, \leq)$ is a set X with a relation $\leq$ on $X \times X$, obeying

- Reflexivity : $\forall x \in X,\ x \leq x$
- Antisymmetry : $\forall x, y \in X,\ x \leq y \wedge y \leq x \to x = y$
- Transitivity : $\forall x, y, z \in X,\ x \leq y \wedge y \leq z \to x \leq z$

As a category, a partial order is simply defined by $\mathrm{Obj}(\mathcal{C}) = X$. Its morphisms are defined by the relation $\leq$: if $x \leq y$, there is a morphism between x and y, which we will call $\leq_{x,y}$ formally, or simply $\leq$ if there is no risk of confusion. This obeys the categorical axioms for morphisms as the identity morphism Id_x is simply given by reflexivity, $\leq_{x,x}$, and obeys the triangular identities by transitivity :

$$\leq_{x,x} \circ \leq_{x,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq x \wedge x \leq y \to x \leq y \tag{3.4}$$
$$\leq_{x,y} \circ \leq_{y,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq y \wedge y \leq y \to x \leq y \tag{3.5}$$

Morphisms : $f : X \to Y$ means $X \leq Y$. Between any two objects, there are exactly zero or one morphisms. The identity is $X \leq X$, composition is transitivity

Example 10 *Given a topological space* (X, τ)*, its* category of open $\mathbf{Op}(X)$ *is given by its set of open sets* τ *with the partial order of inclusion* $\subseteq$*.*

Example 11 *The integers* $\mathbb{Z}$ *as a totally ordered set* $(\mathbb{Z}, \leq)$ *forms a category.*

$$\ldots \xrightarrow{\leq} -2 \xrightarrow{\leq} -1 \xrightarrow{\leq} 0 \xrightarrow{\leq} 1 \xrightarrow{\leq} 2 \xrightarrow{\leq} \ldots$$

The real line is such a category

The category of (locally small) categories **Cat**

Another type of such categories is the *simplicial category* Δ. This can be considered as a category of categories, where each object of Δ is the finite total order $\vec{\mathbf{n}}$, and the morphisms are order-preserving functions,

$$\forall f : \vec{m} \to \vec{n},\ f(m_i \to m_j) = f(m_i) \to f(m_j) \tag{3.6}$$

Example 12 *The* interval category I *is composed of two elements* $\{0, 1\}$ *and a morphism* $0 \to 1$*.*

$$I = \{0 \to 1\} \tag{3.7}$$

3.2 Morphisms

As our categories are fundamentally built from sets and classes, we can look at specific subsets of our set of morphisms, $\mathrm{Mor}(\mathbf{C})$, with specific properties.

A simple example is simply the set of all morphisms between two objects :

Definition 13 *The* hom-set *between two objects* $X, Y \in \mathbf{C}$ *is the set of all morphisms between* X *and* Y *:*

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathrm{Mor}(\mathbf{C}) | s(f) = X,\ t(f) = Y\} \tag{3.8}$$

This notion is more rigorously only valid for what are called *locally small categories*, where the cardinality of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is small enough to be a set. If the cardinality is too large, and could only fit in say a class, it is more properly described as a *hom-object*. This will not affect our discussion much as most categories of interest are locally small.

Example 14 *The hom-set on the category of sets* $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ *is the set of all functions between* X *and* Y

Example 15 *The hom-set on the category of vector spaces* $\mathrm{Hom}_{\mathbf{Vec}}(X, Y)$ *is the set of all linear functions between* X *and* Y*, also known as* $L(X, Y)$*.*

Example 16 *The hom-set on the category of topological spaces* $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ *is the set of all continuous functions between* X *and* Y*, also known as* $C(X, Y)$*.*

Induced morphism on hom-sets : for a morphism $f : X \to Y$, the induced function f_* on the hom-sets

$$f \tag{3.9}$$

Beyond that, there are special types of morphisms

Definition 17 *A* monomorphism $f : X \to Y$ *is a morphism such that, for every object* Z *and every pair of morphisms* $g_1, g_2 : Z \to X$*,*

$$f \circ g_1 = f \circ g_2 \to g_1 = g_2 \tag{3.10}$$

Left-cancellability

alternatively, a morphism is mono if its induced morphism f_* on hom-sets is

Example 18 *on* **Set***, a monomorphism is an injective function.*

Proof 1 *If* $f : X \to Y$ *is a monomorphism, then given two elements* $x_1, x_2 \in X$*, represented by morphisms* $x_1, x_2 : \{\bullet\} \to X$*, then we have*

$$f \circ x_1 = f \circ x_2 \to x_1 = x_2 \tag{3.11}$$

So that we can only have the same value if the arguments are the same, making it injective. Conversely, if f *is injective, take two functions* $g_1, g_2 : Z \to X$*. As the functions in sets are defined by their values, we need to show that*

$$\forall z \in Z,\ f(g_1(z)) = f(g_2(z)) \to g_1(z) = g_2(z) \tag{3.12}$$

By injectivity, the only way to have $f(g_1(z)) = f(g_2(z))$ *is that* $g_1(z) = g_2(z)$*. As this is true for every value of* z*,* $g_1 = g_2$*.*

It is tempting to try to view monomorphisms as generalizing injections, but categories may lack both elements on which to define such notions and

In terms of hom sets : for every Z, the hom functor $\mathrm{Hom}(Z, -)$ maps monomorphisms to injective functions between hom sets :

$$\mathrm{Hom}(Z, X) \hookrightarrow \mathrm{Hom}(Z, Y) \tag{3.13}$$

Definition 19 *An* epimorphism $f : X \to Y$ *is a morphism such that, for every object Z and every pair of morphisms g_1, g_2*

Mono and epi on sets, etc

Definition 20 *An* isomorphism $f : X \to Y$ *is a morphism with a two-sided inverse, ie there exists a morphism $f^{-1} : Y \to X$ such that*

$$f \circ f^{-1} = \mathrm{Id}_Y \tag{3.14}$$

$$f^{-1} \circ f = \mathrm{Id}_X \tag{3.15}$$

Example 21 *In the category of sets, isomorphisms are bijections.*

Despite the most typical examples, it is not in general true that a morphism that is both a monomorphism and an epimorphism is an isomorphism.

Definition 22 *For a given object X, the subset of all its endomorphisms which are isomorphisms are called its* core *:*

$$\mathrm{core}(X) = \{f \in \mathrm{Hom}_{\mathbf{C}}(X, X) \mid f \text{ isomorphism}\} \tag{3.16}$$

Core has a group structure

Theorem 23 *The composition of two monomorphisms is a monomorphism.*

Proof 2

Theorem 24 *The composition of two epimorphisms is an epimorphism.*

Proof 3

Theorem 25 *For f, g two morphisms that can be composed as $g \circ f$, if $g \circ f$ and g are monomorphisms, then so is f.*

Proof 4

As monomorphisms represent embeddings and inclusions, this means in particular that if we have the inclusion $S \hookrightarrow X$ and

Split monomorphisms and split epimorphisms

3.3 Functors

Functors are roughly speaking functions on categories that preserve their structures. In more details,

Definition 26 *A* functor $F : \mathbf{C} \to \mathbf{D}$ *is a function between two categories* $\mathbf{C}, \mathbf{D}$*, mapping every object of* $\mathbf{C}$ *to objects of* $\mathbf{D}$ *and every morphism of* $\mathbf{D}$ *to morphisms of* $\mathbf{D}$*, such that categorical properties are conserved :*

$$\begin{aligned} s(F(f)) &= F(s(f)) && (3.17)\\ t(F(f)) &= F(t(f)) && (3.18)\\ F(\mathrm{Id}_X) &= \mathrm{Id}_{F(X)} && (3.19)\\ F(g \circ f) &= F(g) \circ F(f) && (3.20) \end{aligned}$$

Example 27 *The* Identity functor $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ *maps every object and morphism to themselves.*

Example 28 *The* constant functor $\Delta_X : \mathbf{C} \to \mathbf{D}$ *for some object* $X \in \mathbf{D}$ *is the functor mapping every object in* $\mathbf{C}$ *to* X *and every morphism to* Id_X*.*

Example 29 *For a category where the objects can be built from sets (such as* $\mathbf{Top}$ *or* $\mathbf{Vect}_k$*), the* forgetful functor $U_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set}$ *is the functor sending every object to their underlying set, and every morphism to their underlying function on sets.*

A common functor is the *forgetful functor*, which maps a category that is composed of a set with extra structure its underlying set. For instance, the category **Top** has a forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$, which maps every topological space to its set, and every continuous function to the corresponding function on sets. If we have a set X and on it are all the different topologies (X, τ_i), then the forgetful functor maps

$$U((X, \tau_i)) = X \tag{3.21}$$

The forgetful functor on **Top** is obviously not injective, as two topological spaces with the same underlying set (such as any set of cardinality ≥ 1 with the discrete or trivial topology) will map to the same set.

Functors on partial orders : Order isomorphism

Example : negation

Definition 30 *A* contravariant functor *is a functor from the opposite category, ie $F : C \to D$ is a contravariant functor equivalently to a functor $F : C^{\text{op}} \to D$ is a functor. In particular, this changes the rules as*

$$\begin{aligned} s(F(f)) &= F(t(f)) & (3.22)\\ t(F(f)) &= F(s(f)) & (3.23)\\ F(g \circ f) &= F(f) \circ F(g) & (3.24) \end{aligned}$$

Theorem 31 *The composition of contravariant and covariant functors works as follow :*

$$C \tag{3.25}$$

A generalization of functors is the notion of multifunctors (we mean here specifically the *jointly functorial* multifunctor)

Definition 32 *A multifunctor $F : \mathbf{C} \to \mathbf{D}$ is a function from a product of category to another category.*

"Functor categories serve as the hom-categories in the strict 2-category Cat."

3.3.1 The hom-functor

The hom-bifunctor $\text{Hom}_{\mathbb{C}}(-,-)$ is the map

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(-,-) : \mathbf{C} \times \mathbf{C} &\to \mathbf{Set} & (3.26)\\ (X,Y) &\mapsto \text{Hom}_{\mathbf{C}}(X,Y) & (3.27) \end{aligned}$$

mapping objects of $\mathbf{C}$ to their hom-sets. Given a specific object X, we can furthermore define two types of hom functors : the covariant functor $\text{Hom}(X,-)$, also denoted by h^X, and the contravariant functor $\text{Hom}(-,X)$, denoted by h_X.

Example :

3.3.2 Full and faithful functor

Full, faithful functor

Definition 33 *A functor $F : \mathbf{C} \to \mathbf{D}$ induces the function*

$$F_{X,Y} : \text{Hom}_{\mathbf{C}}(X,Y) \to \text{Hom}_{\mathbf{D}}(F(X), F(Y)) \tag{3.28}$$

F is said to be

- faithful *if* $F_{X,Y}$ *is injective*
- full *if* $F_{X,Y}$ *is surjective*
- fully faithful *if* $F_{X,Y}$ *is bijective*

Despite the analogy of full and faithful functors with surjective and injective functions,

Functors that are fully faithful but not injective/surjective on objects or morphisms

3.3.3 Subcategory inclusion

An important type of functor is the inclusion of a subcategory. If we take a category $\mathbf{C}$, and then create a new category $\mathbf{S}$ for which $\mathrm{Obj}(\mathbf{S}) \subseteq \mathrm{Obj}(\mathbf{C})$

Definition 34 *An inclusion of a subcategory* $\mathbf{S}$ *in a category* $\mathbf{C}$ *is a functor* $\iota : \mathbf{S} \hookrightarrow \mathbf{C}$*, such that* $\iota(\mathrm{Obj}(\mathbf{S})) \subseteq \mathrm{Obj}(\mathbf{C})$, $\iota(\mathrm{Mor}(\mathbf{S})) \subseteq \mathrm{Mor}(\mathbf{C})$*, and*

- *If* $X \in \mathbf{S}$*, then* $\mathrm{Id}_X \in \mathbf{S}$
- *For any morphism* $f : X \to Y$ *in* $\mathbf{S}$*, then* $X, Y \in \mathbf{S}$.
-

For discrete categories, $\mathbf{n}$ in $\mathbf{m}$ if $n \leq m$

Full subcategories

Example 35 *The linear order of the integers* $\mathbb{Z}$ *has inclusion functors to* $\mathbb{R}$

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{R} \tag{3.29}$$

If treated as Canonical inclusion :

$$\iota_h : \mathbb{Z} \to \mathbb{R} \tag{3.30}$$
$$k \mapsto h + k \tag{3.31}$$

Example 36 *The category of finite sets* **FSet** *is a subcategory of* **Set***, via the identity functor restricted to finite groups.*

Example 37 *The category of Abelian group* **Ab** *in* **Grp**

3.4 Natural transformations

Natural transformations are a type of transformations on functors (2-morphisms in **Cat**)

Definition 38 *For two functors* $F, G : \mathbf{C} \to \mathbf{D}$*, a* natural transformation η *between them is a map* $\eta : F \to G$ *which induces for any object* $X \in \mathbf{C}$ *a morphism on* $\mathbf{D}$

$$\eta_X : F(X) \to G(X) \tag{3.32}$$

and for every morphism $f : X \to Y$ *the identity*

$$\eta_Y \circ F(f) = G(f) \circ \eta_X \tag{3.33}$$

[Commutative diagram]

Example 39 *The* identity transformation $\mathrm{Id}_F : F \to F$ *on the functor* $F : \mathbf{C} \to \mathbf{D}$ *is the natural tranformation for which every component* $\mathrm{Id}_{F,X} : F(X) \to F(X)$ *for* $X \in \mathbf{D}$ *is the identity map. This obeys the identity as*

$$\begin{aligned} \eta_Y \circ F(f) &= \mathrm{Id}_Y \circ F(f) && (3.34) \\ &= F(f) && (3.35) \\ &= F(f) \circ \mathrm{Id}_X && (3.36) \end{aligned}$$

Example 40 *The category of groups* **Grp** *has a functor to the category of Abelian groups* **AbGrp***, the Abelianization functor*

$$\begin{aligned} \mathrm{Ab} : \mathbf{Grp} &\to \mathbf{AbGrp} && (3.37) \\ G &\mapsto G/[G, G] && (3.38) \end{aligned}$$

[show functoriality] There is a natural transformation from the identity functor on groups to the abelianization endofunctor

$$\eta : \mathrm{Id}_{\mathbf{Grp}} \to \mathrm{Ab} \tag{3.39}$$

Example 41 *Given the category* $\mathbf{Vect}_k$*, for any vector space* V *we have the dual space* V^* *[see later in the internal hom section for why] of linear maps* $V \to k$*, and its double dual* V^** *of linear maps* $V^* \to k$*. We would like to show that there is an equivalence between* V *and* V^{**}*.*

Example 42 *The opposite group functor is simply given by the opposite category functor on* Grp*. Groups to opposite group*

For constant F and G : cone and cocone

3.5 Yoneda lemma

One of the common philosophical idea underlying category theory is that of the

"An object is completely determined by its relationships to other objects"

For a functor $F : \mathbf{C} \to \mathbf{Set}$, for any object $X \in \mathbf{C}$

$$よ : \mathbf{C} \to [\mathbf{C}, \mathbf{Set}] \tag{3.40}$$

Functor lives in the functor space $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

$$\mathrm{Nat}(h_A, F) \cong F(A) \tag{3.41}$$

Example 43 *Consider the category of a single group ($G = \mathrm{Aut}(*)$). A functor $F : \mathrm{G} \to \mathbf{Set}$ is a set X and a group homomorphism to its permutation group $G \to \mathrm{Perm}(X)$ (A G-set). Natural transformation is an equivariant map Cayley's theorem*

Example 44

3.6 Enriched categories

By default, we consider the hom sets of a category $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ to be sets, but many categories may have additional structure on the hom set. For instance, if we consider the category $\mathbf{Vect}_k$ of vector spaces over the field k, its morphisms are k-linear maps, and its hom-sets are

$$\mathrm{Hom}_{\mathbf{Vect}_k}(V, W) = L_k(V, W) \tag{3.42}$$

However, in addition to being a set, $L_k(V, W)$, the k-linear maps, also form themselves a vector space, ie we can define the sum $f + g$ of two linear maps, and the scaling αf, $\alpha \in k$, of a linear map.

Definition 45 *An* enriched category $\mathbf{C}$ *over* $\mathbf{V}$ *a monoidal category* $(\mathbf{V}, \otimes, I)$ *is a category such that each hom-set* $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ *is associated to a* hom-object $C(X, Y) \in \mathbf{V}$*, such that every hom-object in* $\mathbf{V}$ *obeys the same rules regarding composition and identity, which are*

$$\circ_{X,Y,Z} : C(Y, Z) \otimes C(X, Y) \to C(X, Z) \tag{3.43}$$

$$j_X : I \to C(X, X) \tag{3.44}$$

with the following commutation diagrams :

[composition is associative]

[Composition is unital]

Example 46 *A category enriched in* **Set** *is a locally small category.*

Example 47 *A k-*linear category *is enriched over* Vect_k*.*

3.7 Comma categories

The notion of a comma category can be used to describe categories whose objects are the morphisms of another category.

Definition 48 *The* comma category $(f \downarrow g)$ *of two functors* $f : C \to E$ *and* $g : D \to E$ *is the category composed of triples* (c, d, α) *such that* $\alpha : f(c) \to g(d)$ *is a morphism in* E*, and whose morphisms are pairs* (β, γ)

$$\beta \quad : c_1 \quad \to c_2 \tag{3.45}$$

$$\gamma \quad : d_1 \quad \to d_2 \tag{3.46}$$

that are morphisms in C *and* D*, such that* $\alpha_2 \circ f(\beta) = g(\gamma) \circ \alpha_1$ *[Commutative diagram]*

Composition

Def via pullback

Comma categories are rarely used directly, but are more typically used to define more specific operations. The three important one we will see are arrow categories, slice categories and coslice categories.

3.7.1 Arrow categories

Arrow categories

3.7.2 Slice categories

Slice categories

Given an object $X \in \mathbf{C}$, we can define the *over category* (or *slice category*) $\mathbf{C}_{/X}$ by taking all morphisms emanating from X as objects :

$$\mathrm{Obj}(\mathbf{C}_{/X}) = \{f | s(f) = X\} \tag{3.47}$$

As a comma category, this is the comma category of the two functors $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ and $\Delta_X : \mathbf{1} \to \mathbf{C}$, of the identity functor on $\mathbf{C}$ and the inclusion of the object X, in which case $\mathbf{C}_{/X} = (\mathrm{Id}_{\mathbf{C}} \downarrow \Delta_X)$ is defined by the triples $(c, *, \alpha)$ of objects $c \in \mathbf{C}$, the unique object $* \in \mathbf{1}$, and morphisms in $\mathbf{C}$ $\alpha : c \to X$. As there is only one object in the terminal category, we can drop it as it is isomorphic to simply (c, α), and furthermore, c is implied by α as simply being the source term. Our slice category is therefore indeed just defined by the set of morphisms from objects of the category to our selected object.

Slice categories are useful to consider objects in a category as a category in themselves, where the objects are simply all the relations they have with all other objects in the category.

Example 49 *In* **Set***, given a set* X*, the slice category* $\mathbf{Set}_{/X}$ *has as its objects all functions with codomain* X*,* $f : Y \to X$*, and as morphisms all functions between sets* $g : Y \to Y'$ *for which*

$$f'(g(y)) = f(y) \tag{3.48}$$

Category of X-indexed collections of sets, object $f : Y \to X$ is the X-indexed collection of fibers $\{Y_x = f^{-1}(\{x\})| \in X\}$, morphisms are maps $Y_x \to Y'_x$

[fiber product $Y \times_X Y'$ is the product in the slice category]

If we look for instance at $\mathbb{N}$ as a set (the natural number object of sets), its slice category $\mathbf{Set}_{/\mathbb{N}}$ is the category of all functions to numbers

Example 50 *For a poset* P*, the slice category* $\mathbf{P}/p$ *is isomorphic to the downset of* p*, ie the subcategory of every element* $\{q | q \leq p\}$*.*

Objects : Id_p*, and every map* $\leq_{q,p}$ *(corresponding to* p *and object inferior to* p*), morphisms are*

Example 51 $\mathbf{Top}/X$ *is the category of covering spaces over* X*.*

3.7.3 Coslice categories

Coslice categories

comma categories, under categories?

Dependent sum, dependent product, indexed sets

3.7.4 Base change

Base change functor

3.8 Limits and colimits

In category theory, a limit or a colimit are roughly speaking a construction on a category. For some given set of objects $A, B, \ldots$ in our category C, and some morphisms between them, a limit or colimit of those objects will be some construction performed using those. Those constructions can be quite different, but overall, a limit will often be like a ”subset”, while a colimit is more of an ”assemblage” of those.

Definition 52 *For a functor $F : \mathbf{C} \to \mathbf{D}$, a* universal property *from an object $X \in \mathbf{D}$ to F*

Universal property, kan extension?

A (co)limit is done using an indexing category, which is roughly the ”shape” that our construction will take. An indexing category is a small category, ie it has a countable number of objects and morphisms small enough that you could fit them into sets. Typically they are fairly simple ones. As we are only interested in their shape, it's common to denote the objects by simple dots. Examples include the discrete categories of n elements $\mathbf{n}$,

$$\bullet \qquad \bullet \qquad \bullet \qquad \ldots$$

The span category :

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

and the cospan :

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

and the parallel pair :

$$\bullet \rightrightarrows \bullet$$

Directed and codirected set, sequential and cosequential limit

To do our constructions, we need to send this diagram's shape into our category C, which we do by using some functor $F : I \to C$, producing a diagram :

Definition 53 *A* diagram *of shape I into C is a functor $F : I \to C$.*

The image of this functor in C will then be some subset of C that "looks like" I, although we are not guaranteed that the objects and morphisms of I will be mapped injectively into C (this simply corresponds to cases where our construction will use the same object or morphisms several times).

Let's now consider the constant functor $\Delta_X : I \to C$, which for $X \in C$ sends every object of I to X. If we can find a natural transformation between Δ_X and our diagram $F : I \to C$, we will have either a cone over F $\eta : \Delta_X \Rightarrow F$ or a cone under F $\eta : F \Rightarrow \Delta_X$.

Definition 54 *The* limit *of a diagram $F : I \to M$ is an object* $\lim F \in \mathrm{Obj}(C)$ *and a natural transformation $\eta : \Delta_{\lim F} \to F$, such that for any $X \in \mathrm{Obj}(C)$ and any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f : X \to \lim F$ such that $\alpha = \eta \circ F$. The cone of $\Delta_{\lim F}$ over F is the* universal cone *over F.*

In other words, if we pick any object X in our category C and define some collection of morphisms from X to other objects

Let's consider for instance the case of the trivial category $\mathbf{1}$. Any functor $F : \mathbf{1} \to C$ is simply a choice of an object in C, mapping $\bullet$ to $F(\bullet) = A$, ie it is just the constant functor Δ_A for some A. A natural transformation $\eta : \Delta_X \to F$ is them simply $\eta : \Delta_X \to \Delta_A$, and conversely, $\eta : F \to \Delta_X$ is $\eta : \Delta_A \to \Delta_X$. The components of this natural transformations are simply a morphism from X to A (and a morphism from A to X).

Limit and colimit

Types of indexing category I and their limit and colimit :

Definition 55 *Given the empty category $\mathbf{0}$, the limit* Lim$F$ *of a diagram $F : \mathbf{0} \to C$ is its* terminal object*, and the colimit is its* initial object*.*

As there exists only one functor from the empty category to any other category (the empty functor $\varnothing$), the initial and terminal objects do not depend on specific objects and are simply special objects of the category. Every "constant functor" Δ_X sending objects of I to $X \in \mathrm{Obj}(C)$ is also the empty functor, sending them trivially to X by simply not having any objects to send. This is therefore also true of the constant functor to the limit $\lim \varnothing$, meaning that the natural transformation $\eta : \varnothing \to \varnothing$ is simply the identity transformation

$\mathrm{Id}_{\varnothing^\varnothing}$. This means that the limit of the empty diagram in a category C is the object (defined by no other objects in the category) $\lim \varnothing$ such that for any natural transformation $\alpha : \Delta_X \to F$ (as we've seen, only possibly the identity transformation), there exists a unique morphism $f : X \to \lim \varnothing$

[...]

The terminal object $t = \lim \varnothing$ of a category, if it exists, is therefore an object for which there exists only one morphism from any object $X \in \mathrm{C}$ to t

Dually, the *initial object* $i = \mathrm{colim}\varnothing$ of a category, if it exists, is an object for which there exists only one morphism from i to any object $X \in \mathrm{Obj}(C)$.

Theorem 56 *Initial and terminal objects are unique in a category up to isomorphisms.*

Proof 5

Initial and terminal objects occur in quite a lot of important categories, and tend to be somewhat similar objects. In **Set**, the

3.8.1 Products and coproducts

The product and coproduct are the limits where the diagrams are the discrete categories of n elements $\mathbf{n}$. This means obviously that the trivial case $\mathbf{0}$ of the diagram of zero object is the initial and terminal object, and this will correspond to the trivial product and coproduct as we will see later :

$$\sum_{\varnothing} = 0, \prod_{\varnothing} = 1 \tag{3.49}$$

Any diagram of shape $\mathbf{n}$ simply selects n objects in the category (and their identity functions),

$$\forall F : \mathbf{n} \to \mathbf{C},\ \exists X_1, \ldots X_n \in \mathbf{C},\ \mathrm{Im}(F) = (X_1, \ldots, X_n) \tag{3.50}$$

The constant functor Δ_X is the functor sending each of those points to X, $\bullet_i \to X$, and the identity of those points to the identity on X. We will denote the limit of a diagram F on the discrete category, selecting the objects $\{X_i\}$, by $\prod_i X_i$. This product is therefore such that for any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f = \Delta_X \to \prod X_i$ such that $\alpha = \eta \circ F$.

What this property means for the product is that given any object X picked in $\mathbf{C}$,

3.8.2 Spans and cospans

The span and cospan are dual diagrams, with **Span** = **Cospan**$^{\mathrm{op}}$, having an object connecting to two others for the span

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

or $\bullet \to \bullet \leftarrow \bullet$ for short, and two objects connecting to one for the cospan :

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

or $\bullet \leftarrow \bullet \to \bullet$.

The span diagram can be mapped to any three objects connected thusly. For $A, B, C \in \mathbf{C}$, and $f : A \to C$, $g : B \to C$, the diagram of shape I will be

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

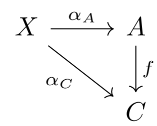
If we now look at the constant functor $\Delta_X : I \to \mathbf{C}$, this will map our three objects A, B, C to X, and f and g to Id_X. To find the limit of F, we therefore need to find $\eta : \Delta_{\lim F} \to F$

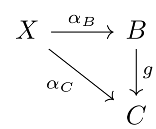
For any $X \in \mathbf{C}$, and any $\alpha : \Delta_X \to F$, there is a unique morphism $h : X \to \lim F$ such that $\alpha = \eta \circ F$.

Components of α : for every $Y \in I$, a morphism $\alpha_Y : \Delta_X(Y) \to F(Y)$, so

$$\alpha_Y : X \to F(Y) \tag{3.51}$$

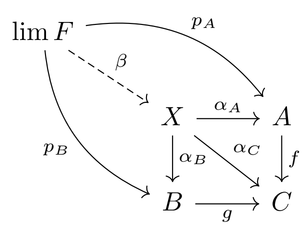
$F(Y)$ can only be A, B, C, so we have three components for objects, and for $f : A \to C$ and $g : B \to C$, then using what we know of Δ_X we have the following commuting diagrams :



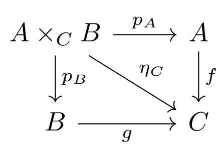


$$F(f) \circ \eta \tag{3.52}$$

The resulting cone is



The limit is the pullback, denoted as $A \times_C B$, along with the two and the universal cone that we have constructed gives us the commutative square



This means that our pullback diagram is given by this object and the two projectors, obeying

$$f \circ p_A = g \circ p_B \tag{3.53}$$

The interpretation of this is the *dependent sum* of the equality

$$\sum_{a:A} \sum_{b:B} (f(a) = g(b)) \tag{3.54}$$

Example 57 *In* **Set***, the pullback by* $f : A \to C$*,* $g : B \to C$ *is the set*

$$A \times_C B = \{(a, b) \in A \times B | f(a) = g(b)\} \tag{3.55}$$

Semantics of an equation

Dependent product/sum, indexed objects

Fiber :

Definition 58 *The* fiber *of a morphism $f : A \to B$ is the pullback of the cospan with the terminal object :*

$$A \xrightarrow{f} B \xleftarrow{point} * \tag{3.56}$$

Ex : Fiber of a bundle $p : E \to B$ and $x : 1 \to B$ is the fiber at x, E_x.

"In an additive category fibers over the zero object are called kernels."

Cofiber

Definition 59 *A* cofiber *of a morphism $f : A \to B$ is the pushout of the span with the terminal object*

$$* \longleftarrow A \xrightarrow{f} B \tag{3.57}$$

$$\mathrm{cofib}(f : A \to B) = \tag{3.58}$$

Cokernels in additive categories

Pushout :

$$* \longleftarrow \square_{\varnothing} X \longrightarrow X \tag{3.59}$$

Pushout $A \sqcup_0 B \cong A \sqcup B$, therefore

$$\square_{\varnothing} X = X \sqcup 1 \tag{3.60}$$

3.8.3 Equalizer and coequalizer

$\bullet \rightrightarrows \bullet$

Constant functor : $\Delta_X : I \to \mathbf{C}$ maps A, B to X, f, g to Id_X

Equalizer corresponds roughly to a solution of an equation, coequalizer to a quotient?

3.8.4 Directed limits

Given some directed set

3.8.5 Properties of limits and colimits

Theorem 60 *For two diagrams I, J,*

$$f : \operatorname{colim} \lim D \tag{3.61}$$

Commutation

Effective morphisms

$$\bigsqcup_i U_i \to U \tag{3.62}$$

3.8.6 Limits and functors

A common tool in category theory to use is the behaviour of limits under the action of a functor.

Definition 61 *For a functor $F : \mathbf{C} \to \mathbf{D}$ and a diagram $J : \mathbf{I} \to \mathbf{C}$, a functor is said to preserve the limit* $\lim_J$ *if*

$$F \circ \lim_J \cong \lim_{F \circ J} \tag{3.63}$$

Preservation of limits and colimits

Left and right exact functors :

Definition 62 *A functor is* left exact *(resp.* right exact*)*

Maps inital objects to initial objects, products to products, and equalizers to equalizers

Even if we do not know the explicit limits and colimits of a category, we can verify that a functor preserves them, using the notion of a flat functor.

Definition 63 *A functor is* flat

[Flat functors preserve any limit and colimit]

Example 64 *The covariant hom-functor preserve limits :*

$$h^X(\lim F) = \lim(h^X F) \tag{3.64}$$

and the contravariant hom-functor preserves limits in the category $\mathbf{C}^{\mathrm{op}}$, ie colimits in $\mathbf{C}$:

$$h_X() \tag{3.65}$$

Proof 6

Example : hom-sets of vector spaces with limits?

Example 65 *In the category of vector spaces* **Vec***, the covariant hom functor* h^V *gives us the set of linear transformations* $L(V,-)$*. If we take a look at various cases, we have for the terminal object* k^0 *:*

$$\begin{aligned} h^V(k^0) &= L(V,k^0) &(3.66)\\ &= \{0\} &(3.67)\\ &\cong \{\bullet\} &(3.68) \end{aligned}$$

The product of two vector spaces is the direct sum

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \quad (3.69)$$

The kernel of a linear map f *can be described as the equalizer of this map with the* 0 *map, Equalizer : for* $f, 0 : X \to Y$*, the equalizer is* $\ker(f)$*. The two functions* f, g *map to*

$$h^V(f) = \{a \in L(X,Y)\} \quad (3.70)$$

$$h^V(0) = \{0\} \quad (3.71)$$

$$\begin{aligned} h^V(\ker(f)) &= L(V,\ker(f)) &(3.72)\\ &= \mathrm{eq}(h^V(f), h^V(0)) &(3.73) \end{aligned}$$

$$\begin{aligned} h^V(f) &= L(V,\ker(f)) &(3.74)\\ &= &(3.75) \end{aligned}$$

The contravariant one : initial object is also k^0*, therefore terminal object in op*

$$\begin{aligned} h_V(k^0) &= L(k^0,V) &(3.76)\\ &= \{0\} &(3.77)\\ &\cong \{\bullet\} &(3.78) \end{aligned}$$

Same deal with the coproduct in op, which is also $\oplus$

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \quad (3.79)$$

but $\lim(h^V \circ *) = \lim \varnothing = *$

3.9 Monoidal categories

It is common in categories to have some need of defining a binary operation, ie some function of the type

$$A \times B = C \tag{3.80}$$

It is common in categories as well to have this notion applied to objects. Given two objects $A, B \in \mathbf{C}$, we want to find a third object C, such that there exists a bifunctor $(-) \times (-) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$

$$A \times B = C \tag{3.81}$$

While the notion of bifunctor covers this well enough, we often need to have additional conditions. A common case is that of a *monoid*, where we ask that the operation be associative and unital, ie

$$\begin{aligned} (A \times B) \times C &= A \times (B \times C) && (3.82) \\ \exists I \in \mathbf{C},\ I \times A &= A \times I = A && (3.83) \end{aligned}$$

To categorify this notion, we define monoidal categories

Definition 66 *A* monoidal category $(\mathbf{C}, \otimes, I)$ *is a category* $\mathbf{C}$*, a bifunctor* $\otimes$*, and a specific object* $I \in \mathbf{C}$*, along with three natural transformations :*

$$\begin{aligned} a : ((-) \otimes (-)) \otimes (-) &\stackrel{\cong}{\to} (-) \otimes ((-) \otimes (-)) && (3.84) \\ (X \otimes Y) \otimes Z &\mapsto X \otimes (Y \otimes Z) && (3.85) \\ \lambda : (I \otimes (-)) &\stackrel{\cong}{\to} (-) && (3.86) \\ I \otimes X &\mapsto X && (3.87) \\ \rho : ((-) \otimes I) &\stackrel{\cong}{\to} (-) && (3.88) \\ (X \otimes I) &\mapsto X && (3.89) \end{aligned}$$

which obey the following rules :

$$\begin{array}{ccccc} W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{k} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{k} & ((W \otimes X) \otimes Y) \otimes Z)) \\ \downarrow{\scriptstyle I_W \otimes a_{X,Y,Z}} & & & & \downarrow{\scriptstyle f_i} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\qquad a_{W,X \otimes Y,Z} \qquad} & & W \otimes (X \otimes Y)) \otimes Z \end{array}$$

We do not ask the equality for the associator and unitors, as equivalence is typically what we ask in general for a category. If in addition those equivalences are equality, we say that this is a *strict monoidal category.*

Example 67 *The tensor product of two vector spaces is a monoid in* $\mathbf{Vect}_k$

Proof 7 *First we need to show that the tensor product is functorial. Given two morphisms* $f : X \to X'$, $g : Y \to Y'$, *the product* $f \otimes g : X \otimes Y \to X' \otimes Y'$

Proof that it is functorial, unit k, associator, unitor, obeys the identities

Example 68 *The product and coproduct are both monoidal in* **Set**.

Cartesian monoidal category

Bimonoidal categories

3.10 Internalization

If a category admits set-like properties, typically properties such as finite limits, monoidal structures or Cartesian closedness, it is possible to recreate many types of mathematical structures inside the category itself.

Definition 69 *In a category* $\mathbf{C}$ *with finite products, a group object* G *is an object* $G \in \mathbf{C}$ *equipped with the morphisms*

- *The unique map to the terminal object* $p : G \to 1$
- *A neutral element morphism from the terminal object :* $e : 1 \to G$
- *An inverse endomorphism :* $(-)^1 : G \to G$
- *A binary morphism on the product :* $m : G \times G \to G$

such that all the following diagrams commute

Example 70 *Every category with finite product has the trivial group object* $\{e\}$ *which is the terminal object and the unique map to itself.*

Example 71 *The set of two elements* 2, *along with a given morphism* $e : 1 \to 2$, *the automorphism* $f : 2 \to 2$ *that is not the identity (the one that exchanges the two elements) and the morphism* $2 \times 2 \to 2$

form the internalized group $\mathbb{Z}_2$ *in* **Set** *:*

$$\mathbb{Z}_2 \cong (2, e, f, f) \tag{3.90}$$

Example 72 *As groups can be defined using sets, the category of sets contains every group as group objects using the traditional definition of groups.*

Example 73 *The group objects in the category* Top *are the topological groups, where all group operations are continuous functions.*

Example 74 *The group objects in the category of smooth manifolds are the Lie groups, where the group operations are smooth maps.*

Definition 75 *A* ring object

Delooping

3.11 Subobjects

Given an object X in a category $\mathbf{C}$, a *subobject* S of X is an isomorphism class of monomorphisms $\{\iota_i\}$

$$\iota_i : S_i \hookrightarrow X \tag{3.91}$$

so that the equivalence classes of $\{\iota_i\}$ is given by any two such monomorphisms if there exists an isomorphism between the two subobjects S_i;

$$S = [S_i]/(S_i \cong S_j \leftrightarrow \exists f : S_i \to S_j,\ \exists f^{-1} S_j \to S_j,\ f \circ f^{-1} = \mathrm{Id}) \tag{3.92}$$

Isomorphisms given by the automorphism group Aut()

This is meant to define the common mathematical notion of an object being *part* of another object in some sense.

Examples :

Example 76 *On* **Set***, subobjects are subsets (defined by injections up to the symmetric group)*

Example 77 *On* $\mathbf{Vect}_k$*, subobjects are subspaces (defined by injections up to the general linear group?)*

On **Top**, subobjects are subspaces with the subspace topology

On **Ring**,

On **Grp**, subobjects are subgroups

Example 78 *On the category of smooth manifolds* **SmoothMan**, *subobjects are* submanifolds, $\iota : S \hookrightarrow M$, *where the set of all submanifolds with the same image up to diffeomorphism of the base* Diff(S) *are equivalent.*

"Let C_c be the full subcategory of the over category C/c on monomorphisms. Then C_c is the poset of subobjects of c and the set of isomorphism classes of C_c is the set of subobjects of c. "

3.12 Simplicial categories

Simplicial category Δ is made of simplicial objects, ie

$$\left[\vec{\mathbf{0}}\right] = \{\bullet\} \tag{3.93}$$
$$\left[\vec{\mathbf{1}}\right] = \{\bullet \to \bullet\} \tag{3.94}$$
$$\left[\vec{\mathbf{2}}\right] = \{\bullet \to \bullet \to \bullet\} \tag{3.95}$$
$$\left[\vec{\mathbf{3}}\right] = \{\bullet \to \bullet \to \bullet \to \bullet\} \tag{3.96}$$

3.13 Equivalences and adjunctions

Like many objects in mathematics, it is possible to try to define some kind of equivalence between two categories. Like most such things, we try to consider two mappings between our categories. Let's consider two categories $\mathbf{C}, \mathbf{D}$, and two functors F, G between them,

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D} \tag{3.97}$$

The usual process of finding equivalent objects in such cases is to have those two maps be inverses of each other, ie $FG = \mathrm{Id}_{\mathbf{D}}$ and $GF = \mathrm{Id}_{\mathbf{C}}$. The composition functor FG maps every object and morphism of D to itself and likewise for GF on C, so that in some sense, the objects X and $F(X)$ are the same objects, and likewise, $f : X \to Y$ and $F(f) : F(X) \to F(Y)$ represent the same morphism.

If two categories admit such a pair of functors, we say that they are *isomorphic*. While this is the most obvious definition of equivalence, it is in practice not commonly used, as very few categories of interest are actually isomorphic, and this is generally considered too strict a definition in the philosophy of category theory, as we are generally more interested in the relationships between objects rather than the objects themselves. A good example of this overly strict definition is that the product of two objects done in a different order will not be isomorphic.

If we weaken the notion of equivalence, we can look at the case where our two functors are merely isomorphic to the identity, $FG \cong \mathrm{Id}_{\mathbf{D}}$ and $GF \cong \mathrm{Id}_{\mathbf{C}}$, where there exists a natural transformation η taking FG to $\mathrm{Id}_{\mathbf{D}}$.

$$\eta : FG \rightarrow \mathrm{Id}_{\mathbf{D}} \tag{3.98}$$
$$\epsilon : \mathrm{Id} \tag{3.99}$$

$$\mathbf{C} \qquad \mathbf{D}$$

Triangle identities

For every objects $X \in C, Y \in D$, the components of the relevant natural transformations are

$$\mathrm{Id}_{F(Y)} = \epsilon_{F(Y)} \circ F(\eta_Y) \tag{3.100}$$
$$\mathrm{Id}_{G(X)} = G(\epsilon_X) \circ \eta_{G(X)} \tag{3.101}$$

$F \dashv G$

Adjunction between two categories

Example 79 *A basic non-trivial example of adjoint functors is the even and odd functors. If we consider* $\mathbb{Z}$ *as a linear order category, with* $\leq$ *as its morphisms, functors are its order-preserving functions. Two specific functors that we have are the even and odd functors, give by*

$$\forall k \in \mathbb{Z},\ \mathrm{Even}(k) = 2k,\ \mathrm{Odd}(k) = 2k+1 \tag{3.102}$$

These do indeed preserve the order so that for the unique morphism $k_1 \leq k_2$*, it is mapped to the unique morphism* $2k_1 \leq 2k_2$ *and similarly for the odd functor. The corresponding "inverse" functor is the functor mapping any integer to the floor of its division by* 2 *:*

$$\lfloor -/2 \rfloor : \mathbb{Z} \to \mathbb{Z} \tag{3.103}$$

Even *is then the left adjoint and* Odd *the right adjoint of* $\lfloor -/2 \rfloor$*. The unit and counit of* Even *are :*

$$\varepsilon_l = f \circ \lfloor -/2 \rfloor \tag{3.104}$$

[...]

This is however not an equivalence, as the floor function is not strictly an inverse of the even and odd functor (and not being a faithful functor to begin with), as $\lfloor (2n)/2 \rfloor = \lfloor (2n+1)/2 \rfloor$*, and we have*

Example 80 *Take the two linear order categories of $\mathbb{Z}$ and $\mathbb{R}$, with their elements being the objects and their order relations are the morphisms. The inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$, mapping $n \in \mathbb{Z}$ to its equivalent real number, is a functor*

We can try to define left and right adjoints for it, by finding two functions $f, g : \mathbb{R} \to \mathbb{Z}$ for which there exists

- *A left and right counit $\epsilon_l : f\iota \to \mathrm{Id}_{\mathbb{Z}}$ and $\epsilon_r : \iota g \to \mathrm{Id}_{\mathbb{R}}$*
- *A left and right unit $\eta_l : \mathrm{Id}_{\mathbb{R}} \to \iota f$ and $\eta_r : \mathrm{Id}_{\mathbb{Z}} \to g\iota$*

And all these must obey the triangle identities

If we take for instance the left adjoint, we need that our function f be such that there exists a natural transformation between the identity on $\mathbb{R}$ ($\mathrm{Id}_{\mathbb{R}}(x) = x$) and our function f reinjected into $\mathbb{R}$: $\iota(f(x))$. For every $x, y \in \mathbb{R}$, there's a morphism

$$\begin{array}{ccccc} x & & x & \xrightarrow{\eta_{l,x}} & \iota(f(x)) \\ \downarrow{\scriptstyle \leq} & \overset{\eta}{\Longrightarrow} & \downarrow{\scriptstyle \leq} & & \downarrow{\scriptstyle \iota(f(\leq))} \\ y & & y & \xrightarrow{\eta_{l,y}} & \iota(f(y)) \end{array}$$

In the context of our linear order, this means that for any two numbers x, y such that $x \leq y$, we have $x \leq \iota(f(x))$, $y \leq \iota(f(y))$ and $\iota(f(x)) \leq \iota(f(y))$

$$x \leq y \leq \iota(f(y)) \tag{3.105}$$

and for the counit, we need a natural transformation between the mapping of an integer into $\mathbb{R}$ and then back into $\mathbb{Z}$ via f with $f(\iota(n))$, and the identity on $\mathbb{Z}$, $\mathrm{Id}_{\mathbb{Z}}(n) = n$. For every n, m, $n \leq m$,

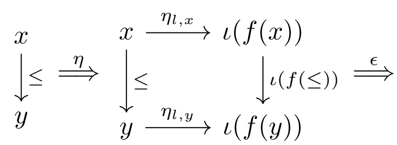
$$\begin{array}{ccccc} n & & f(\iota(n)) & \xrightarrow{\epsilon_{r,n}} & n \\ \downarrow{\scriptstyle \leq} & \overset{\epsilon}{\Longrightarrow} & \downarrow{\scriptstyle f(\iota(\leq))} & & \downarrow{\scriptstyle \leq} \\ m & & f(\iota(m)) & \xrightarrow{\epsilon_{r,m}} & m \end{array}$$

We have the condition that if we inject n into $\mathbb{R}$, its left adjoint will be such that $f(\iota(n)) \leq n$, $f(\iota(m)) \leq m$ and $f(\iota(n)) \leq f(\iota(m))$. If we take the case $m = n+1$ and ignoring the injection ι for now, this means that $f(n) \leq f(n+1) \leq n+1$ and $f(n) \leq n$. As $f(n) \leq n+1$ cannot be equal to $n+1$, $f(n)$ can only be equal to n or smaller. If we pick the case $n-1 \leq n$ instead, the natural transformation implies $f(n-1) \leq f(n) \leq n$ and $f(n-1) \leq n-1 \leq n$, so that $f(n-1) < n$ and

Triangle identities : for any $n \in \mathbb{Z}$,

$$\mathrm{Id}_{\iota(n)} = \iota(\epsilon_{l,n}) \circ \eta_{l,\iota(n)} \tag{3.106}$$

The components of this natural transformations give us that, if we transform our integer n to a real and back,

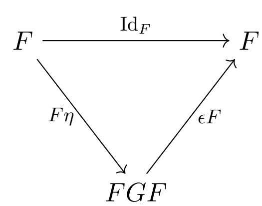


Different definitions of adjunction

Adjoint functors : For two functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$, the functors form an *adjoint pair* $F \dashv G$, F the left adjoint of G and G the right adjoint of F, if there exists two natural transformations, η and ϵ

$$\eta : \mathrm{Id}_{\mathbf{C}} \to G \circ F \tag{3.107}$$
$$\epsilon : F \circ G \to \mathrm{Id}_{\mathbf{D}} \tag{3.108}$$

obeying the triangle equalities



Adjunct : for an adjunction of functors $(L \dashv R) : \mathbf{C} \leftrightarrow \mathbf{D}$, there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(LX, Y) \cong \mathrm{Hom}_{\mathbf{D}}(X, RY) \tag{3.109}$$

Two morphisms $f : LX \to Y$ and $g : X \to RY$ identified in this isomorphism are *adjunct*. g is the right adjunct of f, f is the left adjunt of g.

$$g = f^\sharp,\ f = g^\flat \tag{3.110}$$

Adjunction for vector spaces

Galois connection

3.14 Grothendieck construction

3.15 Reflexive subcategories

Given two categories **C**, **D**, **C** is a reflexive subcategory of **D** if it is a full subcategory, $\mathbf{C} \hookrightarrow \mathbf{D}$, ie every morphism of **C** is a morphism of **D**, and such that every object $d \in \mathrm{Obj}(\mathbf{D})$ and morphism $(f : d \to d') \in \mathrm{Mor}(\mathbf{D})$ have a reflection in **C**.

Def : The inclusion functor $\iota : \mathbf{C} \hookrightarrow \mathbf{D}$ has a left adjoint T, the *reflector* :

$$(T \dashv \iota) : \mathbf{C} \overset{\iota}{\hookrightarrow} \mathbf{D} \tag{3.111}$$

T is the *reflector*

[...]

Examples : **Ab** $\hookrightarrow$ **Grp**, reflection is the abelianization

Example 81 *The category of metric spaces with isometries as morphisms has as a full subcategory the category of complete metric spaces. The reflector associates the completion of the metric space to any metric space.*

3.16 Monads

Definition 82 *A* monad *is a category* **C** *with*

- *An object* $A \in \mathbf{C}$
- *An endomorphism* $t : A \to A$
- *A natural transformation* $\eta : \mathrm{Id}_A \to t$
- *A natural transformation* $\mu : t \circ t \to t$

Monads from adjunctions

Kleisli category

Algebra over a monad

Definition 83 *For a monad* (T, η, μ) *on a category* **C**, *a* T*-algebra is a pair* (X, f) *of an object* $X \in \mathbf{C}$ *and a morphism* $f : TX \to X$ *making the following diagrams commute :*

Example 84 *The basic adjoint modality example is the* even/odd modality *pair,*

$$\text{Even} \dashv \text{Odd} \tag{3.112}$$

This is done on the category of integers as an ordered set, $(\mathbb{Z}, \leq)$, for which the morphisms are the order relations, and endofunctors are order-preserving functions.

The functor we consider here is the largest integer which is smaller to $n/2$:

$$\begin{aligned} \lfloor -/2 \rfloor : (\mathbb{Z}, \leq) &\to (\mathbb{Z}, \leq) & (3.113)\\ n &\mapsto \lfloor n/2 \rfloor & (3.114) \end{aligned}$$

This functor has a left and right adjoint functor,

$$\begin{aligned} \text{even} : (\mathbb{Z}, \leq) &\hookrightarrow (\mathbb{Z}, \leq) & (3.115)\\ n &\mapsto 2n & (3.116)\\ \text{odd} : (\mathbb{Z}, \leq) &\hookrightarrow (\mathbb{Z}, \leq) & (3.117)\\ n &\mapsto 2n+1 & (3.118)\\ & & (3.119) \end{aligned}$$

Proof :

$\lfloor -/2 \rfloor$ has as a domain the whole category

For a total order, the hom-set $\text{Hom}(X, Y)$ *is simply empty if $X > Y$ and has a single element otherwise. For $\lfloor -/2 \rfloor$, the hom-set*

The left adjoint of $\lfloor -/2 \rfloor$ is a functor such that

$$\text{Hom}_{\mathbb{Z}}(L(-), -) \cong \text{Hom}_{\mathbb{Z}}(-, \lfloor -/2 \rfloor) \tag{3.120}$$

In the case of a total order, the isomorphism simply means that both sets have the same cardinality, ie they either have no elements (the two objects are not ordered) or one (the two objects are ordered). So

$$\text{Hom}_{\mathbb{Z}}(L(n), m) \cong \text{Hom}_{\mathbb{Z}}(n, \lfloor m/2 \rfloor) b \Leftrightarrow L(n) \leq m \leftrightarrow n \leq \lfloor m/2 \rfloor \tag{3.121}$$

If we have $L(n) = 2n$, we need to show this equivalence both ways.

If $2n \leq m$, then dividing by 2, we have $n \leq m/2$, which we can then apply the floor to both sides (it is monotonous), so $\lfloor n \rfloor \leq \lfloor m/2 \rfloor$. As n is an integer, $n \leq \lfloor m/2 \rfloor$

Converse : If $n \leq \lfloor m/2 \rfloor$ *: From properties of floor :*

$$n \leq \lfloor \frac{m}{2} \rfloor \leftrightarrow 2 \leq \frac{\lceil m \rceil}{n} \tag{3.122}$$

As m *is an integer,* $2n \leq m$*.*

So the even function is indeed left adjoint.

Odd function is right adjoint :

From these three functions, we can define adjoint monads :

$$(\mathrm{Even} \vdash \mathrm{Odd}) \tag{3.123}$$

which send numbers to their half floor and then to their corresponding even and odd number :

$$\mathrm{Even}(n) = 2\lfloor n/2 \rfloor \tag{3.124}$$

$$\mathrm{Odd}(n) = 2\lfloor n/2 \rfloor + 1 \tag{3.125}$$

n	Even(n)	Odd(n)
-2	-2	-1
-1	-2	-1
0	0	1
1	0	1
2	2	3
3	2	3

Table 3.1: Caption

Monad and comonad, unit and counit, multiplication

Example 85 *Integrality modality : Given the two total order categories* $(\mathbb{Z}, \leq)$ *and* $(\mathbb{R}, \leq)$*, the inclusion functor*

$$\iota : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{R}, \leq) \tag{3.126}$$

$$n \mapsto n \text{ (as a real number)} \tag{3.127}$$

Left and right adjoints :

An important class of monads are the ones which are associated (in the internal logic of the category, cf. [X]) to the classical *modalities*, ie necessity and possibility.

Kripke semantics

Example 86 *Necessity/Possibility modalities*

3.17 Linear and distributive categories

Chapter 4

Spaces

One of the main type of category we will use for objective logic are categories which are *spaces* or relate to spaces, in a broad sense, such as frames, sheaves and toposes.

4.1 General notions of a space

Before looking into how spaces work in category theory, let's first look at how spaces are treated both intuitively, in philosophical analysis, and the most common ways to treat spaces in mathematics.

4.1.1 Mereology

One important aspect of a space in philosophical terms is that of *mereology*. The mereology of a space is the study of its parts, where we can decompose a space into regions with some specific properties. A space X is composed of a collection of regions $\{U_i\}$, which are ordered by a relation of inclusion $(\{U_i\}, \subseteq)$, called *parthood*, which obeys the usual partial order relations :

- Reflexion :
$$U \subseteq U$$
- Symmetry :
$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_1 \to U_1 = U_2$$
- Transitivity :
$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_3 \to U_1 \subseteq U_3$$

Those are the typical notion of a partial order : reflexivity (a region is part of itself), antisymmetry (if a region is part of another, and the other region is part of the first, they are the same region) and transitivity (if a region is part of another region, itself part of a third region, the first is part of the third). The antisymmetry allows us to define equality in terms of parthood, simply as $U_1 = U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_2 \subseteq U_1$.

We also have the *proper parthood* relation $\subset$, which is defined simply as parthood excluding equality :

$$U_1 \subset U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_1 \neq U_2 \tag{4.1}$$

Other relations we can define are proper extension, underlap, disjointness, indiscernibility ,

Definition 87 *The* overlap *of two regions is the existence of a third region which is a part of both :*

$$U_1 \circ U_2 \leftrightarrow \exists U_3,\ [U_3 \subseteq U_1 \wedge U_3 \subseteq U_1] \tag{4.2}$$

Unless a mereological nihilist, we also typically define an operation to turn several regions into one, the *fusion* :

Definition 88 *Given a set of regions $\{U_i\}_{i\in I}$, we say that U is the* fusion *of those region, $\sum(U, \{U_i\})$,*

Mereologies can vary quite a lot depending on what you wish to model or your own philosophical bent. *Mereological nihilism* will assume for instance that there are no objects with proper parts (so $U_1 \subseteq U_2$ implies $U_1 = U_2$), and we can only consider a collection of atomic points with no greater structure (in particular, there is no *space* itself which is the collection of all its regions), while on the other end of the spectrum is *monism* (such as espoused by Parmenides[1]), where the only region is the whole space itself, with no subregion.

Typically however, we tend to consider some specific base axioms for a mereology. Beyond the partial ordering axioms (M1 to M3), we also have

- M4 - Weak supplementation : if U_1 is a proper part of U_2, there's a third region U_3 which is part of U_2 but does not overlap with U_1 : $U_1 \subset U_2 \to \exists U_3,\ [U_3 \subseteq U_2 \wedge \neg U_3 \circ U_1]$

Systems[27, 28] : mereology **M**, minimal mereology **MM**, extensional mereology **EM**, classical extensional mereology **CEM**, general mereology **GM**, general extensional mereology **GEM**, atomic general extensional mereology **AGEM**

In terms of categories, the various formalizations of mereology are expressed by different types of algebraic structures on posets. **M** is simply a poset with no extra structure.

The TOP axiom corresponds to the existence of a greatest element in this partial order (if we consider this applying to spaces, this is the object X of the space itself), BOTTOM to a least element (the empty set).

Most axiomatizations of mereology do not include the bottom element, but we will keep it for a better analogy with spaces in terms of a category, as they typically include one.

4.1.2 Topology

A common approach for space in mathematics is the notion of *topology*. We have already briefly defined the category **Top** of topological spaces, but as they form the basis for most of the common understanding of spaces in math, we should look into them more deeply : what their motivations are, what they are for, and how they may relate to other objects.

If we look at one of the common structuration of mathematics, popularized by Bourbaki[ref on structuralism], spaces are built first as sets, then as topological spaces, and they may afterward get further specified into other structures, such as metric spaces, etc.

This is only a convention, as there are many other structures one may choose, that can be easier, more general, more specific to a given property, etc. The point of topological spaces is that they are a good compromise between those constraints, being fairly easy to define and allowing to talk about quite a lot of properties.

First, let's look at the basic structure of sets. From the perspective of mereology, sets are a rather specific choice of structure, corresponding to an atomic unbounded relatively complemented distributive lattice 1, [see definition of points in philosophy too etc]

Historically, the notion that spaces are made of point is quite ancient [cf. Sextus Empiricus], but it has not had the modern popularity it now has until the works of [Riemann?] Cantor, Hausdorff, Poincaré, etc, and the notion was put into the modern mathematical canon with such works as Bourbaki, etc.

If we consider spaces only as sets however, the informations we can derive from them is rather limited. This is an observation from antiquity [Sextus again]

In modern terms, we can talk about subsets, cardinalities, overlap and unions, but we would be missing on quite a lot of intuitively important properties of a space. If we consider physical space as our example, as we've seen from mereology, some points seem to "belong together" more than other points, some may be "next to" a give subset even if they do not belong to it, two subsets may "touch" without any overlap, and so on.

To illustrate those notions, we can consider some subsets of the plane. If we compare let's say some kind of continuous shape, a disk and an uncountable set of points sprinkled in an area (for instance the two-dimensional Cantor dust),

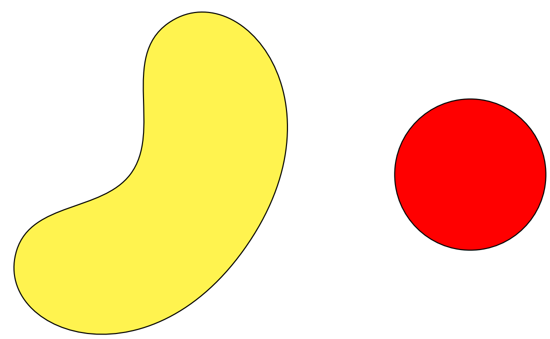


Figure 4.1: Three subsets of the plane of identical cardinality

We would expect the first two objects to have more in common than with the third, having what we will later call the same *shape*, but as sets, they are all isomorphic, simply by the virtue of containing the same amount of points. The same goes for comparing one connected shape and one composed of two disconnected shapes

[...]

If we consider the set D of all points that are at a distance strictly inferior to a given value r from a central point o, we would like to say that a point at exactly a distance of r is somehow closer to D than other points, but as a set, this point is simply equivalent to every point outside of D, as $D \cap \{p\} = \varnothing$.

Convergence

quasitopological spaces, approach spaces, convergence spaces, uniformity spaces, nearness spaces, filter spaces, epitopological spaces, Kelley spaces, compact Hausdorff spaces, δ-generated spaces Cohesion, pretopology, proximity spaces, convergence spaces, cauchy spaces, frames, locales

Approaches via open/closed sets, closure operators, interior operators, exterior operators, boundary operators, derived sets

Definition 89 *Given a set X, a* net *on X is a directed set $(A, \leq)$ and a map $\nu : A \to X$*

Example 90 *A* sequence *is a net with directed set $(\mathbb{N}, \leq)$*

Example 91 *Net of neighbourhoods*

Definition 92 *A net is said to* converge *to an element $x \in X$*

Definition 93 *A* filter

4.2 Frames and locales

All those notions of mereology and topology can be formalized within the context of category theory using the notion of frames and locales.

As we've seen, any formalization of a space can be at least formalized as a poset ordered by inclusion, already a category. All further notions relating to spaces will therefore be extra structures on posets, typically relating to their limits.

First we need to define the notion of semilattices for joins and meets.

Definition 94 *A meet-semilattice is a poset $(S, \leq)$ with a meet operation $\wedge$ corresponding to the greatest lower bound of two elements (which is assumed to always exist in a meet-semilattice) :*

$$m = a \wedge b \leftrightarrow m \leq a \wedge m \leq b \wedge (\forall w \in S,\ w \leq a \wedge w \leq b \rightarrow w \leq m) \tag{4.3}$$

Example 95 *In $\mathbb{Z}$ and $\mathbb{R}$ (in fact for any total order), the meet of two numbers is the min function :*

$$k_1 \wedge k_2 = \in (k_1, k_2) \tag{4.4}$$

Theorem 96 *In a poset category, the meet is the coproduct.*

Example 97 *In the partial order defined by the power set of a set, the meet is the intersection of two sets.*

Definition 98 *A join-semilattice is a poset $(S, \leq)$ with a join operation $\vee$ corresponding to the least upper bound of two elements (which is assumed to always exist in a join-semilattice) :*

$$m = a \vee b \leftrightarrow a \leq m \wedge b \leq m \wedge (\forall w \in S,\ a \leq w \wedge b \leq w \rightarrow m \leq w) \tag{4.5}$$

Theorem 99 *In a poset category, the join is the product.*

Proof 8 *Product : $\prod_i a_i$*

Natural transformation (by components) :

$$\eta_X : \prod_i a_i \to a_i \tag{4.6}$$

There is one morphism from the join to each element, therefore $\prod_i a_i \leq a_i$

Upper bound is least : for any other b *such that* $b \leq a_i$ *(ie the natural transformation* $\alpha_b : b \to a_i$ *for some* a_i*), then the unique morphism* $f : b \to \prod a_i$ *(*b *smaller than* a_i*)*

Properties :

Proposition 100 *The meet is commutative :* $a \wedge b = b \wedge a$.

Proof 9 *As the roles of* a *and* b *in the definition of the meet are entirely symmetrical, due to the commutativity of the logical conjunction, this is true.*

Proposition 101 *The meet is associative :* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof 10 *If* $m = b \wedge c$*, then* $a \wedge (b \wedge c) = a \wedge m$*, meaning that the meet can be defined by some element* m' *such that*

$$(m' \leq a) \wedge (m' \leq m) \wedge (m \leq b) \wedge (m \leq c) \tag{4.7}$$
$$\wedge (\forall w \in S,\ w \leq b \wedge w \leq c \to w \leq m) \tag{4.8}$$
$$\wedge (\forall w' \in S,\ w' \leq a \wedge w \leq m \to w \leq m') \tag{4.9}$$

as $(m' \leq m) \wedge (m \leq b)$

Definition 102 *A* lattice *is a poset that is both a meet and join semilattice, such that* $\wedge$ *and* $\vee$ *obey the* absorption law

$$a \tag{4.10}$$

Definition 103 *A* Heyting algebra *is a bounded lattice*

A common poset structure that we will use is the algebra generated by a family of subsets. If we have a set X, and a family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, $\mathcal{B}$ forms a poset by the inclusion relation ordering, $(\mathcal{B}, \subseteq)$. Some common families of subsets of interest are the power set $\mathcal{P}(X)$, and more generally the set of opens $\mathrm{Op}(X)$ for a given topology.

It is a directed set with X the top element

In this context, the meet is the intersection

Theorem 104 *The intersection of two sets is their meet.*

Proof 11 *As the intersection of A and B is defined via*

$$A \cap B = \{x | x \in A \wedge x \in B\} \tag{4.11}$$

We can

We already know that $A \cap B \subseteq A, B$. If we assume a set $C \neq A \cap B$ such that $A \cap B \subseteq C$ and $C \subseteq A, B$, this means that C contains all the same elements as $A \cap B$ with some additional elements (since $A \cap B$ is a subset, we have $C = (A \cap B) \cup (A \cap B)^C$, and as they are different, $(A \cap B)^C \neq \varnothing$). However, as C is a subset of A and B, that complement can only contains elements of A and B

Theorem 105 *The union of two sets is their join.*

Proof 12 *As the union of A and B is defined via*

$$A \cup B = \{x | x \in A \vee x \in B\} \tag{4.12}$$

A family of sets is therefore a meet semilattice if it is closed under intersection, and a join semilattice if it is closed under union. If it is both, it is automatically a lattice as the absorption laws are obeyed by union and intersection.

[proof]

If the empty set is furthermore included, it is a bounded lattice, with $1 = X$, $0 = \varnothing$

Semi-lattice, lattice, Heyting algebra, frame (complete Heyting algebra)

Definition 106 *A* Heyting algebra *H is a bounded lattice for which any pair of elements $a, b \in H$ has a greatest element x, denoted $a \to b$, such that*

$$a \wedge x \leq b \tag{4.13}$$

Definition 107 *The* pseudo-complement *of an element a of a Heyting algebra is*

$$\neg a = (a \to 0) \tag{4.14}$$

Example 108 *A bounded total order $0 \to 1 \to \ldots \to n$ is a Heyting algebra given by*

$$a \to b = \begin{cases} n & a \leq b \\ b & a > b \end{cases} \tag{4.15}$$

The pseudo-complement is therefore just $\neg a = 0$.

Example 109 *For the power set $\mathcal{P}(X)$ poset, the relative pseudo-complement of two sets A, B is*

$$C = (X \setminus A) \cup B \tag{4.16}$$

This follows the property as $A \cap C = A \cap (B \setminus A)^C$

(eq. to the discrete topology)

Definition 110 *A Heyting algebra is* complete *if*

Definition 111 *A* frame $\mathcal{O}$ *is a poset that has all small coproducts (called joints $\vee$) and all finite limits (called meets $\wedge$), and satisfied the distribution law*

$$x \wedge (\bigvee_i y_i) \leq \bigvee (x \wedge y_i) \tag{4.17}$$

Frames define a mereology by considering its objects as regions, its poset structure by the parthood relation, and joins and meets by

Mereological axiom for distribution law?

Category of frames : **Frm**

Dual category : the locales $\mathbf{Frm}^{\mathrm{op}} = \mathbf{Locale}$

Definition 112 *A* boolean algebra $a \wedge \neg a = 0$

Example 113 *A power set is a boolean algebra*

The basic example of a frame in math is that of the frame of opens for a topological space (X, τ).

Example 114 *The category of open sets of a topological space X,* $\mathrm{Op}(X)$*, is a frame.*

Proof 13 *If we consider the poset of opens, as a union and intersection of open sets is itself an open set, we have a lattice, which is bounded by X itself and the empty set $\varnothing$.*

The frame of open is not boolean typically, as the negation $\neg$ can be defined as $\neg a \to 0$, and the implication

$$\begin{aligned} U \to V &= \bigcup \{W \in \mathrm{Op}(X) | U \cap W \subseteq V\} && (4.18) \\ &= (U^c \cup V)^\circ && (4.19) \end{aligned}$$

$$\begin{aligned}\neg U &= (U^c \cup \varnothing)^\circ && (4.20)\\ &= (U^c)^\circ && (4.21)\\ &= X \setminus \mathrm{cl}(U \cap X) && (4.22)\\ &= X \setminus \mathrm{cl}(U) && (4.23)\end{aligned}$$

The interior of the complement

$$\begin{aligned}U \cup (X \setminus U)^\circ &= (X \cup U) \setminus (\mathrm{cl}(U) \setminus U) && (4.24)\\ &= X \setminus \partial U && (4.25)\\ & && (4.26)\end{aligned}$$

Therefore a frame of open is boolean if open sets never have a boundary, which is that every open set is a clopen set.

Stone theorem

Theorem 115 *The category* **Sob** *of sober topological spaces with continuous functions and the category* **SFrm** *of spatial frames are dual to each other.*

Examples :

Example 116 *For a given set X, the partial order defined by inclusion of the power set $\mathcal{P}(X)$, is a complete atomic Boolean algebra.*

Sober space

4.2.1 Sublocales

Moore closure

double negation sublocale

Consider the map

$$\begin{aligned}\neg\neg : L &\to L && (4.27)\\ U &\mapsto \neg\neg U && (4.28)\end{aligned}$$

A nucleus on L (a frame) is a function $j : L \to L$ which is monotone ($j(a \wedge b) = j(a) \wedge j(b)$), inflationary ($a \leq j(a)$) and $j(j(a)) \leq j(a)$

A meet-preserving monad.

Properties :

- $j(\top) = \top$
- $j(a) \leq j(b)$ if $a \leq b$
- $j(j(a)) = j(a)$

Quotient frames : L/j is the subset of L of j-closed elements of L (such that $j(a) = a$).

Double negation sublocale :

4.3 Coverage and sieves

To define a space in categorical terms, we need to have some formalization of an equivalent notion to mereology, open sets, frames or such that we saw earlier. The notion of *coverage* that we will see will be more general than that (in particular not necessarily be about subregions) but contain those as a special case.

[define cover/covering family first?]

Definition 117 *Given an object X in a category $\mathbf{C}$, a* coverage *J is a collection of morphisms to that object indexed by some indexing set I,*

$$J = \{U_i \to X\}_{i \in I} \tag{4.29}$$

such that morphisms between two objects of $\mathbf{C}$ induce a coverage. For $g : Y \to X$, there exists a covering family $\{h_j : V_j \to Y\}_{j \in J}$ such that gh_j factors through f_i for some i :

$$\begin{array}{ccc} V_j & \xrightarrow{k} & U_i \\ \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow{g} & X \end{array}$$

If we take the case of topology that we've seen as an example, we define the standard coverage of a space X to be the collection of all families of open subsets that cover it, ie

$$J(X) = \{\{U_i \to X\} \mid U_i \subseteq X, \bigcup_i U_i = X\} \tag{4.30}$$

Its stability under pullback corresponds to the fact that for any continuous function $f : Y \to X$, as the pre-image of any open set is itself an open set, we can define a family

$$\{f^{-1}(U_i) \to Y\} \tag{4.31}$$

and as any point in X is covered by some U_i, any point in Y will similarly be covered by $f^{-1}(U_i)$, obeying the properties of a coverage.

"Another perspective on a coverage is that the covering families are "postulated well-behaved quotients." That is, saying that $\{f_i : U_i \to U\}_{i\in I}$ is a covering family means that we want to think of U as a well-behaved quotient (i.e. colimit) of the U_i. Here "well-behaved" means primarily "stable under pullback." In general, U may or may not actually be a colimit of the U_i; if it always is we call the site subcanonical. " To define spaces in the mathematical sense of the word, we need to have some sort of equivalent definition of a *topology*.

If $\mathbf{C}$ has pullback : the family of pullbacks $\{g^*(f_i) : g^*U_i \to V\}$ is a covering family of V.

Grothendieck topology :

An important class of coverage is the *Grothendieck topology*

Cech nerve

Sieve

Definition 118 *For a covering family $\{f_i : U_i \to U\}$ in a coverage J, its* sieve *is the coequalizer*

$$\coprod_{j,k} j(U_j) \times_{j(U)} j(U_k) \rightrightarrows \coprod_i j(U_i) \to S(\{U_i\}) \tag{4.32}$$

with j the Yoneda embedding $j : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C})$

Other definition : A sieve $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ on $X \in \mathbf{C}$ is a subfunctor of $\mathrm{Hom}_{\mathbf{C}}(-, X)$

Objects $S(Y)$ are a collection of morphisms $Y \to X$, and for any morphism $f : Y \to Z$, $S(f)$

Pullback by a sieve :

Ordering : $S \subseteq S'$ if $\forall X$, $S(X) \subseteq S'(X)$

Category of sieves is a partial order, with intersection and union, it is a complete lattice

Grothendieck topology : covering sieves

4.3.1 Cech nerves

Given a covering sieve $\{U_i \to X\}$ with respect to a coverage,

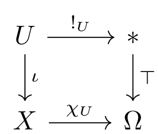
$$C(U) = \left(\cdots U \times_X U \times_X U \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} U \times_X U \rightrightarrows U \right) \tag{4.33}$$

4.4 Subobject classifier

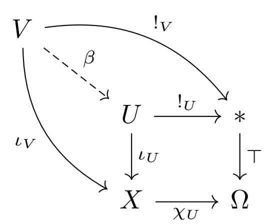
In a category $\mathbf{C}$ with finite limits, a subobject classfier is an object Ω (the object of truth values) and a monomorphism

$$\top : * \to \Omega \tag{4.34}$$

from the terminal object $*$, such that for every monomorphism [inclusion map] $\iota : U \hookrightarrow X$, there is a unique morphism $\chi_U : X \to \Omega$ such that there is a pullback of $* \to \Omega \leftarrow X$



ie this diagram commutes and is universal, in the sense that for any other subobject V of X, with $\iota_V : V \to X$, the following diagram only commutes if V is itself a subobject of U :



ie that V has the same type of valuation in Ω as U through the characteristic function χ_U. This is best exemplified by the simple case for sets :

Example 119 *In* **Set**, *Ω is the set containing the initial object, $\Omega = \{\varnothing, \{\bullet\}\}$, also noted as $2 = \{0, 1\}$.*

For a subset $S \subseteq X$ with an inclusion map $\iota : S \hookrightarrow X$, the characteristic function $\chi_S : X \to 2$ is the function defined by $\chi_S(x) = 1$ for $x \in S$ and $\chi_S(x) = 0$ otherwise. The truth function simply maps $*$ to 1 in Ω.

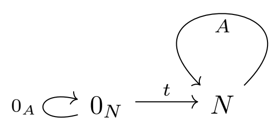
$$\forall x \in U, \ \chi_U(\iota_U(x)) = 1 \tag{4.35}$$

And conversely, if we look at another subobject $V \subseteq X$, the pullback works out

$$\forall x \in V, \ \chi_U(\iota_U(x)) = 1 \tag{4.36}$$

only if $V \subseteq U$, ie there exists a monomorphism from $V \to U$

Subobject classifiers can be more complex than the simple boolean domain true/false. A good illustration of this is the subobject classifier in the category of graphs[29]. A graph is composed by two sets, those of nodes N and of arrows A, with two functions s, t for the source and target of each arrow.



For a subgraph $\iota : S \hookrightarrow G$, the classifying map χ_S has the following behaviour :

- If a node in G is not in S, it is mapped to 0_N.
- If a node in G is in S, it is mapped to N.

Subobject classifier for a topological space

4.4.1 In a sheaf topos

Given the sheaf topos $\mathbf{H} = \mathrm{Sh}(\mathbf{C}, J)$, there is a natural subobject classifier

4.5 Elements

One of the important difference between set theory and category theory is that while sets are typically composed of elements, as defined by the $\in$ relation, categories (for which the objects are often somewhat similar to sets themselves) do not seem to have a naturally equivalent notion.

If we wish to define elements of a set in terms of the morphisms of sets (functions), this is best done via the use of functions from the singleton set $\{\bullet\}$, as those functions are in bijection with the elements of a set

$$\mathrm{Fun}(\{\bullet\}, X) \cong X \tag{4.37}$$

As functions from the singleton are all of the form $\{(\bullet, x)\}$ for every $x \in X$ (From the properties of the Cartesian product)

Generalized elements : Given the yoneda embedding $Y : C \hookrightarrow [C^{\text{op}}, \mathbf{Set}]$, the representable functor

$$\text{GenEl}(X) : C^{\text{op}} \to \mathbf{Set} \tag{4.38}$$

Sends each object U of C to the set of generalized elements of X at stage U.

For an object $X \in \text{Obj}(\mathbf{C})$, its *global elements* are morphisms $x : 1 \to X$. It's a generalized element at stage of definition 1.

Definition 120 *An object $S \in \mathbf{C}$ is a* separator *if for every pair of morphisms $f : X \to Y$, and every morphism $e : S \to X$, then $f \circ e = g \circ e$ implies $f = g$.*

In other words, the global elements generated by the separator are enough to entirely define the morphisms. For instance, in the case of **Set**, $\{\bullet\}$ is a separator, essentially saying that the elements of a set entirely define its functions : a function $f : X \to Y$ is defined by its value $f(x)$ for every $x \in X$.

If we have a topos E such that its terminal object 1 is a separator, and $1 \neq 0$, we say that the topos is *well-pointed*, meaning that

Other definitions : global section functor is faithful

Prop : well-pointed topos are boolean, its subobject classifier is two-valued,

Definition 121 *A* concrete category $\mathbf{C}$ *is such that there exists a faithful functor F*

$$F : \mathbf{C} \to \mathbf{Set} \tag{4.39}$$

What the faithful functor implies

[Concrete categories and well-pointed ones do not imply each other in any direction, depends on if elements are global elements?]

Given a set-valued functor $F : \mathbf{C} \to \mathbf{Set}$, its *category of elements*

Category of elements

The elements of an object in a category are not necessarily best modeled by global elements, however. The archetypical example of this being the category of vector spaces **Vec**, for which the terminal object, being a zero object, only has one possible morphism to any other object, giving them only a single "point" if we try to go with that kind of element. The better choice here would be k, the field of the vector space and tensor unit, as the maps $k \to V$ are isomorphic to V itself as sets.

This is commonly the case in monoidal categories, such as **Hilb**

[...]

What is the relation of global elements wrt the subobject classifier?

4.5.1 Points

As we have seen, we can represent a general notion of space as that of a frame, but a more contentious issue is how to define *points* in a space. This is an issue that goes back all the way to the foundation of geometry [30], and to this day is not an uncontentious one, as the assumption of point-like structures in space is still a thorny issue.

Categories for which we have fairly simple notions of discrete elements such as finite sets do not have too much trouble defining what a point could be, corresponding in some sense to the notion of discrete objects that were used uncontroversially in antiquity, but given a frame like that of continuous physical space, this becomes a more complex notion to define, as there is no internal notion of what a point is in the context of the frame of opens $\mathcal{O}(X)$.

In the abstract, we can define a point just as we would define an element for another category, simply as morphisms from some terminal object to the regions of space, but we are neither guaranteed the existence of such an object nor that the space is in some sense *composed* by those points rather than just those merely inhabiting it.

The intuitive notion, going back at least as far as [30], would be to consider a point as the limit of a shrinking family of open sets, but we could have for instance a family of regions $\{U_i\}$ which converge to another (non-point like) region, such as a family of disks of radius $r_n = 1 + 2^{-n}$. Furthermore, two different such sequences can converge to the same point so that we also need to be able to define the equivalence of such sequences.

To represent the notion of several sequences of regions converging to the same result, we need to use the notion of *filter*

A subset F of a poset L is called a filter if it is upward-closed and downward-directed; that is:

If $A \leq B$ in L and $A \in F$, then $B \in F$; for some A in L, $A \in F$; if $A \in F$ and $B \in F$, then for some $C \in F$, $C \leq A$ and $C \leq B$.

Points given a locale?

Given a locale X, a concrete point of X a completely prime filter on $O(X)$. [Show equivalence with a continuous map $f : 1 \to X$: treat $f^* : O(X) \to O(1)$ as a characteristic function]

Completely prime filter :

A filter F is prime if $\perp \notin F$ and if $x \vee y \in F$, then $x \in F$ and $y \in F$. For every finite index set I, $x_k \in F$ for some k whenever $\bigvee_{i \in I} x_i \in F$.

[Some descent of open sets for a topological space?]

4.6 Internal hom

One component of the definition of a topos regards the behaviour of its *internal homs*, which are a way with which to include the hom-set of the category in its objects. In other words, every space of morphisms between two objects of the topos is itself an object of the topos.

Definition 122 *In a symmetric monoidal category* $(\mathbf{C}, \otimes, I)$*, an* internal hom *is a bifunctor*

$$[-,-] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C} \tag{4.40}$$

such that for any object $X \in \mathbf{C}$*, the functor* $[X,-]$ *is right adjoint to the functor* $(-) \otimes X$ *:*

$$((-) \otimes X) \dashv [X,-]) : \mathbf{C} \to \mathbf{C} \tag{4.41}$$

As an adjunction, we have two natural transformations

$$\eta : \mathrm{Id}_{\mathbf{C}} \to [X, ((-) \otimes X)] \tag{4.42}$$

$$\epsilon : ([X,-] \otimes X) \to \mathrm{Id}_{\mathbf{C}} \tag{4.43}$$

The simplest definition of the internal hom is via the adjunction of hom sets :

$$\mathrm{Hom}_{\mathbf{C}}(Y \otimes X, Z) \cong \mathrm{Hom}_{\mathbf{C}}(Z, [X,Y]) \tag{4.44}$$

If we have a morphism $Y \otimes X \to Z$, there is equivalently some morphism $Z \to [X,Y]$

Example : take $Z = [X,Y]$, take the morphism $\mathrm{Id}_{[X,Y]} : [X,Y] \to [X,Y]$. Its adjunct is

$$Y \otimes X \to [X,Y] \tag{4.45}$$

Evaluation map :

Internal hom bifunctor

Adjunction $- \times A \dashv (-)^A$

4.7 Presheaves

Definition 123 *A* presheaf *on a small category* **C** *is a functor* F

$$F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set} \tag{4.46}$$

This definition also generalizes to any category. If we replace **Set** with any category **S**, we speak of an S-valued presheaf, defined as

$$F : \mathbf{C}^{\mathrm{op}} \to \mathbf{S} \tag{4.47}$$

In a similar manner, we have the dual of presheaves, called *copresheaves*, and defined as sheaves on the opposite category :

$$F : \mathbf{C} \to \mathbf{Set} \tag{4.48}$$

And similarly, for an S-valued copresheaf,

$$F : \mathbf{C} \to \mathbf{S} \tag{4.49}$$

Fundamentally, any functor can be described as a (co)presheaf, as any functor from a category **C** (or its opposite) fits the definition, but a presheaf is typically gonna be studied with more specific goals in mind, ie to turn them into sheaves or topos.

An example for the motivation of (co)presheaves is to consider a topological space (X, τ). The category of interest here is the frame of opens $\mathrm{Op}(X)$. A sheaf on the frame of open is some functor associating a set to every open set :

$$\forall U \in \mathrm{Op}(X),\ F(U) = A \in \mathbf{Set} \tag{4.50}$$

In a way that preserves the functions in some sense. In particular, if we have an inclusion $\iota : U \hookrightarrow U'$, its opposite is $\iota^{\mathrm{op}} : U' \to U$, and the functor maps it to

$$F(\iota^{\mathrm{op}}) : F(U') \to F(U) \tag{4.51}$$

We will see in the section on sheaves the meaning of this construction.

Example 124 *An S-valued presheaf on* **C** *is a* constant presheaf *if it is a constant functor, ie for some element $X \in \mathbf{S}$, the presheaf is just*

$$\Delta_X : \mathbf{C}^{\mathrm{op}} \to \mathbf{S} \tag{4.52}$$

interpretation of presheaves $X(U)$ as a function U to X via Yoneda

4.7.1 Simplexes

A basic example of presheaves is the simplexes, which are a presheaf over the simplex category.

$$X : \Delta^{\mathrm{op}} \to \mathbf{Set} \tag{4.53}$$

By the Yoneda embedding [representable presheaves etc], any object in the simplex category is a simplex. Furthermore, we can consider simplexes which are constructed from the combination of different simplexes

Example : the triangle. Take the two simplexes $\vec{\mathbf{2}}$, $\vec{\mathbf{1}}$

4.8 Sheaves

The more important construction based on (co)presheaves is that of (co)sheaves, which are (co)presheaves with some additional conditions, meant to signify the spatial nature of the construction : the category corresponds in some sense to the piecing together of regions.

[...]

Consider the Yoneda embedding of $\mathbf{C}$:

$$j : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C}) \tag{4.54}$$

Definition 125 *Given a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, and given a coverage J of $\mathbf{C}$, F is a sheaf with respect to J if*

- *for every covering family $\{p_i : U_i \to U\}_{i \in I}$ in J*
- *for every compatible family of elements $(s_i \in F(U_i))_{i \in I}$,*

there is a unique element $s \in F(U)$ such that $F(p_i)(s) = s_i$ for all $i \in I$.

If we consider the case where a covering family is composed of monomorphisms with subobjects (assuming the equivalence something something), then $p_i : U_i \hookrightarrow U$ can be considered [something], and the morphism generated by the sheaf is understood to be a *restriction* : $F(p_i)(s) = s_i$, we are restricting the section s on U to the subobject on U_i.

[The section of a sheaf is defined by its local elements]

The easiest example of this is to pick once again a presheaf on the frame of open $F \in \mathrm{Psh}(\mathrm{Op}(X))$, and as the coverage, pick the subcanonical coverage. For any open set $U \subseteq X$, a subcanonical coverage is a family of open sets $\{U_i\}$ such that

$$\bigcup_i U_i = U \tag{4.55}$$

[...]

A nice class of such sheaves are the sheaves given by function spaces over the appropriate sets, *sheaves of functions.*

Example 126 *For an S-valued sheaf on (C, J), the* constant sheaf

Examples with sheaves on frames of opens

For $\mathrm{Op}(X)$ with the canonical coverage, a presheaf F is a sheaf if for every complete subcategory $\mathcal{U} \hookrightarrow \mathrm{Op}(X)$,

$$F(\lim_{\rightarrow} \mathcal{U}) \cong \lim_{\leftarrow} F(\mathcal{U}) \tag{4.56}$$

Proof 14 *Complete full subcategory is a collection $\{\iota_i : U_i \hookrightarrow X\}$ closed under intersection. The colimit*

$$\lim_{\rightarrow}(\mathcal{U} \hookrightarrow \mathrm{Op}(X)) \cong \bigcup_i U_i \tag{4.57}$$

is the union of these open subsets. By construction,

Empty sheaf, unit sheaf

4.9 Topos

One important type of categories that will be the main focus of study here is that of a *topos*. There are many possible definitions and intuitions of what a topos is, many of them listed in [31], but for our purpose, a topos will mostly be about

- A universe of types in which to do mathematics
- A category of spaces
- A categorification of some types of logics

There are a few different nuances to what a topos can be, but the most general case we will look at for now (disregarding things such as higher topoi) is that of an elementary topos.

Definition 127 *An* elementary topos $\mathbf{H}$ *is a category which has all finite limits, is Cartesian closed, and has a subobject classifier.*

An elementary topos' definition fits best in the first sense of the definitions, in that it is a universe in which to do mathematics. These properties are overall modeled over **Set**, and in some sense it is the generalization of a set. As we will see [], **Set** itself is a topos.

Its use as a mathematical universe is given by its closure under limits (and as we will show, colimits) and exponentiation. We can easily talk about any (finite) construction of objects in a topos, as well as any function between two elements of a topos, without leaving the topos itself, and the subobject classifier [...]

Theorem 128 *An elementary topos has all finite colimits.*

Proof 15 *Contravariant power set functor :*

$$\Omega^{(-)} : \mathbf{H}^{op} \to \mathbf{H} \tag{4.58}$$

Properties : locally Cartesian closed, finitely cocomplete, Heyting category,

4.9.1 Grothendieck topos

The most common type of topos, and the one we will typically use, is the Grothendieck topos.

Definition 129 *A* Grothendieck topos E *on a site* $\mathbf{C}$ *with coverage* J *is a sheaf over the site* $\mathbf{C}$

$$\mathcal{E} \cong \mathrm{Sh}(\mathbf{C}, \mathcal{J}) \tag{4.59}$$

Proposition 130 *A Grothendieck topos is an elementary topos*

Proof 16

Example 131 *A trivial example of a Grothendieck topos is the initial topos, which is the sheaf topos over the empty category with the empty topology (which is the maximal topology on this category),* $\mathrm{Sh}(\mathbf{0})$*. The only element of this topos is the empty functor, with the identity natural transformation on it (as there is no possible components to differentiate them on the empty category, this is the only one). We therefore have*

$$\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1} \tag{4.60}$$

Its only object $*$ *is both the initial and terminal object (therefore a zero object), its product and coproduct are simply* $* + * \cong *$ *and* $* \times * \cong *$

One important nuance in topos theory is that a topos can be considered alternatively as a space in itself, or as a category in which every object is a space. The former is referred to as a *petit topos*, while the latter is a *gros topos*. A typical example of this would be for instance the topos of smooth spaces **Smooth**, which contains (among other things) all manifolds, as a gros topos, while the topos of the site of opens on a topological space $\mathrm{Sh}(\mathrm{Op}(X))$ would be an example of a petit topos.

Theorem 132 *For any topos* $\mathbf{H}$*, the slice category given by one of its object* $\mathbf{H}_{/X}$ *is itself a topos.*

This construction allows us to bridge the gros and petit topos, in that given a space X in a gros topos $\mathbf{H}$, its corresponding petit topos will be $\mathbf{H}_{/X}$.

"Also in 1973 Grothendieck says the objects in any topos should be seen as espaces etales over the terminal object of the topos, in a generalized sense that includes saying any orbit of a group action lies "etale" over a fixed point. "

Subobject classifier of a sheaf topos :

Example 133 *In the sheaf topos* $\mathbf{Set} \cong \mathrm{Sh}(\mathbf{1})$

Sheaf topos is Cartesian closed, internal hom from this monoidal structure

4.10 Site

A site is roughly speaking the elements from which a (Grothendieck) topos is stitched together.

A site (C, J) is a category C equipped with a coverage J

Sieve

Example :

Example 134 *The terminal category* 1 *is a site. The covering family is simply the only function,* $\{\mathrm{Id}_* : * \to *\}$*. As there are no other objects in the category, we only need to check the induced coverage on itself. Given the morphism* $\mathrm{Id}_* : * \to *$*, the diagram commutes trivially by using the identity function everywhere.*

$$\begin{array}{ccc} * & \xrightarrow{\mathrm{Id}_*} & * \\ \downarrow{\scriptstyle \mathrm{Id}_*} & & \downarrow{\scriptstyle \mathrm{Id}_*} \\ * & \xrightarrow{\mathrm{Id}_*} & * \end{array}$$

Example 135 *The category of opens of a topological space is a site*

In particular, Cartesian spaces?

"Every frame is canonically a site, where U is covered by $\{U_i\}$ precisely if it is their join."

Is there some kind of relationship between the sheaves of a Grothendieck topos, and the elements of the site taken as (representable) sheaves + coproduct and equalizer

4.10.1 Site morphisms

Site morphism : for C, D sites, a functor $f : C \to D$ is a morphism of sites if it is covering-flat and preserves covering families : for every covering $\{p_i : U_i \to U\}$ of $U \in C$, $\{f(p_i) : f(U_i) \to f(U)\}$ is a covering of $f(U) \in D$.

Covering-flat :

For a set-valued functor $F : C \to \mathbf{Set}$,

Filtered category : A filtered category is a category in which every diagrgam has a cocone.

For any finite category D and functor $F : D \to C$, there exists an object $X \in C$ and a nat. trans. $F \to \Delta_X$.

Simpler version :

- There exists an object of C (non-empty category)
- For any two objects $X, Y \in C$, there is an object Z and morphisms $X \to Z$, $Y \to Z$
- For any two parallel morphism, $f, g : X \to Y$, there exists a morphism $h : Y \to Z$ such that $hf = hg$.

Every category with a terminal object is filtered.

Every category which has finite colimits is filtered.

Interpretation : for any limit that the site has, they are preserved.

4.11 Stalks and étale space

[32]

4.11.1 In a topological context

In **Top**, consider an object B (the base space), and take the slice category $\mathbf{Top_{/B}}$, the category of bundles $\pi : E \to B$ over B.

If π is a local homeomorphism, ie for every $e \in E$, there is an open neighbourhood U_e such that $\pi(U_e)$ is open in B, and the restriction $\pi|_{U_e} : U_e \to \pi(U_e)$ is a homeomorphism, then we say that $\pi : E \to B$ is an *etale space*, with $E_x = \pi^{-1}(x)$ the *stalk* of π over x.

For $\mathrm{Sh}(\mathbf{C}, J)$ a topos, if F is a sheaf on $(\mathbf{C}, J)$, the slice topos $\mathrm{Sh}(\mathbf{C}, J)/F$ has a canonical étale projection $\pi : \mathrm{Sh}(\mathbf{C}, J)/F \to \mathrm{Sh}(\mathbf{C}, J)$, a local homeomorphism of topoi, the étale space of F.

For any object $X \in \mathbf{C}$, $y(X)$ the Yoneda embedded object,

$$U(X) = \mathrm{Sh}(\mathbf{C}, J)/y(X) \tag{4.61}$$

Sections of π_F over $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ are in bijection with elements of $F(X)$.

If $(\mathbf{C}, J)$ is the canonical site of a topological space, each slice $\mathrm{Sh}(\mathbf{C}, J)/F$ is equivalent to sheaves on the etale space of that sheaf. In particular, $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ corresponds to the inclusion of an open subset.

4.12 Topological spaces

While the category of topological spaces **Top** is *not* a topos, I feel like I should bring up some comments on this.

The common notion of a "space" in mathematics (common as in defined on set theory) is usually done using that of a topological space, for various reasons such as history, ease of use, broadness, etc. The common grounding as with the rest in general is done using set theory as

Definition 136 *A topological space (X, τ) is a set X and a set of subsets $\tau \subset \mathcal{P}(X)$, called* open sets*, such that for any collection of open sets $\{U_i\} \subseteq \tau$, we have*

- *The entire space X and the empty set $\varnothing$ are both open sets : $X, \varnothing \in \tau$*
-

In terms of category, we have the category of topological spaces **Top**, with objects the class of all topological spaces, and morphisms the class of all continuous functions.

Concrete category

Limits and colimits

Subobject classifier

Problem with exponential object

Convenient category of topological spaces : subcategory of Top such that

Every CW complex is an object C is Cartesian closed C is complete and co-complete Optional : C is closed under closed subspaces in Top : if X in C and $A \subseteq X$ is a closed subspace, then A belongs to C.

4.13 Geometry

The broad notion of "geometry" in a topos involved the use of so-called *geometric morphisms* (although in terms of the topos itself, those are actually functors)

For two toposes E, F, a geometric morphism $f : E \to F$ is a pair of adjoint functors (f^*, f_*)

$$f_* : E \to F \tag{4.62}$$
$$f^* : F \to E \tag{4.63}$$

such that the left adjoint f^* preserves finite limits. f_* is the *direct image functor*, while f^* is the *inverse image functor*.

To get a better idea of what a geometric morphism is, let's look at the more concrete case of a Grothendieck topos. If the sites are $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$, with a morphism of site $f : X \to Y$, inducing a functor by precomposition :

$$(-) \circ f : \mathrm{PSh}(Y) \to \mathrm{PSh}(X) \tag{4.64}$$
$$(F : Y^{\mathrm{op}} \to \mathbf{Set}) \mapsto (G : X^{\mathrm{op}} \to \mathbf{Set}) \tag{4.65}$$

ie for some element $y \in Y$, and a presheaf F, we have the map

$$F \circ f : Y \to \mathrm{Set} \tag{4.66}$$

Upon restriction to act on sheaves, this is our inverse image functor f^*, with the right adjoint to this being the direct image functor.

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y) \tag{4.67}$$
$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X) \tag{4.68}$$

Example 137 *The basic example which gives the morphisms their names is the case of a sheaf over a topological space X, where the site is the category of opens* $\mathrm{Op}(X)$*, and a common type of sheaf is the sheaf of functions to some set* A *:* $\mathrm{Sh}(\mathrm{Op}(X)) = C(X, A)$*.*

$$[X] \tag{4.69}$$

A morphism of site in this case is a continuous function $f : X \to Y$, which induces a functor on $\mathrm{Op}(X)$ *by restriction :*

$$f(U \in \mathrm{Op}(X)) = \tag{4.70}$$

$$f_*F(U) = F(f^{-1}(U)) \tag{4.71}$$

Example 138 *Another similar example to look at the basic functioning of the geometric morphisms is to consider two slice topos from the category of sets. Taking two sets X and Y, consider the slice topoi* $\mathbf{Set}_{/X}$ *and* $\mathbf{Set}_{/Y}$*.*

Example : One of the most common type of geometric morphism on a (Grothendieck) topos is the case of global sections. The site morphism involved is from whichever site we decide on our topos X to the trivial site $*$, so that our geometric morphism is between our topos and the topos of sets, $\mathbf{Set} = \mathrm{Sh}(*)$. The only site morphism available here is the constant functor

$$p : X \to * \tag{4.72}$$

which is a site morphism as any covering family of X is sent to $\mathrm{Id}_* : * \to *$, which is the only covering of 1. As the terminal category does not have much in the way of limits, we will have to show that this functor is filtered.

The induced functor is therefore some functor from the category of sets to our topos, so that for any object $F : X^{\mathrm{op}} \to \mathbf{Set}$ in our topos, and any object $x \in X$ in the site, the precomposition becomes

$$(-) \circ p : \mathrm{Set} \quad \to \quad \mathrm{Sh}(X) \tag{4.73}$$

$$(A : * \to \mathrm{Set}) \quad \mapsto \quad (A \circ p : X^{\mathrm{op}} \to * \to \mathbf{Set}) \tag{4.74}$$

ie for any "sheaf" $* \to$ A (a set), we obtain a sheaf on X simply giving us back this set.

there is only one morphism between two sites, $\mathrm{Id}_* : * \to *$. The induced functor on the sheaf is

$$F \circ \mathrm{Id}_* : * \tag{4.75}$$

direct image functor is

$$x \tag{4.76}$$

If f^* has a left adjoint $f_! : E \to F$, f is an essential geometric morphism.

Direct image functor :

$$f_* F(U) = F(f^{-1}(U)) \tag{4.77}$$

Global section : if $p : X \to *$, $*$ the terminal object of the site

Inverse image functor :

$$f^{-1} G(U) = G(f(U)) \tag{4.78}$$

Local geometric morphism

4.14 Subtopos

Definition 139 *Given a topos* $\mathbf{H}$*, a subtopos* $\mathbf{S}$ *is a topos for which there exists a geometric morphism* $\iota : \mathbf{S} \hookrightarrow \mathbf{H}$

Slice topos, over topos, comma topos?

Dense subtopos, Lawvere-Tierney topology j

Level of a topos : an essential subtopos $H_l \hookrightarrow H$ is a *level* of H.

"the essential subtoposes of a topos, or more generally the essential localizations of a suitably complete category, form a complete lattice"

"If for two levels $H_1 \hookrightarrow H_2$ the second one includes the modal types of the idempotent comonad of the first one, and if it is minimal with this property, then Lawvere speaks of "Aufhebung" (see there for details) of the unity of opposites exhibited by the first one."

4.15 Motivic yoga

[33, 34] Six functor formalism

[35]

4.16 Lawvere-Tierney topology

On a topos **H**, a *Lawvere-Tierney topology* is given by a morphism j on the subobject classifier, ie

$$j : \Omega \to \Omega \tag{4.79}$$

Analog of Grothendieck topology for a topos? [32]

First to define a sheaf on a topological space (X, τ) (the category $\mathrm{Op}(X)$ with the canonical coverage). Take a collection $C = \{U_i\}_{i \in I}$.

The locality operator j maps C to the open sets covered by C.

$$j(C) = \{U \in \mathrm{Op}(X) | U \subseteq \bigcup_{i \in I} U_i\} \tag{4.80}$$

Properties :

If $U \in C$, $U \in j(C)$

$j(\mathrm{Op}(X)) = \mathrm{Op}(X)$

$j(j(C)) = j(C)$

$j(C_1 \cap C_2) \subseteq j(C_1) \cap j(C_2)$

If C_1, C_2 are sieves, $j(C_1 \cap C_2) = j(C_1) \cap j(C_2)$

If $C = S(U)$, $j(C)$ is also a sieve.

If C is a sieve, it is an element of $\Omega(U)$, the subobject classifier on the topos of presehaves on X.

Generalization : j is a map $j : \Omega \to \Omega$ on the subobject classifier of a topos, the *Lawvere-Tierney topology*, with properties

- $j \circ \top = \top$
- $j \circ j = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

j is a modal operator on the truth values Ω.

Example 140 *In the Grothendieck topos $E = \mathrm{Set}^{\mathrm{Op}(X)^{\mathrm{op}}}$ of presheaves on X a topological space, Classifier object $U \mapsto \Omega(U)$, the set of all sieves S on U, a set of open subsets $V \subseteq U$ such that $W \subseteq V \in S$ implies $W \in S$.*

Each open subset $V \subseteq U$ determines a principal sieve $\hat{V}$ consisting of all opens $W \subseteq V$

The map $\top_U : 1 \to \Omega(U)$ is the map that picks the maximal sieve $\hat{U}$ on U.

$$J(U) = \{S | \ S \text{ is a sieve on } U \text{ and } S \text{ covers } U \ \} \tag{4.81}$$

S covers U means

4.17 Localization

In some cases we wish to

[36] For a category C and a collection of morphisms $S \subseteq \text{Mor}(C)$, an object $c \in C$ is S-local if the hom-functor

$$C(-, c) : C^{\text{op}} \to \text{Set} \tag{4.82}$$

sends morphisms in S to isomorphisms in Set, so that for every $(s : a \to b) \in S$,

$$C(s, c) : C(b, c) \to C(a, c) \tag{4.83}$$

is a bijection

"localization of a category consists of adding to a category inverse morphisms for some collection of morphisms, constraining them to become isomorphisms"

"In homotopy theory, for example, there are many examples of mappings that are invertible up to homotopy; and so large classes of homotopy equivalent spaces"

Example 141 *The basic example of a localization is that of a commutative ring. localizing with prime* 2 *:* $\mathbb{Z}[1/2]$*, localization away from all primes :* $\mathbb{Q}$

Example 142 *Localization of* $\mathbb{R}[x]$ *away from* a *: rational functions defined everywhere except at* a

Localization at a class of morphisms W : reflective subcategory of W-local objects (reflective localization).

Localization of an internal hom : localization of the morphisms defined by $\prod_{X,Y}[X, Y]$?

Localization of a topos corresponds to a choice of Lawvere topology, localization of a Grothendieck topos to a Grothendieck topology.

Duality of a localization?

4.18 Number objects

One benefit of topoi as a category is the guaranteed existence of a natural number object[37]

Definition 143 *A* natural number object *for a topos is an object denoted* $\mathbb{N}$ *such that there exists the morphisms*

- *The morphism* $z : 1 \to \mathbb{N}$
- *The* successor morphim $s : \mathbb{N} \to \mathbb{N}$

such that for any diagram $q : 1 \to X$*,* $f : A \to A$*, there is a unique morphism* u

$$\begin{array}{ccccc} 1 & \xrightarrow{z} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow^{q} & \downarrow{u} & & \downarrow{u} \\ & & A & \xrightarrow{f} & A \end{array}$$

z : zero object(?), successor map : $s(n) = n + 1 = \mathrm{Maybe}(n)$

f defines a sequence, such that $a_0 = q$ and $a_{n+1} = f(a_n)$

Relation to maybe monad

Show that a morphism $\mathbb{N} \to A$ induces a diagram $A \to A \to A \to \ldots$, which induces a limit

$$\lim_{f} A \tag{4.84}$$

Topos also induce a *real number object*

4.19 Ringed topos

As with sheaves in general, we do not have to consider our topos to be exclusively set-valued, and we can give it a variety of other types. One of the most commonly used is that of a ringed topos.

Definition 144 *A* ringed topos $(X, \mathcal{O}_X)$ *is*

Chapter 5

Example categories

For the consideration of the methods to be studied, we need to look at a few good examples of appropriate categories. We will mostly look at topos (the main focus of this), more specifically Grothendieck topos, as well as

All the cool categories we consider here are topos

Quantales? Topos of the sheaves of commutative algebras on Hilbert space? Effectus categories?

Why **Top** isn't a topos : Not balanced, not Cartesian closed or localy Cartesian closed[38]

Simplicial category? Sierpinski topos?

5.1 Category of sets

The most basic topos (outside of the initial topos $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$) is the category of sets **Set**, with objects made from the class of all sets and morphisms the class of all functions.

In terms of a sheaf topos, sets can be defined as the sheaf on the terminal category : $\mathbf{Set} = \mathrm{Sh}(\mathbf{1})$, which is the functor from $*^{\mathrm{op}} = *$ to Set. As the set of all functions from $\{\bullet\}$ to any set X is isomorphic to X itself, this is easily seen to be isomorphic to **Set**. We only need to consider sieves on its unique object $*$, and as there is only one possible morphism there, there are only two possible sieves : the empty sieve $S_\varnothing$ which maps *ast* to the empty set, and the maximal sieve S_* which maps it to the singleton containing the identity map.

This allows us two possible topologies, the *chaotic topology*

As the site only has a trivial coverage, there is only a fairly limited amount of assembly that we can do from it.

Also **FinSet** as a subcategory

5.1.1 Limits and colimits

Theorem 145 *The empty set* $\varnothing$ *is the initial object of* **Set***.*

Proof 17 *We need to show that for any set* $X \in \mathrm{Obj}(\mathbf{Set})$*, there is a unique function* $f : \varnothing \to X$*. A function* $f : A \to B$ *is a subset of* $A \times B$ *obeying some properties, therefore we need to look at the set of subsets of* $\varnothing \times A$*. By properties of the Cartesian product,*

$$\varnothing \times A = \{\varnothing\} \tag{5.1}$$

There is therefore only one element to choose from, $\varnothing$*, which is indeed a function since it obeys (vacuously) the constraints on functions.*

Theorem 146 *Any singleton set* $\{\bullet\}$ *is a terminal object of* **Set***, all isomorphic.*

Proof 18 *We need to show that for any set* $X \in \mathrm{Obj}(\mathbf{Set})$*, there is a unique function* $f : X \to \{\bullet\}$*.*

$$X \times \{\bullet\} \tag{5.2}$$

As a category of sets, which are fundamentally defined by $\in$, **Set** has global elements $x : I \to X$. Those global elements are separators

Well-pointed topos

Products and coproducts :

Theorem 147 *The product on* **Set** *is isomorphic to the Cartesian product.*

Proof 19

Theorem 148 *The coproduct on* **Set** *is isomorphic to the disjoint union.*

Proof 20

Theorem 149 *Given two functions* $f, g : A \to B$*, the equalizer in* **Set** *is the subset* $C \subseteq A$ *on which those functions coincide,*

$$\mathrm{eq}(f, g) = \{c \in A | f(c) = g(c)\} \tag{5.3}$$

Theorem 150 *The equalizer of two functions* $f, g : A \rightrightarrows B$ *is the set of elements of* A *whose image agree :*

$$\mathrm{eq}(f, g) = \{x \in A | f(x) = g(x)\} \tag{5.4}$$

Theorem 151 *The coequalizer of two functions $f, g : A \rightrightarrows B$ is the quotient set on A by the equivalence relation*

$$x \sim y \leftrightarrow f(x) = f(y) \tag{5.5}$$

Proof 21

$$A \to B \to C \tag{5.6}$$

Definition 152 *The pullback of the (co?)span $A \to C \leftarrow B$ is the indexed set*

Theorem 153 *The pushout of the (co?)span $A \leftarrow C \to B$ is the*

Given these, we can see that **Set** has all small limits and colimits.

5.1.2 Elements

Fairly obviously, given its status as the model for it, **Set** has generalized elements : $x : \{\bullet\} \to X$, corresponding to the functions

$$\forall x \in X,\ x(\bullet) = x \tag{5.7}$$

so that explicitely, $x = \{(\bullet, x)\}$ (this set can be shown to exist with the axiom of pairing)

5.1.3 Subobject classifier

For **Set**, the subobject classifier is the set $\{\varnothing, \{\bullet\}\}$, also denoted by $\{0, 1\}$ or $\{\bot, \top\}$, corresponding to the two valuations of a subobject : either being a subset or not being a subset.

For a subset $\iota : S \hookrightarrow X$

$$\begin{array}{ccc} S & \xrightarrow{!} & * \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \text{true}} \\ X & \xrightarrow{\chi_S} & \Omega \end{array}$$

The function χ_S is more typically called the characteristic function and uses the notation $\chi_S : U \to \mathbb{B}$

$$\chi_U(x) = \begin{cases} 0 \\ 1 \end{cases} \tag{5.8}$$

Internal hom : The set of functions

$$
\begin{aligned}
[X, Y] &= \{f | f : X \to Y\} && (5.9) \\
&= \{f \subseteq X \times Y | \forall x \in X,\ \exists y \in Y,\ (x, y) \in f \wedge ((x, y) \in f \wedge (x, z) \in f \to y = z)\} && (5.10)
\end{aligned}
$$

Natural number object : the natural number construction.

The maybe monad is simply

$$\text{Maybe}(X) = X \sqcup \{\bullet\} \tag{5.11}$$

Coproduct for natural number :

$$0 \sqcup 0 = \{\} \tag{5.12}$$

Lawvere-Tierney topology : some morphism $j : \mathbf{2} \to \mathbf{2}$

Properties : $j(\{\bullet\}) = \{\bullet\}$, $j(j(x)) = j(x)$, $j(a \wedge b) =$

$j(\{\bullet\}) = \{\bullet\}$ reduces the choice to $j = \text{Id}_\Omega$ and $j = \{\bullet\}$, the constant map.

For the identity map : Given a subset $\iota : S \hookrightarrow X$, with classifier $\chi_S : A \to \Omega$, the composition $j \circ \chi_S$ defines another subobject $\bar{\iota} : \overline{S} \hookrightarrow A$ such that s is a subobject of $\bar{\iota}$, $\bar{s}$ is the j-closure of s

Identity map closure : every object is its own closure. This is the *discrete topology*.

Constant map $j(x) = \{\bullet\}$: the composition $j \circ \chi_S$ is the "always true" characteristic function, which is just χ_A. The closure of a set S in A is the entire set A. This is the *trivial* or *codiscrete* topology.

Those are the only two allowed topologies in **Set**.

Relation to $\text{loc}_{\neg\neg}$: $\neg : \Omega \to \Omega$ is

$$
\begin{aligned}
\neg(\{\bullet\}) &= \varnothing && (5.13) \\
\neg(\varnothing) &= \{\bullet\} && (5.14)
\end{aligned}
$$

$\neg\neg$ is simply the identity on **Set**. The j-closure associated to it is the identity map.

Localization?

5.1.4 Closed Cartesian

Internal hom : for any two sets A, B, the hom-set $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ is itself a set, by the traditional set definition of functions

$$f : A \to B \leftrightarrow f = \{(a, b) \subseteq A \times B | f(a) = b\} \tag{5.15}$$

The evaluation map of a function in **Set** is given by the traditional formulation of function evaluations in set theory. For a function $f : A \to B$, its evaluation by an element $x \in A$ is the unique element y in Y for which $(x, y) \in R_f$. In set theoretical terms, this can be defined Russell's iota operator,

$$\begin{aligned} \mathrm{ev}(x, f) &= \iota y,\ x R_f y && (5.16) \\ &= \bigcup \{z \mid \{y \mid x R_f y\} \ = \{z\}\} && (5.17) \end{aligned}$$

Counit of the tensor product/internal hom adjunction?

$$S \times (-) \dashv [S, -] \tag{5.18}$$

currying

5.2 Topos on a set

As any set forms a site with the power set as coverage, we can consider the Grothendieck topos

$$\mathrm{Sh}(X, \mathcal{P}(X)) \tag{5.19}$$

for some set X.

5.3 Topos of a topological space

Another common example of a topos is the sheaf of a topological space $\mathrm{Sh}(\mathrm{Op}(X))$ with the subcanonical coverage, simply written as $\mathrm{Sh}(X)$ for short. This is the type of topos we originally saw in the definition of a sheaf as an ur-exemple. If we call our topos $E = \mathrm{Sh}(X)$, then we have the interpretation that we saw earlier.

Restriction maps, gluing, locality

$\mathrm{Sh}(\mathrm{Op}(X))$

5.4 Category of smooth spaces

A more geometric category for a topos is the category of smooth spaces **Smooth**, which is defined as the sheaf over the category of smooth Cartesian spaces,

$$\mathbf{Smooth} = \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{Smooth}}) \tag{5.20}$$

The category of smooth Cartesian spaces is composed of objects from the open sets of $\mathbb{R}^n$, with morphisms being smooth maps between such objects.

The coverage of this site is slightly tricky. The most obvious cover is simply the coverage by open sets, where we consider the lattice generated by all open sets of **CartSp**. While we can construct a sheaf over this coverage [equivalent to good open cover?], there are coverages with better properties with what will follow.

Definition 154 *A* good open cover *is an open cover for which any finite intersection of open sets is contractible, ie a good open cover $\{f_i : U_i \to X\}$ of X*

$$\int \prod_{i\in I,\ X} U_i \cong \star \tag{5.21}$$

Properties with respect to the Cech nerves

This implies a homeomorphism to the open ball

Definition 155 *A good open cover $\{f_i : U_i \to X\}$ is a* differentially good open cover *if finite intersections of the cover are all diffeomorphic to the open ball.*

$$\prod_{i\in I,\ X} U_i \cong B^k \tag{5.22}$$

Theorem 156 *All three coverage of $\mathbf{CartSp}_{\mathrm{smooth}}$ lead to isomorphic sheaves :*

$$\mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{open}) \cong \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{good}) \cong \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{smooth}}, \mathcal{J}_{diff})$$

Therefore for our purpose we can pick the best behaved coverage.

Sheaves on the category of Cartesian spaces is best understood, in the context of geometry, as being *plots.*

Definition 157 *A* plot *is a map between an open set of a Cartesian space $\mathcal{O} \subseteq \mathbb{R}^n$ and a topological space X*

From the Yoneda lemma, we have that for any sheaf $X \in \textbf{Smooth}$ and Cartesian space U, we have the isomorphism

$$X(U) \cong \mathrm{Hom}_{[\textbf{CartSp},\textbf{Set}]}(y(U), X) \tag{5.23}$$

$$y(U) : \textbf{CartSp}^{\mathrm{op}} \rightarrow \textbf{Set} \tag{5.24}$$

$$O \mapsto \tag{5.25}$$

While this is a good intuitive way to understand the spaces probed by plots, it can be useful to know that in fact the topological space X itself is not necessary as a data to define a space cattaneo.

Theorem 158 *Given the set of transition functions on a manifold, the topological space can be reconstructed as*

$$M = \bigsqcup O_i / \sim \tag{5.26}$$

where two points in $O_i \sqcup O_j$ are equivalent if $\tau_{ij}(x_i) = x_j$

This is what we do with the smooth sets topos, as we are only considering the existence of those maps (as the set $F(\textbf{Cartsp})$), and the behaviour of those plots over overlapping regions.

Example 159 *Take the circle S^1, which we will defined a bit simplistically as a plot over $I = (0, 1)$*

$$S^1(I) = \{\varphi^+, \varphi^-\} \tag{5.27}$$

In terms of an atlas, if we considered our circle as the interval $[0, 2]$ with ends identified, those would be the charts

$$\varphi^\pm : [0, 2] \rightarrow U^\pm \subset S^1 \tag{5.28}$$

$$x \mapsto x \pm 1 \tag{5.29}$$

Those two coordinate neighbourhood overlap, as

$$U^\pm = U^+ \cap U^- = (0, 1) \cup (1, 2) \tag{5.30}$$

with transition functions [...]

In terms of overlap, we have $U^+ \cap U^- \cong I \sqcup I$, so that we need to consider additionally the plot of that Cartesian space (slightly complicated by the non-connected aspect of it, but we can consider the open set of the line $(-1, 0) \cup (0, 1)$. While we can do it this is partly why we generally consider good open covers)

$$S^1(I \sqcup I) = \{\varphi^\pm\} \tag{5.31}$$

which maps this overlap region onto S^1. The inclusion of this overlap area is done as

$$\iota_+(x \in I \sqcup I) = x \tag{5.32}$$
$$\iota_-(x \in I \sqcup I) = 2 - x \tag{5.33}$$

The overlap in terms of the plot is that we map $(-1,0) \cup (0,1)$ to the interval I as

Those morphisms on **CartSp** *are mapped onto opposite mappings on* **Set** *:*

$$S^1(\iota_+) : \{\varphi^+, \varphi^-\} \to \{\varphi^\pm\} \tag{5.34}$$

If we take the less abstract case of a concrete sheaf to look at smooth spaces, considering CartSp is a concrete site, the concrete presheaf of Sh(CartSp) is the category of *diffeological spaces* DiffeoSp, where each global element $X : 1 \to$ DiffeoSp is a diffeological space.

[Diff is a quasitopos]

An important subcategory is also the category of smooth manifolds **SmoothMan**.

$$\mathbf{SmoothMan} \subseteq \mathbf{DiffeoSp} \subseteq \mathbf{Smooth} \tag{5.35}$$

SmoothMan is not itself a topos, as it lacks an exponential object (Hom sets between manifolds are not themselves manifolds, although they are close to it [39]), and the quotients or equalizers of manifolds are not themselves manifolds [examples]

Smooth manifolds are locally representable objects of **Smooth**. If $X : 1 \to$ **Smooth** is a concrete smooth space (diffeological space), it is locally representable if there xists $\{U_i \hookrightarrow X\}$, $U_i \in$ **Smooth** such that the canonical morphism out of the coproduct

$$\bigsqcup_i U_i \to X \tag{5.36}$$

Is an effective epimorphism in **Smooth**.

$$\bigsqcup_i U_i \times_X \bigsqcup_j U_j \rightrightarrows \bigsqcup_i U_i \to X \tag{5.37}$$

By commutativity of coproduct and pullback [prove it]

$$\coprod_{i,j}(U_i \times_X U_j) \rightrightarrows \coprod_i U_i \to X \tag{5.38}$$

Theorem 160

$$\mathbf{Smooth} \cong \mathrm{Sh}(\mathbf{SmoothMan}) \tag{5.39}$$

An important property of **Smooth** is that it contains a large proportion of **Top**, more specifically the category of

Status wrt top, delta generated top, etc

Due to this wide variety of physically important objects in **Smooth**, it will typically be (or at least some wider categories that we will define later) the topos serving as the setting for physics in general.

5.4.1 Limits and colimits

Theorem 161 *The initial object of* **Smooth** *is the constant functor*

$$\Delta_\varnothing : \mathbf{CartSp}_{\mathrm{Smooth}} \to \mathbf{Set} \tag{5.40}$$

which maps every cartesian space to the empty set $\varnothing$*.*

Proof 22

The interpretation of this is that the

Theorem 162 *The terminal object of* **Smooth** *is the constant functor*

$$\Delta_{\{\bullet\}} : \mathbf{CartSp}_{\mathrm{Smooth}} \to \mathbf{Set} \tag{5.41}$$

which maps every cartesian space to the singleton $\{\bullet\}$*.*

The interpretation of this is that the terminal object of **Smooth** is a *point*, as in particular the plot of points $p : \mathbb{R}^0 \to \{\bullet\}$ is the only one which is

Important functors :

Forgetful functor to **Set** $U_{\mathbf{Set}} : \mathbf{Smooth} \to \mathbf{Set}$

Logic?

5.4.2 Subcategories of smooth sets

Concrete sheaves (diffeological space)

Example 163

5.4.3 Non-concrete objects

As we've seen, the concrete sheaves in **Smooth** do not form the entire topos, leaving non-concrete sheaves.

The archetypical example of this is the smooth set of differential k-forms,

$$\begin{aligned} \Omega^k : \mathbf{CartSp} &\to \mathbf{Set} && (5.42)\\ U &\mapsto \Omega^k(U) && (5.43) \end{aligned}$$

which associates to every Cartesian space the set of k-forms over that space.

If we attempt to look at the set of "points" of this space, if that term can be applied here, that would be the plot of the terminal object in the site, $\mathbb{R}^0$. But of course, in the sense of the sheaf as described here, this will just be the set of all k-forms over the point $\Omega(\mathbb{R}^0)$, which will just include the zero section, so that if we try to consider this plot as the "point content" of the space, there is but a single point :

$$\Omega(\mathbb{R}^0) = \{0\} \tag{5.44}$$

As we would not really consider the elements of this space to be that single section, it is therefore important to be mindful of what the plots of the sheaf represent.

Global sections :

$$\Gamma(\Omega^k) = \mathrm{Hom}_{\mathbf{Smooth}}(1, \Omega) \tag{5.45}$$

5.4.4 Important objects

The category of smooth spaces contains most of the objects of importance in physics and other fields, so that it is useful to look at the various types of objects within it.

First, as a topos, it has a terminal object as we've seen (the constant sheaf 1 which maps all probes to a single element, the constant plot). From this and the coproduct, we can construct objects similar to sets as we wish (this is in fact what the discrete functor will be later on), and as with any topos, a natural number object in particular.

As the coverage is subcanonical, the Yoneda embedding makes any Cartesian space a smooth space via its representable presheaf,

$$U \mapsto \mathrm{Hom}_{\mathbf{CartSp}}(-, U) \tag{5.46}$$

As we have seen, any diffeological space is a smooth space, in fact every concrete smooth space is a diffeological space.

Manifolds

By the Cartesian closed character of the topos, for any pair of manifolds, the set of all smooth maps between them is itself a smooth space, ie

$$C^\infty(M, N) \in \mathbf{Smooth} \tag{5.47}$$

Important classes of non-concrete sheaves are the *moduli spaces*, which are sheaves giving back appropriate function spaces on a Cartesian space. For instance the moduli space of Riemannian metrics Met is a sheaf

$$\begin{aligned} \mathrm{Met} : \mathbf{CartSp}^{\mathrm{op}} &\to \mathbf{Set} && (5.48) \\ U \subseteq \mathbb{R}^n &\mapsto && (5.49) \end{aligned}$$

where $\mathrm{Met}(U)$ is the set of all Riemannian metrics on U. For instance, as there is only one metric on a point (since the tangent bundle there is zero dimensional), we have

$$\mathrm{Met}(\mathbb{R}^0) = \{0\} \tag{5.50}$$

And there is only one component to the metric on the line which must also be positive, so its set of metric is that of the positive definite smooth functions.

Moduli space of differential forms

Moduli space of symplectic forms

True for any section?

Theorem 164 *The moduli spaces of sections is a smooth space*

5.5 Category of classical mechanics

The exact category to give to classical mechanics is somewhat controversial, due to the difficulties of finding an appropriate notion of morphisms, but a common pick is the *category of Poisson manifolds*

Definition 165 *A* Poisson manifold (P, π) *is a manifold* P *equipped with a Poisson bivector* $\pi \in \Gamma(\bigwedge^2 P)$

Poisson bracket :

$$\{f, g\} = \langle df \otimes dg, P\rangle \tag{5.51}$$

Definition 166 *An* ichtyomorphism *is a smooth map preserving the Poisson bivector :* $f^*\pi = \pi$

From this, the category of Poisson manifolds is the category with the

Category of Poisson manifolds **Poiss**

To consider our category within the context of a topos, it is useful to look at the moduli space of symplectic forms

Slice topos $\mathrm{Smooth}_{/\Omega^2}$?

Poisson manifold : locally representable concrete object?

5.5.1 Logic

The logic of classical mechanics is tied to the logic of measurement of observables. If we have some classical theory, with a Poisson manifold

Example : phase space of a point particle in n dimensions $\mathbb{R}^{2n}$, with the Poisson bracket

If we have some observable

$$f_o : \mathbb{R}^{2n} \to \mathbb{R} \tag{5.52}$$

Inversely, f_o selects a subset of the Poisson manifold. The statement that the measurement m_o is in the Borel subset $\Delta_o \subseteq \mathbb{R}$ is equivalent to a subobject of **Poiss**

$$S_o = f_o^{-1}(\Delta_o) \tag{5.53}$$

Limits and colimits :

$$S_{o_1} \sqcup S_{o_2} \tag{5.54}$$

What is the topos

Logic and presheaf etc

[40, 41, 42]

5.6 Category of spectral presheaves for quantum theories

[Difference between the spectral presheaf approach and Bohr topos aproach]

[43, 44, 45, 46, 47, 48, 49, 50, 51]

Another topos of interest is the main topos relating to quantum theory, the spectral presheaf [Bohr topos?].

5.6.1 Quantum mechanics as a symmetric monoidal category

The basic formulation of quantum mechanics in terms of category theory is to simply look at the categories of its main objects, which are Hilbert spaces and C^*-algebras.

Definition 167 *The category* **Hilb** *of Hilbert spaces has as its objects Hilbert spaces and as morphisms bounded linear maps between two Hilbert spaces.*

The condition of bounded linear maps is here to guarantee the existence of a dual on every operator.

Hilb can be entirely defined in categorical terms etc

C^*-algebra : Internal hom of $\mathcal{H}$? $[\mathcal{H}, \mathcal{H}]$

von Neumann algebra :

Definition 168 *A von Neumann algebra (or W^*-algebra) is a C^*-algebra A that admits a predual, a complex Banach space A_* with an isomorphism of complex Banach spaces*

$$* : A \to (A_*)^* \tag{5.55}$$

5.6.2 Daseinisation

While monoidal categories are a perfectly serviceable setting for dealing with quantum mechanics, it has a few issues making it unsuitable for this analysis. In some sense it corresponds to the construction of an actual "quantum object" with an existence independent of measurement, giving it fairly problematic properties from a logical perspective (this is the content of the Kochen-Specker theorem). Due to this it also famously fails to be a topos, which is the main object we are concerned with here.

To deal with those problems, we have to deal with the *Daseinisation*[52] of the category, where rather than deal with some quantum object directly, we only consider its measurements in some context.

The simplest way to consider a measurement in quantum mechanics is to look at the projectors P of the theory. If we ignore the wider case of positive operator-valued measure and only look at projection-valued measure (we will assume no additional source of uncertainty beyond quantum theory), every measurement in a quantum theory can be modelled by this. If a measurement is associated with an observable A with spectrum $\sigma(A)$, and of projection-valued measure

$$E : \Sigma(\sigma(A)) \rightarrow \mathrm{Proj}(\mathcal{H}) \tag{5.56}$$
$$\Delta \mapsto E(\Delta) \tag{5.57}$$

The Born rules is that the probability of the measurement lying in some measurable subset of the spectrum Δ is

$$P(X \in \Delta|\psi) = \langle \psi, E(\Delta)\psi \rangle \tag{5.58}$$

After said measurement the system will collapse to the state $E(\Delta)\psi$. Our logic is that a system is indeed such that $X \in \Delta$ if it was last measured to be so. The creation of a *context* from there is to consider the set of all measurements composed from commutative operators so that they can be said to be both true at the same time in a manner consistent with classical logic. If we have another measurement derived from an observable A' with a projection-valued measure E', the two PVM commute, in the sense that for any two measurable subsets of their spectra, $\Delta \subset \sigma(A), \Delta' \subset \sigma(A')$, we have

$$E(\Delta)E(\Delta') = E(\Delta')E(\Delta) \tag{5.59}$$

Meaning that if we have done a first measure $E(\Delta)$ (meaning $x \in \Delta$), and a second measure $E'(\Delta')$ ($x' \in \Delta'$), a third measure of the original quantity will yield the same result :

First measurement : Collapse

$$\psi \rightarrow \frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \tag{5.60}$$

Second measurement : Collapse $E(\Delta)\psi$ to $E'(\Delta')E(\Delta)\psi$

$$\frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \rightarrow \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|} \tag{5.61}$$

Third measurement :

$$\begin{aligned} P(X \in \Delta | \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}) &= \langle \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}, E(\Delta)\frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|} \rangle \\ &= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2} \langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)E(\Delta)\psi \rangle \\ &= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2} \langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)\psi \rangle \\ &= 1 \end{aligned} \tag{5.62}$$

Therefore in a context, we can say that the measured values are "real" in that they do not depend on the measurement.

As the identity and the zero projector both commute with every operator, they are a part of every context.

We will furthermore need the notion of ordering of projectors, which corresponds to the ordering of the lattice in quantum logic, ie we say that two projectors P_1, P_2 are ordered if

$$P_1 \leq P_2 \leftrightarrow \mathrm{im}(P_1) \subseteq \mathrm{im}(P_2) \tag{5.63}$$

or equivalently, $P_1 P_2 = P_2 P_1 P = P_1$. This means

Example : Given a projection-valued measure P and a measurable set of its spectrum Δ, with some subset $\Delta' \subseteq \Delta$, by the rules

$$E(\Delta') = E(\Delta' \cap \Delta) = E(\Delta')E(\Delta) \tag{5.64}$$

We therefore have $E(\Delta') \leq E(\Delta)$.

In terms of interpretation, this means that for $P \leq P'$, P' is *weaker* : we only know that our state is in some subspace larger than for P. This can be seen in the case of projection-valued measures on some interval, where the weaker statement is $x \in [a-\varepsilon_1, b+\varepsilon_2]$ compared to the more precise statement $x \in [a, b]$. The best one could find is in fact the 1-dimensional projector, as no projector is smaller than that (except for the zero projector which cannot provide any information), and corresponds to the measurement of the exact state. Due to this, two 1-dimensional projectors are never ordered, unless they are the same

$$\forall P, P',\ \dim(\mathrm{im}(P)) = \dim(\mathrm{im}(P')) = 1 \to (P \leq P' \leftrightarrow P = P') \tag{5.65}$$

Properties :

$$\forall P,\ 0 \leq P \tag{5.66}$$

$$\forall P,\ P \leq \mathrm{Id} \tag{5.67}$$

The point of daseinisation is to consider measurements in general not as projectors in the category of Hilbert spaces, but spread onto all possible contexts that a system may have by considering the closest approximation of that measurement in a given context. This approximation is given by the narrowest projection that is superior to our projector, ie for all the projectors P' in the context, we wish to find the one such that $P \leq P'$, and for any other projector P'' which also obeys $P \leq P''$, $P' \leq P''$. This projector is denoted by, for a context V, $\delta(P)_V$, the V-support of P. In terms of lattice notation, this is given by

$$\delta(P)_V = \bigwedge \{P' \in \mathrm{proj}(V) \mid P \leq P'\} \tag{5.68}$$

As Id is always part of every context and the supremum of any context, we are always guaranteed to have such a projector more precise or equal to the identity projector, which merely informs us that the state is in the Hilbert space at all and nothing more. If $P \in \mathrm{proj}(V)$, $\delta(P)_V = P$.

Example of a subset again

We will need to consider the approximation of $E(\Delta)$ in every possible contexts

$$P \to \{\delta(E(\Delta))_V | V \in \mathcal{V}(\mathcal{H})\} \tag{5.69}$$

Why is this a sheaf? Contexts are ordered

von Neumann algebras

To formalize this idea, we will need to use the notion of von Neumann algebra. While we could merely use C^* -algebras, there will be a difficulty if we do so : the projectors of C^*-algebras do not form a complete lattice, ie there may be subsets $S \subseteq \mathrm{proj}(A)$ which lack a lower or upper bound.

An example for this would be the algebra of compact operators $K(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$

Every projection in $K(\mathcal{H})$ has finite rank :

$$\dim(\mathrm{im}(P)) < \infty \tag{5.70}$$

If we consider an infinite dimensional Hilbert space, like $L^2(\mathbb{R})$, consider this subset : $\{P_i\}_{i\in\mathbb{N}}$, such that P_i maps to an i-dimensional subspace, and we have

$$\mathrm{im}(P_i) \subset \mathrm{im}(P_{i+1}) \tag{5.71}$$

Union of these is dense in $\mathcal{H}$?

$$\overline{\bigcup_{i\in\mathbb{N}} \mathrm{im}(P_i)} = \mathcal{H} \tag{5.72}$$

Supremum : $\sup(\{P_i\}) = I$, but I is not a compact operator.

[Why it happens? Relation to topology]

This possible lack of supremum and infimum would lead to the absence of disjunctions and conjunctions in our category [cf logic chapter]. While not tragic (this would only affect infinite conjunctions of propositions), we will try to keep things complete.

To insure the completeness of the lattice, we will use instead von Neumann algebras

Definition 169 *A* von Neumann algebra *A is a C^*-algebra with a predual A_*, a Banach space dual to A.*

Weak operator topology : The basis of neighbourgoods of 0 given by sets of the form

$$U(x, f) = \{A \in L(V, W) | f(A(x)) < 1\} \tag{5.73}$$

for $x \in V$, $f \in W^* = \mathrm{Hom}_{\mathrm{TVS}}(W, k)$. A sequence of operators (A_n) converges to A iff $(A_n(x))$ in the weak topology on W.

von Neumann algebras are closed in weak operator topology : any limit of net converges.

[...]

Definition 170 *An Abelian von Neumann algebra*

Definition 171 *For any Abelian von Neumann algebra over $\mathcal{B}(\mathcal{H})$, there exists a self-adjoint operator generating it as [...]*

The Bohr topos

With those notions, we can now proceed with the construction of the Bohr

(in the usual category of compact symmetric monoidal objects etc of quantum logic) is transformed to a (clopen) sub-object $\delta(P)$ of the spectral presheaf in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$

Kochen-Specker theorem : equivalent to the presheaf on the category of self-adjoint operator has no global element

Take a C^*-algebra (von Neumann?) A.

Subcategory of commutative subalgebras ComSub(A) is the poset wrt inclusion maps

for any operator (self-adjoint?) A, let W_A be the spectral algebra. W_A is the boolean algebra of projectors $E(A \in \Delta)$ that projects onto the eigenspaces associated with the Borel subset Δ of the spectrum $\sigma(A)$. $E[A \in \Delta]$ represents the proposition $A \in \Delta$

Spectral theorem : for all Borel subsets J of the spectrum of $f(A)$, the spectral projector $E[f(A) \in J]$ for $f(A)$ is equal to the spectral projector $E[A \in f^{-1}(J)]$ for A. In particular, if $f(\Delta)$ is a Borel subset of $\sigma(f(A))$, since $\Delta \subseteq f^{-1}(f(\Delta))$,

$$E[A \in \Delta] \leq E[A \in f^{-1}(f(\Delta))] \tag{5.74}$$

$$E[A \in \Delta] \leq E[f(A) \in f(\Delta)] \tag{5.75}$$

This means $f(A) \in f(\Delta)$ is weaker than $A \in \Delta$. $f(A) \in f(\Delta)$ is a *coarse graining* of $A \in \Delta$.

If $A \in \Delta$ has no truth value defined, $f(A) \in f(\Delta)$ may have for some f

Relations between two logical systems here :

First, any proposition corresponding to the zero element of the Heyting algebra should be valued as false, $\nu(0_L) = 0_{T(L)}$.

If $\alpha, \beta \in L$, $\alpha \leq \beta$, then α implies β. Ex : $A \in \Delta_1$, $A \in \Delta_2$, $\Delta_1 \subseteq \Delta_2$. Valuation should be $\nu(\alpha) \leq \nu(\beta)$ (monotonicity).

If $\alpha \leq \alpha \vee \beta$, $\beta \leq \alpha \vee \beta$, then $\nu(\alpha) \leq \nu(\alpha \vee \beta)$ and $\nu(\beta) \leq \nu(\alpha \vee \beta)$, and therefore

$$\nu(\alpha) \vee \nu(\beta) \leq \nu(\alpha \vee \beta) \tag{5.76}$$

Not as strong as $\nu(\alpha) \vee \nu(\beta) = \nu(\alpha \vee \beta)$. For instance for $A = a_1$, $A = a_2$, the projection operator for both of these proposition projects on the 2D span of the eigenvectors, not their union.

Similarly,

$$\nu(\alpha \wedge \beta) \leq \nu(\alpha) \wedge \nu(\beta) \tag{5.77}$$

Exclusivity : a condition and its complementation cannot both be totally true :

$$\alpha \wedge \beta = 0_L \wedge \nu(\alpha) = 1_{T(L)} \to \nu(\beta) \leq 1_{T(L)} \tag{5.78}$$

Unity condition : $\nu(1_L) = 1_{T(L)}$

Take the boolean subalgebra W of the lattice $P(H)$ of projection operators. Forms a poset under subalgebra inclusion. W is a poset category.

Take the set $\mathcal{O}$ of all bounded, self-adjoint operators on $\mathcal{H}$. Spectral representation :

$$A = \int_{\sigma(A)} \lambda dE^A_\lambda \tag{5.79}$$

$\sigma(A) \subseteq \mathbb{R}$ the spectrum of A, $\{E^A_\lambda | \lambda \in \sigma(A)\}$ a spectral family of A.

$$E[A \in \Delta] = \int_\Delta dE^A_\lambda \tag{5.80}$$

for Δ a borel subset of $\sigma(A)$. If a belongs to the discrete spectrum of A, the projector ontop the eigenspace with eigenvalue a is

$$E[A = a] := E[A \in \{a\}] \tag{5.81}$$

for $f : \mathbb{R} \to \mathbb{R}$ any bounded Borel function,

$$f(A) = \int_{\sigma(A)} f(\lambda) dE^A_\lambda \tag{5.82}$$

Categorification of $\mathcal{O}$: Objects are elements of $\mathcal{O}$, morphisms from B top A if an equivalence class of Borel functions $f : \sigma(A) \to \mathbb{R}$ exists such that $B = f(A)$, ie

$$B = \int_{\sigma(A)} f(\lambda) dE^A_\lambda \tag{5.83}$$

Definition 172 *The spectral algebra functor $W : \mathcal{O} \to W$ is*

- *Objects mapped $W(A) = W_A$, W_A is the spectral algebra of A*
- *Morphisms : if $f : B \to A$, then $W(f) : W_B \to W_A$ is the subset inclusion of algebras $i_{W_B W_A} : W_B \to W_A$.*

Spectral algebra for $B = f(A)$ is naturally embedded in the spectral algebra for A since $E[f(A) \in J] = E[A \in f^{-1}(J)]$ for all Borel subsets $J \subseteq \sigma(B)$

$$i_{W_{f(A)} W}(E[f(A) \in J]) = E[A \in f^{-1}(J)] \tag{5.84}$$

[...]

Category $\mathcal{O}_d$ of discrete spectra self-adjoint operators

Definition 173 *The spectral presheaf on $\mathcal{O}_d$ is the contravariant functor $\Sigma : \mathcal{O}_d \to \mathbf{Set}$*

- $\Sigma(A) = \sigma(A)$ *(spectrum of A)*
- *if $f_{\mathcal{O}_d} : B \to A$, so that $B = f(A)$, then $\Sigma(f_{O_d}) : \sigma(A) \to \sigma(B)$ is defined by $\Sigma(f_{\mathcal{O}_d})(\lambda) = f(\lambda)$ for all $\lambda \in \sigma(A)$*

Works because on discrete spectrum $\sigma(f(A)) = f(\sigma(A))$.

$$\Sigma(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d}) = \Sigma(f_{\mathcal{O}_d}) \circ \Sigma(g_{\mathcal{O}_d}) \tag{5.85}$$

global section : function γ that assigns for every object of the site an element γ_A of the topos, such that if $f : B \to A$, then $H(f)(\gamma_A) = \gamma_b$.

For the spectral functor, a global section / element is a function that assigns to each self-adjoint operator A with a discrete spectrum a real number $\gamma_A \in \sigma(A)$, such that if $B = f(A)$, then $f(\gamma_A) = \gamma_B$.

Kochen-Specker theorem : if $\mathrm{Dim}(\mathcal{H}) > 2$, there are no global sections of the spectral presheaf.

Continuous case

By Gelfand duality, the presheaf topos PSh(ComSub(A)) contains a canonical object, the presheaf

$$\Sigma : C \mapsto \Sigma_C \tag{5.86}$$

which maps a commutative C^*-algebra $C \hookrightarrow A$ to (the point set underlying) its Gelfand spectrum Σ_C.

[53]

projection

5.6.3 The finite dimensional case

Take the case $\mathbb{C}^n$ of the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$, which is for instance used in quantum computing. The C^*-algebra is just

$$C^*(\mathcal{H}) = L(\mathcal{H}, \mathcal{H}) \tag{5.87}$$

(denoted $L(\mathcal{H})$ for short), with operator composition as its algebraic operation and complex conjugate as involution, as all finite-dimensional linear maps are bounded. If we pick a specific basis, this is the algebra of $n \times n$ matrices on $\mathbb{C}^n$ with matrix multiplication.

The algebra $L(\mathcal{H})$ is also a von Neumann algebra [proof]

any subalgebra is a von Neumann subalgebra

The projections of this algebra are formed by the orthogonal projections, as any oblique projection would not be self-adjoint, classified by the Grassmannians of the space,

$$\bigoplus_{i=0}^{n} \mathrm{Grass}(i, \mathcal{H}) \tag{5.88}$$

$$P = I_r \oplus 0_{d-r} \tag{5.89}$$

An operator A will simply be one of the linear map $A \in L(\mathcal{H})$

A context here is an Abelian von Neumann subalgebra of $L(\mathcal{H})$. The category of contexts $\mathcal{V}(L(\mathcal{H}))$, equivalently a set of commuting matrices

"An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator."

Theorem 174 *Any abelian von Neumann algebra on a separable Hilbert space is $*$-isomorphic to either*

- $\ell^\infty(\{1, 2, \ldots, n\})$
- $\ell^\infty(\mathbb{N})$
- $L^\infty([0, 1])$
- $L^\infty([0, 1] \cup \{1, 2, \ldots, n\})$
- $L^\infty([0, 1] \cup \mathbb{N})$

There is therefore some surjection from the self-adjoint operators to commutative von Neumann algebras :

$$f : \mathcal{B}_{\mathrm{sa}}(\mathcal{H}) \to \mathcal{V}(W^*(\mathcal{H})) \tag{5.90}$$

Spectral theorem :

Theorem 175 *For a bounded self-adjoint operator, there is a measure space (X, Σ, μ) and a real-valued essentially bounded measurable function f on X and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that*

$$U^\dagger T U = A \tag{5.91}$$

$$[T\varphi](x) = f(x)\varphi(x) \tag{5.92}$$

and $\|T\| = \|f\|_\infty$

Finite dimensional :

Theorem 176 *There exists eigenvalues* $\{\lambda_i\}$ *(ordered by value by i) of A and eigen subspaces* $V_i = \{\psi \in \mathcal{H} | A\psi = \lambda_i \psi\}$ *such that*

$$\mathcal{H} = \bigoplus_{i=1}^{n} V_j \tag{5.93}$$

Theorem 177 *For self-adjoint A, there exists an orthonormal basis of eigenvectors of A.*

Theorem 178 *For a self-adjoint operator A with respect to an orthogonal matrix, there exists an orthogonal matrix T such that* $T^{-1}AT$ *is diagonal.*

Theorem 179 *For a self-adjoint operator A, there exists different eigenvalues* $\{\lambda_i\}$, $i \leq j \to \lambda_i \leq \lambda_j$, *and eigen subspaces,*

$$W_i = \{\psi \in \mathcal{H} \mid A\psi = \lambda_i \psi\} \tag{5.94}$$

Let P_i *be the orthogonal projection of* $\mathcal{H}$ *onto* W_i*, then*

- $\mathcal{H}$ *is an orthogonal direct sum of* W_i : $\mathcal{H} = \bigoplus_{i=1}^{n} W_i$, *and* $W_i \perp W_j$ *for* $i \neq j$
- $P_i P_j = \delta_{ij} P_i$ *and* $\mathrm{Id}_{\mathcal{H}} = \sum_i P_i$
- $A = \sum_i \lambda_i P_i$

Theorem 180 *For a normal operator A (ie, commutes with its adjoint), there is a spectral resolution of A.*

Spectrum in finite dimension :

$$\sigma(A) = \tag{5.95}$$

For our observable A,

$$A = \sum_i \lambda_i^m P_i \tag{5.96}$$

The commutative algebra generated is that which is spanned by those projective operators, ie

$$\forall B \in \mathrm{ComSub}(A),\ \exists\{c_i\} \in \mathbb{C}^k,\ B = \sum_i^m c_i P_i \tag{5.97}$$

All those operators are commutative, simply by the commutativity of the projectors between themselves.

Example of two operators with the same commutative subalgebra : any two operators with the same projectors but different eigenvalues

Alternatively : define them entirely by sets of projectors (up to a scale?), ie some subset of commutative projector (between 0 and n)

The Gelfand spectrum of a von Neumann algebra is the unique measurable space we define

"The predual of the von Neumann algebra B(H) of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm ——A——= Tr(—A—). The Banach space of trace class operators is itself the dual of the C*-algebra of compact operators (which is not a von Neumann algebra)."

Self-duality in finite dimension due to every operator being trace-class

Spectral measure [54] (1) :

For (X, Ω) a Borel space, a spectral measure is

$$\Phi : \Omega \to \mathcal{B}(\mathcal{H}) \tag{5.98}$$

- $\Phi(U)$ is an orthogonal projection for all U, $\Phi(U)^2 = \Phi(U) = \Phi(U)^*$
- $\Phi(\varnothing) = 0$ and $\Phi(X) = \mathrm{Id}$
- $\Phi(U \cap V) = \Phi(U)\Phi(V)$
- For a sequence (U_i) of pairwise disjoint Borel subsets,

$$\Phi(\bigcup_i U_i) = \sum_i \Phi(U_i)$$

(convergence wrt strong operator topology)

[...]

Spectral measure for finite dimensional case : Given the Abelian von Neumann algebra generated by

$$A = \sum_i \lambda_i P_i \tag{5.99}$$

with functions

$$f(A) = \sum f(\lambda_i) P_i \tag{5.100}$$

$$\int f d\mu_\psi = \langle \psi, f(A)\psi \rangle \quad (5.101)$$

$$= \sum_i f(\lambda_i)\langle \psi, P_i\psi \rangle \quad (5.102)$$

$$= \sum_i f(\lambda_i)\|P_i\psi\|^2 \quad (5.103)$$

measure is the counting measure

$$\mu_\psi = \sum_i \|P_i\psi\|\delta_{\lambda_i} \quad (5.104)$$

Gelfand dual :

- The space is the discrete space $\sigma(A)$
- The sigma-algebra is the discrete sigma algebra given by $\mathcal{P}(\sigma(A))$
- The measure is the counting measure

The spectral presheaf is then the presheaf

$$\underline{\Sigma} : \mathcal{V}(\mathrm{VNA}(\mathcal{H}))^{\mathrm{op}} \to \mathbf{Set} \quad (5.105)$$

which maps

Decomposition of operators : Given a set of n 1-dimensional orthogonal projectors, $\{P_i\}$, $P_iP_j = 0$,

The two-dimensional case

The simplest case we can use is the one-dimensional case, $\mathbb{C}$, but having only a single state in its projective Hilbert space, is a bit too trivial, so let's look at $\mathbb{C}^2$.

To classify its orthogonal projectors, let's look at the Grassmannians of various dimensions for $\mathbb{C}^2$:

- $\mathrm{Gr}_0(\mathbb{C}^2) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^2) \cong \mathbb{C}P^1$
- $\mathrm{Gr}_2(\mathbb{C}^2) = \{\mathbb{C}^2\}$

The zero and two dimensional cases are simple enough, the zero-dimensional projection operator being the zero operator 0, with Abelian von Neumann algebra the trivial algebra $\{0\}$, and the two-dimensional projection operator is the identity map $\mathrm{Id}_{\mathbb{C}^2}$, with Abelian von Neumann algebra the scaling matrices, $c\mathrm{Id}_{\mathbb{C}^2}$

The one-dimensional case will contain most of the cases of interest. for some point $p \in \mathbb{C}P^1$, ie a point on the Riemann sphere $p \in S^2$, $p = (\theta, \phi)$, there is a projector to that line in the complex plane.

Given any self-adjoint operator $\mathcal{B}_{\mathrm{sa}}(\mathbb{C}^2)$, the finite-dimensional spectral theorem tells us that the Hilbert space can be decomposed into orthogonal subspaces $\{W_i\}$ which each contain one or more of the eigenvectors of the operator. As there can only be as many orthogonal spaces as the sum of their dimension being inferior or equal to the total dimension, this will only allow the trivial case (Just the 0-dimensional subspace), a single 1-dimensional subspace, two 1-dimensional subspace, or a single 2-dimensional subspace. The first case is simply the projector 0, corresponding only to the 0 operator. The second case is, for the choice of a point (θ, ϕ) on the Riemann sphere,

$$A = \lambda P_{(\theta,\phi)} \tag{5.106}$$

The third case is

$$A = \lambda_1 P_{(\theta_1,\phi_1)} + \lambda_2 P_{(\theta_1,\phi_2)} \tag{5.107}$$

And the last case is a diagonal operator,

$$A = \lambda \mathrm{Id}_{\mathbb{C}^2} \tag{5.108}$$

The Abelian von Neumann algebras are therefore classified by those two points on the Riemann sphere,

$$((\theta_1, \phi_1), (\theta_2, \phi_2)) \to \mathrm{VNA}(\mathbb{C}^2) \tag{5.109}$$

The category of contexts is therefore such that

- The trivial von Neumann algebra is included in all algebras
- The scaling von Neumann algebra is not included in any other algebra?
- The von Neumann algebra constructed from a single one dimensional projection $P_{(\theta,\phi)}$ is included in any von Neumann algebra constructed from two one-dimensional projections, as long as they share that projection.

Diagram of the category

$\mathrm{VNA}(P_{(\theta,\phi)}) \leq \mathrm{VNA}(P_{(\theta,\phi)}, P_{(\theta',\phi')})$

Approximation of a projection : For any projection P, there is only two possible cases :

- The projection is 0, and the V-

Kochen-Specker : $\mathbb{C}^2$ is not concerned by this.

The three-dimensional case

To have a case that is actually covered by the big quantum theorems properly, we will have to consider the case of the Hilbert space $\mathbb{C}^3$. This is for instance the case given by massive spin 1 particles.

The classification of projectors is much the same as previously, thanks to the duality of Grassmannians,

- $\mathrm{Gr}_0(\mathbb{C}^3) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_2(\mathbb{C}^3) = \mathrm{Gr}_{3-2}(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_3(\mathbb{C}^3) = \{\mathbb{C}^3\}$

The orthogonal subspaces of an operator will be

1. The empty subspace 0
2. One 1-dimensional subspace
3. Two 1-dimensional subspace
4. Three 1-dimensional subspace
5. One 2-dimensional subspace
6. One 1-dimensional subspace and one 2-dimensional subspace
7. One 3-dimensional subspace

As before, the first and last case are trivial, consisting of the trivial subspace and the whole subspace.

5.6.4 The infinite-dimensional case

For a Bohr topos with a more interesting structure, such as a differential cohesive structure, let's consider instead a simple infinite dimensional case, of the theory of a quantum particle on the unit interval,

$$\mathcal{H} = L^2([0,1],\ell) \tag{5.110}$$

with the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_0^1 \psi_1^\dagger \psi_2 \mu_\ell \tag{5.111}$$

where two functions ψ, ψ' are identified if they have the same inner product with all other functions, ie up to differences on a set of measure zero.

This is the Hilbert space used for the particle in a box problem. The boundedness of the underlying space allows us to freely use the position operator

$$\hat{x}\psi(x) = x\psi(x) \tag{5.112}$$

as it is a bounded operator in this case, using the inequality $x\|\psi(x)\| \leq \|\psi(x)\|$ for all $x \in [0,1]$

$$\|\hat{x}\| \leq 1 \tag{5.113}$$

proof self-adjoint

1-dimensional classification : every projector is part of the set of all 1-dimensional subspaces of $\mathcal{H}$, is it the projective limit $\mathcal{C}P^\infty$? The Eilenberg-MacLane space $K(\mathbb{Z},2)$, classifier of U(1) bundles

Kuiper's theorem?

Due to its much more complex nature, the full classification of projection operators, and therefore contexts, is not gonna be attempted here, so that only a few representative examples will be look at here.

Examples of operators [projectors?] with continuous spectrum

As a continuous operator, $\hat{x}$ does not have an eigenbasis (outside of the more general case of the Gelf'and triple rigged Hilbert space), but we can instead compute its projection-valued measure.

If we consider our position operator $\hat{x}$ as the generator of an Abelian von Neumann algebra,

Chapter 6

Logic

One element of interest of topoi as a good foundation for math is that there is a connection between topos and logical theories.

Definition 181 *In a logical theory, a (first-order) signature is composed of*

- *A set* Σ_0 *of* sorts
- *A set* Σ_1 *of* function

Definition 182 *For a category* **C** *with finite products, and a signature* Σ*, a* Σ-structure M *in* **C** *defines :*

- *A function between sorts in* Σ_0 *and objects in* **C**
- *A function between functions in* Σ_1 *and morphisms in* **C**
- *A function between relations and subobjects*

[internal v. external logic]

[55]

Logic from types, logic from topos, Heyting algebra [56] The subobjects of objects X in a topos H form a Heyting algebra, with operations $\cap, \cup, \to$ the partial ordering $\subseteq$ and the greatest and smallest elements 1_A, 0_A.

The language $L(H)$ of a topos is a many-sorted first-order language having the objects $X \in H$ as types for the terms of $L(H)$, there is a type operator τ which assigns to any term of $L(H)$ an object $\tau(p)$ of H called the type of p.

- 0_H is a constant term of type 1.

- For any object A of E, there is a countable number of variables of type A
- For any map $f : A \to B$, there is an "evaluation operator" $f(-)$ for terms of type A to terms of type B : p of type $A \Rightarrow f(p)$ of type B
- For any ordered pair (A, B) of H, there is an ordered pair operator $\langle -, - \rangle$
- For any subobject $M : A \to \Omega$, there is a unary "membership-predicate $(-) \in M$ for elements of A. $x \in M$ is an atomic formula provided $x \in A$.
- The propositional connectives $\neg$, $\wedge$, $\vee$ and $\to$ are allowed for new formulas
- For any object A and variable $x \in A$, the quantifier $\exists x \in A$ and $\forall x \in A$ are allowed

Definition 183 *Two objects $x, y \in A$ are equal if*

$$x = y \leftrightarrow \langle x, y \rangle \in \Delta_A \tag{6.1}$$

$\Delta_A : A \times A \to \Omega$ *the diagonal operator*

Unique equality :

$$(\exists ! x \in A)\phi(x) \leftrightarrow \exists x \in A, \forall y \in A, (\phi(y) \leftrightarrow x = y) \tag{6.2}$$

Membership :

$$x \in y \leftrightarrow \langle y, x \rangle \in (\mathrm{ev} : PA \times A \to \Omega) \tag{6.3}$$

For $x \in A$ and $F \in B^A$,

$$F(x) = (\mathrm{ev} : B^A \times A \to B)\langle F, x \rangle \tag{6.4}$$

For any map $f : A \to B$ with exponential adjoint $\overline{f} : 1 \to B^A$, we define an element $f_e = \overline{f}(0_e) \in B^A$ which represents f internally.

6.1 The internal logic of Set

The internal language $L(\mathbf{Set})$ roughly corresponds to classical logic (as applied to sets). Many-sorted first order language having the objects of the topos as types

Boolean algebra of subsets : for $X \in \mathbf{Set}$, we consider the boolean algebra of subobjects $\mathrm{Sub}(X)$ with the correspondences

$\to$: every element except the elements of A that aren't also elements of B.

Boolean algebra	Set operator
$a, b, c, \ldots$	$A, B, C \subseteq X$
$\wedge$	$\cap$
$\vee$	$\cup$
$\leq$	$\subseteq$
0	$\varnothing$
1	X
$\neg A$	$X \setminus A = A^c$
$A \to B$	$(X \setminus A) \cup B = A^c \cup B$

Table 6.1: Caption

Boolean algebra identities :

$$A \cup (B \cup C) = (A \cup B) \cup C \quad (6.5)$$
$$A \cup B = B \cup A \quad (6.6)$$
$$A \quad (6.7)$$

[...]

A basic example of statement in our topos is given by the morphism $1 \to \Omega$, which trivially factors through itself,

$$1 \xrightarrow{\mathrm{Id}_1} 1 \xrightarrow{\top} \Omega \quad (6.8)$$

which corresponds to the trivial statement

$$\vdash \top \quad (6.9)$$

Simply stating that truth is always internally valid. Conversely, 0, representing falsity, will not be, as $0 \to \Omega$ but on the other hand, the negation of falsity, $[0, 0]$,

Axioms :

Negation : for $p : A \hookrightarrow X$, ie

$$A \overset{p}{\hookrightarrow} X \xrightarrow{\chi_A} \Omega \quad (6.10)$$

The negation of the set $\neg A$ is such that $\neg p : \neg A \hookrightarrow X$ factors through the negation and p,

$$\chi_{\neg A} = \chi_A \circ \neg \quad (6.11)$$

Statement with context : $p, p \to q \dashv q$: slice category $\mathbf{Set}_{/(p:A\to X)\times[]}$.

Localization modality : $\bigcirc_j$ for $j = \mathrm{Id}_\Omega$:

6.2 The internal logic of a spatial topos

Logic of $\mathrm{Sh}(\mathbf{X})$, $\mathrm{Sh}(\mathbf{CartSp}_{\mathbf{Smooth}})$

Locality modality j

6.3 The internal logic of smooth spaces

Subobject classifier : Ω is the sheaf associating to any $U \subseteq \mathbb{R}^n$ such that

$$\Omega(U) = \{S | S \text{ is a } J\text{-closed sieve on } U\} \tag{6.12}$$

$$\Omega(f) = f^* \tag{6.13}$$

$\top : 1 \to \Omega$: maximal sieve on each object

Points in Ω : all the maps $1 \to \Omega$, all the J-closed sieves on $\mathbf{R}^0$.

6.4 The internal logic of classical mechanics

Internal logic for Poisson manifolds

Symmetric monoidal category with projection

A particular Poisson structure we can give is the trivial Poisson structure,

$$\{f, g\} = 0 \tag{6.14}$$

If we consider the map $\mathbb{R} \to P$ corresponding to this trivial structure on the real line, we will get our local example of a real line object

Are the measurement given by $\mathbb{R}$ with the trivial Poisson structure

types given by morphisms to R?

6.5 The internal logic of quantum mechanics

There are three possible internal logics that we can consider for quantum mechanics here. If we consider it as a symmetric monoidal category, this is a form of *linear logic*. If we consider a given Hilbert space $\mathcal{H}$, the logic of the slice category $\mathbf{Hilb}_{\mathcal{H}}$ is *quantum logic*. And finally, we will look at the internal logic of the Bohr topos that we have constructed.

6.5.1 Linear logic

The category of Hilbert spaces and linear logic are not quite like the other ones that we have looked into so far, not forming a topos.

As we do not have a subobject classifier here, we will not be able to use a Mitchell-Benabou language. But we can still perform that translation using the basic translation as a type theory.

If we pick a given Hilbert space $\mathcal{H}$ as a reference, propositions are given by monomorphisms $\iota : W \hookrightarrow \mathcal{H}$ (up to isomorphisms). In the category of Hilbert spaces, all monomorphisms are split, meaning that there exists a retraction P_W

$$W \overset{\iota_W}{\hookrightarrow} \mathcal{H} \xrightarrow{P_W} W \tag{6.15}$$

such that $P_W \circ \iota_W = \mathrm{Id}_W$. This is the notion we've seen before for the state of a system depending on a projection of the Hilbert space via some measurement operator.

The category of Hilbert spaces, like the category of vector spaces, has an initial and terminal object that are the same, the *zero object* 0, corresponding to the Hilbert space $\mathbb{C}^0$. Being the subobject of any Hilbert space, the unique map $0 : 0 \to \mathcal{H}$ has an interpretation as a proposition for any Hilbert space,

Interpretation of daggers logically

Chapter 7

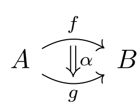
Higher categories

To generalize categories, we can introduce the concept of *higher categories*.

Definition 184 *A k-morphism is defined inductively as an arrow for which the source and target is a* $(k-1)$*-morphism, and an object is a* 0*-morphism.*

[Identity of indistinguishables]

Therefore the morphisms we saw are 1-morphisms, 2-morphisms are morphisms between two 1-morphisms, etc etc



Globular 2-morphism

"In the 2-category Cat, 2-morphisms are natural transformations between functors."

"The objects in the hom-category C(x,y) are the 1-morphisms in C from x to y, while the morphisms in the hom-category C(x,y) are the 2-morphisms of C that are horizontally between x and y."

Arrow category? Over category?

Example 185 *In the category* **Top***, homotopies between two continuous functions are* 2*-morphisms.*

Gauge example?

For $n, r \in \mathbb{N} \cup \{\infty\}$, we say that **C** is an (n, r)-category "An (n,r)-category is an r-directed homotopy n-type." Ex : a $(0, 0)$-category is isomorphic to a set (the set of all objects), a $(1, 0)$-category is a groupoid, a $(1, 1)$-category is a category

$(\infty, 0)$: ∞-groupoid (∞, ∞) :

Descent to negative degrees : $(-1, 0)$-category : truth values $(-2, 0)$-category : Point

n-truncation : a category is n-truncated if it is an n-groupoid

Loop space object and suspension object

Homotopy limit

Definition 186 *A* weighed limit *over a functor $F : K \to C$ with respect to a weight $W : K \to V$ is*

$$[K, V] \left(W, C\left(c, F(-)\right)\right) \tag{7.1}$$

Chapter 8

Moments

The formalization of qualities of an instance of an object are given by the concept of *moments* of such an instance, given by some "projection operator" $\bigcirc : C \to C$, such that

$$\bigcirc \bigcirc X \cong \bigcirc X \tag{8.1}$$

If we reduce the object X to merely the qualities given by $\bigcirc$, there is nothing left to remove so that any subsequent projection will be isomorphic to it. In categorical term, we also demand that the projection given as $X \to \bigcirc X$ be, within the category of qualities $\bigcirc$, an equivalence :

$$\bigcirc(X \to \bigcirc X) \in \mathrm{core}(X) \tag{8.2}$$

In terms of types, this is an idempotent monad.

Dually, we can also define comonads $\Box$, $\Box X \to X$, with $\Box(\Box X \to X)$ is an equivalence

Definition 187 *A* moment *on a type system/topos* **H** *is either an idempotent monad or comonad.*

Inclusion of the image :

$$\mathbf{H}_{\bigcirc} \hookrightarrow \mathbf{H} \tag{8.3}$$

$$\mathbf{H}_{\bigcirc} \hookrightarrow \mathbf{H} \tag{8.4}$$

Semantics as similarity : two objects are $\bigcirc$-similar if their modality is identical

$$X \cong_{\bigcirc} Y \leftrightarrow \bigcirc X = \bigcirc Y \tag{8.5}$$

$\mathbf{H}_{\bigcirc}$ is the Eilenberg-Moore category of $\bigcirc$

"we may naturally make sense of "pure quality" also for (co-)monads that are not idempotent, the pure types should be taken to be the "algebras" over the monad."

Accidence : A moment $\bigcirc$ is exhibited by a type J if $\bigcirc$ is a J-homotopy localization :

$$\bigcirc \cong \mathrm{loc}_J \tag{8.6}$$

$\bigcirc J \cong *$

Homotopy localization : for an object $A \in \mathbf{C}$, take the class of morphisms W_A

$$X \times (A \overset{\exists!}{\to} *) : X \times A \xrightarrow{p_1} X \tag{8.7}$$

what means

"The idea is that if A is, or is regarded as, an interval object, then "geometric" left homotopies between morphisms $X \to Y$ are, or would be, given by morphisms out of $X \times A$, and hence forcing the projections $X \times A \to X$ to be equivalences means forcing all morphisms to be homotopy invariant with respect to A."

Example using the real line for homotopy localization?

Notation : $\bigcirc$

Example 188 *The adjunction* Even $\dashv$ Odd *is an opposition of the form* $\square \dashv \bigcirc$

As monads	As moments	Notation
Comonad	p-moment	$\square$
Monad	s-moment	$\bigcirc$

Table 8.1: Caption

8.1 Unity of opposites

Given an adjoint pair of modal operators, ie an adjoint triple

$$F \dashv G \dashv H : C \underset{G}{\overset{F}{\rightleftarrows}} D \tag{8.8}$$

where $F, H : C \to D$ and $G : D \to C$

The two adjunctions imply that G preserves all limits and colimits in D

Gives rise to an adjoint pair of monads,

$$(GF \dashv GH) : C \underset{GF}{\overset{GH}{\rightleftarrows}} C \tag{8.9}$$

and the pair

$$(FG \dashv HG) : D \underset{FG}{\overset{HG}{\rightleftarrows}} D \tag{8.10}$$

Theorem 189 *For $F \dashv G \dashv H$, F is fully faithful iff H is.*

F being fully faithful is equivalent to $\eta : \mathrm{Id} \to GF$ being a natural isomorphism.

H being fully faithful is equivalent to $\varepsilon : GH \to \mathrm{Id}$ being a natural isomorphism.

GF is isomorphic to the identity if GH is

$F \dashv G \dashv H$ is a fully faithful adjoint triple in this case. "This is often the case when D is a category of "spaces" structured over C, where F and H construct "discrete" and "codiscrete" spaces respectively."

The opposite of a moment $\bigcirc$ is a moment $\Box$ such that they form either a left or right adjunction, ie :

$$(\Box \dashv \bigcirc) : \mathbf{H}_{\Box} \simeq \mathbf{H}_{\bigcirc} \underset{\hookrightarrow}{\overset{\hookrightarrow}{\longleftarrow}} \mathbf{H} \tag{8.11}$$

or

$$(\bigcirc \dashv \Box) : \mathbf{H}_{\Box} \simeq \mathbf{H}_{\bigcirc} \underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}} \mathbf{H} \tag{8.12}$$

We will denote the unity of opposites $\Box \dashv \bigcirc$ as a unity of a preceding to a successive moment, or *ps* unity, and $\bigcirc \dashv \Box$ as an *sp* unity.[8]

Theorem 190 *A ps unity defines an essential subtopos.*

(level of a topos)

Theorem 191 *As sp unity defines a bireflective subcategory*

"$\square \dashv \bigcirc$ – Here are two different opposite "pure moments" ."

"$\bigcirc \dashv \square$ – Here is only one pure moment, but two opposite ways of projecting onto it."

8.2 Negation

In addition to opposition, monads and comonads can also have *negations*, what are called determinate negations. The negation of a moment will be, if it exists, an operator which removes specifically the attributes of a given moment. In other words, if we have both the moment and its negation acting on an object,

$$\square\overline{\square}X = 1 \tag{8.13}$$

We are left with the terminal object with no specific properties. Equivalently, $\square\overline{\square} = \bigcirc_*$, the modality of being that we will see later on.

To do this, we need to find a map from the category to the subcategory containing only objects that the moment map to the terminal object. Given the counit $\square X \to X$, this is the cofiber :

Definition 192 *The determinate negation of a comonadic moment is the cofiber of its counit :*

$$\overline{\square}X = \mathrm{cofib}(\square X \to X) \tag{8.14}$$

Dually we can also define the determinate negation of a monadic moment, but while a cofibration only involves the pushout $1 \leftarrow \square X \to X$, where the morphism $\square X \to 1$ is unique, the fibration is the pullback $1 \to X \leftarrow \bigcirc X$, which depends on a specific choice of a point $1 \to X$ in the object. This means that this negation will either be defined if the object contains a single point, if we are given a specific choice of a point, if the result is independent from the choice of basepoint, or if we allow more flexible negations such as the homotopy fiber of a connected object.

Definition 193 *The determinate negation of a monadic moment*

$$\overline{\bigcirc}X = \mathrm{fib}(X \to \bigcirc X) \tag{8.15}$$

$$\Box\overline{\Box} = * \tag{8.16}$$

Show that the intersection of subcategories is something

Example 194

Definition 195 Determinate negation *of a unity of opposite moments* $\bigcirc \dashv \Box$ *if* $\Box, \bigcirc$ *restrict to* 0*-types and*

- $\bigcirc * \cong *$
- $\Box \to \bigcirc$ *is an epimorphism.*

"For an opposition with determinate negation, def. 1.14, then on 0-types there is no $\bigcirc$-moment left in the negative of $\Box$-moment"

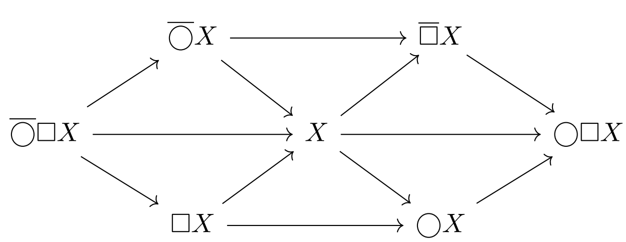
$$\bigcirc\overline{\Box} \cong * \tag{8.17}$$

Proof 23 *By left adjoints preserving colimits,*

$$\bigcirc\overline{\Box}X = \bigcirc\mathrm{cofib}(\Box X \to X) \cong \mathrm{cofib}(\Box X \to \bigcirc X) \tag{8.18}$$

Since $\Box X \to \bigcirc X$ *is epi, which is preserved by pushout, this is an epimorphism from the terminal object, therefore the terminal object itself.*

"For opposite moments of the form $\bigcirc \dashv \Box$, then for stable types X the hexagons"



are homotopy exact in that

- both squares are homotopy Cartesian, hence are fracture squares
- the boundary sequences are long homotopy fiber sequences

"In particular every stable type is the fibered direct sum of its pure $\bigcirc$-moment and its pure $\overline{\Box}$-moment:"

$$X \simeq (\bigcirc X) \underset{\bigcirc\overline{\Box}X}{\oplus} (\overline{\Box}X)\,. \tag{8.19}$$

"However, this fiber depends on a chosen basepoint, so it only makes sense on types which have only one constituent (but possibly this constituent has (higher) equalities), or, thought of homotopically, have only one connected component. In this case,$\bigcirc T$ contains that part of the structure of T that is trivialized by $T \to \bigcirc T$. Note that $\Box$ and $\overline{\Box}$ (or the dual notions) do not form a unity of oppositions. However, for each object T , a sequence $\Box T \to T \to \overline{\Box}T$ exists and this sequence decomposes each T , in the sense that T could be reconstructed from its aspects under a moment and its negative, as well as their relation. This is not generally true for unities of oppositions."

8.3 Sublation

Sublation (or Aufhebung in the original German), levels of a topos

§180 The resultant equilibrium of coming-to-be and ceasing-to-be is in the first place becoming itself. But this equally settles into a stable unity. Being and nothing are in this unity only as vanishing moments; yet becoming as such is only through their distinguishedness. Their vanishing, therefore, is the vanishing of becoming or the vanishing of the vanishing itself. Becoming is an unstable unrest which settles into a stable result.

§181 This could also be expressed thus: becoming is the vanishing of being in nothing and of nothing in being and the vanishing of being and nothing generally; but at the same time it rests on the distinction between them. It is therefore inherently self-contradictory, because the determinations it unites within itself are opposed to each other; but such a union destroys itself.

§182 This result is the vanishedness of becoming, but it is not nothing; as such it would only be a relapse into one of the already sublated determinations, not the resultant of nothing and being. It is the unity of being and nothing which has settled into a stable oneness. But this stable oneness is being, yet no longer as a determination on its own but as a determination of the whole.

§183 Becoming, as this transition into the unity of being and nothing, a unity which is in the form of being or has the form of the onesided immediate unity of these moments, is determinate being.

Chapter 9

Objective logic

Yoneda v.

"These many different things stand in essential reciprocal action via their properties; the property is this reciprocal relation itself and apart from it the thing is nothing"

As there will be many notations for very similar concepts of different types, we will require the following convention :

- Unless a more specific unambiguous symbol exists, monads will be denoted by $\bigcirc_{(-)}$, with its subscript to differentiate it
- Similarly, comonads will be denoted by $\Box_{(-)}$, with its subscript to differentiate it
- A generic topos $\mathbf{H}$
- The terminal object is 1
- The initial object is 0

This to avoid circumstances such as $*$ to represent both an object, functor, category and monad.

A trivial opposition we have in the objective logic is

$$\mathrm{Id} \dashv \mathrm{Id} \tag{9.1}$$

This is an opposition defined by the triple of endofunctors

$$(\mathrm{Id} \dashv \mathrm{Id}) : \mathbf{H} \overset{\hookleftarrow}{\hookrightarrow} \tag{9.2}$$

Representing three identity functors, composing into the two identity monad and comonad, with the subtopi being **H** itself. (Moment of identity?)

9.1 Being and nothingness

"Being, pure being, [...] it has no diversity within itself nor any with a reference outwards"

"Nothing, pure nothing: it is simply equality with itself, complete emptiness"

The most basic type of moments are the monads of *being* (*Sein*) and *nothingness* (*Nichts*). This can be seen easily enough by considering that the smallest subtopos is the initial topos $\mathrm{Sh}(\varnothing) \cong \mathbf{1}$. Let's consider the constant functor on this subtopos,

$$\Delta_* : \mathbf{C} \to 1 \tag{9.3}$$

which maps every object of $\mathbf{C}$ to the unique object $*$ of the terminal category 1. This corresponds to the *unit type* in type theory term, and this is what is referred to as *The One* (*Das Eins*) in objective logic. From this already we can see that the opposition will be a *ps*-unity.

We give this functor a left and right adjoint, which as we will see are the constant functors of the initial and terminal object, Δ_0 and Δ_1, forming the adjoint cylinder

$$(\Delta_0 \dashv \Delta_* \dashv \Delta_1) : \mathbf{C} \xrightarrow{\Delta_*} \mathbf{1} \underset{\Delta_0}{\overset{\Delta_1}{\rightrightarrows}} \mathbf{C} \tag{9.4}$$

An easy way to see this is via the adjunction of hom-sets :

$$\mathrm{Hom}_{\mathbf{C}}(\Delta_0(*), X) \cong \mathrm{Hom}_{\mathbf{1}}(*, \Delta_* X) \tag{9.5}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between $\Delta_0(*)$ and any object X, making it the initial object of the topos. Similarly,

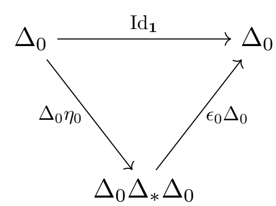
$$\mathrm{Hom}_{\mathbf{1}}(\Delta_*(X), *) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Delta_1(*)) \tag{9.6}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between any object X and $\Delta_1(*)$, making it the terminal object of the topos, confirming our choice of those adjoints as constant functors.

We can also look at this in terms of the unit and counit of the adjunction. First if we look at the adjunction $\Delta_0 \dashv \Delta_*$, as adjoint functors, they are equipped with the unit and counit natural transformations,

$$\eta_0 \quad : \quad \mathrm{Id}_{\mathbf{1}} \Rightarrow \Delta_* \circ \Delta_0 \tag{9.7}$$
$$\epsilon_0 \quad : \quad \Delta_0 \circ \Delta_* \Rightarrow \mathrm{Id}_{\mathbf{C}} \tag{9.8}$$

they have to obey the triangle identities



In terms of components, this means that for any object $X \in \mathbf{C}$ (and the only object $*$ in $\mathbf{1}$),

$$\mathrm{Id}_{\Delta_0(*)} \quad = \quad \epsilon_{\Delta_0(*)} \circ \Delta_0(\eta_*) \tag{9.9}$$
$$\mathrm{Id}_{\Delta_*(X)} \quad = \quad \Delta_*(\epsilon_X) \circ \eta_{\Delta_*(X)} \tag{9.10}$$

We have the identities $\Delta_*(X) = *$, and any component of the counit can only be the identity morphism on $*$, so that

$$\mathrm{Id}_{\Delta_0(*)} \quad = \quad \epsilon_{\Delta_0(*)} \circ \mathrm{Id}_{\Delta_0(*)} \tag{9.11}$$
$$\mathrm{Id}_* \quad = \quad \mathrm{Id}_* \tag{9.12}$$

The second line is trivial, but the first line tells us that ...

for any object $X \in \mathbf{C}$, there exists an object $\Delta_*(X) \in \mathbf{1}$ and a morphism $\epsilon_X : \Delta_0 \circ \Delta_*(X) \to X$ such that for every object in $\mathbf{1}$ (so only for $*$), and every morphism $f : \Delta_0(*) \to X$, there exists a unique morphism $g : * \to \Delta_*(X) = *$ with $\epsilon_X \circ \Delta_0(g) = f$.

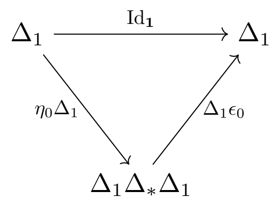
The unique morphism g is simply the identity function Id_1, and our natural transformation at c is simply $\epsilon_X : \Delta_0(*) \to X$. We therefore need the constraint

$$\epsilon_X \circ \Delta_0(\mathrm{Id}_1) = f \tag{9.13}$$

However, as g can only be one function, we cannot have more than one such possible morphism f, and we need exactly one to map to Id_1. This means that the empty functor Δ_0 maps the single object of 1 to the initial object 0 in $\mathbf{C}$ (if the category contains one), as its name indicates, justifying our notation of the constant functor Δ_0.

[Do the other triangle?]

Conversely, the right adjoint Δ_1 has to obey



Δ_1 maps the unique object of the terminal category to the terminal object 0 of $\mathbf{C}$, if this object exists.

From the adjoint cylinder $(\Delta_0 \dashv \Delta_* \dashv \Delta_1)$, we can see that this is a *ps*-type unity of opposites, giving rise to an adjoint modality that we will write as $\Box_\varnothing \dashv \bigcirc_*$, with $\Box_\varnothing$ the modality of nothingness (or empty comonad) and $\bigcirc_*$ the modality of being (or unit monad), defined by

$$\Box_\varnothing = \Delta_0 \circ \Delta_* \tag{9.14}$$
$$\bigcirc_* = \Delta \circ \Delta \tag{9.15}$$

where the unit and counit of the adjunction are given by those that we have seen,

In terms of their modal action, the empty monad maps any object of the category to its initial element,

$$\Box_\varnothing(X) = 0 \tag{9.16}$$

while the unit monad maps any object of the category to its terminal element,

$$\bigcirc_*(X) = 1 \tag{9.17}$$

In the Hegelian sense : $\bigcirc_*$ maps every object of $\mathbf{C}$ to a single object (they all share the same characteristic of existence, "pure being", and there is nothing differentiating them in that respect, no further qualities). Conversely, $\Box_\varnothing$ maps every object to nothing, the opposition of being.

The unit of the monad and counit of the comonad are given in terms of components by the typical morphisms of the terminal and initial object,

$$\epsilon_X : X \rightarrow \bigcirc_* X \cong 1 \quad (9.18)$$
$$\eta_X : 0 \cong \Box_\varnothing X \rightarrow X \quad (9.19)$$

Both of those adjoint functors roughly reflect the fact that each has to map elements to a single element and morphisms between that element and every other element to a single morphism.

The composition of the unit and counit give us the *unity of opposites* for being and nothingness

$$\varnothing \rightarrow X \rightarrow * \quad (9.20)$$

"there is nothing which is not an intermediate state between being and nothing."

An alternative interpretations of this modality is given by the opposition of the dependent sum and dependent product on the empty context

$$\sum_\varnothing (-) \vdash \prod_\varnothing (-) \quad (9.21)$$

Cartesian product v. internal home adjunction of the unit type

$$((-) \times \varnothing) \dashv (\varnothing \rightarrow (-)) \quad (9.22)$$

Negation in categories : internal hom to the initial object : $\neg = [-, \varnothing]$

Examples of those two modalities on a topos will not give us very different results overall, as they all roughly have the same behaviour. For **Set** for instance,

$$\forall A \in \text{Obj}(\mathbf{Set}),\ \Box_\varnothing(A) = \varnothing \quad (9.23)$$
$$\forall f \in \text{Mor}(\mathbf{Set}),\ \Box_\varnothing(f) = \text{Id}_\varnothing \quad (9.24)$$

$$\forall A \in \text{Obj}(\mathbf{Set}),\ \bigcirc_*(A) = \{\bullet\} \quad (9.25)$$
$$\forall f \in \text{Mor}(\mathbf{Set}),\ \bigcirc_*(f) = \text{Id}_{\{\bullet\}} \quad (9.26)$$

[...]

As the examples we have given thus far do not have particularly varied definitions for the initial and terminal object (being mostly concrete categories where $I = \varnothing$ and $T = \{\bullet\}$, the modality of being and nothingness do not offer particularly more insight in those topos. Smooth sets simply get mapped to the empty space and the one point space, etc

"In geometric language these are categories equipped with a notion of discrete objects and codiscrete objects."

As any further adjoint would have to be a new functor from **H** to **1**, they would simply be just the functor Δ_* again, so that the further adjoints would simply be the same monads again. There is therefore no further moments at this level.

9.1.1 Negations

The modality of being admits a determinate negation

Theorem 196 *The negation of the monad of being is the identity monad :*

$$\overline{\bigcirc}_* = \mathrm{Id} \tag{9.27}$$

Proof 24 *The computation is simple enough, as*

$$\begin{aligned}
\overline{\bigcirc}_*(X) &= \mathrm{Fib}(X \to \bigcirc_* X) && (9.28)\\
&= \mathrm{Fib}(X \to 1) && (9.29)\\
&= \mathrm{Fib}(!_X) && (9.30)\\
&= X \times_* * && (9.31)\\
&= X && (9.32)
\end{aligned}$$

This is independent of the choice of basepoint, so that the negation of being is therefore the identity.

Interpretation? "contains that part of the structure that is trivialized by $X \to \bigcirc X$". As the unit monad removes all struture from the object, the trivialized part is all of it.

Theorem 197 *The left dual*

On the other hand, the modality of nothingness's cofibration does not give us exactly a determinate negation :

Theorem 198 *The negation of the comonad of nothingness is the maybe monad :*

$$\overline{\Box}_\varnothing = \mathrm{Maybe} \tag{9.33}$$

Proof 25

$$\begin{aligned}
\overline{\Box}_\varnothing &= \mathrm{cofib}(\Box_\varnothing X \to X) && (9.34)\\
&= X +_{\Box_\varnothing X} 1 && (9.35)\\
&= X +_\varnothing 1 && (9.36)\\
&= X + 1 && (9.37)
\end{aligned}$$

While the determinate negation is well-defined, it is not an idempotent monad, as it simply adds a new element to any object.

$$\text{Maybe}^2 X = (\text{Maybe} X) \sqcup \{\bullet\} = X \sqcup \{\bullet_1, \bullet_2\} \tag{9.38}$$

$$\overline{\Box}_\varnothing \Box_\varnothing X = 1 \tag{9.39}$$

$\Box_\varnothing \overline{\Box}_\varnothing = 0$ Interpretation?

Is the hexagonal diagram valid?

9.1.2 Algebra

As a monad, the unit monad has an associated algebra. For a given element X, and a morphism $x : \bigcirc_* X \cong 1 \to X$, ie a point of X,

Free algebra : algebra on 1 with morphism $\text{Id}_1 : 1 \to 1$, trivial algebra

Coalgebra of the empty comonad $\Box_\varnothing$: given X and a morphism $f : X \to \Box_\varnothing X \cong 0$

If there is no such map : empty coalgebra? Except on $\varnothing$, the cofree coalgebra, which is the coalgebra with

9.1.3 Logic

In terms of logic, those two monads will correspond to modalities

For sets : given a proposition $X \to \Omega$, that factors through either 0 or 1, the corresponding modality is

$$\Box_\varnothing (X \to 1 \to \Omega) = \tag{9.40}$$

subset relation defined by $\chi_U : X \to \Omega$,

$$\bigcirc_* (\chi_U) = \tag{9.41}$$

Projection to the trivial logic?

9.1.4 Sublation

If we wish to sublate this initial opposition to find higher ones, we need the resolution of $\Box_\varnothing \dashv \bigcirc_*$ to some higher adjunction $(\Box'_\varnothing \dashv \bigcirc'_*)$, so that we need

$$\Box'_\varnothing \Box_\varnothing = \Box_\varnothing \tag{9.42}$$
$$\bigcirc'_* \bigcirc_* = \bigcirc_* \tag{9.43}$$

In other words, each of these preserve the initial and terminal object

$$\Box'_\varnothing 0 = 0 \tag{9.44}$$
$$\bigcirc'_* 1 = 1 \tag{9.45}$$

We will look here more specifically at a *right sublation*, with the additional property

$$\bigcirc'_* \Box_\varnothing \cong \Box_\varnothing \tag{9.46}$$

From the properties of the empty comonad, this simply means that the sublated monad preserves the terminal product :

$$\bigcirc'_* 0 \cong 0 \tag{9.47}$$

which is the exact property of $\mathbf{H}_{\bigcirc'_*}$ being a dense subtopos. Fortunately there is a natural choice for this, given by this theorem

Theorem 199 *The smallest dense subtopos of a topos is that of local types with respect to double negation $\sharp = \mathrm{loc}_{\neg\neg}$. (Johnstone 02, corollary A4.5.20)*

From this, we have that the natural sublation of the ground opposition can be constructed from the localization by the double negation. This is called the *sharp modality.*

To understand the role of the sharp modality here, let's look at the internal logic, the most natural setting for the double negation. As a proposition in the internal logic corresponds to some subobject relation,

$$\begin{array}{ccc} U & \xrightarrow{!_U} & 1 \\ \downarrow{\scriptstyle \iota_U} & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[\chi_U]{} & \Omega \end{array}$$

In logical term, the subobject U associated to a proposition p is the largest context in which p holds, and its negation $\neg_X U$ the largest context in which $\neg p$ holds, but as our categories are not required to be boolean, the proposition $p \vee \neg_X p$ is not required to be "true", in the sense that the object associated to it is not X itself. For instance in a Grothendieck topos over a topological space, given some open subset $U \subseteq X$,

$$\neg U = \tag{9.48}$$

The point structure is preserved however, as for any point $x : 1 \to X$, we have that either this point belongs to U

[diagram of $1 \to U \to X$]

or to its negation

[diagram of $1 \to \neg U \to X$]

[...]

"The double negation subtopos is Boolean topos." (Johnstone 02, lemma A4.5.21)

The subtopos $\mathbf{H}_\sharp$ is therefore a *Boolean topos*. It can be understood in terms of the existence of a complement for any subobject, where every object can be split exactly in two parts by a subobject, one part which contains every point of A, and another part which contains every other point. [etc etc]

Any adjoint modality $\Box \dashv \bigcirc$ that includes the modalities $\varnothing \dashv *$, ie $\varnothing \subset \Box$, $* \subset \bigcirc$, formalizes a more determinate being (Dasein)

Sublation by $\sharp$ always exists for any topos?

"A topos E is Boolean iff E has exactly one dense subtopos, namely E neg neg = E."

Left sublation?

$$\Box'_{\varnothing} \bigcirc_* \cong \bigcirc_* \tag{9.49}$$

Preserves the unit monad

$$\Box'_{\varnothing} 1 \cong 1 \tag{9.50}$$

This is true of $\sharp$ but is it the first possible sublation?

9.2 Necessity and possibility

Before looking further into the first sublation of the ground opposition, let's briefly look at another direction to generalize.

The interpretation of being and nothingness as the duality between the dependent product and sum on the empty context gives us a possibility of generalization in this direction, in which we simply generalize to an arbitrary context.

As we've seen before, the context Γ of the internal logic corresponds to the slice category $\mathbf{C}_\Gamma$. If we wish to change our context, this is done via a display morphism $f : X \to Y$ which induces the functor

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{9.51}$$

The ground that we've seen is done on the empty context, which is given by the terminal object 0. The corresponding context is that of the display morphism $!_0 : 0 \to 1$, changing the context from 0, falsity, to 1, truth. The corresponding contexts are $\mathbf{C}_1 \cong \mathbf{C}$ and $\mathbf{C}_0 = \mathbf{1}$, giving the base change functor

$$f^* : \mathbf{C} \to \mathbf{1} \tag{9.52}$$

which is exactly the functor that we used as its basis.

The interpretation in this sense is therefore that $\bigcirc_*$ is adding the context

For a morphism $f : X \to 1$ (what context is that?) :

$$f^* : \mathbf{C} \to \mathbf{C}_{/X} \tag{9.53}$$

Adjoints : $(f_! \dashv f^* \dashv f_*)$

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}_X \underset{\Sigma_f}{\overset{\Pi_f}{\rightrightarrows}} \mathbf{H}_X \xrightarrow{f^*} \mathbf{H}_X \tag{9.54}$$

for $f : X \to 1$:

[...]

Adjoint modality : $(f_! f^* \dashv f_* f^*)$

Writer comonad and reader monad

Possibility comonad and necessity monad $(\Box \dashv \Diamond)$

[...]

The interpretation of this modality in terms of standard modal logic (the modality of necessity and possibility) can be understood using the Kripke semantics of modal logic.

interpretation of the ground in this context : $\bigcirc_*$ sends every proposition to true and $\Box_\varnothing$ to false. Every proposition is *possibly* true and none are *necessarily* true.

9.3 Determinate being

Determinate being (dasein), being in a certain place (hacceity?)

Sublation of being/non-being into non-becoming/becoming? idk via localization of $\neg\neg$

The localization of $\neg\neg$ allows for a boolean topos : we only consider subobjects which have a complement. For any object X and subobject $\iota : S \hookrightarrow X$, there is a complementary object $\bar{\iota} : \overline{S} \hookrightarrow$. We can *separate* S from X.

9.4 Cohesion

"Quantity is the unity of these moments of continuity and discreteness"

As we've seen, the natural sublation of the ground is given by the localization by the double negation, $\mathrm{loc}_{\neg\neg}$, the sharp modality $\sharp$. To get the full sublation of the adjunction, we will need also the existence of an adjoint modality, called the *flat modality* $\flat$. If this adjoint exists, the given adjoint cylinder

$$\mathbf{H}_\sharp \hookrightarrow \mathbf{H} \xrightarrow{\Gamma} \mathbf{H} \tag{9.55}$$

As a boolean topos, $\mathbf{H}_\sharp$ is typically either the topos of sets **Set** or some variant thereof, such as some ETCS variant of it or plus minus the axiom of choice. We will generally assume that we are using **Set** here, but most of the properties used here should generalize to any boolean topos.

Qualität/Quality, Etwas/Something, Die Endlichket/Finitude, Etwas und ein Anderes/Something and another

[57, 58]

The opposition generated here is $\flat \dashv \sharp$, called *cohesion.* As $\sharp$ can be understood as the

In the context of a topos,

Negation : internal hom into the initial object, $\neg = [-, \varnothing]$. In a topos, $\neg A$ is the internal hom object 0^A with 0 the initial object

Double negation : 0^{0^A}

$$\neg\neg = [[-, \varnothing], \varnothing] \tag{9.56}$$

For an object $X \in \mathbf{H}$, $\neg X = [X, \varnothing]$, the internal hom of all morphisms $X \to \varnothing$

Topos localized by $\neg\neg$, the new topos is $H_\sharp$. The opposition $\sharp \dashv \flat$ is the ground topos of $H_\sharp$ Localization of a topos

Adjoint triple

: based on the functor Γ, the direct image functor

Definition 200 *A topos* $\mathbf{H}$ *is* cohesive *over a base topos* $\mathbf{B}$ *if it is equipped with the geometric morphisms*

$$(f^* \dashv f_*) : \mathbf{H} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbf{B} \tag{9.57}$$

and obeys the following properties :

- *It is a locally connected topos :*

For most of our cases here, the base topos will be consider will be the topos of sets, **Set**, so that it is best to understand cohesion in terms of functors to sets. The archetypical cohesion is done using the global section functor Γ :

Cohesion is a sublation of the ground opposition

Do we have

$$\Gamma(A + \neg_X A) \cong \Gamma(X) \tag{9.58}$$

Preservation of pushout requires the geometric morphism to be surjective idk

Similarly $\flat(A + \neg_X A) \cong \flat(X)$?

If this flat modality admits a further adjunction, we will call its left adjoint the *shape modality*, $\int$. This will allow us to define the full notion of cohesion properly by adding some appropriate requirements to it.

First, we ask that $\int$ admits a definite negation. As this is an *sp*-unity, the two moments are meant to project on the same moment, so that the negation [...]

$$\int * \quad \cong \quad * \tag{9.59}$$

and $\flat \to \int$ is an epimorphism.

Vague : $\flat$ maps to the points of X, and every point of $\int X$ is an image of one of those points. (point to pieces transform)

To give it its proper meaning of being about the connected components of

$$(\Pi_0 \dashv \mathrm{Disc}) : \mathbf{H} \xrightarrow{\Pi_0} \mathbf{Set} \xrightarrow{\mathrm{Disc}} \mathbf{H} \tag{9.60}$$

$$\mathrm{Hom}_{\mathbf{Set}}(\Pi_0(X), A) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{Disc}(A)) \tag{9.61}$$

The hom-set of functions from our space to the discrete space from a set A is isomorphic to the set of functions from $\Pi_0(X)$ to that set.

Interpretation : Π_0 send each element - subobject in the same "connected component" to a different point.

Connected object : X is a connected object if the hom-set functor preserves coproducts

Shape preserves connected objects?

Shape modality : $\int = \mathrm{Disc} \circ \Pi_0$. *sp*-unity, so $\int$ and $\flat$ share the same space : $\int X$ is a discrete space : $\flat \int = \int$

In many cases, we will ask that the shape modality correspond to a retraction to a point of the (path)-connected components of the space.

Path space object I, localization by I, typically $\mathbb{R}$.

[...]

Hierarchy :

Topos (every topos has a terminal geometric morphism with adjoint?), sheaf topos (geometric morphism?)

splitting : existence of a further left / right adjoint :

local topos (codisc adjoint), locally connected topos ()

essential topos? (???)

thm : if a topos has a site with an initial object / terminal object, Proposition 4.3

Cohesive site

9.4.1 Negation

The negation of the sharp modality :

$$\bar{\sharp} X = \mathrm{Cofib}(\sharp X \to X) \tag{9.62}$$

$$X \sqcup_{\sharp X} * \tag{9.63}$$

Universal property with this object : for any object Y such that there is a morphism $f : Y \to X$, there exists a unique function $\beta : Y \to \overline{\sharp} X$ such that

Negation of the flat modality?

$$\overline{\flat} = \mathrm{Fib}(X \to \flat X) \tag{9.64}$$

Negation of the shape modality

$$\overline{\int} X = \mathrm{Cofib}(\int X \to X) \tag{9.65}$$

9.4.2 Algebra

9.4.3 Concrete objects

Definition 201 *An object in a cohesive topos is* concrete *if the unit of the adjunction* $(\Gamma \dashv \mathrm{CoDisc})$ *is a monomorphism.*

If a morphism has a concrete codomain, ie $f : X \to Y$ is such that

Concrete objects and separated presheaves

"is such that the $X \to \sharp X$ (unit or counit?) is a monomorphism"

Definition 202 *A morphism* $f : X \to Y$ *is said to be* intensive *if its codomain* X *is concrete.*

Definition 203 *For an intensive morphism, we have [some isomorphism idk]*

Extensive objects : maximally non-concrete codomin, ie X is

9.4.4 Logic

Definition 204 *A proposition* $p : A \hookrightarrow X$ *is* discretely true *if in the pullback*

$$\begin{array}{ccc} \sharp A\big|_X & \longrightarrow & \sharp A \\ \downarrow & & \downarrow \\ X & \xrightarrow{\eta_X} & \sharp X \end{array}$$

$\sharp A\big|_X \to X$ *is an isomorphism*

Proposition that is true over discrete spaces.

Theorem 205 *If* p

9.4.5 Cohesion on sets

Set is not a cohesive topos, as we will see (or more specifically, it does not have the strongest cohesion that can be had, being *sufficiently cohesive*). If we consider the case where its base topos is itself, then we need to investigate the adjoint cylinder from its functor of global sections, ie

$$(\mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc}) : \mathbf{Set} \underset{\mathrm{Disc}}{\overset{\mathrm{CoDisc}}{\rightrightarrows}} \mathbf{Set} \xrightarrow{\Gamma} \mathbf{Set} \tag{9.66}$$

The global section functor, as the hom functor from the terminal object, is simply the identity on sets, as sets are entirely defined by their points

As the identity, there are indeed a left and right adjoint to this functor, but they will both be the identity as well, so that $\mathrm{Disc} = \mathrm{CoDisc} = \mathrm{Id}_{\mathbf{Set}}$. This is about the behaviour we would expect from the discrete object, sending an object to the coproduct of terminal objects over its point content, in other words a set with the same cardinality. However, the codiscrete object, being the same, does not seem to follow what we would expect of a codiscrete object, of being "one whole" in some sense.

Let's push on and see the issue however. The same way as before, the discrete functor does have a further left adjoint, again the identity, $\Gamma_! = \Pi_0 = \mathrm{Id}_{\mathbf{Set}}$. This can be understandable in that the connected components of a set are indeed the same as the set itself, sets being "totally disconnected" objects. We also have that it is locally connected, in that sets are indeed the coproduct of an object over the connected components, ie

$$X \cong \coprod_{i \in \Pi_0(X)} \{\bullet\} \tag{9.67}$$

It is indeed connected, in that Π_0 preserves the terminal object (being the identity), and strongly connected, preserving finite products. And it local, as we can extend the identity again with codisc etc [redo]

The shape and flat modality $\mathrm{Id} \dashv \mathrm{Id}$ are indeed an epimorphism (points to pieces transformation), being just the identity

The natural transformation $\flat \to \int$ (the point-to-pieces transform) is indeed an epimorphism (an isomorphism even), still being the identity, and there is a sublation of the initial opposition ($\sharp 0 = 0$, as this is just the identity).

Pieces of powers are powers of pieces : obviously true again because identity

The one issue is that of the connectedness of the subobject classifier, 2, as trivially, $\Pi_0 2 = 2$, being the identity map, and not 1 as we would need to. This stems from an obstruction found in [59], saying that a localic topos cannot both have a shape modality over sets that preserves the product and have a connected subobject classifier. [proof?]

[local topos?]

As we've seen in the case of **Set**, there are two possible Lawvere-Tierney closure operators we could try over sets, and the $\mathrm{loc}_{\neg\neg}$ closure is the discrete topology, in the sense that every subset $S \subseteq X$ is its own closure, including singletons $\{\bullet\}$. No point is "in contact" with another, we can entirely separate a given element from the whole. The other closure operator is the trivial one $j(x) = 1$, giving us the *chaotic* topology on 1.

Equivalence between the lawvere-tierney topology and Grothendieck topology, chaotic grothendieck topology, collapse of sets into triviality?

As $\neg\neg = \mathrm{Id}_\Omega$, the localization does not do anything and the smallest dense subtopos is simply itself, there are no levels in between Set and the ground.

The global section functor $\Gamma(-) \cong \mathrm{Hom}_{\mathbf{Set}}(1, -)$ is simply the identity, as

$$\Gamma(S) \cong \mathrm{Hom}_{\mathbf{Set}}(1, S) \cong S \tag{9.68}$$

Since the hom-set of the point to a set is of the same cardinality as the set itself, and

$$\Gamma(f) \tag{9.69}$$

The left adjoint functor Disc here will work out as

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Disc}(-), -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, -) \tag{9.70}$$

while the right adjoint is

$$\mathrm{Hom}_{\mathbf{Set}}(-, -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{9.71}$$

As far as the objects go, these are identities, so that the discrete and codiscrete objects of sets are the same objects (they have the same point content). However, the actions on morphisms does change, and most importantly, on the Lawvere-Tierney topology of the topos. For the morphism $j : \Omega \to \Omega$, $j \in \mathrm{Hom}_{\mathbf{Set}}(2, 2)$, we have

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(-), -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{9.72}$$

The left adjoint modality $\flat \dashv \sharp$

"Note that in this example, the "global sections" functor $S \to Set$ is not the forgetful functor $Set/U \to Set$ (which doesn't even preserve the terminal object), but the exponential functor $\Pi U = \mathrm{Hom}(U, -)$. This is the direct image functor in the geometric morphism $Set/U \to Set$, whereas the obvious forgetful functor is the left adjoint to the inverse image functor that exhibits S as a locally connected topos."

Negation of sharp modality :

$$\begin{aligned} \bar{\sharp}X &= X \sqcup_{\sharp X} * & (9.73) \\ &= X \sqcup_{\sharp X} * & (9.74) \end{aligned}$$

Is it just $*$? The negation of the moment of continuity (maximally non-concrete object)

9.4.6 Cohesion of a spatial topos

Spatial topos always has disconnected truth values hence not a sufficiently cohesive topos cf Lawvere

9.4.7 Cohesion of smooth spaces

As a Grothendieck topos, the cohesiveness of smooth spaces relies on the cohesiveness of its site, the category of Cartesian spaces.

CartSp has a terminal object ($\mathbb{R}^0$)

CartSp is cosifted in that it has finite products, ie for any $U_1, U_2 \subseteq \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$,

$$U_1 \times U_2 \subseteq \mathbb{R}^{n_1+n_2} \tag{9.75}$$

CartSp with its coverage (differentiably good cover) is locally connected, ie for any object [...]

As a cohesive site, **CartSp** has the global section functor Γ which simply assigns to each open set of $\mathbb{R}^n$ its set of points

$$\Gamma(U_{\mathbf{CartSp}}) = U_{\mathbf{Set}} \tag{9.76}$$

and as its left adjoint the functor $\Gamma^* \cong \mathrm{LConst} \cong \mathrm{Disc}$ which associate to any set the smooth space composed by an equinumerous number of disconnected copies of $\mathbb{R}^0$,

$$\mathrm{Disc}(X) = \bigsqcup_{x \in X} \mathbb{R}^0 \tag{9.77}$$

This is equivalently the discrete (fine?) diffeology on a set, given by some

Its right adjoint is given by the coarse diffeology on a set

Right adjunction :

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(U), X) = \mathrm{Hom}_{\mathbf{CartSp}}(U, \mathrm{CoDisc}(X)) \tag{9.78}$$

Every possible function (as a set) between some Cartesian space U and our codiscrete space X correspond to a valid plot.

In terms of intuition, this means that every point is "next to" every other point in some sense. For instance, given the probe $[0,1]$, there is a smooth curve for every possible combination of points, ie the set of smooth curves is just $X^{[0,1]}$.

There is therefore no meaningful way to separate points (as we would expect from the smooth equivalent of the trivial topology).

Given these two functors, we can construct our two modalities. Our flat modality is

$$\flat = \mathrm{Disc} \circ \Gamma \tag{9.79}$$

which first maps a smooth space to its points and then to the coproduct of $\mathbb{R}^0$ over those points, giving us the discrete space $\flat X$, and the sharp modality $\sharp$ is

$$\flat = \mathrm{CoDisc} \circ \Gamma \tag{9.80}$$

which maps a smooth space to its points and then to the

Are all smooth spaces in the Eilenberg category diffeological spaces?

fine diffeology v. coarse diffeology

"Every topological space X is equipped with the continuous diffeology for which the plots are the continuous maps."

Negation of

Extensive and intensive quantities

9.4.8 Cohesion of classical mechanics

9.4.9 Cohesion of quantum mechanics

The terminal object in the Bohr topos is given by the constant sheaf which maps all commutative operators in a context to a spectrum of a single element, the singleton $\{\bullet\}$ in **Set**. This is the spectral presheaf which has a spectrum of a single value for all

Global section functor : hom-set of

$$\Gamma : \mathbb{C}^0 \to \tag{9.81}$$

$$\Gamma : \mathbb{C}^0 \to \tag{9.82}$$

9.4.10 Sublation

9.5 Elastic substance

[60]

Sublation of $\int \dashv \flat$ to $\Im \dashv \&$, $\bigcirc \dashv \Box$ opposition

$$X \to \Im X \to \int X \tag{9.83}$$

Differential cohesion

None of the topos we have seen thus far are differentially cohesive, but it is simple enough to extend them to be. This is generally done concretely by changing the site to include an infinitesimal structure. The basic example for this is the site of *formal Cartesian spaces*, **FormalCartSp**, which is the site with objects being open sets of $\mathbb{R}^n$ composed with an *infinitesimally thickened point*, D. As we will see, D is a point if we forget the elastic structure, $\Re(D) \cong *$, but in basic mathematical terms, this relates to Weil algebras, the algebras of infinitesimal objects.

The basic example of a Weil algebra is the algebra of *dual numbers*, a vector space composed by a pair of real numbers, $\mathbb{R}[\varepsilon]/\varepsilon^2$

$$(x + \epsilon y)^2 = x^2 + xy\varepsilon \tag{9.84}$$

The topos from that site is the *Cahier topos*,

$$\mathbf{Cahier} = \mathrm{Sh}(\mathbf{FormalCartSp}) \tag{9.85}$$

[61, 62, 63]

9.5.1 Synthetic infinitesimal geometry

Koch-Lawvere axioms

9.5.2 Differential cohesion of the Cahier topos

Formal Cartesian space **FormalCartSp**

9.5.3 Crystalline cohomology

9.6 Solid substance

Sublation of $\Re$ and $\Im$, ($\rightsquigarrow \dashv$ Rh)

Bosonic v. fermionic spaces

Chapter 10

Nature

10.1 Mechanics topos

The general topos typically used by nlab to respond to the requirements of all those modalities is the *super formal smooth infinity-groupoid*. This is the ∞-sheaf on a special site composed from the category of Cartesian spaces (to give the cohesion), the category of infinitesimally thickened points (to give the elasticity), and the category of superpoints (to give the solidity).

We have already seen the category of Cartesian spaces in detail, so let's now look into infinitesimally thickened points.

Infinitesimally thickened points are a geometrical realization of the formal notion of infinitesimals as provided by Weil algebras.

Example : dual numbers

$$(x, \epsilon) \mapsto f(x, \epsilon) = f(x) \tag{10.1}$$

Chapter 11

The Ausdehnungslehre

One early attempt at mathematization of similar philosophical notions was the Ausdehnungslehre of Grassmann[5]

[64, 65, 66]

Chapter 12

Kant's categories

Quantity : unity, plurality, totality

Quality : reality, negation, limitation

Relation : inherence and subsistence, causality and dependence, community

Modality : possibility, actuality, necessity

Chapter 13

Lauter einsen

Cantor's original attempt at set theory[67] involved the notion of *aggregates* (*Mengen*), which is what we would call a sequence today, some ordered association of various objects. If we have objects $a, b, c, \ldots$, their aggregate M is denoted by

$$M = \{a, b, c, \ldots\} \tag{13.1}$$

Despite the notation reminiscent of sets, the order matters here. The notion closer to that of a set is given by an abstraction process $\overline{M}$, in which the order of elements is abstracted away (this would be something akin to an equivalence class under permutation nowadays).

The *cardinality* of the aggregate is given by a further abstraction, $\overline{\overline{M}}$, given by removing the nature of all of its element, leaving only "units" behind :

$$\overline{\overline{M}} = \{\bullet, \bullet, \bullet, \ldots\} \tag{13.2}$$

$\bullet$ here is an object for which all characteristics have been removed, and all instances of $\bullet$ are identical. In some sense this is the application of the being modality on its objects : we only have as its property that the object exists, similarly to *das eins*.

Comment from Zermelo [68, p. 351]:

"The attempt to explain the abstraction process leading to the "cardinal number" by conceiving the cardinal number as a "set made up of nothing but ones" was not a successful one. For if the "ones" are all different from one another, as they must be, then they are nothing more than the elements of a newly introduced set that is equivalent to the first one, and we have not made any progress in the abstraction that is now required."

Relation to the discrete/continuous modality

From Lawvere : The maps between the *Menge* and the *Kardinal* is the adjunction

$$(\text{discrete} \dashv \text{points}) : M \underset{\text{discrete}}{\overset{\text{points}}{\rightleftarrows}} K \tag{13.3}$$

The functor points maps all elements of an object in M to the "bag of points" of the cardinal in K, the functor discrete sends back

[69]

Chapter 14

Spaces and quantities

From Lawvere[70]

Distributive v. other categories

Intensive / extensive

"The role of space as an arena for quantitative "becoming" underlies the qualitative transformation of a spatial category into a homotopy category, on which extensive and intensive quantities reappear as homology and cohomology."

Definition 206 *A* distributive category **C** *is a category with finite products and coproducts such that the canonical distributive morphism*

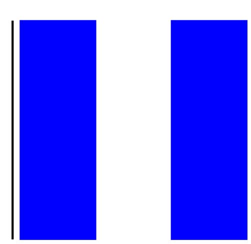
$$(X \times Y) + (X \times Z) \to X \times (Y + Z) \tag{14.1}$$

is an isomorphism, ie there exists a morphism

$$X \times (Y + Z) \to (X \times Y) + (X \times Z) \tag{14.2}$$

that is its inverse.

Distributive categories are typically categories that are "like a space" in some sense, in the context that concerns us. In terms of physical space, we can visualize it like this :



Whether we compose this figure by first spanning each red line along the black line and then summing them, or first summing the two red lines and spanning them along the black line does not matter and will give the same figure.

Example 207 *If a category is Cartesian closed and has finite coproducts, it is distributive.*

Proof 26 *In a Cartesian category, the Cartesian product functor $X \times -$ is a left adjoint to the internal hom functor $[X, -]$. As colimits are preserved by left adjoint tmp, we have*

$$(X + Y) \tag{14.3}$$

This means in particular that any topos is distributive, such as **Set** and **Smooth**.

Example 208 *The category of topological spaces* **Top** *is distributive.*

Proof 27 *In* **Top***, the product is given by spaces with the product topology,*

$$(X_1, \tau_1) \prod (X_2, \tau_2) = (X_1 \times X_2, \tau_1 \prod \tau_2) \tag{14.4}$$

where the product of two topologies is the topology generated by products of open sets in X_1, X_2 :

$$\tau_1 \prod \tau_2 = \{U \subset\} \tag{14.5}$$

[...] and the coproduct is the disjoint union topology :

[Category of frames?]

As can be seen, those are some of the most archetypical categories of spaces.

Definition 209 *A* linear category *is a category for which the product and coproduct coincide, called the* biproduct. *For any two objects $X, Y \in \mathbf{C}$, the biproduct is*

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y \tag{14.6}$$

Proposition 210 *A linear category has a zero object 0, which is both the initial and terminal object.*

Proof 28 *This stems from the equivalence of initial and terminal objects as the product and coproduct over the empty diagram.*

Proposition 211 *In a linear category, there exists a zero morphism $0 : X \to Y$ between any two objects X, Y, which is the morphism given by the terminal object map $0 \to Y$ and the initial object map $X \to 0$:*

$$0_{X,Y} : X \to 0 \to Y \tag{14.7}$$

A linear category is so called due to its natural enriched structure over commutative monoids.

Proposition 212 *For any two morphisms $f, g : X \to Y$ in a linear category, there exists a morphism $f \oplus g$ defined by the sequence*

$$X \to X \times X \cong X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \cong Y + Y \to Y \tag{14.8}$$

Proposition 213 *The sum of two morphisms is associative and commutative*

Proposition 214 *The zero morphism is the unit element of the sum.*

"in any linear category there is a unique commutative and associative addition operation on the maps with given domain and given codomain, and the composition operation distributes over this addition; thus linear categories are the general contexts in which the basic formalism of linear algebra can be interpreted."

Definition 215 *If in a linear category every morphism $f : X \to Y$ has an inverse denoted $-f : X \to Y$, such that $f \oplus -f = 0$, then it is enriched over the category of Abelian groups Ab, and is called an* additive category*.*

Chapter 15

Parmenides

Arguments from Zeno & Parmenides :

"All objects are similar to each other and all objects are different from each other"

Parmenides proceeded: If one is, he said, the one cannot be many? Impossible. Then the one cannot have parts, and cannot be a whole? Why not? Because every part is part of a whole; is it not? Yes. And what is a whole? would not that of which no part is wanting be a whole?

Certainly. Then, in either case, the one would be made up of parts; both as being a whole, and also as having parts?

To be sure. And in either case, the one would be many, and not one? True. But, surely, it ought to be one and not many? It ought. Then, if the one is to remain one, it will not be a whole, and will not have parts?

No. But if it has no parts, it will have neither beginning, middle, nor end; for these would of course be parts of it.

Right. But then, again, a beginning and an end are the limits of everything?

Certainly. Then the one, having neither beginning nor end, is unlimited? Yes, unlimited. And therefore formless; for it cannot partake either of round or straight.

But why? Why, because the round is that of which all the extreme points are equidistant from the centre?

Yes. And the straight is that of which the centre intercepts the view of the extremes?

True. Then the one would have parts and would be many, if it partook either of a straight or of a circular form?

Being is indivisible, since it is equal as a whole; nor is it at any place more, which could keep it from being kept together, nor is it less, but as a whole it is full of Being. Therefore it is as a whole continuous; for Being borders on Being.

Chapter 16

Bridgman and identity

what the fuck is that about [12, 71]

”We must, for example, be able to look continuously at the object, and state that while we look at it, it remains the same. This involves the possession by the object of certain characteristics — it must be a discrete thing, separated from its surroundings by physical discontinuities which persist.”

Definition of a system?

Chapter 17

Ludwig Gunther

Relation of moments to Ludwig's structuralism?

Axiomatization from Ludwig [72]

[73]

Chapter 18

Dialectics of nature

When we look at dialectical logic in practice, ie [14], the examples given are much more concrete. We are considering the identity of some *entity*, such as an object, a group, etc, and considering what it means for at entity to be identical to itself. All concrete entities are never identical to themselves, either in time, context, etc.

From Engels :

The law of the transformation of quantity into quality and vice versa; The law of the interpenetration of opposites; The law of the negation of the negation.

Example : an object moving in space, an organization changing, ship of Theseus, etc

To keep things concrete, let's try to consider a simple example of both category theory and dialectical logic, which is an object in motion. We will simply consider the kinematics here and not the dynamics as this is unnecessary.

The simplest case is the $(1+1)$-dimensional case, of a point particle moving along a line $x : L_t \to L_s$.

First notion : We are considering the "identity" of an object under a certain *lens* (ie with respect to its relations with a number of other objects). Its identity is only assured under the full spectrum of those relationships, ie we say that two objects A, B are identical if

$$\forall X \in \mathbf{H},\ \mathrm{Hom}_{\mathbf{H}}(A, X) \cong \mathrm{Hom}_{\mathbf{H}}(B, X),\ \mathrm{Hom}_{\mathbf{H}}(X, A) \cong \mathrm{Hom}_{\mathbf{H}}(X, B) \quad (18.1)$$

Relation to Yoneda? Relation to Hegel's "only the whole is true" thing?

This corresponds to the hom covariant and contravariant functor

$$h^A \cong h^B,\ h_A \cong h_B \tag{18.2}$$

Two objects are identical if their hom functors are naturally isomorphic, ie if there exists a natural transformation

$$\eta : h^A \Rightarrow h^B,\ \epsilon : h_A \Rightarrow h_B \tag{18.3}$$

with two-sided inverses each.

Yoneda embedding :

$$\begin{aligned}
\mathrm{Nat}(h_A, h_B) &\cong \mathrm{Hom}_{\mathrm{H}}(B, A) && (18.4)\\
\mathrm{Nat}(h^A, h^B) &\cong \mathrm{Hom}_{\mathrm{H}}(A, B) && (18.5)\\
& && (18.6)
\end{aligned}$$

those isomorphisms are elements of this, therefore isomorphic to morphisms from A to B. Therefore A and B are identical in term of their relationships to every other object if they are identical in the more typical sense of the existence of an isomorphism.

As those are functors, this analysis also applies to elements of the objects X, or subobjects. For any two monomorphisms

$$x, y : S \to X \tag{18.7}$$

We say that those elements are identical if the hom functors applied to them [...]

The breakage of the law of identity comes by considering different perspectives. If given an object X [or element $x : \bullet \to X$?], we attempt to use different relations with other objects to "probe" it, its moments and the assessment of its identity will change.

[Abstract example ?]

Expression via the subobject classifier of the topos

Example : case of motion. What does it means for a moving object to *be* in different positions.

Ex : consider the position of an object wrt two different time intervals $[t_a, t_b]$, $[t'_a, t'_b]$, rather than all possible intervals.

Characterization by observables at those different times

Observables : $O_i : \mathrm{Conf} \to \mathbb{R}$ [Isbell duality thing idk]

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