Commentary on nLab's Science of Logic and other matters

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Couples are things whole and things not whole, what is drawn together and what is drawn asunder, the harmonious and the discordant. The one is made up of all things, and all things issue from the one.

Heraclitus

Contents

1 Introduction

While not a new phenomenon by any mean, there is a certain recent trend of trying to mathematize certain philosophical theories, in particular ideas relating to dialectics. Dialectics can refer to quite a lot of somewhat related topics, such as a general method to arrive at the truth by trying to reconcile contradictions between opposing ideas, but in this case we are talking more specifically about the focus of dialectics on oppositions between concepts and their contradictions. In the case of metaphysics, those oppositions can be for instance

- One / Many
- Sameness / Difference
- Being / Nothing
- Space / Quantity
- General / Particular
- Objective / Subjective
- Qualitative / Quantitative
- Finite / Infinite

The oldest of such thoughts goes back to ancient greek traditions, to such philosophers as Parmenides[1] and Heraclitus. Heraclitus in particular was very much involved in the notion that oppositions such as these do not form the division that people would typically expect, and in fact are very much related to each other.

"What opposes unites, and the finest attunement stems from things bearing in opposite directions, and all things come about by strife." [Aristotle's Nicomachaen Ethics, book VIII]

The contradictions involved would be for instance that a collection of objects is both one and many at the same time,

> PARMENIDES: If one is, the one cannot be many?
> SOCRATES: Impossible.
> PARMENIDES: Then the one cannot have parts, and cannot be a whole?
> SOCRATES: Why not?
> PARMENIDES: Because every part is part of a whole; is it not?
> SOCRATES: Yes.
> PARMENIDES: And what is a whole? would not that of which no part is wanting be a whole?
> SOCRATES: Certainly.
> PARMENIDES: Then, in either case, the one would be made up of parts; both as being a whole, and also as having parts?
> SOCRATES: To be sure.
> PARMENIDES: And in either case, the one would be many, and not one?
> SOCRATES: True.
> PARMENIDES: But, surely, it ought to be one and not many?
> SOCRATES: It ought.
> PARMENIDES: Then, if the one is to remain one, it will not be a whole, and will not have parts?
> SOCRATES: No.
> PARMENIDES: But if it has no parts, it will have neither beginning, middle, nor end; for these would of course be parts of it.
> SOCRATES: Right.
> PARMENIDES: But then, again, a beginning and an end are the limits of everything?
> SOCRATES: Certainly.
> PARMENIDES: Then the one, having neither beginning nor end, is unlimited?
> SOCRATES: Yes, unlimited.
> PARMENIDES: And therefore formless; for it cannot partake either of round or straight.

[Plato's Parmenides]

with a similar notion in Heraclitus

Couples are things whole and things not whole, what is drawn together and what is drawn asunder, the harmonious and the discordant. The one is made up of all things, and all things issue from the one. [Pseudo-Aristotle, On The Cosmos]

But while Parmenides' solution was to deny plurality, Heraclitus was to welcome the opposition.

Sameness and difference?

Other such examples can be found in the medieval era with the work of Nicholas of Cusa, De Docta Ignorantia (on learned ignorance) [2].

"Now, I give the name "Maximum" to that than which there cannot be anything greater. But fullness befits what is one. Thus, oneness—which is also being—coincides with Maximality. But if such oneness is altogether free from all relation and contraction, obviously nothing is opposed to it, since it is Absolute Maximality. Thus, the Maximum is the Absolute One which is all things. And all things are in the Maximum (for it is the Maximum); and since nothing is opposed to it, the Minimum likewise coincides with it, and hence the Maximum is also in all things. And because it is absolute, it is, actually, every possible being; it contracts nothing from things, all of which [derive] from it."

The first major author for the exact field that we will broach here is Kant and his transcendental logic [3]

The main author for these recent trends target is Hegel and his Science of Logic [4, 5], where he describes his *objective logic*. The "classical" logic originally described by Aristotle, the logic of propositions and such, is described under the term of *subjective logic*, the logic of merely assembling

Heidegger, being and time?

The original attempt at the formalization of those ideas (or at least ideas similar to it) was by Grassmann[6], giving his theory of extensive quantities [vector spaces], which while it was commented on and inspired some things, mostly did not go much further.

In modern time, this programme was originally started by Lawvere[7]

"It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy."

This is a somewhat recurring theme in nlab[8]

[9, 10, 11, 12, 13, 14]

As is traditional for such types of philosophy, the writings are typically fairly abstract and lacking example. For a more pedagogical exposition, I have tried here to include more examples and demonstrations to such ideas. I am not an expert in algebraic topology by any mean and have tried my best to explain those notions without using notions from this field. These are mostly written from the perspective of a physicist, so while I may not expect prior knowledge in schemes and type theory, I will likely commonly use notions of quantum theory or mechanics.

There are a few caveats to bring up here. First, Hegel's system is primarily about *thought*. Although it tries its best to explain the modern science of its era (the 19^{th} century) through this framework, the focus is more on the way that the mind can conceptualize those notions. This may not on the other hand be the primary goal of the categorical approach here, or at least a very difficult goal to attain, although that is certainly true of Hegel himself as he has not been known to be the most obvious to read.

Furthermore, while some of these ideas could be argued to be fairly faithful translations of the philosophical ideas, others seem more to be generally inspired by them, the original notion of unity of opposites being more of a collection of different ideas on that theme than a rigorous construction. While such notions as quantity from the abstraction to pure being seem to have some parallels, I do not believe that Hegel had particularly in mind the concept of a graded algebra when he spoke of the opposition between das Licht and die Körper des Gegensatzes (in particular, this opposition is true regardless of dynamics, and therefore should not be particularly relevant to the solidity of a body), but if the opposition can somehow mirror the adjunction of bosonic and fermionic modalities, why not look into it. The notion of being-for-itself and being-for-one are [according to X] more related to consciousness than etc.

And in the other direction, as is common in the context of formal systems, this is merely one semantics that we can apply to it.

The notions described here are furthermore not firmly rooted in the formalism but merely described by it, as many of these notions are already needed to *define* the formalism. In particular, it is difficult to define any theory without the concept of discrete objects (in our case, by rooting the formalism of categories in the notion of sets, and in fact in general in any thought process requiring to have different ideas as discrete entities). The rooting of the notion of oneness in terminal objects are somewhat superficial, as the very notion of a terminal object already requires the notion of oneness, ie in the unique morphism demanded by universal properties.

Actual applications of dialectical logic [15, 16] also seem fairly disconnected from the formalism developed here, so that we will have to look into it separately.

It is therefore best to keep in mind that this is not a faithful formalization of Hegel's system but merely reflects some of its ideas.

[13, 17, 18, 19, 20, 21, 22, 23, 24]

Transitions Into, With, and From Hegel's Science of Logic

Before going into those various formalizations, we will first have a rather in depth look at the formalisms on which these are based, which are type theory and category theory.

Resources : [25]

Part I

Types

The first element of the theory discussed is that of *types*[26], which will relate to the notion of categories and logic later on, through the notion of *computational trinitarianism*.

Relation with whatever Kant idk

A *type* is, as the name implies, a sort that some mathematical object can be. We denote that an object c is of type C by

$$c : C \tag{1.1}$$

and say that c is an *instance* of type C.

From the computational trinitarianism interpretation, there are roughly three main interpretations of a type. In terms of logic, a type represents a proposition. In terms of category theory, a type is an object, and in our focus in particular, a space. And in terms of type theory, a type is a construction.

The typical simple example, as used in mathematics and computation theory, is that of integers. An integer n is an instance of the integer type, $n : \mathbb{N}$.

As a space :

Types being themselves a mathematical object, we also have some type for types, denoted Type, although to avoid some easily foreseeable Russell style paradox (called the Girard paradox[27]), we will instead use some hierarchy of such types, called type universes :

$$C : \mathrm{Type}_0, \mathrm{Type}_0 : \mathrm{Type}_1, \mathrm{Type}_1 : \mathrm{Type}_2, \ldots \tag{1.2}$$

Although as we will not really require much foray into the hierarchy of type universes, we will simply refer Type_0 as Type from now on.

"A proposition is interpreted as a set whose elements represent the proofs of the proposition"

The notation of instances belonging to types that we've seen is one example of such a judgment, called a *typing judgment*. A typing judgment is any

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Formulas and judgements

The basic element of type theory is the *formula*, similarly to the case of logic, which is some statement about the type theory.

Definition 2.0.1. A *formula* is a finite sequence of symbols from a set of symbols S.

The basic formulas that we will deal with are the declaration of type $c : C$, and the equivalence $c \equiv d$. The main structure we will use on formulas is that of a *judgement*. Given a type theory TT, a judgment is given by two lists of formulas as

$$A_1, A_2, \ldots, A_n \vdash_{\mathrm{TT}} B_1, B_2, \ldots, B_m \tag{2.1}$$

with the semantics of "within the type theory TT, and assuming all the formulas (A_i), then at least one formula of (B_j) is true". We will not often work in different type theories at the same time, so that we can omit the subscript on $\vdash_{\mathrm{TT}}$.

It should be noted that those lists are indeed lists and not sets. While many type theories do not place any importance on the ordering of those formulas, some (such as quantum logic) do, so that the commutativity of formulas is a specific axiom of the system.

To shorten notation, as we may be typically dealing with rather long lists of arbitrary formulas, we will use the notion of *context*, which is defined as a (possibly empty) list of formulas.

$$\Gamma = A_1, A_2, \ldots, A_n \tag{2.2}$$

where the concatenation of contexts and formulas is understood to mean the obvious concatenation of all formulas within :

$$\Gamma, A = A_1, A_2, \ldots, A_n, A \tag{2.3}$$

$$\Gamma_1, \Gamma_2 = A_{1,1}, A_{1,2}, \ldots, A_{1,n}, A_{2,1}, A_{2,2}, \ldots, A_{2,n} \tag{2.4}$$

As with formulas in general, we can perform a substitution operation. If a formula A contains a variable x of type X, we denote its substitution by a term t of type X by

$$A[t/x] \tag{2.5}$$

Example 2.0.1. If we have a formula P with free variable such as a function f with variable x, $f(x)$, we can substitute a term t for it,

$$P[t/x] \equiv f(t) \tag{2.6}$$

3

Sequent calculus

The transformation rules of statements about types are given by the sequent calculus. In a type theory, the basic entities that we manipulate are the *formulas* and the *judgements*. A formula is simply some statement we have on our type theory, the basic one being the typing of a term. For instance, the statement "a is of type A" is a formula :

$$a : A \tag{3.1}$$

A judgement of a formula is the notion that the formula that we have can be proven in our system. For instance, if we consider the formula that 0 is an integer, $0 : \mathbb{N}$, our judgement is that this is indeed true in our system. If we write down our specific system by S, this is denoted by the turnstyle $\vdash_S$:

$$\vdash_S 0 : \mathbb{N} \tag{3.2}$$

As we will here not typically work with many different systems however, we will keep S implicit unless necessary, so that we will just use $\vdash$.

Judgements in general are done with specific assumptions, that is, a formula is valid in the system only assuming another formula. We write this assumption on the left as

$$A \vdash B \tag{3.3}$$

The left side of the judgement is called the *antecedent*, while the right side is called the *consequent*.

In general, in Gentzen style sequent calculus, we can have multiple formulas on both sides.

$$A_1, A_2, \dots, A_n \vdash B_1, B_2, \dots, B_m \tag{3.4}$$

The semantics of which are meant to be read as if *all* the formulas A_i are valid, then at least one of the formula B_j is valid. This specific semantics is meant to emulate the notion of implication in propositional logic, as

$$A_1 \wedge A_2 \wedge \dots \wedge A_n \to B_1 \vee B_2 \vee \dots \vee B_m \tag{3.5}$$

As we can have quite a lot of formulas on either side, it is common to write large numbers of formulas in a variable called a *context*.

$$\Gamma = \{A_i\}_{i \in I} \tag{3.6}$$

where I is a possibly empty finite set.

Above : premises, below : conclusion

$$\frac{\Gamma \vdash P}{\Gamma \vdash P}$$

A simple universal inference rule in sequent calculus for instance is that a formula entails itself, with no premises required :

$$A \vdash A$$

Meaning that even assuming no previous judgments, we can deduce that the hypothesis of assuming A entails A.

Example of judgement : type judgment C : Type, type equality judgment $A \equiv B$Type, element $c : C$, equal element judgment $c = c' : C$

Context as a list of type instance declarations? $a : A,\ b : B,\ c : C, \dots$

Definition 3.0.1. An *equality type*, denoted by $a = b$, is a formula indicating the equality between two terms.

Example 3.0.1. A useful example of a type theory in our context is that of intuitionist logic, where the basic type is that of propositions, Prop : Type, with two terms $\top, \bot$: Prop

Substitution rule : If we replace a variable in a formula

$$\frac{\Gamma, x : X \vdash A : \mathrm{Type} \qquad \Gamma \vdash t : X}{\Gamma \vdash A[t/x] : \mathrm{Type}}$$

[...]

For the various types and constructions involved in type theory, we can generally split the rules as follow.

Formation rules allow the existence of a given type from other types (possibly none if they are fundamental). If we have some list of types $(A, B, C, \dots)$, then a formation rule is the formation of a new type from those :

$$\frac{\vdash A : \mathrm{Type} \qquad \vdash B : \mathrm{Type} \qquad \vdash C : \mathrm{Type} \qquad \dots}{F(A, B, C, \dots) : \mathrm{Type}}$$

Likewise, *introduction rules* give us a way to construct terms from existing terms.

Elimination rules

Computation rules

To give an example of a formalism we will not detail later on, let's see how this definition applies to functions in typed lambda calculus. As we will use dependent typing to define our functions, this is not quite how we will define our functions later on.

A lambda term in type theory is given by some formula f and variable x

$$\lambda x. f(x) \tag{3.7}$$

In lambda calculus, we have three different types of rules. First is the alpha conversion, where we can relabel any formula with different symbols for variables :

$$\lambda x.\ f(x) \to \lambda y.\ f(y) \tag{3.8}$$

we can apply another variable to that lambda function, if the expression has the bound variable in it, leading to a substitution of this variable, called beta reduction

$$(\lambda x. f(x))y = f(x)[x/y] \tag{3.9}$$

and if the variable does not appear in the expression, we simply ignore the substitution of the beta case, called eta conversion :

$$(\lambda x.y)z = y \tag{3.10}$$

4

Equality type

Separate from the notion of judgmental equality, where we judge two terms or types to be equal by definition, somewhat externally to the theory, is the notion of equality type, also called identity type. An equality type is a type associated to the equality between two terms of the same type, ie for $a : A$ and $a' : A$, we have the existence of a type $a =_A a'$. In terms of formation rule, any type induces an equality type :

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, a : A, a' : A \vdash (a =_A a') : \mathrm{Type}}$$

Unlike the judgmental equality, the equality type is meant to represent an actual equality of the theory and not merely a definition of terms.

Two terms are equal if their equality type is inhabited, ie if we have

$$a : A, b : A \vdash c : a =_A b \tag{4.1}$$

Like any definition of equality, an equality type has some notion of reflexivity, symmetry and transitivity. The reflexivity is given by its introduction rule :

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, a : A \vdash \mathrm{refl}_A(a) : (a =_A a)}$$

In the interpretation of the unit type as truth, this means that a term is equal to itself.

5
Dependent types

We speak of *dependent types* for a type that depends on a value, ie a "type" that is actually a function from one type to the universe of types. If we pick for instance the function type $B : A \to \text{Type}$, its evaluation for each different instance of A may lead to a different type :

$$a : A \vdash B(a) : \text{Type} \tag{5.1}$$

B as a whole is the dependent type of A, with each instance $B(a)$

An example of this is the dependent type of vector spaces, where for some integer type $n : \mathbb{N}$, we associate a type of n-dimensional vector space, $\text{Vect}(n)$, where we have the series of types

$$\text{Vect}(0) : \text{Type}, \text{Vect}(1) : \text{Type}, \text{Vect}(2) : \text{Type}, \ldots \tag{5.2}$$

Example 5.0.1. An indexed set, given by a function mapping elements of the indexing set I to some set of sets X,

$$f : I \quad \to \quad X \tag{5.3}$$

$$i \quad \mapsto \quad f(i) = X_i \tag{5.4}$$

can be described as a term of the dependent type

$$f : \text{Set} \to \text{Set} \tag{5.5}$$

Example 5.0.2. The type of $n \times n$ matrices is a dependent type indexed by an integer type $n : \mathbb{N}$:

$$n : \mathbb{N} \vdash \mathrm{Mat}(n) : \mathrm{Type} \tag{5.6}$$

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Function types

Given any two types $A, B \in \text{Type}$, we can define another type called the *function type* of A to B :

$$f : A \to B \tag{6.1}$$

which are meant to model functions, ie given a term $a : A$, we will have a corresponding term in B given by something of the form $f(b) : B$.

As we can form a function type for any two types, the formation rule is given by

$$\frac{\Gamma \vdash A : \text{Type} \qquad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A \to B : \text{Type}}$$

The introduction rule is given by the association of a term $x : X$ and some dependent type $f(x) : Y$, associating

$$\frac{\Gamma, x : X \vdash f(x) : Y}{\Gamma \vdash (x \mapsto f(x)) : X \to Y}$$

The elimination rule is the evaluation of the function, where given a function f and a term of its domain $a : A$, we obtain the evaluation of f at a :

$$\frac{\Gamma \vdash f : X \to Y \qquad \Gamma \vdash x : X}{\Gamma \vdash f(x) : Y}$$

7
Sums and products

The sum type constructor takes two types A, B : Type and combines it in a single type $A + B$: Type, which can be understood as a type containing the terms of both A and B. A term of $A + B$ will therefore correspond to either a term of A or a term of B.

This means that there is some map from each of those type to the sum type, denoted

$$\iota_A : A \to A + B \tag{7.1}$$
$$\iota_B : B \to A + B \tag{7.2}$$

We therefore have the formation rule that given two types, there exists a sum type for those types

$$\frac{\Gamma \vdash A : \text{Type} \qquad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A + B : \text{Type}}$$

and given this, we also have the existence of our injection maps

Example 7.0.1. If we consider the type of even number Even and odd numbers Odd, then integers are the sum type of both :

$$\mathbb{N} \equiv \text{Even} + \text{Odd} \tag{7.3}$$

The sum of two types gives us a pair of the individual types, ie

$$a : A, b : B \vdash (a, b) : A \times B \tag{7.4}$$

Dependent sum : the type of the second element might depend on the value of the first.

$$\sum_{n:\mathbb{N}} \mathrm{Vect}(\mathbb{R}, n) \tag{7.5}$$

Dually to the sum type is the product type, where given two types A, B, we have the product type $A \times B$.

8
Inductive type

One of the primary construction to create a type is that of *induction*, by which we define a type by declaring the existence of a term in that type, and by declaring functions mapping terms of that type to other terms.

There are two different ways to deal with inductive types

[28]

[...]

The classic example of an inductive type is given by the integers $\mathbb{N}$, which is a type constructed inductively by a single object $0 \in \mathbb{N}$ and a function type

$$S : \mathbb{N} \to \mathbb{N} \tag{8.1}$$

9

Martin-Löf type theory

1

The basis for our type theory will usually be some Martin-Löf type theory[29], which corresponds to intuitionistic logic in the trinitarianism view, and is generally a rather universal sort of approach to logic. This will be the basic form of logic for any topos later on.

A few rules exist which are entirely independent from the specific types we will define later on. These are the *structural rules*, which tell us how the judgements we do depend on the context. While obvious enough in a context of classical logic, these are not in fact universal in type theory, and will in fact be broken in the case of quantum logic later on.

First is the identity rule, that a formula entails itself :

$$A \vdash A \text{ (I)}$$

The weakening rules are given by the property that additional context preserves judgement. For the *left weakening rule*, we add additional context to the antecedent :

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (WL)}$$

In other words, an additional hypothesis does not change the validity of the deduction. Conversely we have the *right weakening rule*

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ (WR)}$$

The contraction rules allow us to remove duplicated context without changing judgement. For the *left contraction rule*,

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (CL)}$$

right contraction rule,

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ (CR)}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}$$

There are three basic types for it, called the *finite types* :

- The zero type $\mathbf{0}$, or empty type $\varnothing$ or $\bot$, which contains no terms.
- The one type $\mathbf{1}$, or unit type $*$, which contains one canonical term.
- The two type $\mathbf{2}$, which contains two canonical term.

Formation rule :

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbf{1} : \text{Type}}$$

Introduction rule :

$$\frac{\Gamma \vdash}{\Gamma \vdash * : \mathbf{1}}$$

Empty type for nothingness, something that doesn't exist

Unit type for existence

Two type for a choice between two values, such as boolean values.

As with any type theory, those types also give us function types

$\nvdash \to \nvdash$: the empty function (no term or 1 term?) $\nVdash \to \nVdash : \nVdash \to \nvDash$: Two functions (can be interpreted as some boolean?)

$\nvDash \to \nvDash$: unary boolean functions

In addition to these types, we have a variety of *type constructors*, which allow us to construct additional types from those basic types.

First are the constructions which just combine different types together. These are given by the sum type and the product type.

Indexed sets

Equality type

Inductive type

10

Classical logic as a type theory

A useful example of a type theory is that of classical logic. There are a few different methods to implement logic in type theory. We will use the idea of *propositions as types*, where any type can be evaluated as a proposition, where from the various constructions we have seen, we can ask "Is there any term in a type constructed in such a way?". Any term of a type is therefore called a *witness* of the proposition, a proof that the proposition is true, simply by giving an example of the proposition being true.

The basic two propositions for this are simply the two constant types, the empty type 0 and the unit type 1. The empty type 0 has no term and is therefore always false, and the type 1 always has a term, and is therefore true. We have the correspondence $1 \sim \top$ and $0 \sim \bot$. To consider the truth of any other proposition, we define its propositional truncation :

Definition 10.0.1. The *support* of a type A, or its *bracket type*, denoted by $[A]$, is a type that is the unit type if A contains a term, and the empty type otherwise. Its formation rule is that any type A defines a bracket type :

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash [A] : \text{Type}}$$

Its introduction rules is that if A has a term, $[A]$ has a term :

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma,\ x : A \vdash [x] : [A]}$$

and that for any two terms of A, the corresponding term of $[A]$ is the same (their equality type is inhabited by a term) :

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma,\ x : A,\ y : A \vdash \text{trunc}_A(x, y) : ([x] =_{[A]} [y])}$$

Its elimination rule is that given a type A and dependent type $B(a)$ for a term of A,

Given this, we can consider the product and sum types.

Theorem 10.0.1. The sum types of 0 and 1 are the following :

$$[0 + 0] = [0] \tag{10.1}$$
$$[0 + 1] = [1] \tag{10.2}$$
$$[1 + 0] = [1] \tag{10.3}$$
$$[1 + 1] = [1] \tag{10.4}$$
$$\tag{10.5}$$

Proof. □

Theorem 10.0.2. The product types of 0 and 1 are the following :

$$[0 \times 0] = [0] \tag{10.6}$$
$$[0 \times 1] = [0] \tag{10.7}$$
$$[1 \times 0] = [0] \tag{10.8}$$
$$[1 \times 1] = [1] \tag{10.9}$$
$$\tag{10.10}$$

Proof. □

This makes the sum and product of truncated types the equivalent of *or* and *and*.

Definition 10.0.2. The negation of a type is the function type into the empty type :

$$\neg A : A \to 0 \tag{10.11}$$

Definition 10.0.3. An implication between two type is simply their function type :

$$(A \to B) : A \to B \tag{10.12}$$

Theorem 10.0.3. Ex falso quodlibet : A falsehood entails anything :

$$\Gamma \vdash 0 \to A \tag{10.13}$$

Proof. □

The basic type in classical logic is the *boolean type*, which is the two type we saw in Martin-Löf type theory. This is the type

11
Homotopy types

[30]

A further refinement of type theory is the notion of *homotopy type*, where in addition to identity types, we also include the more general notion of equivalence types, which can be thought of as equality types with possibly more than one term.

Definition 11.0.1.

Definition 11.0.2. Two types A, B : Type are said to be *equivalent*, denoted $A \cong B$, if there exists an equivalence between them.

$$(A = B) \to (A \cong B) \tag{11.1}$$

Univalence axiom :

$$(A = B) \cong (A \cong B) \tag{11.2}$$

Correspondence between type theory and category theory :

- A universe of types is a category
- Types are objects in the category $T \in \mathrm{Obj}(C)$
- A term $a : A$ of A is a generalized element of A

- The unit type $*$ if present is the terminal object
- The empty type $\varnothing$ if present is the initial object
- A dependent type $x : A \dashv B(x) : \mathrm{Type}$ is a display morphism $p : B \to A$, the fibers $p^{-1}(a)$ being the dependent type at $a : A$.

12
Modalities

Definition 12.0.1. A *modality* $\square$ on a type theory is a unary operator between two types

$$\square : \text{Type} \to \text{Type} \tag{12.1}$$

along with a modal unit Induction principle Computation rule Equivalence

The classic example of a modal theory is that of the necessity monad, or S4 modal logic.

Introduction rule :

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

Elimination rule :

$$\frac{\Gamma \vdash \Diamond A \qquad \Gamma, A \vdash \Diamond B}{\Gamma \vdash \Diamond B}$$

13
Interpretation

Notions and ideas?

Part II

Categories

As is often the case in foundational issues in mathematics, the foundations used to define mathematics can easily become circular. In our case, although category theory can be used to define set theory and classical logic, as well as your other typical foundational field like model theory, type theory, computational theory, etc, we still need those concepts to define category theory itself. In our case we will simply use implicitly classical logic (what we will call the *external logic*, as opposed to the internal logic of a category we will see later on) and some appropriate set theory like ETCS[31] (as ZFC set theory will typically be too small to talk of important categories). We will not get too deeply into this, but it can be an important issue.

Definition 13.0.1. A *category* $\mathbf{C}$ is a structure composed of a class of objects $\mathrm{Obj}(\mathbf{C})$ and a class of morphisms $\mathrm{Mor}(\mathbf{C})$ such that

- There exists two functions $s, t : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{C})$, the *source* and *target* of a morphism. If $s(f) = X$ and $t(f) = Y$, we write the morphism as $f : X \to Y$.
- For every object $X \in \mathrm{Obj}(\mathbf{C})$, there exists a morphism $\mathrm{Id}_X : X \to X$, such that for every morphism g_1 with $s(g_1) = X$ and every morphism g_2 with $t(g_2) = X$, we have $g_1 \circ \mathrm{Id}_X = g_1$ and $\mathrm{Id}_X \circ g_2 = g_2$.
- For any two morphisms $f, g \in \mathrm{Mor}(\mathbf{C})$ with s

To simplify notation, if there is no confusion possible, we will write the set of objects and the set of morphisms as the category itself, ie :

$$X \in \mathrm{Obj}(\mathbf{C}) \quad := \quad X \in \mathbf{C} \tag{13.1}$$

$$f \in \mathrm{Mor}(\mathbf{C}) \quad := \quad f \in \mathbf{C} \tag{13.2}$$

Categories are often represented, in totality or in part, by *diagrams*, a directed graph in which objects form the nodes and morphism the edges, such that the direction of the edge goes from source to target. For instance, if we consider some category of two objects A, B with some morphism f with $s(f) = A$ and $t(f) = B$, we can write it as (neglecting the implicit identity morphisms)

$$A \xrightarrow{f} B$$

We will also use a lot the function notation, where this morphism is denoted by $f : A \to B$.

Throughout this section we will use a variety of common categories for examples. Some of them will be seen in more details later on V, using all the tools we have accumulated. For now, we will just mostly make our intuition on those categories by either considering categories with sets and functions for objects and morphisms, or elements and partial order relations.

Before we go on detailing examples of categories, first a quick note on skeletonized categories. It is common in category theory to more or less assume the identity of objects that are isomorphic (we will see the exact definition of isomorphism later on but we can assume the usual definition here). This is not necessarily the case formally speaking (the category of sets can be seen as having multiple isomorphic sets in it, like $\{0,1\}$, $\{1,2\}$, etc), but for some purposes (such as trying to get a broad view of that category) it will be useful to consider the category where the set of objects is given as the equivalence class up to isomorphism.

Definition 13.0.2. A category $\mathbf{C}$ is *skeletal* if all isomorphisms are identities, and the *skeleton* of a category $\mathbf{C}$, written $\mathrm{sk}(\mathbf{C})$, is given by the equivalence class of objects up to their isomorphisms, ie

$$\mathrm{Obj}(\mathrm{sk}(\mathbf{C})) = \{[X] \mid \forall X, X' \in [X],\ \exists f \in \mathrm{iso}(\mathbf{C}),\ f : X \to X'\} \tag{13.3}$$

It will be pretty typical as we go on to implicitely consider the skeletal version of whatever category we talk about, as we will talk about *the* set of one element, *the* vector space of n dimensions, etc. If not specified, just assume that it is implicitly "up to isomorphism".

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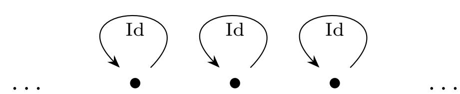
Examples

A few categories can easily be defined in categorical terms alone, such as the *empty category* **0**, which is the category with no objects and no morphisms (with the empty diagram as its diagram). We also have the *discrete categories* **n** for $n \in \mathbb{N}$, which consist of all the categories of exactly n objects and n morphisms (the identities of each object)



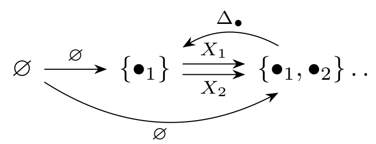
The empty category is in fact itself the discrete category **0**.

Many categories can also be defined using typical mathematical structures built on set theory, using sets as objects and functions as morphisms.

As long as such a structure is closed under function composition and possessing the identity function, it will obey all the properties of 13.0.1, with its objects the sets, its morphisms the functions, s, t the domain and codomain of the functions.

Example 14.0.1. The category of sets **Set** has as its objects all sets (Obj(**Set**) is the *class* of all sets), and as its morphisms the functions between those sets.

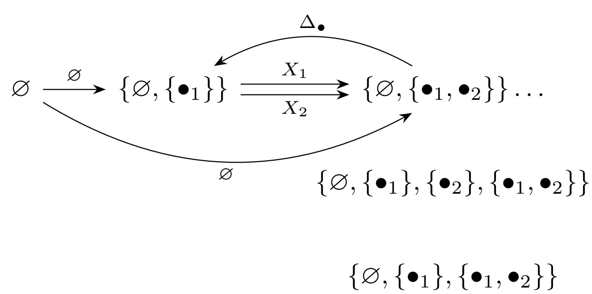
If we consider the skeletonized version of **Set**, where we only consider unique sets of a given cardinality, its diagram will look something like this for the first three elements classified by cardinality :



where $\varnothing$ is the empty function, X_1, X_2 are the functions that map the only element of $\{\bullet_1\}$ to either $\bullet_1$ or $\bullet_2$, and $\Delta_\bullet$ maps all elements to $\bullet_1$. The rest of the elements of the category follow similar patterns.

Example 14.0.2. The category of topological spaces **Top** has as its objects the topological spaces (X, τ_X), and as morphisms $f : (X, \tau_X) \to (Y, \tau_Y)$ the continuous functions between two such spaces.

As isomorphic sets can have different topologies, we can see more branching in **Top**. If we denote topologies on finite sets by their open sets, the first few elements of **Top** look like

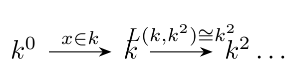


starting with the empty topological space, the singleton topology, and the topologies on the set of two elements : the trivial, discrete and Sierpinski topology.

Example 14.0.3. The category of vector spaces $\mathbf{Vec}_k$ over a field k has as its objects the vector spaces over k, and as its morphisms the linear maps between them. The hom-set between V_1 and V_2 is therefore the set of linear maps $L(V_1, V_2)$ (see later for enriched category)

Diagram : finite dimensional case classified by dimension

For a few examples of the diagram of $\mathbf{Vec}_k$, we can look at the vector spaces of its first few dimensions (up to isomorphism), with the morphisms $k^0 \to k^n$ being only the zero linear map 0, $k^n \to k^0$ likewise being the zero linear map 0, the morphisms $k^1 \to k^2$ being isomorphic to the vectors of k^2, while the morphisms $k^2 \to k_1$ are the linear maps $L(k^2, k^1)$



Example 14.0.4. The category of rings **Ring** has as its objects rings, and as morphisms ring homomorphisms.

Example 14.0.5. The category of groups **Grp** has as its objects groups G, and as its morphisms group homomorphisms.

The first few groups of the category are the trivial group and the two cyclic groups,

$$\{e\} \underset{0}{\overset{0}{\rightleftarrows}} C_2 \longrightarrow C_3$$

As any group homomorphism must map identity elements together, there is only one such morphism from the trivial group to any other group, and likewise all objects have a unique map to the trivial group. This leaves the homomorphisms between C_2 and C_3, which are counted by $\gcd(2,3)$ and $\gcd(3,2)$, in this case one homomorphism each,

An important category for geometry is the one given by Cartesian spaces. There's a few different ways this can be interpreted. The objects can be either the real spaces $\mathbb{R}^n$ themselves, the open subsets of $\mathbb{R}^n$, or open subsets obeying certain properties, such as connectedness, simple connectedness, etc. The morphisms can be continuous maps, smooth maps, etc.

Choice :

Example 14.0.6. The category of smooth Cartesian spaces $\mathbf{CartSp}_{\mathrm{Smooth}}$ has as its objects the real spaces $\mathbb{R}^n$, and as morphisms smooth maps between them.

Besides concrete categories, another common type of category is *partial orders*, which are defined as usual in terms of sets, ie a partial order $(X, \leq)$ is a set X with a relation $\leq$ on $X \times X$, obeying

- Reflexivity :
$$\forall x \in X,\ x \leq x$$
- Antisymmetry :
$$\forall x, y \in X,\ x \leq y \wedge y \leq x \to x = y$$
- Transitivity :
$$\forall x, y, z \in X,\ x \leq y \wedge y \leq z \to x \leq z$$

As a category, a partial order is simply defined by $\mathrm{Obj}(\mathcal{C}) = X$. Its morphisms are defined by the relation $\leq$: if $x \leq y$, there is a unique morphism between x and y, which we will call $\leq_{x,y}$ formally, or simply $\leq$ if there is no risk of confusion. Otherwise, there is no morphisms between the two. This obeys the categorical axioms for morphisms as the identity morphism Id_x is simply given by reflexivity, $\leq_{x,x}$, and obeys the triangular identities by transitivity :

$$\leq_{x,x} \circ \leq_{x,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq x \wedge x \leq y \to x \leq y \tag{14.1}$$

$$\leq_{x,y} \circ \leq_{y,y} = \leq_{x,y} \quad \leftrightarrow \quad x \leq y \wedge y \leq y \to x \leq y \tag{14.2}$$

Example 14.0.7. Given a topological space (X, τ), its *category of opens* $\mathbf{Op}(X)$ is given by its set of open sets τ with the partial order of inclusion $\subseteq$.

Example 14.0.8. The integers $\mathbb{Z}$ as a totally ordered set $(\mathbb{Z}, \leq)$ forms a category.

$$\ldots \xrightarrow{\leq} -2 \xrightarrow{\leq} -1 \xrightarrow{\leq} 0 \xrightarrow{\leq} 1 \xrightarrow{\leq} 2 \xrightarrow{\leq} \ldots$$

Other total orders of interest are the rational numbers $(\mathbb{Q}, \leq)$ and real numbers $(\mathbb{R}, \leq)$.

Another type of such categories is the *simplicial category* Δ. There are a few different interpretation for it, one of them being that of totally ordered finite sets with monotone functions between them, ie

$$\forall f : \vec{m} \to \vec{n},\ f(m_i \to m_j) = f(m_i) \to f(m_j) \tag{14.3}$$

or alternatively, as its equivalent poset category, where each object of Δ is the finite total order $\vec{\mathbf{n}}$.

Example 14.0.9. The *interval category* I is composed of two elements $\{0, 1\}$ and a morphism $0 \to 1$.

$$I = \{0 \to 1\} \tag{14.4}$$

Since categories are constructed to be mathematical structures, there is in fact a category of categories, **Cat**. To avoid any Russell-type paradox, we will only consider specific kinds of categories here, typically the category of small categories, where $\mathrm{Obj}(\mathbf{C})$ is small enough to be a set, and larger categories fit into some hierarchy. As the category of categories require a bit more machinery from category theory, we will look at its definition in more details later on.

As our categories are simply built from sets, one thing we can do is also construct a product of categories.

Definition 14.0.1. A product of two categories $\mathbf{C} \times \mathbf{D}$ is the category given by the objects

$$\mathrm{Obj}(\mathbf{C} \times \mathbf{D}) = \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{D}) \tag{14.5}$$

with the morphisms

$$\mathrm{Mor}(\mathbf{C} \times \mathbf{D}) = \mathrm{Mor}(\mathbf{C}) \times \mathrm{Mor}(\mathbf{D}) \tag{14.6}$$

with every object a pair (X, Y) and every morphism (f, g), such that the source and target obey

$$s((f,g)) = (s(\pi_1((f,g))), s(\pi_2((f,g)))) \quad (14.7)$$
$$t((f,g)) = (t(\pi_1((f,g))), t(\pi_2((f,g)))) \quad (14.8)$$
$$(14.9)$$

or more succinctly,

$$s((f,g)) = (s(f), s(g)) \quad (14.10)$$
$$t((f,g)) = (t(f), t(g)) \quad (14.11)$$
$$(14.12)$$

While we can do this comfortably outside the categorical framework here simply with set theory, it is also something that we will be able to do internally later on as in fact a product category is given by a product 20.4 in the category of categories, **Cat**.

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Morphisms

As our categories are fundamentally built from sets and classes, we can look at specific subsets of our set of morphisms, $\mathrm{Mor}(\mathbf{C})$, with specific properties.

A simple example is simply the set of all morphisms between two objects, called the hom-set :

Definition 15.0.1. The *hom-set* between two objects $X, Y \in \mathbf{C}$ is the set of all morphisms between X and Y :

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathrm{Mor}(\mathbf{C}) \mid s(f) = X,\ t(f) = Y\} \tag{15.1}$$

This notion is more rigorously only valid for what are called *locally small categories*, where the cardinality of $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is small enough to be a set. If the cardinality is too large, and could only fit in say a class, it is more properly described as a *hom-object*. This will not affect our discussion much as most categories of interest are locally small.

Example 15.0.1. The hom-set on the category of sets $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the set of all functions between X and Y

Example 15.0.2. The hom-set on the category of vector spaces $\mathrm{Hom}_{\mathbf{Vec}}(X, Y)$ is the set of all linear maps between X and Y, also known as $L(X, Y)$.

Example 15.0.3. The hom-set on the category of topological spaces $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ is the set of all continuous functions between X and Y, also known as $C(X, Y)$.

A specific type of hom-set is the set of endomorphisms of an object.

Definition 15.0.2. The endomorphisms of an object X are the morphisms in the hom-set

$$\mathrm{End}(X) = \mathrm{Hom}_{\mathbf{C}}(X, X) \tag{15.2}$$

We can also define an induced function given an object between hom-sets. For a morphism $f : X \to Y$, the induced function f_* on the hom-sets for any Z

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \to \mathrm{Hom}_{\mathbf{C}}(Z, Y) \tag{15.3}$$

is defined by composition. In other words, if we have a function $g : Z \to X$, we can map it to a function $g' : Z \to Y$ by post-composition with f :

$$g' = f_*(g) = f \circ g \tag{15.4}$$

Likewise, we have a similar induced function by pre-composition, defined by

$$\begin{aligned} f^* : \mathrm{Hom}_{\mathbf{C}}(Y, Z) &\to \mathrm{Hom}_{\mathbf{C}}(X, Z) && (15.5) \\ g &\mapsto f^*(g) = g \circ f && (15.6) \end{aligned}$$

These will be the transformations induced by the hom-functors 17.1 later on.

15.1 Monomorphisms

Definition 15.1.1. A *monomorphism* $f : X \to Y$ is a morphism such that, for every object Z and every pair of morphisms $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \to g_1 = g_2 \tag{15.7}$$

We say that a monomorphism is *left-cancellable*.

We will write the set of all monomorphisms as Mono(**C**),

$$\mathrm{Mono}(\mathbf{C}) \subseteq \mathrm{Mor}(\mathbf{C}) \tag{15.8}$$

Example 15.1.1. on **Set**, a monomorphism is an injective function.

Proof. If $f : X \to Y$ is a monomorphism, then given two elements $x_1, x_2 \in X$, represented by morphisms $x_1, x_2 : \{\bullet\} \to X$, then we have

$$f \circ x_1 = f \circ x_2 \to x_1 = x_2 \tag{15.9}$$

So that we can only have the same value if the arguments are the same, making it injective. Conversely, if f is injective, take two functions $g_1, g_2 : Z \to X$. As the functions in sets are defined by their values, we need to show that

$$\forall z \in Z,\ f(g_1(z)) = f(g_2(z)) \to g_1(z) = g_2(z) \tag{15.10}$$

By injectivity, the only way to have $f(g_1(z)) = f(g_2(z))$ is that $g_1(z) = g_2(z)$. As this is true for every value of z, $g_1 = g_2$. □

Example 15.1.2. In **Top**, examples of monomorphisms are topological subspaces, and the identity map (set-wise) from a topological space to the same space with a coarser topology.

Proof. □

Theorem 15.1.1. A monomorphism is equivalent to the fact that the induced function f_* is injective on hom-sets :

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \hookrightarrow \mathrm{Hom}_{\mathbf{C}}(Z, Y) \tag{15.11}$$

Proof. If $f : X \to Y$ is a monomorphism, then for some hom-set $\mathrm{Hom}_{\mathbf{C}}(Z, X)$, we need to show that given two functions g_1, g_2 in there, the elements of $\mathrm{Hom}_{\mathbf{C}}(Z, Y)$ obtained by the induced map f_* obey

$$f_*(g_1) = f_*(g_2) \to g_1 = g_2 \tag{15.12}$$

This is true by the definition of f_* and left cancellability. Conversely, if f_* is an injective function on $\mathrm{Hom}_{\mathbf{C}}(Z, X)$, we have that □

Theorem 15.1.2. The composition of two monomorphisms is a monomorphism.

Proof. Given two composable monomorphisms $f : X \to Y$ and $g : Y \to Z$, we must show that the composition $g \circ f : X \to Z$ is a monomorphism, ie for any two morphisms $h_1, h_2 : W \to X$, we have

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \to h_1 = h_2 \tag{15.13}$$

By associativity, we have that this is equivalent to

$$g \circ (f \circ h_1) = g \circ (f \circ h_2) \tag{15.14}$$

and by the mono status of g, this means that $f \circ h_1 = f \circ h_2$. And likewise, as f is a monomorphism, $h_1 = h_2$. □

It is tempting to try to view monomorphisms as generalizing injections, but categories may lack elements on which to define such notions and even if it does possess elements, monomorphisms may fail to be injective.

Counterexample 15.1.1. Take the category of divisible abelian groups, whose objects are Abelian groups G for which for any positive integer $n \in \mathbb{N}$ and any group element $g \in G$, we have existence of some other element h such that

$$h^n = g \tag{15.15}$$

so that any element is the sum of some element arbitrarily many times. Its morphisms are there is a monomorphism from $\mathbb{Q}$ to $\mathbb{Q}/\mathbb{Z}$,

$$\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \tag{15.16}$$

given by

15.2 Epimorphisms

Definition 15.2.1. An *epimorphism* $f : X \to Y$ is a morphism such that, for every object Z and every pair of morphisms g_1, g_2, we have

$$g_1 \circ f = g_2 \circ f \to g_1 = g_2 \tag{15.17}$$

we say that an epimorphism is *right-cancellable*.

We will write the set of all epimorphisms as $\mathrm{Epi}(\mathbf{C})$,

$$\mathrm{Epi}(\mathbf{C}) \subseteq \mathrm{Mor}(\mathbf{C}) \tag{15.18}$$

Example 15.2.1. On **Set**, an epimorphism is a surjective function.

Proof. □

Theorem 15.2.1. The composition of two epimorphisms is an epimorphism.

Proof. Given two composable epimorphisms $f : X \to Y$ and $g : Y \to Z$, we must show that the composition $g \circ f : X \to Z$ is an epimorphism, ie for any two morphisms $h_1, h_2 : Z \to W$, we have

$$h_1 \circ (g \circ f) = h_2 \circ (g \circ f) \to h_1 = h_2 \tag{15.19}$$

By associativity, we have that this is equivalent to

$$(h_1 \circ g) \circ f = (h_2 \circ g) \circ f \tag{15.20}$$

and by the fact that f is an epimorphism, this means that $g \circ h_1 = g \circ h_2$. And likewise, as g is an epimorphism, $h_1 = h_2$. □

15.3 Isomorphisms

Definition 15.3.1. An *isomorphism* $f : X \to Y$ is a morphism with a two-sided inverse, ie there exists a morphism $f^{-1} : Y \to X$ such that

$$f \circ f^{-1} = \mathrm{Id}_Y \quad (15.21)$$
$$f^{-1} \circ f = \mathrm{Id}_X \quad (15.22)$$

We will write the set of all isomorphisms as $\mathrm{Iso}(\mathbf{C})$,

$$\mathrm{Iso}(\mathbf{C}) \subseteq \mathrm{Mor}(\mathbf{C}) \quad (15.23)$$

Example 15.3.1. In the category of sets, isomorphisms are bijections.

Proof. This is the usual proof of bijective functions having an inverse.

For $f : X \to Y$ an injective and surjective function, define the function $f^{-1} : Y \to X$ to be such that, for any $y \in Y$, we associate the value x such that $f(x) = y$, which exists as f is surjective. And since f is injective, this x is unique, so that f^{-1} is well-defined.

Conversely, if $f : X \to Y$ has an inverse $f^{-1} : Y \to X$, for any element of y there is a corresponding element of x mapped onto x, $f^{-1}(y)$, as $f(f^{-1}(y)) = y$, making it surjective, and for any two elements x_1, x_2 of X, if we have $f(x_1) = f(x_2)$, f^{-1} fails to be defined at that point since it will fail to have a unique image, meaning that by contradiction, f is injective. □

Example 15.3.2. In the category of topological spaces, isomorphisms are homeomorphisms.

Proof. □

Example 15.3.3. In the category of smooth manifolds, isomorphisms are diffeomorphisms.

Despite the most typical examples, it is not in general true that a morphism that is both a monomorphism and an epimorphism is an isomorphism.

Counterexample 15.3.1. Take the Sierpinski category $0 \to 1$. Its unique non-trivial morphism $f : 0 \to 1$ is mono (the only other morphism $0 \to 0$ is the identity), and epi (same for $1 \to 1$), but there does not even exist a morphism $1 \to 0$.

Counterexample 15.3.2. Continuous bijections aren't homeomorphisms.

A specific type of isomorphisms are the isomorphic endomorphisms, called the *automorphisms*. Together, they are called the core of an object.

Definition 15.3.2. For a given object X, the subset of all its endomorphisms which are isomorphisms are called its *core* :

$$\mathrm{core}(X) = \mathrm{Iso}_{X,X}(\mathbf{C}) \tag{15.24}$$

Corrolary 1. In a poset category, the core of an object is always the identity morphism.

Theorem 15.3.1. The core of an object has a group structure.

Proof. If we pick as the set of the group $\mathrm{Core}(X)$, as its group product the composition of morphisms $\circ$, as unit the identity morphism and as inverse the function mapping a morphism to its inverse, we have that this fulfills all the group axioms. □

Example 15.3.4. The core of a set is its symmetric group

$$\mathrm{core}(X) = S_X \tag{15.25}$$

Example 15.3.5. The core of a smooth manifold is its diffeomorphism group

$$\mathrm{core}(M) = \mathrm{Diff}(M) \tag{15.26}$$

Definition 15.3.3. A category **C** for which any morphism that is both mono and epi is an isomorphism is called *balanced.*

Example 15.3.6. Set is balanced.

Proof. This simply stems from the equivalence of monos with injections and epis with surjections. □

15.4 Properties

Theorem 15.4.1. For f, g two morphisms that can be composed as $g \circ f$, if $g \circ f$ and g are monomorphisms, then so is f.

Proof. For f to be a monomorphism, we need

$$f \circ h_1 = f \circ h_2 \to h_1 = h_2 \tag{15.27}$$

we can compose this with g to obtain

$$g \circ (f \circ h_1) = g \circ (f \circ h_2) \tag{15.28}$$

and by associativity and given that $g \circ f$ is a monomorphism, this leads to

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \to h_1 = h_2 \tag{15.29}$$

□

As monomorphisms represent embeddings and inclusions, this means in particular that if we have the inclusion $S \hookrightarrow X$ and

15.5 Split morphisms

From 15.3.1, we've seen that the notions of epis and monos do not generally directly correspond to injective and surjective functions, even in the case where objects are actual sets, but we do have a more accurate categorical notion for these, which are split epis and monos.

Definition 15.5.1. A *split monomorphism* $f : X \to Y$ is a monomorphism possessing a left inverse $r : Y \to X$, called its *retraction*, so that

$$r \circ f = \mathrm{Id}_X \tag{15.30}$$

Conversely, we say also that r is a *section* of f.

Theorem 15.5.1. The backward composition of the split monomorphism (f, r) is idempotent, so that for the morphism $f \circ r : Y \to Y$, we have

$$(f \circ r)^{\circ n} = (f \circ r) \tag{15.31}$$

Proof. By recursion, this is simply true for $n = 1$, and for $n + 1$, we have

$$\begin{aligned}
(f \circ r)^{\circ(n+1)} &= (f \circ r)^{\circ(n-1)} \circ (f \circ (r \circ f) \circ r) && (15.32)\\
&= (f \circ r)^{\circ(n-1)} \circ (f \circ \mathrm{Id}_X \circ r) && (15.33)\\
&= (f \circ r)^{\circ n} && (15.34)\\
&= f \circ r && (15.35)
\end{aligned}$$

□

Example 15.5.1. In the category of vector spaces **Vec**, any monomorphism is split. The monomorphism is the inclusion map of a subspace $\iota : W \hookrightarrow V$, and its retract can be constructed by some projection P onto that subspace. More strictly, if we consider the projection $P : V \to V$ of any point in V onto that subspace, and define our vector space via the direct sum $V = W \oplus U$ for a complementary subspace U with projection $1 - P$, the retraction is given by the projection onto the first element of this product.

$$r = p_1 \tag{15.36}$$

The idempotency of the morphism $f \circ r$ stems from that of the projection

Example 15.5.2. In the category of sets **Set**, any monomorphism is split. The monomorphism is a subset relation $\iota : S \hookrightarrow X$, and its retract r is a given choice function, ie some function defined as

$$\forall x \in X,\ \exists s \in S,\ r(x) = s \tag{15.37}$$

which is guaranteed by the axiom of choice. [diagram]

Definition 15.5.2. A *split epimorphism* $f : X \to Y$ is an epimorphism possessing a right inverse $s : Y \to X$, called a *section*, so that

$$f \circ s = \mathrm{Id}_Y \tag{15.38}$$

Example 15.5.3. Given a bundle between two topological spaces $\pi : E \to B$, a section is a function s mapping points $x \in B$ to points of $E_x = \pi^{-1}(X)$. This defines a subspace $s(B) \subseteq E$.

Theorem 15.5.2. A morphism that is both split epi and split mono is an isomorphism.

Proof. If $f : X \to Y$ is split epi and split mono, there is a section $g : Y \to X$ for which $f \circ g = \mathrm{Id}_Y$. Applying f on the left, we have

$$f \circ g \circ f = f \tag{15.39}$$

and furthermore, $f = f \circ \mathrm{Id}_X$. Since f is a monomorphism, it is left cancellable, meaning that

$$f \circ g \circ f = f \circ \mathrm{Id}_X \to g \circ f = \mathrm{Id}_X \tag{15.40}$$

meaning that g is a left inverse in addition to a right inverse, making it a double sided inverse of f. ☐

Left-unique, left-total?

15.6 Subobjects

As monomorphisms are similar to injective functions, it is tempting to see a way to define subobjects with them. To formalize this notion, we define subobjects that way

Definition 15.6.1. A subobject is an equivalence class of monomorphisms to the same object up to isomorphism,

$$[S] \in \mathrm{Sub}_{\mathbf{C}}(X) \leftrightarrow [S] = \{S' \in \mathbf{C} \mid \exists f \in \mathrm{Iso}(\mathbf{C}),\ f : S \to S'\} \tag{15.41}$$

Equivalently, monomorphism in the skeletal category?

Example 15.6.1. A subobject in **Set** is a subset up to isomorphism, ie the equivalence class of sets of some lower cardinality, where each specific subset is related to each other by the symmetric group.

Example 15.6.2. In **Top**, two examples of subobjects are the subspaces,

$$\iota : S \to X \tag{15.42}$$

and the continuous inclusion of a topological space in a space of coarser topology

$$\forall \tau_1 \subseteq \tau_2,\ (X, \tau_2) \hookrightarrow (X, \tau_1) \tag{15.43}$$

Example 15.6.3. A subobject in **Vec** is a subspace up to isomorphism, $\iota : W \hookrightarrow V$.

Example 15.6.4. A subobject in SmoothMan is a submanifold, up to diffeomorphism. This is in particular the definition of a path (as opposed to a curve),

$$[\gamma] = \gamma/\mathrm{Diff}(I) \tag{15.44}$$

Definition 15.6.2. The *image* of a morphism $f : X \to Y$, denoted $\mathrm{im}(f)$, if it exists, is the smallest subobject of Y which factors through f, ie

$$X \overset{f|^{\mathrm{im}(f)}}{\longrightarrow} \mathrm{im}(f) \overset{\iota_{\mathrm{im}(f)}}{\longrightarrow} Y \tag{15.45}$$

Example 15.6.5. In **Set**, this is the ordinary notion of image, where for a function $f : X \to Y$, we can consider the corestriction $f|^{\mathrm{im}(f)}$

For instance, the function $(-)^2 : \mathbb{R} \to \mathbb{R}$ has image $\mathrm{im}((-)^2) = \mathbb{R}_{\geq 0}$, so that we can decompose it as

$$\mathbb{R} \overset{(-)^2|^{\mathbb{R}_{\geq 0}}}{\longrightarrow} \mathbb{R}_{\geq 0} \overset{\iota_{\mathbb{R}_{\geq 0}}}{\longrightarrow} \mathbb{R} \tag{15.46}$$

15.7 Quotient objects

Dual to the subobjects are the quotient objects :

Definition 15.7.1. A *quotient object* Q of an object X is an equivalence class of epimorphisms with source X,

$$q : X \to Q \tag{15.47}$$

where two epimorphisms q, q' are equivalent if there is an isomorphism $f : X \to X'$ such that $q = q' \circ f$.

As the name implies, a quotient object can be defined in terms of a quotient. We will see the categorical notion of a quotient later on 20.6, but for now we can use the standard set-theoretic definition of a quotient as an example.

Example 15.7.1. In $\mathbb{S}\approx$, a quotient object is given by the quotient map

$$q : X \to Q \tag{15.48}$$

which is a surjective function

Example 15.7.2. In $\mathbb{T}\rtimes\shortmid$, a quotient object is

16

Opposite categories

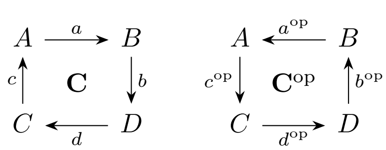
Definition 16.0.1. The *opposite* of a category $\mathbf{C}$, denoted $\mathbf{C}^{\text{op}}$, is a category for which the two categories share the same objects and morphisms

$$\text{Obj}(\mathbf{C}) = \text{Obj}(\mathbf{C}^{\text{op}}) \tag{16.1}$$
$$\text{Mor}(\mathbf{C}) = \text{Mor}(\mathbf{C}^{\text{op}}) \tag{16.2}$$

but for which the source and target map are reversed. That is,

$$s_{\mathbf{C}} = t_{\mathbf{C}^{\text{op}}} \tag{16.3}$$
$$t_{\mathbf{C}} = s_{\mathbf{C}^{\text{op}}} \tag{16.4}$$

Simply speaking, the opposite category is the category for which all arrows are reversed in terms of diagrams.



Theorem 16.0.1. The opposite of a category is an involution,

$$(\mathbf{C}^{\text{op}})^{\text{op}} = \mathbf{C} \tag{16.5}$$

Proof.

$$s_{(\mathbf{C}^{\text{op}})^{\text{op}}} = t_{\mathbf{C}^{\text{op}}} = s_{\mathbf{C}} \tag{16.6}$$
$$t_{(\mathbf{C}^{\text{op}})^{\text{op}}} = s_{\mathbf{C}^{\text{op}}} = t_{\mathbf{C}} \tag{16.7}$$

□

Despite its rather simple definition, the semantics of opposite categories can be rather obscure, which will lead to some difficulties whenever definitions will involve them. This is because for many categories of interest, the morphisms are functions in some sense, and it is not generally clear what the inverse object for a function will be in the general case.

A good example of this is given by the opposite of **Set**, which is one of the simplest category. The more tractable case of this is to consider the category of finite sets, **FinSet**. If we have a function $f : A \to B$, what is the corresponding morphism $f^{\text{op}} : B \to A$ in a category? Functions are not generally invertible. If we attempt to use the preimage of f,

$$f^{-1} : B \to A \tag{16.8}$$

then f will map a point $b \in B$ to a subset of A, which is then not consistent : the objects A and B should be of the same type, otherwise we would run into issues trying to compose f^{-1} with g^{-1}. If we try to expand the opposite category so that each object is the power set of the object in the original category,

$$f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A) \tag{16.9}$$

Example 16.0.1. The opposite category of **FinSet** is equivalent to the category of finite boolean algebras.

Proof. □

Example 16.0.2. Any group interpreted as a category is the opposite group, for which any element g becomes its inverse g^{-1}.

Example 16.0.3. The opposite category of a poset $(X, \leq)$ is the poset of the reverse ordering relation, $(X, \geq)$.

Definition 16.0.2. A *quotient object* Q of an object X, with quotient morphism $q : X \to Q$, is a subobject in the opposite category $\mathbf{C}^{\text{op}}$.

17

Functors

Functors are roughly speaking functions on categories that preserve their structures. In more details,

Definition 17.0.1. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is a function between two categories $\mathbf{C}, \mathbf{D}$, mapping every object of $\mathbf{C}$ to objects of $\mathbf{D}$ and every morphism of $\mathbf{D}$ to morphisms of $\mathbf{D}$, such that categorical properties are conserved :

$$s(F(f)) = F(s(f)) \tag{17.1}$$
$$t(F(f)) = F(t(f)) \tag{17.2}$$
$$F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)} \tag{17.3}$$
$$F(g \circ f) = F(g) \circ F(f) \tag{17.4}$$

Example 17.0.1. The *identity functor* $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ maps every object and morphism to themselves.

Example 17.0.2. The *constant functor* $\Delta_X : \mathbf{C} \to \mathbf{D}$ for some object $X \in \mathbf{D}$ is the functor mapping every object in $\mathbf{C}$ to X and every morphism to Id_X.

Example 17.0.3. A functor between two partial order categories $(X, \leq)$ and $(Y, \leq)$ is simply an order-preserving function (or monotone function).

Example 17.0.4. For a category where the objects are sets and the morphisms are functions (such as **Top** or $\mathbf{Vect}_k$), the *forgetful functor* $U_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set}$ is the functor sending every object to their underlying set, and every morphism to their underlying function on sets.

A common functor is the *forgetful functor*, which maps a category that is composed of a set with extra structure its underlying set. For instance, the category **Top** has a forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$, which maps every topological space to its set, and every continuous function to the corresponding function on sets. If we have a set X and on it are all the different topologies (X, τ_i), then the forgetful functor maps

$$U((X, \tau_i)) = X \tag{17.5}$$

The forgetful functor on **Top** is obviously not injective, as two topological spaces with the same underlying set (such as any set of cardinality ≥ 1 with the discrete or trivial topology) will map to the same set.

Example : negation

Example 17.0.5. The skeletonization functor is a functor $\mathrm{Sk} : \mathbf{Cat} \to \mathbf{Cat}$ which maps categories to their equivalent skeleton category.

Definition 17.0.2. A *contravariant functor* is a functor from the opposite category, so that $F : C \to D$ is a contravariant functor equivalently to a functor $F : C^{\mathrm{op}} \to D$ is a functor. In particular, this changes the rules as

$$s(F(f)) = F(t(f)) \tag{17.6}$$
$$t(F(f)) = F(s(f)) \tag{17.7}$$
$$F(g \circ f) = F(f) \circ F(g) \tag{17.8}$$

Theorem 17.0.1. The composition of contravariant and covariant functors works as follow :

$$C \tag{17.9}$$

A generalization of functors is the notion of multifunctors (we mean here specifically the *jointly functorial* multifunctor)

14.0.1

Definition 17.0.3. A multifunctor $F : \prod_i \mathbf{C}_i \to \mathbf{D}$ is a function from a product of category to another category.

Theorem 17.0.2. Any multifunctor $F : \prod_i \mathbf{C}_i \to \mathbf{D}$ can be constructed by a tuple of functors $F : \mathbf{C}_i \to \mathbf{D}$

Proof. □

"Functor categories serve as the hom-categories in the strict 2-category Cat."

17.1 The hom-functor

The hom-bifunctor $\mathrm{Hom}_{\mathbb{C}}(-,-)$ is the map

$$\mathrm{Hom}_{\mathbf{C}}(-,-) : \mathbf{C} \times \mathbf{C} \to \mathbf{Set} \quad (17.10)$$

$$(X,Y) \mapsto \mathrm{Hom}_{\mathbf{C}}(X,Y) \quad (17.11)$$

mapping objects of $\mathbf{C}$ to their hom-sets. Given a specific object X, we can furthermore define two types of hom functors : the covariant functor $\mathrm{Hom}(X,-)$, also denoted by h^X, and the contravariant functor $\mathrm{Hom}(-,X)$, denoted by h_X.

Example :

Subobject functor :

$$\mathrm{Sub}_{\mathbf{C}} : \mathbf{C} \to \mathbf{Set} \quad (17.12)$$

Theorem 17.1.1. The hom-set of an opposite category is given by

$$\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(X,Y) \cong \mathrm{Hom}_{\mathbf{C}}(Y,X) \quad (17.13)$$

Proof. □

17.2 Full and faithful functor

As functors can be interpreted as functions on categories, we will need some notion similar to injectivity and surjectivity. This is given by the notions of a full functor and faithful functor.

Definition 17.2.1. A functor $F : \mathbf{C} \to \mathbf{D}$ induces the function

$$F_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X,Y) \to \mathrm{Hom}_{\mathbf{D}}(F(X),F(Y)) \quad (17.14)$$

F is said to be

- *faithful* if $F_{X,Y}$ is injective
- *full* if $F_{X,Y}$ is surjective
- *fully faithful* if $F_{X,Y}$ is bijective

Despite the analogy of full and faithful functors with surjective and injective functions, functors being full or faithful does not imply that they are surjective or injective on either objects or morphisms.

Counterexample 17.2.1. Given the terminal category **1** and a category of two objects and one isomorphism between them, $X \overset{\cong}{\rightleftarrows} Y$, then we have a

However, there is some sense in which it is true that such functors are equivalent to injective and surjective functions.

Theorem 17.2.1. On a skeletal category, a conservative functor (preserves isomorphisms) is

Theorem 17.2.2. A fully faithful functor is conservative.

Proof. □

[essentially injective/surjective, pseudomonic]

[image, essential image?]

Monomorphisms are reflected by faithful functors.

17.3 Subcategory inclusion

An important type of functor is the inclusion of a subcategory. If we take a category **C**, and then create a new category **S** for which $\mathrm{Obj}(\mathbf{S}) \subseteq \mathrm{Obj}(\mathbf{C})$

Definition 17.3.1. An inclusion of a subcategory **S** in a category **C** is a functor $\iota : \mathbf{S} \hookrightarrow \mathbf{C}$, such that $\iota(\mathrm{Obj}(\mathbf{S})) \subseteq \mathrm{Obj}(\mathbf{C})$, $\iota(\mathrm{Mor}(\mathbf{S})) \subseteq \mathrm{Mor}(\mathbf{C})$, and

- If $X \in \mathbf{S}$, then $\mathrm{Id}_X \in \mathbf{S}$
- For any morphism $f : X \to Y$ in **S**, then $X, Y \in \mathbf{S}$.
-

For discrete categories, **n** in **m** if $n \leq m$

Full subcategories

Example 17.3.1. The linear order of the integers $\mathbb{Z}$ has inclusion functors to $\mathbb{R}$

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{R} \tag{17.15}$$

If treated as Canonical inclusion :

$$\begin{aligned} \iota_h : \mathbb{Z} &\to \mathbb{R} & (17.16)\\ k &\mapsto h + k & (17.17) \end{aligned}$$

Example 17.3.2. The category of finite sets **FSet** is a subcategory of **Set**, via the identity functor restricted to finite groups.

Example 17.3.3. The category of Abelian group **Ab** in **Grp**

Full subcategory : ie every morphism of **C** is a morphism of **D**, and such that every object $d \in \mathrm{Obj}(\mathbf{D})$ and morphism $(f : d \to d') \in \mathrm{Mor}(\mathbf{D})$ have a reflection in **C**.

17.4 Pseudofunctors

As it is common in category theory to consider objects up to equivalence, we can also weaken the notion of a functor by merely requiring its functorial property to only hold up to equivalence.

Definition 17.4.1. A *pseudofunctor* $F : \mathbf{C} \to \mathbf{D}$ is a function between two categories similar to a functor but for which the map of identity and composition is only defined up to isomorphism,

$$\exists f \in \mathrm{End}(F(X)),\ F(\mathrm{Id}_X) = f \tag{17.18}$$

$$\exists f \in \mathrm{Iso}(\mathbf{D}),\ F(g \circ f) \tag{17.19}$$

18

Natural transformations

Natural transformations are a type of transformations on functors

Definition 18.0.1. For two functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* η between them is a map $\eta : F \to G$ which induces for any object $X \in \mathbf{C}$ a morphism on $\mathbf{D}$

$$\eta_X : F(X) \to G(X) \tag{18.1}$$

and for every morphism $f : X \to Y$ the identity

$$\eta_Y \circ F(f) = G(f) \circ \eta_X \tag{18.2}$$

called the *naturality condition*, corresponding to the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

Example 18.0.1. The *identity transformation* $\mathrm{Id}_F : F \to F$ on the functor $F : \mathbf{C} \to \mathbf{D}$ is the natural tranformation for which every component $\mathrm{Id}_{F,X} : F(X) \to F(X)$ for $X \in \mathbf{D}$ is the identity map. This obeys the identity as

$$\eta_Y \circ F(f) = \mathrm{Id}_Y \circ F(f) \tag{18.3}$$
$$= F(f) \tag{18.4}$$
$$= F(f) \circ \mathrm{Id}_X \tag{18.5}$$

Example 18.0.2. The category of groups **Grp** has a functor to the category of Abelian groups **AbGrp**, the Abelianization functor

$$\begin{aligned} \mathrm{Ab} : \mathbf{Grp} &\to \mathbf{AbGrp} && (18.6)\\ G &\mapsto G/[G,G] && (18.7) \end{aligned}$$

[show functoriality] There is a natural transformation from the identity functor on groups to the abelianization endofunctor

$$\eta : \mathrm{Id}_{\mathbf{Grp}} \to \mathrm{Ab} \tag{18.8}$$

Example 18.0.3. Given the category $\mathbf{Vect}_k$, for any vector space V we have the dual space V^* [see later in the internal hom section for why] of linear maps $V \to k$, and its double dual V^** of linear maps $V^* \to k$. We would like to show that there is an equivalence between V and V^{**}.

Example 18.0.4. The opposite group functor is simply given by the opposite category functor on Grp. Groups to opposite group

For constant F and G : cone and cocone

A special case of a natural transformation is if the underlying functors are bifunctors $F, G : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$

Dinatural transformations

Definition 18.0.2. Given two bifunctors $F, G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$, a dinatural transformation $\alpha : F \to G$

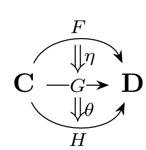
18.1 Composition

There are three ways to compose natural transformations.

Definition 18.1.1. Given two natural transformations $\eta : F \to G$ and $\theta : G \to H$ between three functors $F, G, H : \mathbf{C} \to \mathbf{D}$, the *vertical composition* of those natural transformations is the natural transformation $\theta \circ \eta : F \to H$, defined component-wise by

$$(\theta \circ \eta)_X = \theta_X \circ \eta_X \tag{18.9}$$

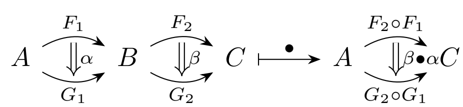
Giving the diagram



Definition 18.1.2. For two natural transformations $\eta : F \to G$ and $\theta : J \to K$, with $F, G : \mathbf{C} \to \mathbf{D}$ and $J, K : \mathbf{D} \to \mathbf{E}$, their *horizontal composition* is given by the composition of their functors $\theta \bullet \eta : J \circ F \to K \circ G$, which is given components-wise as

$$(\theta \bullet \eta)_X = \theta_{G(X)} \circ J(\eta_X) = K(\eta_X) \circ \epsilon_{F(X)} \tag{18.10}$$

Giving the diagram



Finally, a natural transformation can be composed with a functor by pre or post-composition in an operation called *whiskering* :

Definition 18.1.3. For a natural transformation $\eta : F \to G$ between two functors $F, G : \mathbf{C} \to \mathbf{D}$, we talk of *left whiskering* for a functor $H : \mathbf{D} \to \mathbf{E}$ post-composed with it, $H \triangleleft \eta : H \circ F \to H \circ G$, with components

$$(H \triangleleft \eta)_X = H(\eta_X) \tag{18.11}$$

and of *right whiskering* for a functor $K : \mathbf{B} \to \mathbf{C}$ which is pre-composed with it, $\eta \triangleright K : F \circ K \to G \circ K$, with components

$$(\eta \triangleright K)_X = \eta_{K(X)} \tag{18.12}$$

Theorem 18.1.1. Whiskering with respect to a functor F is equivalent to horizontal composition with the identity natural transformation on F,

$$H \triangleleft \eta = \mathrm{Id}_H \bullet \eta \tag{18.13}$$

$$\eta \triangleright K = \eta \bullet \mathrm{Id}_K \tag{18.14}$$

Proof. Given the components of the horizontal composition, we have

$$(\mathrm{Id}_F \bullet \eta)_X = \theta_{G(X)} \circ J(\eta_X) = K(\eta_X) \circ \mathrm{Id}_{F(X)} \tag{18.15}$$

□

In terms of components, left whiskering can be understood as a transformation applied to a diagram where the transformation is applied to the left, for instance given some naturality diagram for $\eta : F \to G$ on a morhism $f : X \to Y$, its whiskering is

$$\begin{CD} F(X) @>\eta_X>> G(X) \\ @VF(f)VV @VVG(f)V \\ F(Y) @>>\eta_Y> G(Y) \end{CD} \qquad \begin{CD} H(F(X)) @>H(\eta_X)>> H(G(Y)) \\ @VH(F(f))VV @VVH(G(f))V \\ H(F(Y)) @>>H(\eta_Y)> H(G(Y)) \end{CD}$$

While a right whiskering can be understood as applying this transformation on the right, as

$$\begin{CD} F(X) @>\eta_X>> G(X) \\ @VF(f)VV @VVG(f)V \\ F(Y) @>>\eta_Y> G(Y) \end{CD} \qquad \begin{CD} F(K(X)) @>\eta_{K(X)}>> G(K(Y)) \\ @VF(K(f))VV @VVG(K(f))V \\ F(K(Y)) @>>\eta_{K(Y)}> G(K(Y)) \end{CD}$$

18.2 2-categories and the category of categories

A good setting for the definition of natural transformations is that of 2-categories, which are categories which also have the possibility to have arrows between arrows, called 2-morphisms.

Definition 18.2.1. A 2-category $\mathbf{C}$ is defined by a class of objects $\mathrm{Obj}(\mathbf{C})$, a class of morphisms $\mathrm{Mor}(\mathbf{C})$,

This notion will be generalized to higher such morphisms in 84.2, but for now we will not need more.

In this context, we can look at the category of categories **Cat**, in which case the objects are (small) categories, morphisms are the functors between them, and natural transformations are the 2-morphisms between those morphisms.

We can reconstruct the categorical notions that we have seen so far. Objects of a category $\mathbf{C}$ are given by maps from the trivial category $\mathbf{1}$ to $\mathbf{C}$

Morphisms as natural transformations between those functors

19
Functor categories

Functors between two categories themselves form a category.

Definition 19.0.1. A *functor category* between two categories $\mathbf{C}$ and $\mathbf{D}$ is a category, denoted by $\mathbf{D}^{\mathbf{C}}$ or $[\mathbf{C}, \mathbf{D}]$, for which the objects are all the functors $F : \mathbf{C} \to \mathbf{D}$ and the morphisms are all the natural transformations, with identity the identity natural transformation and the composition is the vertical composition of natural transformations.

This indeed forms a category as the natural transformations obey all the appropriate requirements of morphisms in a category.

Example 19.0.1. Given the terminal category $\mathbf{1}$, the functor category $[\mathbf{1}, \mathbf{C}]$ is isomorphic to $\mathbf{C}$.

Proof. Any functor of $[\mathbf{1}, \mathbf{C}]$ is constrained to be a function from $* \in \mathbf{1}$ to some object $X \in \mathbf{C}$, mapping Id_* to Id_X. These functions are exactly in bijection with the objects of $\mathbf{C}$, as there is always exactly one such function per object of $\mathbf{C}$. The natural transformations of those functors are

$$\eta : \Delta_X \to \Delta_Y \tag{19.1}$$

To verify the naturality of the morphisms in this category, for the morphism $\mathrm{Id}_* : * \to *$, mapped to $\Delta_X(*) \cong X$, $\Delta_Y(*) \cong Y$ The naturality square is just the unique component

$$\eta_X \cong \eta_Y : X \to Y \tag{19.2}$$

This naturality is obeyed for any morphism $f \in \mathbf{C}$, so that every morphism in $\mathbf{C}$ gives rise to a natural transformation in $[\mathbf{1}, \mathbf{C}]$. □

Example 19.0.2. For the discrete category $\mathbf{n}$, the functor category $[\mathbf{n}, \mathbf{C}]$ is the category of families of objects indexed by $\mathbf{n}$.

Proof. □

Whiskering in a functor category?

Theorem 19.0.1. If a category $\mathbf{D}$ has a given limit or colimit for a diagram I, then so does the functor category $[\mathbf{C}, \mathbf{D}]$.

Proof. □

19.1 Yoneda lemma

One of the common philosophical idea underlying category theory is that of the specific objects involved in a category are not as meaningful as the equivalence of all such objects under isomorphisms. That is, if we have two objects X, X' in a category, such that those two objects have identical behaviour in the category (all morphisms to other objects are mirrored on the other), then they are in essence the same object. In philosophical terms, this

"every individual substance expresses the whole universe in its own manner and that in its full concept is included all its experiences together with all the attendent circumstances and the whole sequence of exterior events. There follow from these considerations several noticeable paradoxes; among others that it is not true that two substances may be exactly alike and differ only numerically, solo numero."

[Leibniz discourse on metaphysics]

This is true in some sense in category theory as expressed by the Yoneda lemma. If we consider all the relationship of an object X to all other objects, this is given by the hom-sets of X to every other objects, that is,

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{19.3}$$

What we mean by two objects having the same relationships to every other objects is that if we consider their respective hom-sets to every other objects, they are isomorphic :

$$\forall Y \in \mathbf{C},\ \mathrm{Hom}_{\mathbf{C}}(X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{19.4}$$

Meaning that for any object Y and any morphism $f : X \to Y$, we will have some corresponding function $f' : X' \to Y'$, and the relationships between all those morphisms reflect each other. If we have $f : X \to Y$ and $g : Y \to Z$, then we have

$$\exists g',\ f' \circ g' : X \tag{19.5}$$

Equivalently (converse)

This equivalence is in fact an equivalence of the hom functors,

$$h_X \cong h_{X'} \tag{19.6}$$

$$h^X \cong h^{X'} \tag{19.7}$$

As an equivalence of functors, this means that we have a pair of natural transformations between them which are two-sided inverses of each other.

For a functor $F : \mathbf{C} \to \mathbf{Set}$, for any object $X \in \mathbf{C}$

$$よ : \mathbf{C} \to [\mathbf{C}, \mathbf{Set}] \tag{19.8}$$

Functor lives in the functor space $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

$$\mathrm{Nat}(h_A, F) \cong F(A) \tag{19.9}$$

Example 19.1.1. Consider the category of a single group ($G = \mathrm{Aut}(*)$). A functor $F : \mathrm{G} \to \mathbf{Set}$ is a set X and a group homomorphism to its permutation group $G \to \mathrm{Perm}(X)$ (A G-set). Natural transformation is an equivariant map Cayley's theorem

Example 19.1.2. Applied to a poset, the hom-functor $\mathrm{Hom}_P(x, -)$ gives us the set containing all the elements superior to x, the upper bounds of x. Then for any presheaf f on P, a function

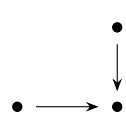
20
Limits and colimits

In category theory, a limit or a colimit are roughly speaking a construction on a category based on some kind of template. For some given set of objects $A, B, \ldots$ in our category C, and some morphisms between them, a limit or colimit of those objects will be some construction performed using those. Those constructions can be quite different, but overall, a limit will often be like a ”subset”, while a colimit is more of an ”assemblage” of those.

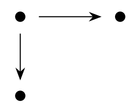
A (co)limit is done using an indexing category, which is roughly the ”shape” that our construction will take. An indexing category is a small category, ie it has a countable number of objects and morphisms small enough that you could fit them into sets. Typically they are fairly simple ones. As we are only interested in their shape, it’s common to denote the objects by simple dots. Examples include the discrete categories of n elements $\mathbf{n}$,

$$\bullet \qquad \bullet \qquad \bullet \qquad \ldots$$

The span category :



and the cospan :



and the parallel pair :

$$\bullet \rightrightarrows \bullet$$

as well as infinite sequences

$$\ldots \to \bullet \to \bullet \to \bullet \to \bullet \to \ldots$$

To do our constructions, we need to send this diagram's shape into our category C, which we do by using some functor $F : I \to C$, producing a diagram :

Definition 20.0.1. A *diagram* of shape I into C is a functor $F : I \to C$.

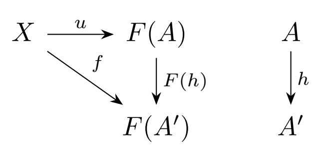
The image of this functor in C will then be some subset of C that "looks like" I, although we are not guaranteed that the objects and morphisms of I will be mapped injectively into C (this simply corresponds to cases where our construction will use the same object or morphisms several times).

There are a few different ways to define limits, so first let's look at a few different formalisms for this.

20.1 Universal constructions

with this diagram, to find a (co)limit, we would like to define a universal construction on it.

Definition 20.1.1. For a functor $F : \mathbf{C} \to \mathbf{D}$, a *universal morphism* from an object $X \in \mathbf{D}$ to F is a unique pair of an object $A \in \mathbf{C}$ and a morphism $u : X \to F(A)$ which obey the universal property : for any object $A' \in \mathbf{C}$ and morphism $f : X \to F(A')$ in $\mathbf{D}$, there is a unique morphism $h : A \to A'$ such that $f = F(h) \circ u$.

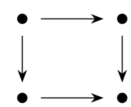


One easy way to interpret a universal construction is in the case of a poset category, where as the morphisms signify an order relation, we can interpret $F(A)$ as the smallest element that is still superior to $F(A')$.

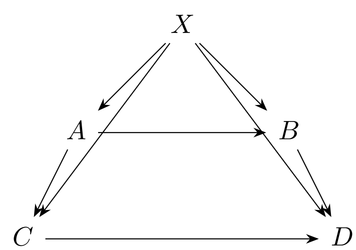
Examples : Smallest object obeying a property

20.2 Cones and cocones

Given some categories $\mathbf{C}, \mathbf{D}$ and some functor $F : \mathbf{C} \to \mathbf{D}$, a cone over F at an object $X \in \mathbf{D}$ is given by a family of morphisms from X to every image of F in $\mathbf{D}$. Its name of cone can be visualized easily enough by considering a category $\mathbf{C}$ with the shape of for instance

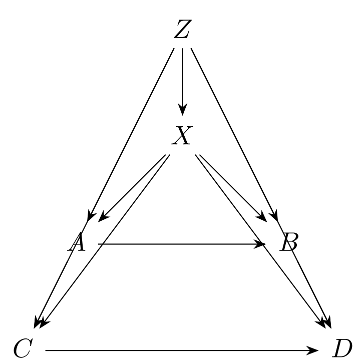


which leads to this diagram in $\mathbf{D}$

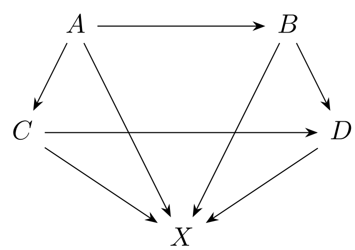


If we take the functor category $\mathbf{D}^{\mathbf{C}}$, and define the embedding functor $\Delta : \mathbf{D} \to \mathbf{D}^{\mathbf{C}}$ which maps any object X of $\mathbf{D}$ to the constant functor $\Delta_X : \mathbf{C} \to \mathbf{D}$, then we can define the category of cones over F as either the category of natural transformations $\alpha_X : \Delta_X \to F$, or as the comma category $(\Delta \downarrow F)$, which has as its objects the triple (c, d, h) of an object $c \in \mathbf{C}$, $d \in \mathbf{D}$, and morphism $h : F(c) \to \Delta_d$ in $\mathbf{D}$.

Map of cones :



Dually, a cocone over F at X is a family of morphisms from every image of F in $\mathbf{D}$ to X



Let's now consider the constant functor $\Delta_X : I \to C$, which for $X \in C$ sends every object of I to X. If we can find a natural transformation between Δ_X and our diagram $F : I \to C$, we will have either a cone over F $\eta : \Delta_X \Rightarrow F$ or a cone under F $\eta : F \Rightarrow \Delta_X$.

Definition 20.2.1. The *limit* of a diagram $F : I \to M$ is an object $\lim F \in \mathrm{Obj}(C)$ and a natural transformation $\eta : \Delta_{\lim F} \to F$, such that for any $X \in \mathrm{Obj}(C)$ and any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f : X \to \lim F$ such that $\alpha = \eta \circ F$. The cone of $\Delta_{\lim F}$ over F is the *universal cone* over F.

In other words, if we pick any object X in our category C and define some collection of morphisms from X to other objects

Let's consider for instance the case of the trivial category $\mathbf{1}$. Any functor $F : \mathbf{1} \to C$ is simply a choice of an object in C, mapping $\bullet$ to $F(\bullet) = A$, ie it is just the constant functor Δ_A for some A. A natural transformation $\eta : \Delta_X \to F$ is them simply $\eta : \Delta_X \to \Delta_A$, and conversely, $\eta : F \to \Delta_X$ is $\eta : \Delta_A \to \Delta_X$. The components of this natural transformations are simply a morphism from X to A (and a morphism from A to X).

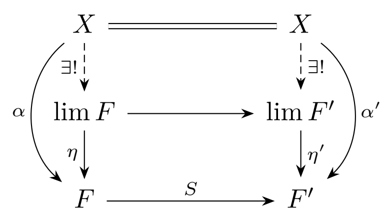
Theorem 20.2.1. If given some equivalence endofunctor S on $\mathbf{I}$, we have for two diagrams F, F' the identity $F' = F \circ S$, then the limits of those diagrams are equivalent.

$$\lim_{\mathbf{I}} F \cong \lim_{\mathbf{I}} F \circ S \tag{20.1}$$

Proof. Any natural transformation $\alpha : \Delta_X \to F$ can be transformed via right whiskering

$$\alpha' = \alpha \triangleright S : \Delta_X \circ S \to F' = F \circ S \tag{20.2}$$

with $\Delta_X \circ S = \Delta_X$. And likewise, any natural transformation to F' corresponds to one to F via $\alpha = \alpha' \triangleright S^{-1}$. Then for any cone over F', there exists



□

20.3 Initial and terminal objects

The simplest kind of limit is the one done on the empty indexing category, **0**, as this leads to a unique diagram in the category **C**, the empty functor.

Definition 20.3.1. Given the empty category **0**, the limit $\lim_{\mathbf{0}} F$ of a diagram $F : \mathbf{0} \to C$ is its *terminal object*, and the colimit is its *initial object*.

Generally we will denote the terminal object as 1 and the initial object as 0, since they often correspond to objects of one and zero elements in concrete categories, except for a few cases where we will use a more specific notation for the given category, such as $\{\bullet\}$ and $\varnothing$ for **Set** (the singleton and empty set), or **1** and **0** for the category of categories **Cat**.

As there exists only one functor from the empty category to any other category (the empty functor $F_\varnothing$), the initial and terminal objects do not depend on specific objects and are simply special objects of the category.

In terms of cones and cocones, the cone of this empty functor will simply be the diagram of the unique object of the limit,

$$\lim_{\mathbf{0}} F$$

and to be a universal cone, we simply need every other cone of apex Z over this diagram to factor uniquely through it. As the cone is empty, this is simply the condition that there is a unique morphism

$$Z \to \lim_{\mathbf{0}} F \tag{20.3}$$

which is the universal property of the terminal object 1 : there exists a unique morphism from any object to it, which we will call $!_Z$.

Dually, the initial object 0 has as its universal property that for

Every "constant functor" Δ_X sending objects of I to $X \in \mathrm{Obj}(C)$ is also the empty functor, sending them trivially to X by simply not having any objects to send. This is therefore also true of the constant functor to the limit $\lim F_\varnothing$, meaning that the natural transformation $\eta : \varnothing \to \varnothing$ is simply the identity transformation $\mathrm{Id}_{\varnothing\varnothing}$. This means that the limit of the empty diagram in a category C is the object (defined by no other objects in the category) $\lim \mathbf{0}$ such that for any natural transformation $\alpha : \Delta_X \to F$ (as we've seen, only possibly the identity transformation), there exists a unique morphism $f : X \to \lim \mathbf{0}$.

This means that the terminal object $1 = \lim \mathbf{0}$ of a category, if it exists, is therefore an object for which there exists only one morphism from any object $X \in \mathrm{C}$ to 1. That is the universal property of the terminal object.

Dually, the *initial object* $0 = \mathrm{colim}\mathbf{0}$ of a category, if it exists, is an object for which there exists only one morphism from i to any object $X \in \mathrm{Obj}(\mathbf{C})$.

Theorem 20.3.1. Initial and terminal objects are unique in a category up to isomorphisms.

Proof. If we have two different terminal objects $1, 1'$, by their universal property, there is exactly one morphism to themselves (the identity), and between them, meaning that there is a morphism $1 \to 1'$ and $1' \to 1$, and their composition in both direction leads to their unique endomorphisms Id_1 and $\mathrm{Id}_{1'}$, meaning those maps are isomorphisms. A similar reasoning shows the same property for the terminal object. □

Initial and terminal objects occur in quite a lot of important categories, and tend to be somewhat similar objects. In **Set**, the initial and terminal objects are the empty set and the singleton set.

Definition 20.3.2. An object that is both an initial and terminal object is called a *zero object*, and denoted by 0.

This type of object is often found in the case of categories for which the objects are understood to have a distinguished element that morphisms are meant to preserve.

Example 20.3.1. The zero dimensional vector space is a zero object of $\mathbf{Vect}_k$, with its unique map in being the projection to 0 and its unique map out the map pointing to 0 in its image.

Example 20.3.2. The trivial group $\{e\}$ in **Grp** is a zero object, with its unique map in being the projection to the neutral element and its unique map out being the map pointing to the neutral element in its image.

Example 20.3.3. The zero ring $\{0\}$ in **Ring** is a zero object, with its unique map in being the projection to the zero element and its unique map out being the map pointing to the zero element in its image.

Definition 20.3.3. In a category with a zero object, there exists for any two objects X, Y a distinguished map called the *zero map* 0_{XY}, defined as

$$0_{XY} : X \xrightarrow{!_X} 0 \xrightarrow{0_Y} Y$$

Theorem 20.3.2. For any objects X, Y and morphism $f : X \to Y$, we have that the terminal morphism is left absorbing :

$$!_Y \circ f = !_X \tag{20.4}$$

while the initial morphism is right absorbing :

$$f \circ 0_X = 0_Y \tag{20.5}$$

Proof. As the composition is always defined but there is always a unique morphism between an object X and 1, that composition can only be $!_Y$. The same reasoning apply to 0_Y. □

Theorem 20.3.3. Any morphism $f : 1 \to X$ is a split monomorphism.

Proof. For any two morphisms $g_1, g_2 : Y \to 1$, by the universal property of the terminal object, we have that $g_1 = g_2$, therefore any such morphism is a monomorphism. Additionally, given the unique map $!_X : X \to 1$, we have $!_X \circ f = \mathrm{Id}_1$, as there is only one endomorphism for 1. □

Theorem 20.3.4. An epimorphism $f : 1 \to X$ is an isomorphism.

Proof. Being a split monomorphism and an epimorphism, it is an isomorphism. □

Theorem 20.3.5. Any morphism to the initial object is an epimorphism.

Proof. This simply stems from the universal property of the initial object, as there is a unique morphism $0 \to Z$ for any Z, therefore $g_1, g_2 : 0 \to Z$ are always identical morphisms. □

It is quite common in categories that there does not exist any morphism from any object to the initial object outside of its own identity map, Id_0, since in many cases the terminal object is "empty" in some sense.

Definition 20.3.4. An initial object is a *strict initial object* if there exists no morphism to it outside of its identity map :

$$\forall f \in \mathbf{C},\ t(f) = 0 \to f = \mathrm{Id}_0 \tag{20.6}$$

Example 20.3.4. In the category of sets **Set**, the empty set $\varnothing$ is a strict initial object.

Theorem 20.3.6. Strict initial objects can only be zero objects if the category is the terminal category.

Proof. If a category has a strict initial object that is also a zero object, we have that any object has a morphism to 0, meaning that all objects must be (up to isomorphism) the initial object. □

20.4 Products and coproducts

The product and coproduct are the limits where the diagrams are the discrete categories of n elements $\mathbf{n}$. This means obviously that the trivial case $\mathbf{0}$ of the diagram of zero object is the initial and terminal object, and this will correspond to the trivial product and coproduct as we will see later :

$$\sum_{\varnothing} = 0,\ \prod_{\varnothing} = 1 \tag{20.7}$$

Any diagram of shape $\mathbf{n}$ simply selects n objects in the category (and their identity functions),

$$\forall F : \mathbf{n} \to \mathbf{C},\ \exists X_1, \dots X_n \in \mathbf{C},\ \mathrm{Im}(F) = (X_1, \dots, X_n) \tag{20.8}$$

The constant functor Δ_X is the functor sending each of those points to X, $\bullet_i \to X$, and the identity of those points to the identity on X. We will denote the limit of a diagram F on the discrete category, selecting the objects $\{X_i\}$, by $\prod_i X_i$. The product is therefore some object $\prod_i X_i \in \mathbf{C}$ along with morphisms $\pi_i : \prod_j X_j \to X_i$

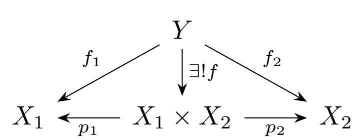
such that for any natural transformation $\alpha : \Delta_X \to F$, there is a unique morphism $f = \Delta_X \to \prod X_i$ such that $\alpha = \eta \circ F$.

For any object $Y \in \mathbf{C}$ with morphisms $f_i : Y \to X_i$ for each object X_i, we therefore have a unique morphism to $\prod_i$ which make the maps commute :

$$\exists ! f : Y \to \prod_i X_i,\ \forall i \in I,\ f_i = \pi_i \circ f \tag{20.9}$$

What this property means for the product is that given any object X picked in $\mathbf{C}$,

Universal property



The semantics of the product is typically that we are considering elements of those objects *together*, as pairs. This can easily be seen in the case of some concrete category where objects are sets

[...]

You should however beware of overextending this interpretation to all categories, as it is typically only really valid in the case of objects being seen as some collection or space. As we've seen, categories such as preorders can have radically different interpretations.

Example 20.4.1. For a preorder category $(X, \leq)$, the product of two objects is their (meet, join?). If we take the product of two elements, the universal property tells us that $\prod_i X_i$ has morphisms to all X_i, so that $\prod_i X_i \leq X_i$ ($\prod_i X_i$ is a lower bound of all elements). Furthermore, for any other object Y with morphisms to all X_i, so that Y is also a lower bound, we have a morphism $Y \to \prod_i X_i$, so that $Y \leq \prod_i X_i$. Therefore any other lower bound is inferior or equal to it. $\prod_i X_i$ is the *greatest lower bound*,

$$\prod_i X_i = \bigwedge_i X_i \tag{20.10}$$

A special case of the product is the case of the product of an object with itself. This leads to a special morphism.

Definition 20.4.1. For any object X in a category with products, the diagonal morphism $\Delta : X \to X \times X$ is the one given by the universal property of the product for the case where the two objects are X : [diagram]

The name of diagonal morphism can be easily seen in the case of $\mathbb{R}$, where it corresponds to

$$\begin{aligned} \delta : \mathbb{R} &\to \mathbb{R}^2 && (20.11) \\ x &\mapsto (x, x) && (20.12) \end{aligned}$$

which corresponds to the graph of the identity function, a diagonal in $\mathbb{R}^2$.

Theorem 20.4.1. The product is commutative

Proof. Given the products $A \times B$ and $B \times A$, which are the limits of the functors $F(\mathbf{2}) = (A, B)$ and $F'(\mathbf{2}) = (B, A)$, those two functors are equivalent. If we consider the endofunctor I on $\nvDash$ exchanging the two elements, we have that $I \cong I^{-1}$ (by some simple arguments on it just being some instance of the symmetric group on $\mathrm{Obj}(\mathbf{2})$), and $F' = F \circ I$. □

Likewise for the coproduct,

Definition 20.4.2. For any object X in a category with products, the codiagonal morphism $\nabla : X + X \to X$ is the one given by the universal property of the product for the case where the two objects are X :

Theorem 20.4.2. The product of an object X by the terminal product is isomorphic to the object itself :

$$X \times 1 \cong X \cong 1 \times X \tag{20.13}$$

Proof. As for any product, there is a morphism

$$p_1 : X \times 1 \to X \tag{20.14}$$

From the properties of the product, for the two morphisms $\mathrm{Id}_X : X \to X$ and $!_X : X \to 1$, we have some unique morphism $f : X \to X \times 1$ such that f

$$X$$

□

Theorem 20.4.3. The product of two monomorphisms is a monomorphisms.

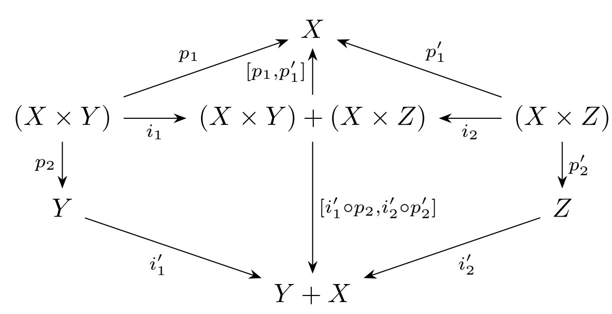
Proof. Given two monomorphisms $f_1 : Y \hookrightarrow X_1$, $f_2 : Y \hookrightarrow X_2$, their product $(f_1, f_2) : Y \to X_1 \times X_2$ □

Theorem 20.4.4. In a category **C** with products and coproducts, there exists a canonical map

$$\delta_{X,Y,Z} : (X \times Y) + (X \times Z) \to X \times (Y + Z) \tag{20.15}$$

called the *distributivity morphism.*

Proof. Given the universal properties of the product and coproduct, we can form the following diagram



from this, using the maps $p''_1 : X \times (Y+Z) \to X$ and $p''_2 : X \times (Y+Z) \to (Y+Z)$, as we have the maps from our original object

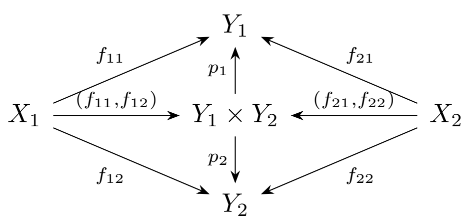
$$\begin{aligned} [p_1, p'_1] : (X \times Y) + (X \times Z) &\to X \\ [i'_1 \circ p_2, i'_2 \circ p'_2] : (X \times Y) + (X \times Z) &\to Y + Z \end{aligned}$$

we can form the product of morphisms

$$([p_1, p_1'], [i_1' \circ p_2, i_2' \circ p_2']) : (X \times Y) + (X \times Z) \to X \times (Y + Z) \qquad (20.16)$$

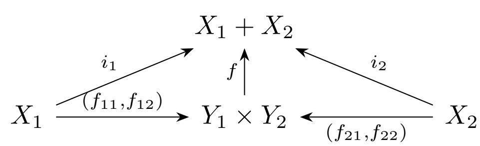
which we will denote as $\delta_{X,Y,Z}$. □

Theorem 20.4.5. Given a group of four morphisms



there exists a canonical map from $X_1 + X_2$ to $Y_1 \times Y_2$.

Proof. Using the universal property of the coproduct,



with

$$f = [(f_{11}, f_{12}), (f_{21}, f_{22})] \qquad (20.17)$$

□

This canonical map is generally denoted in matrix form,

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : X_1 + X_2 \to Y_1 \times Y_2 \qquad (20.18)$$

or larger matrices for the case of $\coprod X_i$ and $\prod Y_j$, using the analogy that, given the following diagram

$$X_a \xrightarrow{i_a} \coprod_i X_i \xrightarrow{f} \prod_j Y_j \xrightarrow{p_b} Y_b$$

we get the identity

$$f_{ab} = i_a \circ f \circ p_b \tag{20.19}$$

so that the injection and projection map form the "basis" of the source and target of this morphism.

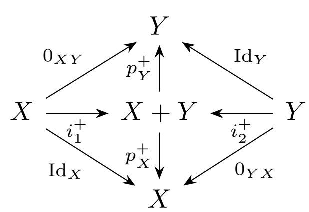
This type of morphism also exists in the opposite direction,

20.4.1 Linear cases

There exists a variety of categories which are in some sense more "linear" than others, in a sense we will explain later on [X]. These will involve some specific properties with the product and coproduct we will see here.

Theorem 20.4.6. In a category **C** with finite products and coproducts and a zero object, there exists a canonical map from the coproduct to its components, and from its components to the product

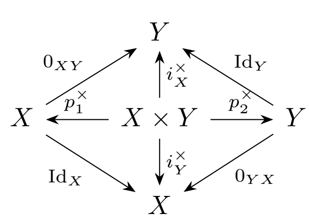
Proof. Using the universal property of the coproduct, we can find two projections for the coproduct



with

$$p_X^+ = [\mathrm{Id}_X, 0_{YX}] \tag{20.20}$$
$$p_Y^+ = [0_{XY}, \mathrm{Id}_Y] \tag{20.21}$$

and likewise for the product,



with

$$i_X^\times = (\mathrm{Id}_X, 0_{YX}) \tag{20.22}$$
$$i_Y^\times = (0_{XY}, \mathrm{Id}_Y) \tag{20.23}$$

□

In matrix notation, the projections of the coproduct are the maps $X+Y$ into the unary product of X or Y alone, corresponding to the column matrix, in the general case

$$p_i^+ = \begin{pmatrix} 0_{1i} \\ \vdots \\ 0_{(i-1)i} \\ \mathrm{Id}_X \\ 0_{(i+1)i} \\ \vdots \\ 0_{ni} \end{pmatrix} \tag{20.24}$$

while the injections of the product are the maps of the unary coproduct

Those maps will correspond, in the linear case, to the actual projections and injections of the direct sum in many cases, but to avoid for now examples where the product and coproduct are isomorphic, let's consider instead the case of pointed sets, which do have zero objects but are not linear :

Example 20.4.2. In the category $\mathbf{Set}_*$ of pointed sets, with $X_{x_0} + Y_{y_0}$ the wedge sum, and the zero map being the function mapping all points to the base point, the projections $X_{x_0} + Y_{y_0} \to X_{x_0}, Y_{y_0}$ correspond to the function mapping all elements of X_{x_0} in the wedge sum to X_{x_0}, and all elements of Y_{y_0} to the basepoint x_0. Likewise for the pointed product,

$$X_{x_0} \times Y_{y_0} = (X \times Y)_{(x_0, y_0)} \tag{20.25}$$

which is just the Cartesian product with the base point (x_0, y_0), the injection $X_{x_0}, Y_{y_0} \to X_{x_0} \times Y_{y_0}$ maps any point x of X_{x_0} (resp. y of Y_{y_0}) to the product pair (x, y_0) (resp. (x_0, y)).

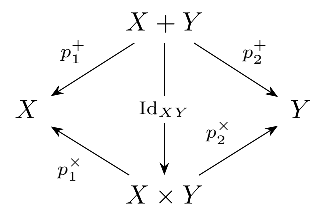
Given these morphisms, we have furthermore some canonical map between the coproduct and product.

Theorem 20.4.7. In a category $\mathbf{C}$ with finite products and coproducts and a zero object, there exists a canonical morphism from the coproduct to the product called the *identity matrix*

$$\mathrm{Id}_{XY} : X + Y \to X \times Y \tag{20.26}$$

defined by

Proof. Using the universal properties of the product and the coproduct's projection,



so that

$$\mathrm{Id}_{XY} = (p_1^+, p_2^+) \tag{20.27}$$

□

As the name implies, the identity matrix takes its name for being the categorical equivalent of an identity matrix, in that

$$\mathrm{Id} = \begin{pmatrix} \mathrm{Id}_{X_1} & 0_{12} & \dots & 0_{1n} \\ 0_{21} & \mathrm{Id}_{X_2} & \dots & 0_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & 0_{n2} & \dots & \mathrm{Id}_{X_n} \end{pmatrix} \tag{20.28}$$

and given the diagram

$$X_i \xrightarrow{i_i} X_i + X_j \xrightarrow{\mathrm{Id}_{ij}} X_i \times X_j \xrightarrow{p_j} X_j$$

obeys the relation

$$p_j \circ \mathrm{Id} \circ i_i = \begin{cases} 0_{ij} & i \neq j \\ \mathrm{Id}_{X_i} & i = j \end{cases} \tag{20.29}$$

The "one dimensional" case of a (co)product of a single element is then simply $\mathrm{Id} = (\mathrm{Id}_X)$.

20.5 Limits of arrows

The simplest non-discrete case of a diagram is that of an arrow

$$a : \bullet_s \to \bullet_t \tag{20.30}$$

This is not commonly considered as a limit, as it can simply be derived as a limiting case of other limits like the (co)equalizer, but let's briefly look at it.

If we have a diagram $a : \bullet_s \to \bullet_t$, denoted by the category $\mathbf{A}$ this maps easily enough to the arrow category of $\mathbf{C}$ 22.1,

$$\mathbf{C}^{\mathbf{A}} \cong \mathrm{Arr}(\mathbf{C}) \tag{20.31}$$

with every functor corresponding to a morphism,

$$F(\bullet_s \to \bullet_t) = (f : X \to Y) \tag{20.32}$$

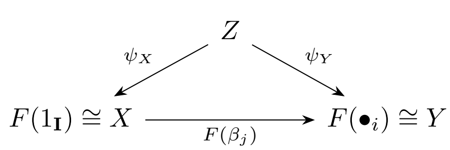
The limit of any such functor is given by the following more general theorem

Theorem 20.5.1. If a diagram $\mathbf{I}$ has an initial object $1_{\mathbf{I}}$, any limit of this diagram is the image of the initial object, ie

$$\lim_I(F) = F(1_{\mathbf{I}}) \tag{20.33}$$

Proof. Let's look at the cones over F at Z for the morphisms starting at the initial object of $\mathbf{I}$. For any object $\bullet_i$ in $\mathbf{I}$, define the unique morphism

$$\beta_i : 1_{\mathbf{I}} \to \bullet_i \tag{20.34}$$



As it is the initial object, for any other morphism of $\mathbf{I}$, $\beta_{ij} : \bullet_i \to \bullet_j$, we can define the morphism

$$\beta_j = \beta_{ij} \circ \beta_j \tag{20.35}$$

defining in turn a cone over F.

Now for any cone over F at Z,

in other words, it is the collection of all the composable functions ψ_X with f with the composite ψ_Y. □

From this, we have that the limit of an arrow is simply its source

$$\lim_{\mathbf{A}}(F) = F(s(a)) = s(F(a)) \tag{20.36}$$

and dually, the colimit of an arrow is its target

$$\operatorname{colim}_{\mathbf{A}}(F) = F(t(a)) = s(t(a)) \tag{20.37}$$

This will generalize for any diagram with an initial or final object, such as a finite chain of arrow, a tree, etc.

20.6 Equalizer and coequalizer

The equalizer and coequalizer are the limits and colimits corresponding to the diagram of a pair of parallel morphisms

$$\bullet_s \overset{a_2}{\underset{a_1}{\rightrightarrows}} \bullet_t \tag{20.38}$$

This diagram maps to some pair of morphisms in our category between two objects,

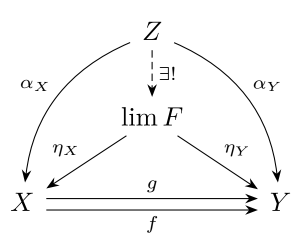
$$X \overset{g}{\underset{f}{\rightrightarrows}} Y \tag{20.39}$$

$$\mathrm{Hom}_{\mathbf{C}}(X, \lim F) \cong \mathrm{Nat}_{\mathbf{C}}(X, F) \tag{20.40}$$

If we consider the constant functor on that diagram, $\Delta_X : I \to \mathbf{C}$, mapping both objects of the diagram to X and the parallel morphisms to Id_X, and the natural transformations $\alpha : \Delta_X \to F$,

$$\alpha = \eta \circ F \tag{20.41}$$

Natural transformation $\eta : \lim F \to F$



So that the equalizer is characterized by an object $E = \lim F$, along with a morphism $e = \eta_X : E \to X$, with the property that

$$g \circ e = f \circ e \tag{20.42}$$

The equalizer of two morphisms corresponds roughly to the notion of solution of an equation, where we try to find some object E whose image into X has the same image through both f and g, and the fact that it is the universal cone for this means that it is in fact the largest object for this. To give this notion more credence, we have that equalizers are always subobjects :

Theorem 20.6.1. The resulting morphism of an equalizer $e : E \to X$ is a monomorphism.

Proof. By the universal property of the equalizer, if for any object Z, we have two morphisms $u, v : Z \to \mathrm{eq}(f, g)$, □

Corrolary 2. If the diagram maps the two arrows of the parallel morphisms to the same arrow, its equalizer, being the limit of an arrow, is simply the source of that arrow :

$$\mathrm{eq}(f, f) \cong s(f) \tag{20.43}$$

with its morphism simply being the identity $\mathrm{eq}(f, f) \xrightarrow{\mathrm{Id}_X} X \xrightarrow{f} \mathrm{Y}$

Example 20.6.1. The equalizer of two morphisms $f, g : X \to Y$ in **Set** is the subset of X on which those morphisms agree, ie

$$\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\} \tag{20.44}$$

Example 20.6.2. In the category of group **Grp**, given a group homomorphism $f : G \to H$ and the trivial homomorphism $\epsilon : G \to H$ (the homomorphism that factors through the zero group $\epsilon : G \to 0 \to H$), the equalizer of those morphisms is the kernel of f :

$$\mathrm{eq}(f, \epsilon) = \ker(f) \tag{20.45}$$

with the morphism to G its inclusion map

$$\iota : \ker(f) \to G \tag{20.46}$$

Proof. As the trivial homomorphism is the one mapping every element to the neutral element,

$$\forall g \in G,\ \epsilon(g) = e_H \tag{20.47}$$

The universal property of the equalizer gives us that

$$f(\iota(\ker(f))) = \epsilon(\iota(\ker(f))) \tag{20.48}$$

so that every element of G in the kernel is mapped to the neutral element/ □

Coequalizer

Example 20.6.3. The coequalizer of two morphisms $f, g : X \to Y$ in **Set** is the quotient set for the equivalence defined by those morphisms :

$$\mathrm{coeq}(f, g) = \{f(x) = g(x)\} \tag{20.49}$$

Example 20.6.4. The coequalizer of a group homomorphism with the trivial group homomorphism ϵ is the cokernel

$$\mathrm{coeq}(f, \epsilon) = \mathrm{coker}(f) \tag{20.50}$$

Theorem 20.6.2. The resulting morphism of a coequalizer $\mathrm{coeq}(f, g) \to X$ is an epimorphism.

Proof. For a coequalizer $X \rightrightarrows Y$ with morphism $q : Y \to Q$, given two morphisms $u, v : Q \to Z$ for any object Z, if $u \circ q = v \circ q$, this means that $u \circ q$ and $v \circ q$ are the same morphism q' in the universal property of the coequalizer. □

Equalizer for terminal object

Generalization : multiple equalizer

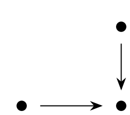
Defined iteratively?

Coequalizer for a quotient by group action?

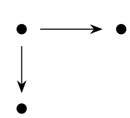
Example 20.6.5. If an object has some endomorphism generated by a group action $\rho_g : X \to X$, its quotient space X/G is the coequalizer

20.7 Pullbacks and pushouts

The pullback and pushout are the limits of dual diagrams, the span and cospan, where the span is given by two morphisms to the same object,



or $\bullet \to \bullet \leftarrow \bullet$ for short, which we will denote by the category $\mathbf{\Lambda}$, and two morphisms from the same object for the cospan :



or $\bullet \leftarrow \bullet \rightarrow \bullet$, which we will denote by the category $\mathbf{V}$. Those are opposite diagrams, in the sense that $\mathbf{\Lambda} = \mathbf{V}^{\text{op}}$.

The pushout is the limit of the span, while the pullback is the limit of the cospan. From their opposition, we can also say that the pushout is the colimit of the cospan and the pullback the colimit of the span.

The span diagram can be mapped to any three objects connected thusly. For $A, B, C \in \mathbf{C}$, and $f : A \to C$, $g : B \to C$, the diagram of shape I will be

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

If we now look at the constant functor $\Delta_X : I \to \mathbf{C}$, this will map our three objects A, B, C to X, and f and g to Id_X. To find the limit of F, we therefore need to find the natural transformation $\eta : \Delta_{\lim F} \to F$

Component-wise :

$$\bullet_1 \xrightarrow{\alpha_1} \bullet_3 \xleftarrow{\alpha_2} \bullet_2 \tag{20.51}$$

$$\eta_{\bullet_i} : \lim F \to A, B, C \tag{20.52}$$

For any morphism, ie either the first or second morphism in I :

$$\eta_C = f \circ \eta_A \tag{20.53}$$

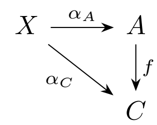
$$\eta_C = g \circ \eta_B \tag{20.54}$$

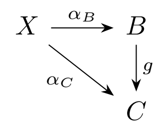
For any $X \in \mathbf{C}$, and any $\alpha : \Delta_X \to F$, there is a unique morphism $h : X \to \lim F$ such that $\alpha = \eta \circ F$.

Components of α : for every $Y \in I$, a morphism $\alpha_Y : \Delta_X(Y) \to F(Y)$, so

$$\alpha_Y : X \to F(Y) \tag{20.55}$$

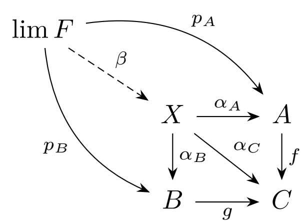
$F(Y)$ can only be A, B, C, so we have three components for objects, and for $f : A \to C$ and $g : B \to C$, then using what we know of Δ_X we have the following commuting diagrams :



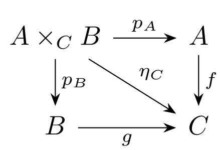


$$F(f) \circ \eta \tag{20.56}$$

The resulting cone is



The limit is the pullback, denoted as $A \times_C B$, along with the two and the universal cone that we have constructed gives us the commutative square



This means that our pullback diagram is given by this object and the two projectors, obeying

$$f \circ p_A = g \circ p_B \tag{20.57}$$

The interpretation of this is the *dependent sum* of the equality

$$\sum_{a:A} \sum_{b:B} (f(a) = g(b)) \tag{20.58}$$

Example 20.7.1. In **Set**, the pullback by $f : A \to C$, $g : B \to C$ is the set

$$A \times_C B = \{(a, b) \in A \times B | f(a) = g(b)\} \tag{20.59}$$

$$\bigcup_{c \in f(A) \cap g(B)} f^{-1} \tag{20.60}$$

Example 20.7.2. A particular type of pullback in **Set** is the pullback of $f : A \to B$ by the identity function on B, $\mathrm{Id}_B : B \to B$

$$A \times_B B = \{(a, b) \in A \times B | f(a) = b\} \tag{20.61}$$

which is the graph of the function f, the pairs $(x, f(x))$.

Definition 20.7.1. For two subobjects $\iota_1 : U_1 \hookrightarrow X$, $\iota_2 : U_2 \hookrightarrow X$, the pullback of $U_1 \to X \leftarrow U_2$ is called the intersection of U_1 and U_2

$$U_1 \cap U_2 = U_1 \times_X U_2 \tag{20.62}$$

This is easy enough to see in terms of **Set**, where this pullback gives us

$$U_1 \times_X U_2 = \{(x_1, x_2) \in U_1 \times U_2 | \iota_1(x_1) = \iota_2(x_2)\} \tag{20.63}$$

which is the set of all points that are in both U_1 and U_2 in X [diagonal morphism?]

Likewise, we can define the intersection of two subobjects thusly

Definition 20.7.2. The intersection of two subobjects $\iota_1 : U_1 \hookrightarrow X$, $\iota_2 : U_2 \hookrightarrow X$ is the pushout of the inclusion maps of their intersection, that is

$$\iota_{1,U_1 \cap U_2} : U_1 \cap U_2 \quad \to \quad U_1 \tag{20.64}$$

$$\iota_{2,U_1 \cap U_2} : U_1 \cap U_2 \quad \to \quad U_2 \tag{20.65}$$

and so

$$U_1 \cup U_2 = U_1 \times_{U_1 \cap U_2} U_2 \tag{20.66}$$

Example 20.7.3. A typical example of the pullback is given in fiber bundles, for instance in the category of manifolds, where given a bundle morphism $\pi : E \to B$ and some morphism $f : B' \to B$, the pullback $B' \times_B E$ is the *pullback bundle*, which is a fiber bundle with the same typical fiber as E but done over B', given by the equation

$$f^* E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \tag{20.67}$$

Semantics of an equation

Properties :

Theorem 20.7.1. The pullback of any morphism $f : X \to Y$ by the identity $\mathrm{Id}_X : X \to X$, $X \times_X Y$, is an isomorphism.

Theorem 20.7.2. The pullback to the terminal object, $X \to 1 \leftarrow Y$, is the product $X \times Y$.

$$\begin{array}{ccc} A \times B & \xrightarrow{p_A} & A \\ \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle !_A} \\ B & \xrightarrow[!_B]{} & 1 \end{array}$$

Proof. As the morphisms from any object to the terminal object are unique, every span $A \to 1 \leftarrow B$ is in unique correspondence with the pair of objects (A, B). As the universal cone over the pullback will induce a universal cone over (A, B) similar to that of the product, it is isomorphic to the product. □

Example 20.7.4. In the context of differential geometry, it is common to call "pullback" simply the act of composition of two maps. That is, for two smooth maps between manifolds $f : M \to N$ and $g : N \to P$, the pullback of g by f is

$$f^* g = g \circ f \tag{20.68}$$

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle f}\downarrow & \searrow^{\alpha_C} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[g]{} & P \end{array}$$

As spans and cospans are dual diagrams, we therefore have a duality between pullback and pushout, ie for a pullback $X \times_Z Y$ in $\mathbf{C}$, there is a dual pushout which is given by $X \sqcup_{X \times_Z Y} Y$, and is isomorphic to Z

Example 20.7.5.

Theorem 20.7.3. Isomorphisms always have a pullback.

Proof. Given some isomorphism $f : X \to Y$ and any other morphism $g : Z \to Y$, we can form a pullback square as

$$\begin{array}{ccc} Z & \xrightarrow{f^{-1} \circ g} & X \\ {\scriptstyle \mathrm{Id}_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

As pullback squares are always unique up to isomorphism, this means that the pullback of any isomorphism along a function will itself be an isomorphism. □

Dually, we define the pushout as the limit of a cospan diagram,

$$\bullet_1 \xleftarrow{\alpha_1} \bullet_3 \xrightarrow{\alpha_2} \bullet_2 \tag{20.69}$$

which corresponds to a functor mapping this diagram to a diagram of the form

$$X \xleftarrow{f} Z \xrightarrow{g} Y \tag{20.70}$$

so that a pushout is an operation on a pair of morphisms with the same target.

[...]

Example 20.7.6. Given the inclusion maps of the intersection of two sets, $\iota_1 : U_1 \cap U_2 \hookrightarrow U_1$ and $\iota_2 : U_1 \cap U_2 \hookrightarrow U_2$, their pushout is the union of the two :

$$U_1 \cap U_2 = U_1 +_{U_1 \cap U_2} U_2 \tag{20.71}$$

In terms of coproduct and coequalizer, this is the disjoint sum of the two sets with their overlaps identified.

Example 20.7.7. Given a subspace $\iota : X \to Y$ in **Top**,

Theorem 20.7.4. Given a pushout $X +_Z Y$ for the functions $f : X \to Z$ and $g : Y \to Z$, if g is an epimorphism, then $f_* g$ is an epimorphism.

Proof. Given two morphisms $h_1, h_2 : X +_Z Y \to W$ such that $h_1 f_* g = h_2 f_* g$, if we precompose it with g, we obtain

$$\begin{aligned} h_1(f_* g)g &= h_2(f_* g)g & (20.72)\\ &= h_1(g_* f)f & (20.73)\\ &= h_2(g_* f)f & (20.74) \end{aligned}$$

Since f is an epimorphism, we have that $h_1(g_* f) = h_2(g_* f)$. By the universal property of the pushout, for the two morphisms $h_1(g_* f) : h_2(g_* f)$ □

20.7.1 Fibers and cofibers

A specific type of pullback that we will commonly use is the fiber of a morphism, which is the pullback by a point :

Definition 20.7.3. The *fiber* of a morphism $f : X \to Y$ by a given base point in Y, $p : 1 \to Y$, is the pullback with the terminal object :

$$X \xrightarrow{f} Y \xleftarrow{p} 1 \tag{20.75}$$

which we denote by

$$\mathrm{Fib}_p(f) = X \times_Y 1 \tag{20.76}$$

As monomorphisms are stable under pullback, and $x : 1 \to Y$ is always a monomorphism, we have that $\mathrm{Fib}(f) \to X$ is always a monomorphism, and therefore corresponds to a subobject of X.

From the definition of pullbacks as equalizers of products, we can rewrite this notion as the equalizer of f and p as factored through the projectors of the product

$$X \times 1 \rightrightarrows Y \tag{20.77}$$

As $X \times 1$ is isomorphic to X itself, this will simply be some subobject of X, given by

$$X \times_Y 1 = \mathrm{eq}(f \circ \mathrm{pr}_1, p \circ \mathrm{pr}_2) \tag{20.78}$$

In other words, the subobject of X which, mapped through f, is the point p of Y.

Example 20.7.8. As the name implies, the classic example of a fiber of a morphism is given by the case of the fiber of a bundle. If f is an epimorphism[?], such as in the case of a bundle space $\pi : E \to B$, the fiber $\mathrm{Fib}(\pi, x) = E \times_B 1$ is the subobject $E_x \hookrightarrow E$ which projects down to the point x :

$$\pi(\iota(E_x)) = x(1) \tag{20.79}$$

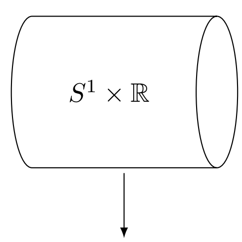
which is the property of the fiber in a bundle.

For instance, if we take the trivial circle bundle on the line, it is given by

$$\pi : S^1 \times \mathbb{R} \to \mathbb{R} \tag{20.80}$$

its fiber at any given point $x \in \mathbb{R}$ is the circle

$$S^1_x = \left\{(\theta, y) \in S^1 \times \mathbb{R} \mid \mathrm{pr}_2(\theta, y) = x\right\} \tag{20.81}$$



Example 20.7.9. More generally, for any function on **Set**,

Example 20.7.10. Another example of a fiber is given by additive categories, such as **Vec** or **Grp**, where the fibers over the zero object are the kernels of the maps.

$$\ker(f) = \mathrm{Fib}(f, 0) = f \times_0 1 \tag{20.82}$$

which are also given by the set of points mapped to the zero subspace by f.

Theorem 20.7.5. The fiber of an isomorphism f is the terminal object

$$\mathrm{Fib}_p(f) \cong 1 \tag{20.83}$$

Proof. Pullback of isomorphisms ☐

Dually to the fiber is also the cofiber, the equivalent for the pushout.

Definition 20.7.4. A *cofiber* of a morphism $f : X \to Y$ is the pushout of the span with the terminal object

$$1 \longleftarrow X \xrightarrow{f} Y \tag{20.84}$$

$$\mathrm{Cofib}(f : X \to Y) = \tag{20.85}$$

Unlike the fiber, the cofiber does not depend on any base point as there is a unique morphism $X \to 1$.

Cokernels in additive categories

Example 20.7.11. In an additive category such as **Vec** or **Grp**, the cofiber of a map $f : X \to Y$ is the *cokernel*, it is the subobject of Y which is

Theorem 20.7.6. The fiber of the identity is the object :

$$\mathrm{Fib}(\mathrm{Id}_X) = X \tag{20.86}$$

Theorem 20.7.7. The cofiber of an epimorphism is the terminal object.

Proof. As pushouts preserve epimorphisms, the pointed object $1 \to \mathrm{Cofib}(f)$ is also an epimorphism. However, the only object for which a morphism out of the terminal object is an epimorphism in a topos is the terminal object itself[?]. ☐

Theorem 20.7.8. The fiber of the terminal morphism $!_X$ is the identity.

Proof. ☐

Theorem 20.7.9. The double fibration of an object is the terminal object.

Proof. Given a morphism $f : X \to Y$,

$$\begin{array}{ccc} \mathrm{Fib}_p(f) & \xrightarrow{!} & 1 \\ {\scriptstyle p^* f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

If we try to further find the fiber of $p^* f$,

$$\begin{array}{ccccc}
\mathrm{Fib}_q(\mathrm{Fib}_p(f)) & \longrightarrow & \mathrm{Fib}_p(f) & \xrightarrow{!} & 1 \\
\downarrow{\scriptstyle !} & & {\scriptstyle p^* f}\downarrow & & \downarrow{\scriptstyle p} \\
1 & \xrightarrow{q} & X & \xrightarrow[f]{} & Y
\end{array}$$

As both squares are pullbacks, we have that the outer square is a pullback as well, so that the double fiber of f is the fiber product

$$\mathrm{Fib}_p(\mathrm{Fib}_p(f)) \cong 1 \times_Y 1 \cong \mathrm{eq}(p, f \circ q) \tag{20.87}$$

For any object Z and morphism $g : Z \to 1$, we therefore have that there must be a unique morphism $u : Z \to \mathrm{Fib}_p(\mathrm{Fib}_p(f))$ such that $\mathrm{eq} \circ u = m$. As there can be only one such morphism from Z to 1, this means $!_{\mathrm{eq}} \circ u =!_Z$. As this is true for any morphism, for it to be unique, we must simply have

$$\mathrm{Fib}_p(\mathrm{Fib}_p(f)) \cong 1 \tag{20.88}$$

[true?] □

20.7.2 Kernel pairs

Definition 20.7.5. For a morphism $f : X \to Y$, its *kernel pair* is the pullback of f with itself :

$$X \times_Y X \tag{20.89}$$

or equivalently, the equalizer of two identical functions on $X \times X$.

Kernel pair can be understood as pairs of elements with the same value under f

Example 20.7.12. For a group homomorphism $f : G \to H$, the kernel pair of f is the set of elements of G which have the same image under f. For instance, given the standard double cover

$$p : \mathrm{Spin}(n) \to \mathrm{SO}(n) \tag{20.90}$$

The kernel pair of p is given by the set of all pairs of elements with the same image, which is given by pairs $(g, -g)$ for all $g \in \mathrm{Spin}(n)$.

Theorem 20.7.10. The kernel pair of the identity morphism is the diagonal morphism

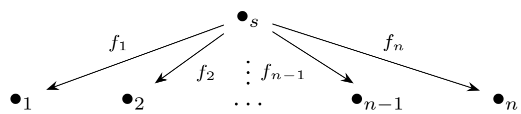
$$\Delta_X : X \overset{(\mathrm{Id}_X, \mathrm{Id}_X)}{\longrightarrow} X \times X \tag{20.91}$$

Cokernel pair

20.7.3 Wide pullbacks and pushouts

The spans and cospans diagrams can be generalized to an arbitrary number of arrows, in what are called sources and sinks

Definition 20.7.6. A *source* is a diagram composed of a central object $\bullet_s$ and a collection of arrows to outward objects $f_i : \bullet_s \to \bullet_i$



with the case $n = 0$ being the terminal diagram (with limit the identity), $n = 1$ the arrow (with limit the source) and $n = 2$ the span (with limit the pullback). Its limits is what is called the *wide span*

Definition 20.7.7. A *wide pullback* is the limit of a functor

$$F : \mathbf{src}^n \to \mathbf{C} \tag{20.92}$$

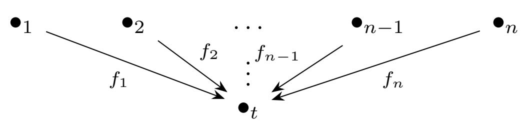
denoted by

$$\lim_{\mathbf{src}^n} F = \prod_{i=1}^{n}{}_{F(\bullet_s)} F(\bullet_i) \tag{20.93}$$

Theorem 20.7.11. A wide cospan can be alternatively written out

Dually, the generalization of a cospan is a *sink*

Definition 20.7.8. A *sink* is a diagram composed of a central object $\bullet_t$ and a collection of arrows from outward objects $f_i : \bullet_i \to \bullet_t$



Definition 20.7.9. A *wide pushout* is the limit of a functor

$$F : \mathbf{snk}^n \to \mathbf{C} \tag{20.94}$$

denoted by

$$\lim_{\mathbf{snk}^n} F = \coprod_{i=1}^{n}{}_{F(\bullet_s)} F(\bullet_i) \tag{20.95}$$

Is there some associator-like functor there

20.7.4 Dependent products and sums

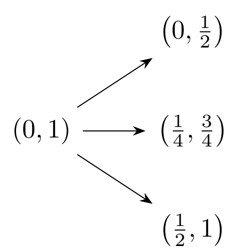
The notion of dependent sum and product from type theory is in fact a variant of that of pullback and pushout[32].

Definition 20.7.10. The dependent product $\prod_f$ for a morphism $f : X \to 1$ is defined as the pullback of

Dependent product/sum, indexed objects

20.8 Directed limits

Given some directed set J forming our diagram, a directed limit is a limit of a functor $F : J \to \mathbf{C}$. As we have seen 20.5, there is little point in considering the limit of a directed set with an initial object, or the colimit of a directed set with a terminal object, so that we will consider only the cases of directed sets with no minimal (resp. maximal) elements, such as for instance the directed set of connected open sets on $(0, 1)$:



which has no

Definition 20.8.1. If the directed set J is furthermore a well-ordered set,

$$\ldots \to 2 \to 1 \to 0 \tag{20.96}$$

the limit of this diagram is a *sequential limit*.

Example 20.8.1. Given a sequence of inclusions

$$\ldots \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow X_0 \tag{20.97}$$

its sequential limit is the union over all indexes

$$\lim_J F = \bigcup_{i \in I} X_i \tag{20.98}$$

Proof. □

Example 20.8.2. Given some directed set in **Set** ordered by inclusion, its directed limit is the union of all sets involved

Projective limit

20.9 Properties of limits and colimits

Theorem 20.9.1. For two diagrams I, J, and a functor $F : I \times J \to \mathbf{C}$,

$$\underset{I}{\mathrm{colim}} \lim_{J} F \to \lim_{J} \underset{I}{\mathrm{colim}} F \tag{20.99}$$

Definition 20.9.1. If we furthermore have the inverse morphism

$$\lim_{J} \underset{I}{\mathrm{colim}} F \to \underset{I}{\mathrm{colim}} \lim_{J} F \tag{20.100}$$

we say that the limit and colimit commute.

The limits and colimits can also be used to define specific morphisms,

Definition 20.9.2. An epimorphism $f : X \to Y$ is *effective* if it is the morphism of the kernel pair

$$X \times_Y X \rightrightarrows X \xrightarrow{f} Y \tag{20.101}$$

"behaves in the way that a covering is expected to behave, in the sense that "Y is the union of the parts of X", identified with each other in some specified way"."

$$\coprod_i U_i \to U \tag{20.102}$$

Limits and colimits in functor categories

Theorem 20.9.2. Given any limit or colimit of $\mathbf{D}$, then $[\mathbf{C}, \mathbf{D}]$ has the same limit or colimit, computed pointwise.

Proof. □

20.10 2-limits

In the case of a 2-category,

20.11 Presentation of a category

While categories can, in set theoretical terms, be quite large, even larger than a set can accomodate, it is common for them to at least be able to be generated from smaller sets of objects. This is the *presentation* of a category.

Definition 20.11.1. A localy small category is said to be κ-accessible if

Example 20.11.1. We can define a variety of subcategories of **Set** as the κ-accessible. **FinSet** is the

Proof. proof? For any set A, we have a directed diagram $\mathrm{Sub}(A)$ of its subset, and the finite subsets given by

$$\mathrm{Sub}^{\aleph_0}(A) \tag{20.103}$$

□

20.12 Limits and functors

A common tool in category theory to use is the behavior of limits under the action of a functor.

Definition 20.12.1. For a functor $F : \mathbf{C} \to \mathbf{D}$ and a diagram $J : \mathbf{I} \to \mathbf{C}$, a functor is said to preserve the limit $\lim_J$ if

$$F \circ \lim_J \cong \lim_{F \circ J} \tag{20.104}$$

Preservation of limits and colimits

Left and right exact functors :

Definition 20.12.2. A functor is *left exact* (resp. *right exact*)

Maps inital objects to initial objects, products to products, and equalizers to equalizers

Even if we do not know the explicit limits and colimits of a category, we can verify that a functor preserves them, using the notion of a flat functor.

Definition 20.12.3. A functor is *flat*

[Flat functors preserve any limit and colimit]

Example 20.12.1. The covariant hom-functor preserve limits :

$$h^X(\lim F) = \lim(h^X F) \tag{20.105}$$

and the contravariant hom-functor preserves limits in the category $\mathbf{C}^{\text{op}}$, ie colimits in $\mathbf{C}$:

$$h_X() \tag{20.106}$$

Proof. □

Example 20.12.2. In the category of vector spaces **Vec**, the covariant hom functor h^V gives us the set of linear transformations $L(V, -)$. If we take a look at various cases, we have for the terminal object k^0 :

$$\begin{aligned} h^V(k^0) &= L(V, k^0) & (20.107)\\ &= \{0\} & (20.108)\\ &\cong \{\bullet\} & (20.109) \end{aligned}$$

The product of two vector spaces is the direct sum

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \tag{20.110}$$

The kernel of a linear map f can be described as the equalizer of this map with the 0 map, Equalizer : for $f, 0 : X \to Y$, the equalizer is $\ker(f)$. The two functions f, g map to

$$h^V(f) = \{a \in L(X, Y)\} \tag{20.111}$$

$$h^V(0) = \{0\} \tag{20.112}$$

$$\begin{aligned} h^V(\ker(f)) &= L(V, \ker(f)) & (20.113)\\ &= \text{eq}(h^V(f), h^V(0)) & (20.114)\\ &= \text{eq}(h^V(f), \{0\}) & (20.115)\\ &= \ker() & (20.116) \end{aligned}$$

$$\begin{aligned} h^V(f) &= L(V, \ker(f)) & (20.117)\\ &= & (20.118) \end{aligned}$$

The contravariant one : initial object is also k^0, therefore terminal object in op

$$\begin{aligned} h_V(k^0) &= L(k^0, V) && (20.119)\\ &= \{0\} && (20.120)\\ &\cong \{\bullet\} && (20.121) \end{aligned}$$

Same deal with the coproduct in op, which is also $\oplus$

$$h^V(W \oplus W') = h^V(W) \times h^V(W') \quad (20.122)$$

but $\lim(h^V \circ *) = \lim \varnothing = *$

Example 20.12.3. In the category of sets, terminal object :

$$h^X(\{\bullet\}) = \{!_X\} \quad (20.123)$$

Product :

$$h^X(Y \times Z) = h^X(Y) \times h^X(Z) \quad (20.124)$$

Definition 20.12.4. Given a functor $F : \mathbf{C} \to \mathbf{D}$, and a diagram $J : \mathbf{I} \to \mathbf{C}$, we say that F reflects the limits of J if for a cone $\eta : \Delta_X \to \mathbf{J}$ over J in $\mathbf{C}$ for which $F(\eta)$ is a limit of $F \circ J$ in $\mathbf{D}$, then η was already a limit of J in $\mathbf{C}$.

Theorem 20.12.1. Given a morphism $f : X + Y \to Z$, there is an equivalent pair of morphisms $f_1 : X \to Z$ and $f_2 : Y \to Z$

Proof. From the contravariance of the hom functor,

$$\mathrm{Hom}_{\mathbf{C}}(X + Y, Z) = \mathrm{Hom}_{\mathbf{C}}(X, Z) \times \mathrm{Hom}_{\mathbf{C}}(Y, Z) \quad (20.125)$$

So that for any morphism $f : X + Y \to Z$, we have the given two morphisms

$$\begin{aligned} f \circ i_1 : f_1 : X &\to Z && (20.126)\\ f \circ i_2 : f_2 : Y &\to Z && (20.127) \end{aligned}$$

which acts functorially in its composition with other morphisms, ie if we have the post-composition function

$$f \circ (-) : \mathrm{Hom}_{\mathbf{C}}(X + Y, s(-)) \to \mathrm{Hom}_{\mathbf{C}}(X + Y, t(-)) \quad (20.128)$$

then for any $g : Z \to W$ in $\mathrm{Hom}_{\mathbf{C}}(Z, W)$, its post-composition with f

$$g \circ f : X + Y \to W \quad (20.129)$$

$\mathrm{Hom}_{\mathbf{C}}(X + Y, g)$ □

Theorem 20.12.2.

Created limits

20.13 Morphisms

Limits also allow for the definition of a few more specific definitions of morphisms.

Definition 20.13.1. An epimorphism $f : X \to Y$ is *effective* if there exists a kernel pair $X \times_Y X$ for it, and

$$X \times_Y X \rightrightarrows X \xrightarrow{f} Y \tag{20.130}$$

The interpretation of an effective epimorphism is that it is meant to convey that

Definition 20.13.2. An epimorphism $f : X \to Y$ is *regular* if it is the coequalizer of some parallel pair $Z \rightrightarrows X$

21

Fibrations

As with the general notion in topology, and the one we defined on objects 20.7.1, there is also a notion of fibration in categories themselves. This notion is simply the application of the categorical notion of fiber to that of the category of categories. So for a functor $\pi : \mathbf{E} \to \mathbf{B}$, and some object of $\mathbf{B}$ defined by the functor $X : \mathbf{1} \hookrightarrow \mathbf{B}$, the fiber $E_X = \pi^{-1}(X)$ is given by the pullback

$$\begin{array}{ccc} E_X & \xrightarrow{!_{E_X}} & \mathbf{1} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle X} \\ \mathbf{E} & \xrightarrow[\pi]{} & \mathbf{B} \end{array}$$

Example 21.0.1. For a forgetful functor to **Set**, the fiber of a set is the category of all objects with the same underlying sets, and their morphisms are [...]

In particular for Top, those are all the topologies on that set

Along with fibrations, we also define opfibrations, which are fibrations on the opposite functor, so that the opfibration

Definition 21.0.1. A fibration of $F : \mathbf{E} \to \mathbf{B}$ is *discrete* if for any $e \in \mathbf{E}$, and any morphism $g : b \to F(e)$, then there is a unique morphism $h : b' \to b$ in $\mathbf{B}$ such that $F(h) = g$.

21.1 Grothendieck fibrations

Definition 21.1.1. A *Grothendieck fibration* of a functor $\pi : \mathbf{E} \to \mathbf{B}$ is a

Example 21.1.1. For the arrow category $[\mathbf{I}, \mathbf{C}]$, with the functor

$$t : [\mathbf{I}, \mathbf{C}] \quad \to \quad \mathbf{C} \tag{21.1}$$

$$(f : X \to Y) \quad \mapsto \quad t(f) = Y \tag{21.2}$$

t is a Grothendieck fibration, called the *codomain fibration*.

22 Comma categories

The notion of a comma category can be used to describe categories whose objects are the morphisms of another category.

Definition 22.0.1. The *comma category* $(F \downarrow G)$ of two functors $F : C \to E$ and $G : D \to E$ is the category composed of triples (c, d, α) such that $\alpha : F(c) \to G(d)$ is a morphism in E, and whose morphisms are pairs (β, γ)

$$\beta \ : c_1 \ \to c_2 \tag{22.1}$$
$$\gamma \ : d_1 \ \to d_2 \tag{22.2}$$

that are morphisms in C and D, such that $\alpha_2 \circ F(\beta) = G(\gamma) \circ \alpha_1$ [Commutative diagram]

Composition

If we look at a comma category in the context of the category of categories **Cat**,

Definition 22.0.2. A comma category $(F \downarrow G)$ of two functors $F : \mathbf{C} \to \mathbf{E}$, $G : \mathbf{D} \to \mathbf{E}$ is the pullback of the following span

$$\begin{array}{ccc} (F \downarrow G) & \longrightarrow & \mathbf{E}^{\mathbf{I}} \\ \downarrow & & \downarrow \\ \mathbf{C} \times \mathbf{D} & \xrightarrow{(F,G)} & \mathbf{E} \times \mathbf{E} \end{array}$$

where $\mathbf{E}^{\mathbf{I}}$ is the functor category of the interval category $\mathbf{I}$

Theorem 22.0.1. The pullback definition is isomorphic to the first one.

It can furthermore be defined entirely as a 2-limit in the context of **Cat** as a 2-category.

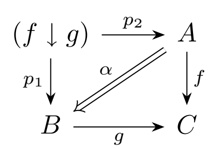
Definition 22.0.3. The *comma object* of two 1-morphisms

$$f : A \to C \tag{22.3}$$
$$g : B \to C \tag{22.4}$$

in a 2-category is an object $(f \downarrow g)$ with two projections

$$p_1 : (f \downarrow g) \to A \tag{22.5}$$
$$p_2 : (f \downarrow g) \to B \tag{22.6}$$

given by the 2-limit of this span



with α a 2-cell isomorphism.

Theorem 22.0.2. A comma category is the comma object in **Cat**.

Proof. □

Theorem 22.0.3. Given a comma category $(F_1 \downarrow F_2)$, with $F_i : X_i \to Y$, if we have a natural transformation

$$\alpha_i : F_i \Rightarrow F_i' \tag{22.7}$$

this induces a canonical functor

$$(\alpha_1 \downarrow F_2) : (F_1' \downarrow F_2) \to (F_1 \downarrow F_2) \tag{22.8}$$
$$(F_1 \downarrow \alpha_2) : (F_1 \downarrow F_2) \to (F_1 \downarrow F_2') \tag{22.9}$$

Proof. If we have a natural transformation $\alpha_i : F_i \Rightarrow F_i'$, the span of the **Cat** 2-category is □

Comma categories are rarely used directly, but are more typically used to define more specific operations. The three important one we will see are arrow categories, slice categories and coslice categories.

22.1 Arrow categories

The simplest kind of comma category is the arrow category, where we just consider the arrows of a category as a category.

Definition 22.1.1. An arrow category is the comma category for the case where the two functors are the identity functors, $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$

$$\mathrm{Arr}(\mathbf{C}) = (\mathrm{Id}_{\mathbf{C}} \downarrow \mathrm{Id}_{\mathbf{C}}) \tag{22.10}$$

Theorem 22.1.1. The arrow category $\mathrm{Arr}(\mathbf{C})$ is equivalent to the category whose objects are the morphisms of $\mathbf{C}$:

$$\mathrm{Obj}(\mathrm{Arr}(\mathbf{C})) = \mathrm{Mor}(\mathbf{C}) \tag{22.11}$$

and its morphisms are given by pairs of morphisms (f, g) in $\mathbf{C}$ obeying the commutative square

22.1.1 Commutative diagrams

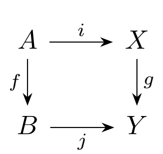
The arrow category $\mathrm{Ar}(\mathbf{C})$ of a category $\mathbf{C}$ gives us a way to describe morphisms on arrows internally, and in particular the notion of commutative diagram.

As the hom-sets of a category are their collections of morphisms, one notion we can define are *morphisms of arrows*, where we define functions on the sets of morphisms between two objects. One example that we have seen already is the notion of pre- and post-composition of morphisms in the hom-set.

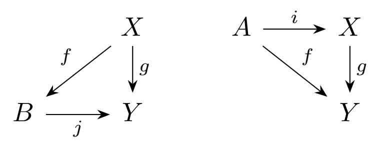
Definition 22.1.2. A *morphism of arrow* from a morphism $f : A \to B$ to a morphism $g : A \to B$ is given by two morphisms $i : A \to X$ and $j : B \to Y$ which obey the morphism equivalence

$$j \circ f \cong g \circ i \tag{22.12}$$

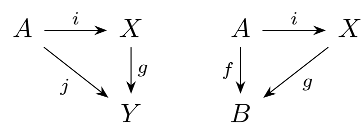
A morphism of arrows in general therefore defines a commutative square diagram,



In particular, we have that our diagram is a commutative triangle if one of the morphism i, j is the identity morphism, for instance for $i = \mathrm{Id}_X$ and $j = \mathrm{Id}_Y$, we get the two diagrams



as well as if either f or g is the identity morphism, leading to the diagrams



The hom-set in the context of the arrow category is a collection of objects which map to

And the action of the hom-set is simply the specific case of a morphism of arrows on the arrows

22.2 Slice categories

Given an object $X \in \mathbf{C}$, we can define the *over category* (or *slice category*) $\mathbf{C}_{/X}$ by taking all morphisms emanating from X as objects :

$$\mathrm{Obj}(\mathbf{C}_{/X}) = \{f \mid s(f) = X\} \tag{22.13}$$

As a comma category, this is the comma category of the two functors $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ and $\Delta_X : \mathbf{1} \to \mathbf{C}$, of the identity functor on $\mathbf{C}$ and the inclusion of the object X, in which case $\mathbf{C}_{/X} = (\mathrm{Id}_{\mathbf{C}} \downarrow \Delta_X)$ is defined by the triples $(c, *, \alpha)$ of objects $c \in \mathbf{C}$, the unique object $* \in \mathbf{1}$, and morphisms in $\mathbf{C}$ $\alpha : c \to X$. As there is only one object in the terminal category, we can drop it as it is isomorphic to simply (c, α), and furthermore, c is implied by α as simply being the source term. Our slice category is therefore indeed just defined by the set of morphisms from objects of the category to our selected object.

Slice categories are useful to consider objects in a category as a category in themselves, where the objects are simply all the relations they have with all other objects in the category.

Example 22.2.1. In **Set**, given a set X, the slice category $\mathbf{Set}_{/X}$ has as its objects all functions with codomain X, $f : Y \to X$, and as morphisms all functions between sets $g : Y \to Y'$ for which

$$f'(g(y)) = f(y) \tag{22.14}$$

Category of X-indexed collections of sets, object $f : Y \to X$ is the X-indexed collection of fibers $\{Y_x = f^{-1}(\{x\})| \in X\}$, morphisms are maps $Y_x \to Y'_x$

[fiber product $Y \times_X Y'$ is the product in the slice category]

If we look for instance at $\mathbb{N}$ as a set (the natural number object of sets), its slice category $\mathbf{Set}_{/\mathbb{N}}$ is the category of all functions to numbers

Example 22.2.2. For a poset P, the slice category $\mathbf{P}_{/p}$ is isomorphic to the down set of p, ie the subcategory of every element $\{q|q \leq p\}$.

Objects : Id_p, and every map $\leq_{q,p}$ (corresponding to p and object inferior to p), morphisms are

Example 22.2.3. $\mathbf{Top}_{/X}$ is the category of covering spaces over X.

Example 22.2.4. The slice category of smooth manifolds $\mathbf{SmoothMan}_{/X}$ is a subcategory of smooth bundles over X, consisting only of its epimorphisms.[?]

Theorem 22.2.1. Given a category $\mathbf{C}$ with pullbacks and a morphism $f : X \to Y$ in that category, there is an induced functor between the slice categories

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{22.15}$$

where for morphisms $p : K \to Y$ in $\mathbf{C}_{/Y}$, we define the equivalent morphism to X in $\mathbf{C}_{/X}$ by pullback. An object of $\mathbf{C}_{/Y}$, some morphism $p : K \to Y$, is mapped to the pullback defined by

$$\begin{array}{ccc} X \times_Y K & \xrightarrow{p_A} & K \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and the bundle morphisms $(p : K \to Y) \to (p' : K' \to Y)$ are given by

Definition 22.2.1. Given a functor $F : \mathbf{C} \to \mathbf{D}$, we can defined a sliced functor for $X \in \mathbf{C}$ via :

$$F_{/X} : \mathbf{C}_{/X} \to \mathbf{D}_{/F(X)} \tag{22.16}$$

Theorem 22.2.2. If a category $\mathbf{C}$ has a limit for a given functor $F : I \to$

Theorem 22.2.3. If $\mathbf{C}$ has an initial object 0, $\mathbf{C}_{/X}$ has the initial object $\varnothing_X : 0 \to X$.

Proof. □

Definition 22.2.2. Given a category $\mathbf{C}$ and its slice category $\mathbf{C}_{/X}$, the subcategory of $\mathbf{C}_{/X}$ which is composed of its monomorphisms is denoted by

$$\mathrm{Mon}(X) \tag{22.17}$$

Theorem 22.2.4. If $\mathbf{C}$ has an initial object, $\mathbf{C}_{/0} \cong \mathbf{1}$.

Proof. □

Theorem 22.2.5. If $\mathbf{C}$ has a terminal object, $\mathbf{C}_{/1} \cong \mathbf{C}$.

Proof. Since every object has exactly one morphism to the terminal object,

$$\mathrm{Obj}(\mathbf{C}_{/1}) \cong \mathrm{Obj}(\mathbf{C}) \tag{22.18}$$

And since the terminal morphisms are left absorbing, any morphism in $\mathbf{C}$ commutes with the slice category triangle, meaning that

$$\mathrm{Mor}(\mathbf{C}_{/1}) \cong \mathrm{Mor}(\mathbf{C}) \tag{22.19}$$

□

22.3 Coslice categories

Definition 22.3.1. Given a category $\mathbf{C}$ and an object $X \in \mathbf{C}$,a coslice category, or under category, $\mathbf{C}^{X/}$, is the comma category of the identity functor $\mathrm{Id}_{\mathbf{C}}$ and the constant functor Δ_X,

$$\mathbf{C}^{X/} = \Delta_X \downarrow \mathrm{Id}_{\mathbf{C}} \tag{22.20}$$

Theorem 22.3.1. A coslice category $\mathbf{C}^{X/}$ is the category whose objects are morphisms in $\mathbf{C}$ with source X, and whose morphisms are morphisms in $\mathbf{C}$ which obey

23

Elements

One of the important difference between set theory and category theory is that while sets are composed of elements, as defined by the $\in$ relation, categories (for which the objects are often somewhat similar to sets themselves) do not seem to have a naturally equivalent notion. While we often have the notion of a concrete category, where objects of a categories are sets, this cannot be generalized to all categories. Poset categories in particular are particularly resistant to this interpretation.

If we wish to define elements of a set in terms of the morphisms of sets (functions), this is best done via the use of functions from the singleton set $\{\bullet\}$, as those functions are in bijection with the elements of a set

$$\mathrm{Fun}(\{\bullet\}, X) \cong X \tag{23.1}$$

As functions from the singleton are all of the form $\{(\bullet, x)\}$ for every $x \in X$ (from the properties of the Cartesian product), and this also works for function composition : for a function $x : 1 \to X$, $x(\bullet) = x$, we have

$$f \circ x = (\bullet, f(x)) \tag{23.2}$$

This sort of notion of elements can be defined for any category with a terminal object under the notion of a *global element*

Definition 23.0.1. For a category **C** with a terminal object 1, a *global element* of X is a morphism from the terminal object to X :

$$x : 1 \to X \tag{23.3}$$

While those kinds of elements are often very important for categories, we would like a broader notion of elements, and in particular, global elements are not necessarily the same as the elements of a set. For instance, for any category involving a preferred element, such as **Grp** or **Vec**, which morphisms have to preserve, we have that the only global element is that point, the neutral element for groups and the zero of a vector space. From group theory, we have that the group whose group homomorphisms are isomorphic to the group's elements is $\mathbb{Z}$

$$|G| \cong \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, G) \tag{23.4}$$

and likewise for sets, we have that the vectors of a k-vector space are isomorphic to the morphisms $v : k \to V$

$$|V| \cong \mathrm{Hom}_{\mathrm{Vec}}(k, V) \tag{23.5}$$

in matrix term, there's a correspondence between the row and column matrices.

Unlike what we might expect from the case of global elements, those two cases are not even monomorphisms,

Definition 23.0.2. A *generalized element* at stage of definition given by U (or a generalized element of shape U) is a morphism

$$x : U \to X \tag{23.6}$$

So that in particular, a global element is simply a generalized element at stage of definition 1.

We will see later on 30.0.3 what exact notion comes into the "good" definition of a generalized object if we want them to be the elements of the underlying set, although generalized objects do not need to be restricted to such notions. Generalized elements can encompass other notions such as curves

$$\gamma : \mathbb{R} \to M \tag{23.7}$$

or loop

$$\ell : S^1 \to M \tag{23.8}$$

Given the yoneda embedding $Y : C \hookrightarrow [C^{\text{op}}, \mathbf{Set}]$, the representable functor

$$\text{GenEl}(X) : C^{\text{op}} \to \mathbf{Set} \tag{23.9}$$

Sends each object U of C to the set of generalized elements of X at stage U.

23.0.1 Separability

Definition 23.0.3. An object $S \in \mathbf{C}$ is a *separator* if for every pair of morphisms $f : X \to Y$, and every morphism $e : S \to X$, then $f \circ e = g \circ e$ implies $f = g$.

In other words, the global elements generated by the separator are enough to entirely define the morphisms. For instance, in the case of **Set**, $\{\bullet\}$ is a separator, essentially saying that the elements of a set entirely define its functions : a function $f : X \to Y$ is defined by its value $f(x)$ for every $x \in X$.

If we have a topos E such that its terminal object 1 is a separator, and $1 \neq 0$, we say that the topos is *well-pointed*, meaning that

Other definitions : global section functor is faithful

Prop : well-pointed topos are boolean, its subobject classifier is two-valued,

Definition 23.0.4. A *concrete category* **C** is such that there exists a faithful functor F

$$F : \mathbf{C} \to \mathbf{Set} \tag{23.10}$$

What the faithful functor implies

$$\text{Hom}_{\mathbf{C}}(X, Y) \overset{F_{X,Y}}{\hookrightarrow} \text{Hom}_{\text{Set}}(F(X), F(Y)) \tag{23.11}$$

ie all the morphisms between two elements are injective. Given any two morphisms, if they are mapped to the same function, then they are the same morphism. Roughly speaking this is to say that we can consider functions on our category to just be a subset of all functions on their underlying set. If we have some function $f : X \to Y$, then there is a corresponding function $|f| : |X| \to |Y|$.

In the general case of a concrete category, we can define its set of points internally to the category by considering its left adjoint to the forgetful functor, the free functor

Theorem 23.0.1. For a concrete category with a forgetful functor $U : \mathbf{C} \to \mathbf{Set}$, and a left adjoint free functor $F : \mathbf{Set} \to \mathbf{C}$, the set of all points of a given object X taken as a set $U(X)$ is equivalent to the hom-set of its free object on a single element $F_1 = F(\{\bullet\})$.

Proof. ☐

Example 23.0.1. For the category of topological spaces, the free functor maps sets to their equivalent discrete topologies, and the free object of a single element F_1 is the terminal object in the category. Therefore

$$U(X) \cong \mathrm{Hom}_{\mathbf{Top}}(F_1, X) \tag{23.12}$$

Example 23.0.2. For the category of vector spaces, the free functor maps sets to the vector space generated by a basis of those elements (a set of n elements generates an n-dimensional vector space). The free object of a single element is the vector space isomorphic to the underlying field, $F_1 = k$, so that

$$U(V) \cong \mathrm{Hom}_{\mathbf{Vec}}(k, V) \tag{23.13}$$

which is just the statement that a vector space is isomorphic to the vector space of linear maps from the field to itself.

Example 23.0.3. For the category of groups, the free functor maps sets to their free group (the group of all strings generated by those elements by concatenation). The free object of one element is the group of integers $\mathbb{Z}$

$$U(G) \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \tag{23.14}$$

This is commonly the case in monoidal categories, such as **Hilb**

[Concrete categories and well-pointed ones do not imply each other in any direction, depends on if elements are global elements?]

Given a set-valued functor $F : \mathbf{C} \to \mathbf{Set}$, its *category of elements*

Category of elements

[...]

Global elements also interact with the subobject classifier.

What is the relation of global elements wrt the subobject classifier?

Definition 23.0.5. We say that a morphism $f : X \to Y$ in a category $\mathbf{C}$ with a terminal object 1 is a *surjection* if for any point $p : 1 \to Y$, there exists a point $q : 1 \to X$ such that $p = f \circ q$.

Example 23.0.4. In **Set**, every epimorphism is a surjection.

24

Monoidal categories

It is common in categories to have some need of defining a binary operation, some function of the type

$$A \cdot B = C \tag{24.1}$$

Given two objects $A, B \in \mathbf{C}$, we want to find a third object C, such that there exists a bifunctor $(-) \cdot (-) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$

$$A \cdot B = C \tag{24.2}$$

While the notion of bifunctor covers this well enough, we often need to have additional conditions. A common case is that of a *monoid*, usually denoted as $\otimes$, where we ask that the operation be associative and unital, so that for any triple of objects A, B, C, we have the isomorphisms

$$(A \otimes B) \otimes C \quad \cong \quad A \otimes (B \otimes C) \tag{24.3}$$

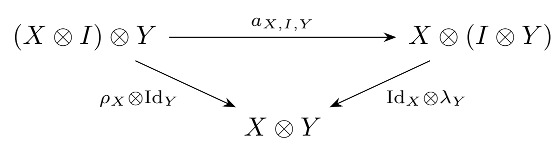
$$\exists I \in \mathbf{C},\ I \otimes A \quad \cong \quad A \otimes I \cong A \tag{24.4}$$

To categorify this notion, we define monoidal categories

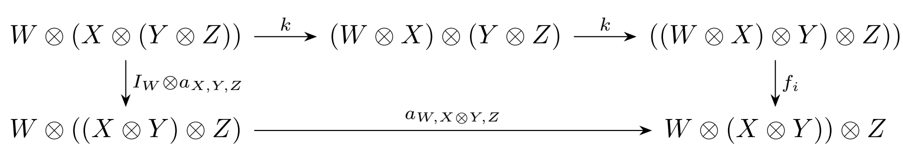
Definition 24.0.1. A *monoidal category* $(\mathbf{C}, \otimes, I)$ is a category $\mathbf{C}$, a bifunctor $\otimes$, and a specific object $I \in \mathbf{C}$, along with three natural transformations :

$$a : ((-) \otimes (-)) \otimes (-) \overset{\cong}{\to} (-) \otimes ((-) \otimes (-)) \tag{24.5}$$
$$(X \otimes Y) \otimes Z \mapsto X \otimes (Y \otimes Z) \tag{24.6}$$
$$\lambda : (I \otimes (-)) \overset{\cong}{\to} (-) \tag{24.7}$$
$$I \otimes X \mapsto X \tag{24.8}$$
$$\rho : ((-) \otimes I) \overset{\cong}{\to} (-) \tag{24.9}$$
$$(X \otimes I) \mapsto X \tag{24.10}$$

called the *associator*, the *left unitor* and the *right unitor*, which obey the following rules, the triangle identity :



and the pentagonal identity :



We do not ask the equality for the associator and unitors, as equivalence is typically what we ask in general for a category. If in addition those equivalences are equality, we say that this is a *strict monoidal category.*

Theorem 24.0.1. Given two morphisms $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, there is a tensor product morphism

$$f_1 \otimes f_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2 \tag{24.11}$$

Proof. By the bifunctoriality (jointly functorial?) of the tensor product. □

Example 24.0.1. The tensor product of two k-vector spaces is a monoid in $\mathbf{Vect}_k$, as $(\mathbf{Vec}_k, \otimes, k)$

Proof. First we need to show that the tensor product is bifunctorial. Given two morphisms $f : X \to X'$, $g : Y \to Y'$, the product $f \otimes g : X \otimes Y \to X' \otimes Y'$

Proof that it is functorial, unit k, associator, unitor, obeys the identities

Given the tensor product with k, consider the map

$$\lambda : k \otimes V \quad \to \quad V \tag{24.12}$$

$$(a, v) \quad \mapsto \quad av \tag{24.13}$$

□

A basic example of monoidal structures on a category is the product and coproduct. If a category **C** has all finite products, it is automatically a monoidal category, called a *Cartesian monoidal category*, given by $(\mathbf{C}, \times, 1)$.

Likewise, a category with all finite coproducts is called a *co-Cartesian monoidal category*, given by $(\mathbf{C}, +, 0)$

Example 24.0.2. The product and coproduct are both monoidal in **Set**, leading to the monoidal category $(\mathbf{Set}, \times, \{\bullet\})$.

A weaker notion to this is that of a *semi-Cartesian* monoidal category, which is when the unit of the tensor product is the terminal object. This is a useful notion to keep in mind due to this example :

Example 24.0.3. The category of Poisson spaces, **Poiss**, is a semi-Cartesian category.

This will have consequences on the differences between classical and quantum mechanics later on.

Definition 24.0.2. A *bimonoidal category*

24.1 Braided monoidal category

From the previous definition, there is a priori no relation between $X \otimes Y$ and $Y \otimes X$. In fact, in some monoidal categories, this can be the case that they are not.

Example 24.1.1. Consider the smallest non-commutative monoid, the monoid on the set $(1, a, b)$ with $a \cdot b = a$, $b \cdot a = b$, and $a^2 = a$, $b^2 = b$. If we pick some category with three objects, I, A, B,

$$A \otimes A \tag{24.14}$$

If we do want to relate those two monoidal products, we need to define the notion of braining on a monoidal category.

Definition 24.1.1. A *braided monoidal category* $(\mathbf{C}, \otimes, I, B)$ is a monoidal category equipped with a natural isomorphism B between the functors

$$B_X : X \otimes (-) \to (-) \otimes X \tag{24.15}$$

with components

$$B_{X,Y} : X \otimes Y \to Y \otimes X \tag{24.16}$$

obeying the commutative diagrams

$$\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{B_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\
{\scriptstyle B_{X,Y}\otimes \mathrm{Id}_Z}\downarrow & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\
(Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[\mathrm{Id}_Y \otimes B_{X,Z}]{} & Y \otimes (Z \otimes X)
\end{array}$$

which says that we can braid X to the right of $Y \otimes Z$ directly, or by braiding first X with Y and then X with Z, and

$$\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{a^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{B_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\
{\scriptstyle \mathrm{Id}_X \otimes B_{Y,Z}}\downarrow & & & & \downarrow{\scriptstyle a^{-1}_{Z,X,Y}} \\
X \otimes (Z \otimes Y) & \xrightarrow[a^{-1}_{X,Z,Y}]{} & (X \otimes Z) \otimes Y & \xrightarrow[B_{X,Z}\otimes \mathrm{Id}_Y]{} & (Z \otimes X) \otimes Y
\end{array}$$

which says that we can braid $X \otimes Y$ to the right of Z directly, or by braiding first Y with Z and then X with Z.

Definition 24.1.2. A monoidal category is *symmetric* if it is a braided monoidal category obeying the commutation law

$$B_{Y,X} \circ B_{X,Y} \cong \mathrm{Id}_{X\otimes Y} \tag{24.17}$$

Example 24.1.2. The category of vector spaces over a field k $\mathbf{Vec}_k$ is a symmetric monoidal category.

Example 24.1.3. An example of a non-symmetric monoidal category is given by the fusion categories, used for instance in the description of anyons. A fusion ring can be described by the lattice $\mathbb{Z}^N$ with the basis

$$\{\phi_i \mid 1 \leq i \leq N\} \tag{24.18}$$

with the structure constants

$$\phi_i \otimes \phi_j = \sum_k N^k{}_{ij} \phi_k \tag{24.19}$$

such that

24.2 Monoids in a monoidal category

Definition 24.2.1. In a monoidal category $(\mathbf{C}, \otimes, I)$, a *monoid* in $\mathbf{C}$ is an object M along with a multiplication map $\mu : M \otimes M \to M$, and a unit $\eta : I \to M$, which satisfy the associative law [...] and the left and right unital law : [...]

24.3 Category of monoids

Definition 24.3.1. Given a monoidal category $(\mathbf{C}, \otimes, I)$, the category of monoids $\mathrm{Mon}(\mathbf{C})$ is the category with objects

24.4 Modules

Definition 24.4.1. A left (resp. right) module over a monoid object A in a monoidal category $(\mathbf{C}, \otimes, I)$ is composed of an object $N \in \mathbf{C}$, and for a left module, a morphism

$$\lambda : A \otimes N \to N \tag{24.20}$$

$$\begin{array}{ccc} A\otimes A\otimes N & \xrightarrow{\mathrm{Id}_A\otimes\lambda} & A\otimes N \\ {\scriptstyle \cdot\otimes \mathrm{Id}_N}\downarrow & & \downarrow{\scriptstyle\lambda} \\ A\otimes N & \xrightarrow[\lambda]{} & Y \end{array}$$

$$\begin{array}{ccc} I\otimes N & \xrightarrow{e\otimes\mathrm{Id}_N} & A\otimes N \\ & \searrow \qquad \swarrow{\scriptstyle\lambda} & \\ & N & \end{array}$$

and for a right module, a morphism

$$\rho : N \otimes A \to N \tag{24.21}$$

$$\begin{array}{ccc} N\otimes A\otimes A & \xrightarrow{\rho\otimes\mathrm{Id}_A} & N\otimes A \\ {\scriptstyle \mathrm{Id}_N\otimes\cdot}\downarrow & & \downarrow{\scriptstyle\rho} \\ N\otimes A & \xrightarrow[\rho]{} & Y \end{array}$$

$$\begin{array}{ccc} N\otimes I & \xrightarrow{\mathrm{Id}_N\otimes e} & N\otimes A \\ & \searrow \qquad \swarrow{\scriptstyle\rho} & \\ & N & \end{array}$$

Module $A \otimes_R B$

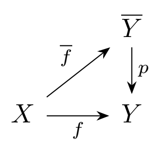
$$A \otimes R \otimes B \rightrightarrows A \otimes B \tag{24.22}$$

given by the action of R on A and on B

25

Lifts and extensions

Definition 25.0.1. A *lift* of a morphism $f : X \to Y$ through some morphism $p : \overline{Y} \to Y$ is a morphism $\overline{f} : X \to \overline{Y}$ such that $f = p \circ \overline{f}$, given by the commutative triangle



As the notation implies here, it is common for p to be an epimorphism, in which case the lift is lifting the morphism f to a "larger" space $\overline{Y}$ that was projected on Y, such that this new function projects to the same one.

Example 25.0.1. For a covering space $p : \overline{Y} \to Y$ in Top, ie such that for any point $y \in Y$, there is a neighbourhood $U_y \subseteq Y$ whose preimage is a discrete union

$$p^{-1}(U_y) = \coprod_i V_i \tag{25.1}$$

with all such subsets homeomorphic to U_y, $U_y \cong V_j$, then the lift of a function $f : X \to Y$ is a function $\overline{f} : X \to \overline{Y}$ which has the same values on those subsets via this same homeomorphism. For instance, the covering space given by the exponential map for a circle is

$$p : \mathbb{R} \to S^1 \tag{25.2}$$

$$x \mapsto e^{ix} \tag{25.3}$$

then the lift of a function on the circle $X \to S^1$ is a periodic function on the real line,

$$\overline{f}(x) = f(\theta) \tag{25.4}$$

Example 25.0.2. For a smooth map $f : X \to Y$ in the category of manifolds, the pushforward

Definition 25.0.2. An *extension* of a morphism $f : X \to Y$ through some morphism $\iota : Y \to \overline{Y}$ is a morphism $\tilde{f} : X \to \overline{Y}$ such that $\tilde{f} \circ \iota = f$, given by the commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{\tilde{f}} & \downarrow \iota \\ & & \overline{Y} \end{array}$$

From what we have seen in [X], commutative diagrams can be expressed in terms of properties of arrow categories. The lift and extension

In terms of arrow categories?

25.1 Lifting property

Example 25.1.1. In **Set**, any morphism that has the right lifting property with respect to the inclusion

$$\iota : \varnothing \hookrightarrow \{\bullet\} \tag{25.5}$$

is a surjective function.

Proof. □

Example 25.1.2. A classic example of an extension is that of a *group extension*, which is an extension in **Grp**, where given Central extension

Example 25.1.3. For a continuous real function on a compact space X, and some inclusion map $\iota : X \hookrightarrow Y$, an extension of f by ι can be the map

$$\tilde{f} = \begin{cases} f(x) & x \in X \\ 0 & \text{else} \end{cases} \tag{25.6}$$

in which case we can easily see that $\tilde{f} \circ \iota = f$.

Definition 25.1.1. For two morphisms i, p,

$$i : A \to B \tag{25.7}$$

$$p : X \to Y \tag{25.8}$$

we say that i has the left lifting property and p has the right lifting property, denoted by $i \boxslash p$, if for every pair of functions f, g

$$f : A \rightarrow X \tag{25.9}$$
$$g : B \rightarrow Y \tag{25.10}$$

that form a commutative diagram with i, p, $p \circ f = g \circ i$, then there exists a function $h : B \rightarrow X$ which commutes with all others

$$h \circ i = f \tag{25.11}$$
$$p \circ h = g \tag{25.12}$$

giving the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

in other words, there is a morphism h which is both a lift of g and an extension of f with respect to p and i respectively.

We also say that i and p are weakly orthogonal. If furthermore, the morphism h is unique, we say that they are *orthogonal*.

Definition 25.1.2. An object B has the left lifting property with respect to some morphism p if its unique morphism $0 \rightarrow B$ is has the left lifting property, and an object X has the right lifting property with respect to some morphism i if its unique morphism $X \rightarrow 1$ has the right lifting property.

Theorem 25.1.1. All isomorphisms have the lifting property with respect to any function.

Proof. Given an isomorphism i or p in a lifting diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

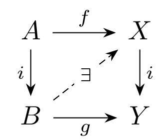
we can always define a morphism $B \rightarrow X$ via the inverse function of i (resp. p), as

$$h = f \circ i^{-1}, \ h = p^{-1} \circ g \tag{25.13}$$

which trivially fulfills the commutativity condition. □

Theorem 25.1.2. A morphism is never weakly orthogonal to itself, unless it is an isomorphism.

Proof. From the previous property it is clear that an isomorphism is equal to itself. For the other case, if we take the square diagram of the same morphism i



□

lifting in term of arrow categories and (co)slice categories?

25.2 Quillen negation

Definition 25.2.1. Given some class of morphisms of a category $\mathbf{C}$,

$$M \subseteq \mathrm{Mor}(\mathbf{C}) \tag{25.14}$$

its left (resp. right) *Quillen negation* (or *weak orthogonal class*) is the class of morphisms which have the left (resp. right) lifting property 25.1.1 with respect to each morphism in M, denoted as $M^{\boxslash l}$ (resp. $M^{\boxslash}r$)

$$M^{\boxslash l} = \{p \in \mathrm{Mor}(\mathbf{C}) \mid \forall l \in M,\ l \boxslash p\} \tag{25.15}$$

$$M^{\boxslash r} = \{i \in \mathrm{Mor}(\mathbf{C}) \mid \forall r \in M,\ i \boxslash r\} \tag{25.16}$$

Iterated Quillen negation is written as a string of l and r, ie

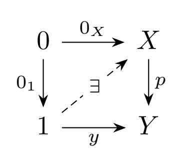
$$M^{\boxslash lrl} = ((M^{\boxslash l})^{\boxslash r})^{\boxslash l} \tag{25.17}$$

The Quillen negation of a morphism is often in some sense a class of morphisms sharing a given property, for which the original class M is a counterexample. M is in some sense some archetypical counterexample of that property.

Example 25.2.1. In **Set**, the non-surjective morphism $f : \varnothing \to \{\bullet\}$ has as its right Quillen negation the class of surjections (or equivalently for sets, the epimorphisms).

$$\{\varnothing \to \{\bullet\}\}^{\boxslash} = \mathrm{Epi}(\mathbf{Set}) \tag{25.18}$$

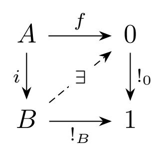
Proof. If we look at the lifting property diagram of $0 \to 1$, using the only function $0_X : 0 \to X$, and $1 \to Y$ being a point of Y



the outer diagram always commutes, the initial morphism always requiring it, so that this diagram simply means that for any point of Y, there is a point x of X of which it is the image of, $p \circ x = y$, which is the property of surjection. The lift of each point in the codomain by the function is its preimage in the domain. And this point is also an extension of the empty function by the empty function. ☐

Example 25.2.2. Left Quillen negation of $0 \to 1$?

Proof. Diagram :



If we are in a category with a strict initial object, there is only a single object for which there exists a morphism to the initial object, which is itself, so that the only type of morphism on the left can be $0_B : 0 \to B$, and likewise the diagonal filler can only exist if $B \cong 0$, so that the only possible left Quillen negation is $\mathrm{Id}_0 : 0 \to 0$. ☐

Alternatively, as we are dealing with conditions on the class of all functions

pullback hom

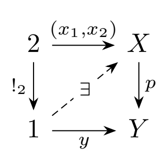
sdv

https://unimath.github.io/agda-unimath/orthogonal-factorization-systems.lifting-structures-on-squares.html

Example 25.2.3. Given the map from the set of two points to the singleton, $!_2 : 2 \to 1$, its right and left Quillen negation is the set of injective functions.

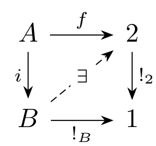
$$\{!_2 : 2 \to 1\}^{\boxslash l} = \{!_2 : 2 \to 1\}^{\boxslash r} = \mathrm{Inj}(\mathbf{C}) \tag{25.19}$$

Proof. If we look at the right lifting property diagram of $2 \to 1$, the upper morphism will be a selection of two points in X, and the lower morphism of a single point in X, with the diagonal filler being a single point in x



where we have by the commutation that $y = p \circ (x_1, x_2)$, so that y is the image of both x_1 and x_2. p therefore only has the right lifting property if the lift of a point in Y (its preimage $x = p^{-1}(y)$) is also the extension of a pair of points in in X by the terminal morphism (those two points are the same point, (x, x)). In other words, for any two points with the same image, they must be the same point.

Likewise for the left lifting property,



where f can be any projection of A to 2, which selects a subset S_A of A and its complement S_A^c.

$$S_A = f^{-1}(1) \tag{25.20}$$
$$S_A^c = f^{-1}(0) \tag{25.21}$$

The upper left triangle is then the existence of a splitting of B into two such disjoint subsets,

$$S_B = h^{-1}(1) \tag{25.22}$$
$$S_B^c = h^{-1}(0) \tag{25.23}$$

in a compatible way with those of A,

$$i(S_A) = S_B \tag{25.24}$$
$$i(S_A^c) = S_B^c \tag{25.25}$$

This property is the preservation of the complement, $i(S_A^c) = i(S_A)^c$ This means in particular that for a singleton subset $S_A = \{a\}$, [see here for more : MO] lower right triangle always true?

□

Example 25.2.4. The left

25.3 Factorization system

If we are given two sets of morphisms (L, R) which are each other's Quillen negation,

$$L^{\perp r} = R \tag{25.26}$$
$$R^{\perp l} = L \tag{25.27}$$

In many categories, there are important pairs of classes of morphisms called *factorization systems*, which are composed of two classes of morphisms, (L, R), where

$$L, R \subseteq \text{Mor}(\mathbf{C}) \tag{25.28}$$

such that every morphism is a composition of those two, ie

$$\forall f \in \text{Mor}(\mathbf{C}),\ \exists l, r \in (L, R),\ f = l \circ r \tag{25.29}$$

[33]

Definition 25.3.1. A *weak factorization system* on a category $\mathbf{C}$ is a pair of classes of morphisms (L, R) such that

- For any morphism $f \in \mathbf{C}$, it can be factored by a left and right morphism $l \in L,\ r \in R:\ f = r \circ l$
- Left and right morphisms have the opposite lifting properties to each other : $R^{\boxslash}$,

Definition 25.3.2. An *orthogonal factorization system* is a weak factorization system for which every left and right lifting is unique.

Definition 25.3.3. An *(epi, mono) factorization system* is an orthogonal factorization system for which the left class is that of epimorphisms and the right class is that of monomorphisms.

Theorem 25.3.1. A category with an (epi, mono) factorization system is balanced.

Proof. □

Example 25.3.1. The category of sets has an (epi, mono) factorization system.

Proof. □

26

The interplay of mathematics and categories

As a field meant to emulate the behavior of mathematical structures, there are many interactions between any given mathematical structure and category theory.

26.1 Decategorification

Definition 26.1.1. For a small category $\mathbf{C}$, its decategorification $K(\mathbf{C})$ is its set of objects quotiented by isomorphisms, or equivalently, the object set of its skeleton.

In terms of internal operation, we can say that decategorification is a functor from the category of categories to that of sets, given by

$$K : \mathbf{Cat} \to \mathbf{Set} \tag{26.1}$$

$$\mathbf{C} \mapsto \mathrm{Obj}(\mathrm{Sk}(\mathbf{C})) \tag{26.2}$$

Furthermore, structures on the category itself may lead to structures on the set. For instance, given any multifunctor on the category

$$F : \prod_i \mathbf{C}_i \to \mathbf{D} \tag{26.3}$$

we have an equivalent function between the decategorified sets,

$$f(c_1, c_2, \ldots, c_n) = d \tag{26.4}$$

Example 26.1.1. The categorification of a monoidal category $(\mathbf{C}, \otimes, I)$ is a monoid $(M, \cdot, e)$ with $M = K(\mathbb{C})$,

$$K(X \otimes Y) = K(X) \cdot K(Y) \tag{26.5}$$

and

$$K(I) = e \tag{26.6}$$

Example 26.1.2. The decategorification of the category of finite sets is the set of integers,

$$K(\mathbf{FinSet}) = \mathbb{N} \tag{26.7}$$

Its monoidal category given by the coproduct is the addition, while the monoidal category of the product is the multiplication, with $K(\{\bullet\}) = 1$ and $K(\varnothing) = 0$, and more generally for any set

$$K(X) = |X| \tag{26.8}$$

The rules of arithmetic are given by the distributivity of the product and coproduct in sets

26.2 Categorification

The reverse process of decategorifcation is that of *categorification.* If we view decategorigication as a functor

$$\begin{aligned} K : \mathbf{Cat} &\to \mathbf{Set} && (26.9) \\ \mathbf{C} &\mapsto \mathrm{Obj}(\mathrm{Sk}(\mathbf{C})) && (26.10) \end{aligned}$$

Then a simple way to view categorification is merely as some functor in the opposite direction,

$$F : \mathbf{Set} \to \mathbf{Cat} \tag{26.11}$$

Such that F is the right inverse of K : $K \circ F = \mathrm{Id}$, if we decategorify our categorification, we should end up on our original set. This notion of categorification is known as *vertical categorification.*

With our earlier example of **FinSet**,

Horizontal categorification

[34]

26.3 Internalization

If a category admits set-like properties, typically properties such as finite limits, monoidal structures or Cartesian closedness, it is possible to recreate many types of mathematical structures inside the category itself. This is called an *internalization* of the structure.

26.4 Microcosm principle

26.5 Algebras

One particular type of categorification is that of the categorification of universal algebras. In this context, ”algebra” is to be understood as a theory of operations of arbitrary arities and not in the sense of a binary operation.

Definition 26.5.1. Given an endofunctor $F \in \mathrm{End}(\mathbf{C})$, an *algebra* (X, α) of F (or F*-algebra*) is composed of an object $X \in \mathbf{C}$ and a morphism $\alpha : F(X) \to X$, the *carrier* of the algebra.

Example 26.5.1. In a given category $\mathbb{C}$, given the endofunctor

$$F(G) = 1 + G + G \times G \tag{26.12}$$

an object G and the morphism $\alpha : F(G) \to G$ defined by

$$\alpha(x) = \begin{cases} e & x \in 1 \\ x^{-1} & x \in G \\ \mu(x) & x \in G \times G \end{cases} \tag{26.13}$$

or $\alpha = [e, (-)^{-1}, \mu]$, for three morphisms obeying the properties of an internal group, define the F-algebra of a group.

Definition 26.5.2. An algebra homomorphism on $\mathbf{C}$ between two algebras (X, α) and (Y, β) is defined by a morphism $m : X \to Y$ such that the following square commute.

Theorem 26.5.1. The algebras of a category along with their homomorphisms form a category

Proof. Given any algebra of a category, we have that the identity morphism always defines an algebra isomorphism □

Example 26.5.2. If we take the functor associating to each object its successor in **Set**,

$$S(X) = X + \{\bullet\} \tag{26.14}$$

corresponding to X with a new element, and which maps morphisms to morphisms identical on all elements of X but preserving the new element,

$$S(f) : S(X) \to S(Y) \tag{26.15}$$

with

$$S(f)(x) = \begin{cases} \{\bullet_Y\} & x = \{\bullet_X\} \\ f(x) & x \neq \{\bullet_X\} \end{cases} \tag{26.16}$$

S-algebras are then given by some set X and function

$$\alpha : X + \{\bullet\} \to X \tag{26.17}$$

By the universal property of the coproduct, α can be written as some pair of function (f, g) such that f is a function $X \to X$ while g is a function $1 \to X$. We will write them as (z, s). This means that any S-algebra is defined by some element in X and some endomorphism.

Given two such S-algebras,

This category of S-algebras has an initial object N, ie some object for which there is only one S-algebra homomorphism from N to any other, meaning that for any other algebra $(X, (z', s'))$, we need

$$\begin{array}{ccc} S(N) & \xrightarrow{(z,s)} & N \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle f} \\ S(X) & \xrightarrow{(z',s')} & X \end{array}$$

So that we need for an arbitrary map $(z', s') : X + \{\bullet\} \to X$ that, composed with the morphism $S(f)$ which acts identical to f on X but preserves the extra element, that

$$\forall n \in N,\ (z', s')(S(f)(n)) = f(\text{succ}(n)) \tag{26.18}$$

now if we pick $n \in N$, the copairing (z, s) will simply be s, and the successor functor $S(f)$ will have the same behaviour as f, so that we get

$$f(s(n)) = f(s'(n)) \tag{26.19}$$

but if we pick $\bullet \in S(N)$, we get $z(\bullet)$ and the successor functor $S(f)(\bullet_N) = \bullet_X$

$$f(z'(\bullet)) = f(\bullet_X) \tag{26.20}$$

For f to be unique,

The initial object in the category of S-algebras is the set of natural numbers. [Because $S(N) = N$? and it's the smallest set to do so or something?]

26.6 Monoid in a monoidal category

The most generic form of internalization is that of an internal monoid (an even more general case would be that of a magma but we will not look at it here). Given some object M in the category, we would like to define what it means for M to be a monoid.

Definition 26.6.1. In a monoidal category $(\mathbf{C}, \otimes, I)$, a monoid in $\mathbf{C}$ is given by a triplet (M, m, e), of an object $M \in \mathbf{C}$, a morphism $m : M \otimes M$, and a morphism $e : I \to M$, such that they make the following diagrams commute :

All internal objects we will see as we go on will be some variation of a monoid in a monoidal category, most of them being more specifically Cartesian monoids, where the monoid will be that of the product, $(\mathbf{C}, \times, 1)$.

A monoid M can always be defined as a category itself, via the construction of a category $\mathbf{M}$ with a single object $\bullet$, and such that the endomorphisms of this object are isomorphic to the monoid as a set,

$$\mathrm{End}(\bullet) \cong |M| \tag{26.21}$$

in which the monoid operation is implemented by the composition rules of those morphisms.

26.7 Internal group

Definition 26.7.1. In a category $\mathbf{C}$ with finite products, a group object G is an object $G \in \mathbf{C}$ equipped with the morphisms

- The unique map to the terminal object $p : G \to 1$
- A neutral element morphism from the terminal object : $e : 1 \to G$
- An inverse endomorphism : $(-)^1 : G \to G$
- A binary morphism on the product : $m : G \times G \to G$

such that all the following diagrams commute

If we want to write it out explicitly, the group object is not merely the object G itself but the quintuple $(G, p, e, (-)^{-1}, m)$.

Example 26.7.1. Every category with finite product has the trivial group object $\{e\}$ which is the terminal object and the unique map to itself, so that

$$\{e\} \cong (1, \mathrm{Id}_1, \mathrm{Id}_1, \mathrm{Id}_1, \mathrm{pr}_1) \tag{26.22}$$

Example 26.7.2. As groups can be defined using sets, the category of sets contains every group as group objects using the traditional definition of groups.

We can work out an explicit example of this. The set of two elements 2 has two functions $f : 1 \to 2$, which are the functions mapping to the first and second element, and has four endomorphisms, corresponding to all unary boolean functions (the constant functions, the identity and the involution $\bullet_0 \to \bullet_1$, $\bullet_1 \to \bullet_0$, equivalent to the negation). The product object $2 \times 2 \cong 4$ has 16 functions, corresponding to all the binary boolean functions.

A possible model of the internal group $\mathbb{Z}_2$ can be defined as an assignment of $\bullet_0$ to the identity 1 and $\bullet_1$ to the element -1, the inverse map as the involutive function $2 \to 2$, and the multiplication map being the equivalent of the XNOR function

$$f(\bullet_0, \bullet_0) = \bullet_1 \tag{26.23}$$
$$f(\bullet_1, \bullet_0) = \bullet_0 \tag{26.24}$$
$$f(\bullet_0, \bullet_1) = \bullet_0 \tag{26.25}$$
$$f(\bullet_1, \bullet_1) = \bullet_1 \tag{26.26}$$

$$\mathbb{Z}_2 \cong (2, e, \neg, \mathrm{XNOR}) \tag{26.27}$$

Example 26.7.3. The group objects in the category Top are the topological groups, where all group operations are continuous functions. This can be done for instance by considering the set of all such quintuples that are mapped to a group object by the forgetful functor to **Set**.

Example 26.7.4. The group objects in the category of smooth manifolds are the Lie groups, where the group operations are smooth maps.

As the notation $(G, p, e, (-)^{-1}, m)$ is fairly cumbersome, from now on internal groups will be simply defined by their object G, with all morphisms left implicit.

Theorem 26.7.1. For a category $\mathbf{C}$ with an internal group G, the hom-set $\mathrm{Hom}_{\mathbf{C}}(X, G)$ for any object X has a group structure. [Abelian only?]

Proof. If we consider two morphisms $f, g \in \mathrm{Hom}_{\mathbf{C}}(X, G)$ in **Set** (in other words an element in the product of two hom-sets), there □

Definition 26.7.2. In a category with internal groups, the left action of an internal group G on an object X is defined by a morphism

$$\rho : G \times X \to X \tag{26.28}$$

such that the following diagrams commute :

As a generalization of internal groups, we also have the notion of an internal groupoid.

Definition 26.7.3. An internal groupoid $\mathcal{G}$ in a category with products is given by three objects in a category (E, G, X), with X the underlying space of the groupoid, G the and the following morphisms :

- The morphism $e : X \to G$ associating the unit element to the underlying object X
- The inverse morphism $(-)^{-1} : G \to G$
- The

26.8 Internal ring

Definition 26.8.1. In a category **C** with finite products, a *ring object* R is an object $R \in \mathbf{C}$ along with the following morphisms :

- The addition morphism $a : R \times R \to R$
- The multiplication morphism $m : R \times R \to R$
- The addition identity morphism $0 : 1 \to R$
- The multiplication identity morphism $1 : 1 \to R$
- The additive inverse morphism : $-(-) : R \to R$

such that all those morphisms make the following diagrams commute :

via the microcosm principle, we can also define it in categorical terms via monoidal categories

Definition 26.8.2. A rig [no n for negation] in a bimonoidal category

$$(\mathbf{C}, \otimes, \oplus, \mathbf{0}, \mathbf{1}) \tag{26.29}$$

Example 26.8.1. The integers object $\mathbb{Z}$

Example 26.8.2. The real line object $\mathbb{R}$ in many categories is an internal ring, with the elements $0, 1 : 1 \to \mathbb{R}$, and the expected morphisms associated to it. Those morphisms are for instance smooth functions and continuous functions, so that both **SmoothMan** and **Top** have the internal ring of the real line.

It is common for many categories to have some variant of the real line, as we can find such an object in **Set**, **Top**, **CartSp**, **SmoothMan**, etc.

27 Groupoids

In many contexts, the objects of a category can themselves be categories, such as the category of categories **Cat**, the category of groupoids **Grpd** (where the monoid structure on the elements of the groupoids is built from morphisms) and so on.

If a category of that sort is equipped with a terminal object 1 which represents a category of a single object $\mathbf{1}$, such as is the case for groupoids for instance, and we have that the object set of some other object category $\mathbf{C}$ is given by the hom-set

$$\mathrm{Obj}(\mathbf{C}) \cong \mathrm{Hom}_?(\mathbf{1}, \mathbf{C}) \tag{27.1}$$

In those circumstances, we can consider the morphisms of those objects by functors preserving 1 and C, as a functor will map those objects $X, Y : 1 \to C$ to each other (associativity?)

From this, we also have that the natural transformations η between the functors (or here morphisms) $f, g \in \mathrm{Mor}(\mathbf{C})$

Definition 27.0.1. A *groupoid* $\mathcal{G}$ is a pair of two sets G_1, G_0, where G_0

28
Subobjects

Given an object X in a category $\mathbf{C}$, a *subobject* S of X is an isomorphism class of monomorphisms $\{\iota_i\}$

$$\iota_i : S_i \hookrightarrow X \tag{28.1}$$

so that the equivalence classes of $\{\iota_i\}$ is given by any two such monomorphisms if there exists an isomorphism between the two subobjects S_i;

$$S = [S_i]/(S_i \cong S_j \leftrightarrow \exists f : S_i \to S_j,\ \exists f^{-1} S_j \to S_j,\ f \circ f^{-1} = \mathrm{Id}) \tag{28.2}$$

Theorem 28.0.1. In a skeletal category, the subobjects are equivalent to the monomorphisms.

This is meant to define the common mathematical notion of an object being *part* of another object in some sense.

Example 28.0.1. On **Set**, subobjects are subsets (defined by injections up to the symmetric group), where

Example 28.0.2. On $\mathbf{Vect}_k$, subobjects are subspaces (defined by injections up to the general linear group?)

So that a subobject is a *line*, $[k] \hookrightarrow V$, and not a specific line given by a specific linear map $k \hookrightarrow V$.

Example 28.0.3. On **Top**, subobjects are subspaces with the subspace topology (defined up to homeomorphisms)

On **Ring**,

On **Grp**, subobjects are subgroups

Example 28.0.4. On the category of smooth manifolds **SmoothMan**, subobjects are *submanifolds*, $\iota : S \hookrightarrow M$, where the set of all submanifolds with the same image up to diffeomorphism of the base $\text{Diff}(S)$ are equivalent.

"Let C_c be the full subcategory of the over category C/c on monomorphisms. Then C_c is the poset of subobjects of c and the set of isomorphism classes of C_c is the set of subobjects of c. "

29

Simplicial categories

The simplicial category Δ is made of simplicial objects, which can be defined in a variety of equivalent ways, either as being finite total orders

[...]

or as being categories themselves for total orders

$$\left[\vec{\mathbf{0}}\right] = \{\bullet\} \tag{29.1}$$

$$\left[\vec{\mathbf{1}}\right] = \{\bullet \to \bullet\} \tag{29.2}$$

$$\left[\vec{\mathbf{2}}\right] = \{\bullet \to \bullet \to \bullet\} \tag{29.3}$$

$$\left[\vec{\mathbf{3}}\right] = \{\bullet \to \bullet \to \bullet \to \bullet\} \tag{29.4}$$

The simplicial morphisms are the order-preserving functions,

$$[\vec{\mathbf{m}}] \to [\vec{\mathbf{n}}] \tag{29.5}$$

Only defined for $m \leq n$

Examples :

$$\mathrm{Hom}_{\Delta}\left(\left[\vec{\mathbf{0}}\right], [\vec{\mathbf{n}}]\right) = n \tag{29.6}$$

This morphism maps the single point to the various objects of $[\vec{n}]$

$$\mathrm{Hom}_{\Delta}\left(\left[\vec{\mathbf{1}}\right], [\vec{\mathbf{n}}]\right) = n \tag{29.7}$$

Theorem 29.0.1. The cardinality of the hom-sets of simplicial objects are

$$\mathrm{Hom}_{\Delta}\left([\vec{\mathbf{n}}], [\vec{\mathbf{m}}]\right) = \tag{29.8}$$

Proof. As we've seen for functions $0 \to m$,

$$|\mathrm{Hom}_{\Delta}\left(\left[\vec{\mathbf{0}}\right], [\vec{\mathbf{m}}]\right)| = m \tag{29.9}$$

By induction, if we know the cardinality of morphisms $n \to m$, the cardinality ∎

interpretation as embedding of simplices

Despite the name, the simplicial categories (and its overarching simplicial 2-category) is not made of simplices, but they will be of use later on to *construct* simplices in the category of simplicial sheaves.

30

Equivalences and adjunctions

Like many objects in mathematics, it is possible to try to define some kind of equivalence between two categories. Like most such things, we try to consider two mappings between our categories. Let's consider two categories $\mathbf{C}, \mathbf{D}$, and two functors F, G between them,

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

The usual process of finding equivalent objects in such cases is to have those two maps be inverses of each other, ie $FG = \mathrm{Id}_{\mathbf{D}}$ and $GF = \mathrm{Id}_{\mathbf{C}}$. The composition functor FG maps every object and morphism of D to itself and likewise for GF on C, so that in some sense, the objects X and $F(X)$ are the same objects, and likewise, $f : X \to Y$ and $F(f) : F(X) \to F(Y)$ represent the same morphism.

If two categories admit such a pair of functors, we say that they are *isomorphic*. While this is the most obvious definition of equivalence, it is in practice not commonly used, as very few categories of interest are actually isomorphic, and this is generally considered too strict a definition in the philosophy of category theory, as we are generally more interested in the relationships between objects rather than the objects themselves. A good example of this overly strict definition is that the product of two objects done in a different order will not be isomorphic, or shifting a total order to the left and back right.

If we weaken the notion of equivalence, we can look at the case where our two functors are merely isomorphic to the identity, $FG \cong \mathrm{Id}_{\mathbf{D}}$ and $GF \cong \mathrm{Id}_{\mathbf{C}}$, where there exists a natural transformation η taking FG to $\mathrm{Id}_{\mathbf{D}}$ and ϵ taking $\mathrm{Id}_{\mathbf{C}}$ to GF (this ordering for the two will make more sense later on)

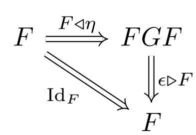
$$\eta \ : \mathrm{Id}_{\mathbf{C}} \to GF \tag{30.1}$$
$$\epsilon \ : FG \to \mathrm{Id}_{\mathbf{D}} \tag{30.2}$$

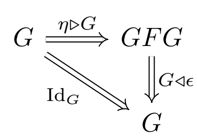
with η and ϵ both being two-sided inverses, that is, for all objects of either category, we have that η_X is an isomorphism, and likewise, ϵ_X is an isomorphism.

C **D**

Furthermore, we say that this equivalence is an *adjoint equivalence* if those natural transformations satisfy the *triangle identities* :

Definition 30.0.1. The triangle identities for two functors F, G is given by the two commutative diagrams





In other words, adding the "almost" identity functors FG or GF on one side by transforming the identity into them, before removing them on the other by transforming it back into the identity is an isomorphism.

$$(\epsilon \triangleright F) \circ (F \triangleleft \eta) \quad = \quad \mathrm{Id}_F \tag{30.3}$$
$$(G \triangleleft \epsilon) \circ (\eta \triangleright G) \quad = \quad \mathrm{Id}_G \tag{30.4}$$

In terms of components, this means that for every objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, the components of the relevant natural transformations are

$$\epsilon_{F(Y)} \circ F(\eta_Y) \quad = \quad \mathrm{Id}_{F(Y)} \tag{30.5}$$
$$G(\epsilon_X) \circ \eta_{G(X)} \quad = \quad \mathrm{Id}_{G(X)} \tag{30.6}$$

Counterexample 30.0.1. An equivalence that is not an adjoint equivalence can be seen in the case of two monoid categories $\mathbf{M}_1$, $\mathbf{M}_2$ with a single object each

Now if we weaken the condition on η and ϵ, merely requiring them to be natural transformations rather than natural isomorphisms, although still obeying the triangle identity, this is what is called an *adjunction* of functors. The adjunction of two functors L and R is denoted by $(L \dashv R)$, where L is called the *left adjoint*, while R is the *right adjoint*, with η and ϵ called the *unit* and *counit* respectively. The diagram for this will be given as

$$(L \dashv R) : \mathbf{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathbf{D}$$

where the left functor L will always be on top of its right adjoint R.

Adjoint categories are not (adjoint) equivalent, they do not share the same objects and morphisms, even up to equivalence

Applied to a morphism :

For some morphism $f \in \mathbf{C}$,

$$L(f) : L(X) \to L(Y) \tag{30.7}$$

For some morphism $g \in \mathbf{D}$,

$$R(g) : R(V) \to R(W) \tag{30.8}$$

$$\epsilon_{L()} \tag{30.9}$$

Definition 30.0.2. Given two objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, with an adjunction $(L \dashv R)$, we say that a morphism $f \in \mathbf{D}$ and $g \in \mathbf{C}$ are *adjunct* if f is of the form

$$f : L(X) \to Y \tag{30.10}$$

and g

$$g : X \to R(Y) \tag{30.11}$$

such that

$$g = R(f) \circ \eta_X \tag{30.12}$$

Theorem 30.0.1. Conversely, we have

$$f = \epsilon_Y \circ L(g) \tag{30.13}$$

Proof. Given the equality

$$g = R(f) \circ \eta_X \tag{30.14}$$

We have

$$\begin{aligned}\epsilon_Y \circ L(g) &= \epsilon_Y \circ L(R(f) \circ \eta_X) && (30.15)\\ &= \epsilon_Y \circ (L \circ R)(f) \circ L(\eta_X) && (30.16)\\ &= \epsilon_Y \circ \epsilon(\mathrm{Id}_{\mathbf{D}})(f) \circ L(\eta_X) && (30.17)\\ &= f \circ \epsilon_{LRL(X)} \circ L(\eta_X) && (30.18)\end{aligned}$$

□

Theorem 30.0.2. For two adjoint functors $(L \dashv R) : \mathbf{C} \rightleftarrows \mathbf{D}$, the triangle identities are equivalent to the identity

$$\mathrm{Hom}_{\mathbf{D}}(L(-), -) \cong \mathrm{Hom}_{\mathbf{C}}(-, R(-)) \tag{30.19}$$

so that for any object $X \in \mathbf{C}$ and $Y \in \mathbf{D}$,

$$\mathrm{Hom}_{\mathbf{D}}(L(X), Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, R(Y)) \tag{30.20}$$

Proof. □

Example 30.0.1. Many categories have what are called *free-forgetful adjunctions*, where we consider the forgetful functor U to some other category (the prototypical example being *Set*, sending a structure to its underlying set), and the left adjoint to that functor F being its left adjoint, called the *free functor*, which constructs an object of this category that is in some sense the "best approximation" of the set from the left, that is, for every other object X in the category, morphisms of the form $FS \to X$ are exactly the image of the functions $S \to UX$, but not the other way around.

These are for instance given by the free topological space on a set (the discrete space on that set), where every function from a discrete space as a space is also a continuous function, the free group FS, where every function from the generators to another group is also a group homomorphism, or the free vector space, where all functions from its generators to another vector space are linear functions.

Adjuncts for forgetful-free adjunction? For the case of topological space, $(F \dashv U)$, given a continuous map $f : F(S) \to Y$, adjunct function $\overline{f} : S \to U(Y)$, and for a function $g : S \to U(Y)$, adjunct continuous map $\overline{g} : F(S) \to Y$

Theorem 30.0.3. Free object of one generator

Example 30.0.2. A basic non-trivial example of adjoint functors is the even and odd functors. If we consider $\mathbb{Z}$ as a linear order category, with $\leq$ as its morphisms, functors are its order-preserving functions. Two specific functors that we have are the even and odd functors, give by

$$\forall k \in \mathbb{Z},\ \mathrm{Even}(k) = 2k,\ \mathrm{Odd}(k) = 2k + 1 \tag{30.21}$$

These do indeed preserve the order so that for the unique morphism $k_1 \leq k_2$, it is mapped to the unique morphism $2k_1 \leq 2k_2$ and similarly for the odd functor. The corresponding "inverse" functor is the functor mapping any integer to the floor of its division by 2 :

$$\lfloor -/2 \rfloor : \mathbb{Z} \to \mathbb{Z} \tag{30.22}$$

Even is then the left adjoint and Odd the right adjoint of $\lfloor -/2 \rfloor$. The unit and counit of Even are :

$$\varepsilon_l = f \circ \lfloor -/2 \rfloor \tag{30.23}$$

[...]

This is however not an equivalence, as the floor function is not strictly an inverse of the even and odd functor (and not being a faithful functor to begin with), as $\lfloor (2n)/2 \rfloor = \lfloor (2n+1)/2 \rfloor$, and we have

Adjuncts :

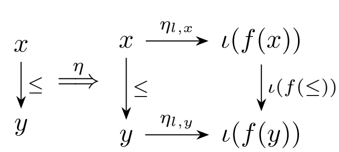
Example 30.0.3. Take the two linear order categories of $\mathbb{Z}$ and $\mathbb{R}$, with their elements being the objects and their order relations are the morphisms. The inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$, mapping $n \in \mathbb{Z}$ to its equivalent real number, is a functor

We can try to define left and right adjoints for it, by finding two functions $f, g : \mathbb{R} \to \mathbb{Z}$ for which there exists

- A left and right counit $\epsilon_l : f\iota \to \mathrm{Id}_{\mathbb{Z}}$ and $\epsilon_r : \iota g \to \mathrm{Id}_{\mathbb{R}}$
- A left and right unit $\eta_l : \mathrm{Id}_{\mathbb{R}} \to \iota f$ and $\eta_r : \mathrm{Id}_{\mathbb{Z}} \to g\iota$

And all these must obey the triangle identities

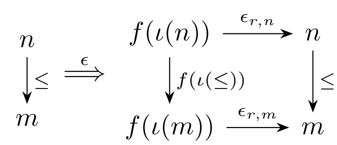
If we take for instance the left adjoint, we need that our function f be such that there exists a natural transformation between the identity on $\mathbb{R}$ ($\mathrm{Id}_{\mathbb{R}}(x) = x$) and our function f reinjected into $\mathbb{R}$: $\iota(f(x))$. For every $x, y \in \mathbb{R}$, there's a morphism



In the context of our linear order, this means that for any two numbers x, y such that $x \leq y$, we have $x \leq \iota(f(x))$, $y \leq \iota(f(y))$ and $\iota(f(x)) \leq \iota(f(y))$

$$x \leq y \leq \iota(f(y)) \tag{30.24}$$

and for the counit, we need a natural transformation between the mapping of an integer into $\mathbb{R}$ and then back into $\mathbb{Z}$ via f with $f(\iota(n))$, and the identity on $\mathbb{Z}$, $\mathrm{Id}_{\mathbb{Z}}(n) = n$. For every n, m, $n \leq m$,

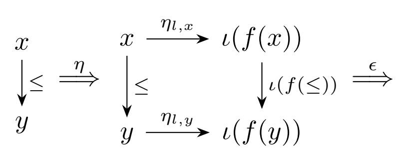


We have the condition that if we inject n into $\mathbb{R}$, its left adjoint will be such that $f(\iota(n)) \leq n$, $f(\iota(m)) \leq m$ and $f(\iota(n)) \leq f(\iota(m))$. If we take the case $m = n+1$ and ignoring the injection ι for now, this means that $f(n) \leq f(n+1) \leq n+1$ and $f(n) \leq n$. As $f(n) \leq n+1$ cannot be equal to $n+1$, $f(n)$ can only be equal to n or smaller. If we pick the case $n-1 \leq n$ instead, the natural transformation implies $f(n-1) \leq f(n) \leq n$ and $f(n-1) \leq n-1 \leq n$, so that $f(n-1) < n$ and

Triangle identities : for any $n \in \mathbb{Z}$,

$$\mathrm{Id}_{\iota(n)} = \iota(\epsilon_{l,n}) \circ \eta_{l,\iota(n)} \tag{30.25}$$

The components of this natural transformations give us that, if we transform our integer n to a real and back,

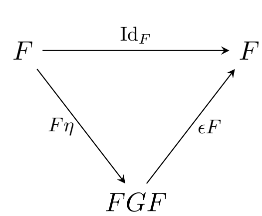


Different definitions of adjunction

Adjoint functors : For two functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$, the functors form an *adjoint pair* $F \dashv G$, F the left adjoint of G and G the right adjoint of F, if there exists two natural transformations, η and ϵ

$$\eta : \mathrm{Id}_{\mathbf{C}} \to G \circ F \tag{30.26}$$
$$\epsilon : F \circ G \to \mathrm{Id}_{\mathbf{D}} \tag{30.27}$$

obeying the triangle equalities



Adjunct : for an adjunction of functors $(L \dashv R) : \mathbf{C} \leftrightarrow \mathbf{D}$, there exists a natural isomorphism

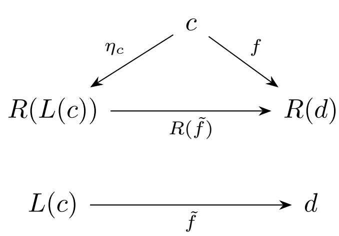
$$\mathrm{Hom}_{\mathbf{C}}(LX, Y) \cong \mathrm{Hom}_{\mathbf{D}}(X, RY) \tag{30.28}$$

Two morphisms $f : LX \to Y$ and $g : X \to RY$ identified in this isomorphism are *adjunct*. g is the right adjunct of f, f is the left adjunct of g.

$$g = f^\sharp, \ f = g^\flat \tag{30.29}$$

Adjunction for vector spaces

Definition 30.0.3. For a functor $R : \mathbf{D} \to \mathbf{C}$ and an object $c \in \mathbf{C}$, a *universal arrow* from c to R is an initial object of the comma category $(\Delta_c \downarrow R)$, which consists of an object $L(c) \in \mathbf{D}$ and a morphism $\eta_c : c \to (R \circ L)(c)$ (the unit) such that for any $d \in \mathbf{D}$ and any $f : c \to R(c)$, f factors through η_c via some morphism $R(\tilde{f})$ for some unique morphism $\tilde{f} : L(c) \to d$, called the *adjunct* of f.



Theorem 30.0.4. For a functor $R : \mathbf{D} \to \mathbf{C}$ and an object $c \in \mathbf{C}$, the following two conditions are equivalent :

- $\eta_c : c \to (R \circ L)(c)$ is a universal arrow.
- (c, η_c) is the initial object in the comma category $(\Delta_c \downarrow R)$

Proof. □

Galois connection

Example 30.0.4. A common type of adjunction (although typically not a formal one) is the free-forgetful adjunction, where given some forgetful functor $U : \mathbf{C} \to \mathbf{D}$, we have the left adjoint of the free functor $F : \mathbf{D} \to \mathbf{C}$, with the adjunction $(F \dashv U)$. In particular, for the case of a concrete category where the forgetful functor maps to **Set**,

Theorem 30.0.5. Given an adjunction

$$(L \dashv R) \tag{30.30}$$

L preserves any colimits, while R preserves any limits.

Proof. As the hom-functor preserves limits, we have that

$$\mathrm{Hom}_{\mathbf{C}}(X, \lim F) \cong \lim \mathrm{Hom}_{\mathbf{C}}() \tag{30.31}$$

□

Example 30.0.5. Given some free-forgetful adjunction, such as some concrete category with $U : \mathbf{C} \to \mathbf{Set}$, the free functor preserves colimits while the forgetful functor preserves limits. For $\mathbf{Top}$: $(\mathrm{Disc} \dashv \Gamma)$, The terminal topological space (point) has a single element :

$$\Gamma(1_{\mathbf{Top}}) = \{\bullet\} \tag{30.32}$$

The free space from the initial set is the empty space

$$\mathrm{Disc}(\varnothing) = 0_{\mathbf{Top}} \tag{30.33}$$

The product of two spaces has the point set of the product of two sets The free space on the sum of two sets is the coproduct of the two free spaces

Overall, we typically expect the forgetful functor to have a left adjoint free functor, as those free spaces behave like the underlying category with respect to the preservation of colimits. Given some forgetful functor to sets, we expect the free object $F(X)$ to preserve coproducts. The terminal set $\{\bullet\}$ leads to the

the coproduct of two sets, a set of cardinality $|A| + |B|$, leads to a free group

Example 30.0.6. For a functor between two poset categories, the adjoints of those functors are Galois connection.

Theorem 30.0.6. A right adjoint functor R preserves monomorphisms, so that

$$f \in \mathrm{Mono} \to R(f) \in \mathrm{Mono} \tag{30.34}$$

Proof. □

General Adjoint Functor Theorem

Theorem 30.0.7. If $\mathbf{C}$ is complete and locally small, and the functor $R : \mathbf{C} \to \mathbf{D}$ preserves all limits, then R admits a left adjoint $L : \mathbf{D} \to \mathbf{C}$ if it satisfies the solution set condition : for any $d \in \mathbb{D}$, there exists a set I and an I-indexed family of objects $(c_i)_{i\in I}$ and morphisms $f_i : d \to F(c_i)$ in $\mathbb{C}$ such that any morphism $h : d \to F(c)$ in $\mathbf{D}$ can be factored for some morphism $t : c_i \to c$ as

$$F(t) \circ f_i : d \xrightarrow{f_i} F(c_i) \xrightarrow{F(t)} F(c) \tag{30.35}$$

Proof. As the existence of an □

"A restatement of this condition is that the comma categories $(Y \downarrow F)$ all admit weakly initial families of objects."

30.1 Dual adjunctions

A dual adjunction between two contravariant functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ is given by a pair of natural transformations

$$\eta : \mathrm{Id}_{\mathbf{C}} \to GF \tag{30.36}$$

$$\theta : \mathrm{Id}_{\mathbf{D}} \to FG \tag{30.37}$$

which obey the triangle equalities $F\eta \circ \theta F = \mathrm{Id}_F$ and $G\theta \circ \eta G = \mathrm{Id}_G$:

$$\begin{array}{ccc} F & \xrightarrow{\theta F} & FGF \\ & \searrow^{\mathrm{Id}_F} & \downarrow F\eta \\ & & F \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{\mathrm{Id}_G} & \downarrow G\theta \\ & & G \end{array}$$

this is a notion that is simply the adjunction of the two covariant functors

$$F : \mathbf{C} \to \mathbf{D}^{\mathrm{op}} \tag{30.38}$$

$$D : \mathbf{D}^{\mathrm{op}} \to \mathbf{C} \tag{30.39}$$

This notion will be mostly useful for the notion of duality of two categories.

30.2 Adjoint transformation

Definition 30.2.1. For two adjunctions $(F \dashv G) : \mathbf{C} \to \mathbf{D}$ and $(F' \dashv G') : \mathbf{C}' \to \mathbf{D}'$, with units and counits (η, ϵ) and (η', ϵ'), a map of adjunction is a pair of functors $K : \mathbf{D} \to \mathbf{D}'$ and $L : \mathbf{C} \to \mathbf{C}'$

$$\begin{array}{ccccc} \mathbf{D} & \xrightarrow{G} & \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle K} \\ \mathbf{D}' & \xrightarrow[G']{} & \mathbf{C}' & \xrightarrow[F']{} & \mathbf{D}' \end{array}$$

such that this diagram commutes, and for any objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, the hom-set diagram commutes

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{D}}(F(X),Y) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathbf{C}}(X,G(Y)) \\
\downarrow{\scriptstyle K} & & \downarrow{\scriptstyle L} \\
\mathrm{Hom}_{\mathbf{D}'}(K(F(X)),K(Y)) & & \mathrm{Hom}_{\mathbf{C}'}(L(X),L(G(Y))) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathrm{Hom}_{\mathbf{D}'}(F'(L(X)),K(Y)) & \xrightarrow[\Phi']{} & \mathrm{Hom}_{\mathbf{C}'}(L(X),G'(K(Y)))
\end{array}$$

30.3 Base change

A useful case of adjunction, relating to the (co)slice categories of a category, is given by the base change functor.

Definition 30.3.1. Given a category $\mathbf{C}$ with pullbacks and a morphism $f : X \to Y$, its induced functor on slice categories 22.2.1 is the *base change functor*

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{30.40}$$

Two useful case to keep in mind from slice categories is that if $\mathbf{C}$ has a terminal object, $\mathbf{C}_{/1} \cong \mathbf{C}$, and if it has a strict initial object, $\mathbf{C}_{/0} \cong \mathbf{1}$ (since only one object has a morphism with 0 as the source).

Theorem 30.3.1. The left adjoint of the base change functor f^* is equivalent to the dependent sum on the morphism f :

$$\sum_f : \mathbf{C}_{/X} \to \mathbf{C}_{/Y} \tag{30.41}$$

Proof. The base change functor transforms our morphism $p : K \to Y$ to the morphism $p^* : X \times_Y K \to X$

Adjoint :

$$\mathrm{Hom}_{\mathbf{C}_{/X}}(A \to X, f^*(B \to Y)) \cong \mathrm{Hom}_{\mathbf{C}_{/Y}}(\sum_f (A \to X), B \to Y) \tag{30.42}$$

□

Theorem 30.3.2. The right adjoint of the base change functor f^* is equivalent to the dependent product of the morphism f

Example 30.3.1. In the category of manifolds **SmoothMan**, the slice categories $\mathbf{SmoothMan}_{/X}$ are the bundles over X, $p : E \to X$.

Example 30.3.2. In a poset category $\mathbf{P}$, the slice category $\mathbf{P}_x$ is the down set $\downarrow(p)$, the subposet of elements $x' \leq x$.

30.4 Kan extension

A lot of concepts in category theory are fundamentally about finding a commutative triangle of functors.

Definition 30.4.1. Given two functor $F : \mathbf{C} \to \mathbf{D}$ and $p : \mathbf{C} \to \mathbf{C}'$, let's consider the induced functor

$$p^* : [\mathbf{C}', \mathbf{D}] \to [\mathbf{C}, \mathbf{D}] \tag{30.43}$$

given by composition,

$$p^*(h : \mathbf{C}' \to \mathbf{D}) = h \circ p \tag{30.44}$$

its *left Kan extension* is a functor $\mathrm{Lan}_p F : \mathbf{C}' \to \mathbf{D}$, given by the left Kan extension operation

Example 30.4.1. The definition of limits and colimits we have seen is the right and left Kan extension of the functor $F : I \to \mathbf{C}$ where $\mathbf{C}'$ is the terminal category $\mathbf{1}$, so that $p : I \to \mathbf{1}$ is simply the constant functor. Its induced functor is then the functor $p^* : [I, \mathbf{C}] \to [\mathbf{1}, \mathbf{C}]$ which maps diagrams on $\mathbf{C}$ to some functor selecting a unique element of $\mathbf{C}$. If we apply it with some object $\bullet$ of the diagram, we find

$$p^*(F)(\bullet) = F(\Delta_*(\bullet)) \tag{30.45}$$

31
Concrete categories

We have seen quite a few times the notion of a category being composed of sets with functions between them, so let's now formalize this notion internally to category theory.

Definition 31.0.1. A category $\mathbf{C}$ is *concretizable* if there exists a faithful functor

$$U : \mathbf{C} \to \mathbf{Set} \tag{31.1}$$

A concrete category is then a pair of the category $\mathbf{C}$ with a specific such functor. Every object X in a concrete category then has an *underlying set* $U(X)$ which we identify in some sense with the object itself. The faithfulness condition lets us associate to every morphism $f : X \to Y$ a unique function on their underlying sets, $U(f) : U(X) \to U(Y)$.

Example 31.0.1. A trivial example of a concrete category is $\mathbf{Set}$ itself, with the identity functor

Definition 31.0.2. Given a concrete category $\mathbf{C}$ with a forgetful functor

$$U : \mathbf{C} \to \mathbf{Set} \tag{31.2}$$

if this functor has a left adjoint $(F \dashv U)$, we say that it is the *free functor*, and for a right adjoint $(U \dashv G)$, we say that it is the *cofree functor*.

Example 31.0.2. For the category of k-vector spaces $\mathbf{Vect}_k$, with the forgetful functor U to the underlying set of the vector space, there is a free functor which maps a set to a corresponding vector space with a basis of the same cardinality,

$$F(S) = \text{Span}(\{e_s\}_{s \in S}) \tag{31.3}$$

Proof. If we take a set S and a vector space X, by the adjunction,

$$\mathrm{Hom}_{\mathbf{Vect}_k}(F(S), X) \cong \mathrm{Hom}_{\mathrm{Set}}(S, U(X)) \tag{31.4}$$

□

Theorem 31.0.1. The opposite of a concrete category is concrete.

Proof. □

Stuff, structure, and properties [21]

32

Relations, spans and Quillen negation

The definition of relations in category theory will lead us to a variety of other concepts important for the further development etc

32.1 Relations

As a reminder, in set theory, a binary relation R on the sets X and Y is given by the subset of the Cartesian product $R \subseteq X \times Y$, where we say that two elements $(x, y) \in X \times Y$ obey the relationship R if $(x, y) \in R$, which is usually denoted by either $R(x, y)$ or xRy, depending on the relationship type. For instance, equality is the relationship given by the image of the diagonal map $\Delta : X \to X \times X$, so that we have

$$x = y \leftrightarrow (x, y) \in \mathrm{Im}(\Delta) \tag{32.1}$$

We can easily generalize this notion from that of a subset (ie an injection $R \hookrightarrow X \times Y$) to some arbitrary function $f : R \to X \times Y$, simply by considering the factorization of functions in sets :

$$R \xhookrightarrow{f|_{\mathrm{Im}(f)}} \mathrm{Im}(f) \xrightarrow{f|^{\mathrm{Im}(f)}} X \times Y$$

so that any function to $X \times Y$ defines a relation, which is given by

$$xRy \leftrightarrow \exists r \in \mathrm{Im}(f),\ x = \mathrm{pr}_1(r) \wedge y = \mathrm{pr}_2(r) \tag{32.2}$$

In terms of logic, in type theory, a relation will be given by some function dependent type

$$\mathrm{Rel}[t_1, t_2] : T_1 \times T_2 \to \mathrm{Prop} \tag{32.3}$$

so that given two objects of the appropriate type, we obtain a proposition based on the terms that compose it.

If we try to define relations in a categorical context, the simplest example for this is given by the category of relations on sets, Rel.

Definition 32.1.1. The category of relations on sets Rel is composed of objects which are sets, and of morphisms $R : X \to Y$ which are composed of subsets of the product $X \times Y$, so that

$$\mathrm{Hom}_{\mathbf{Rel}}(X, Y) = \mathcal{P}(X \times Y) \tag{32.4}$$

with the identity morphism being equality $\mathrm{Id}_X \cong \mathrm{Im}(\Delta_X)$, and composition being defined via the relational composite,

$$R \circ S = \{(x, z) \in X \times Z \mid \exists y \in Y,\ (x, y) \in R \wedge (y, z) \in S\} \tag{32.5}$$

so that x and z are related by this composition if there exists an intermediary y to which they are both related.

This category has the property of being a dagger category, with the dagger operation giving us the converse relation

$$R^\dagger = \{(y, x) \in Y \times X \mid (x, y) \in R\} \tag{32.6}$$

with the usual transpose property : $(S \circ R)^\dagger = R^\dagger \circ S^\dagger$, $\mathrm{Id}_X^\dagger = \mathrm{Id}_X$, $(R^\dagger)^\dagger = R$

Example 32.1.1. The relation $<$ on $\mathbb{N} \times \mathbb{N}$ is the obvious one where $(a, b) \in <$ if $a < b$. Its converse is the set of pairs (b, a) obeying $a < b$, which is just the relation $>$.

Example 32.1.2. The empty relation $\varnothing$ is the relation for which no two elements are related. It is absorbing $\varnothing R = R\varnothing$ (if the two can be composed), and it is self-dual.

In the context of a dagger category, a relation $R : X \to Y$ is self-dual, $R^\dagger \cong R$, if it is symmetric.

The obvious generalization of this notion in some arbitrary category $\mathbf{C}$ is to consider some monomorphism into the product of two objects,

$$R \hookrightarrow A_1 \times A_2 \xrightarrow{p_i} A_i \tag{32.7}$$

We can furthermore generalize this to n-ary relations,

$$R \hookrightarrow \prod_{i=1}^{n} A_i \xrightarrow{p_i} A_i \tag{32.8}$$

where in particular, a nullary relation is a constant subterminal object (which is, as we will see, a truth value), and a unary relation is simply a subobject, where the relationship is true if the object we consider belong to this subobject (it can be given the semantics of a property)

If we have a pair of generalized element

$$a : A \quad \to \quad X \tag{32.9}$$
$$b : B \quad \to \quad Y \tag{32.10}$$

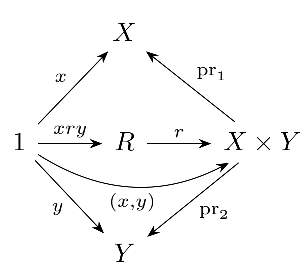
we say that those two elements are related by $R \hookrightarrow A \times B$ if the map

$$(a, b) : A \times B \to X \times Y \tag{32.11}$$

factors through R, ie there exists a factorization of (a, b) as

$$A \times B \xrightarrow{arb} R \xrightarrow{r} X \times Y$$

Example 32.1.3. For any relation in **Set**, we have that xRy simply if the two points $x : 1 \to X$, $y : 1 \to Y$, with $1 \times 1 \cong 1$



Example 32.1.4. To give an example without using global elements, in the category **Grp**, given a group G and its Abelianization $G/[G,G]$, and its embedding in G

$$\iota_{\mathrm{Ab}} : G/[G,G] \hookrightarrow G \tag{32.12}$$

Then ι defines a relation, and given two elements of G,

$$g, h : \mathbb{Z} \to G \tag{32.13}$$

the relation defined by the existence of a factorization by ι_{Ab} means that those two elements commute. For instance, given the smallest non-Abelian group, the dihedral group of order 6, D_6, generated as

$$D_6 = \langle a, b \mid a^2 = b^2 = (ab)^3 = e \rangle \tag{32.14}$$

with elements e, a, b, ab, ba, aba, and inverses

$$\begin{aligned}
a^{-1} &= a && (32.15)\\
b^{-1} &= b && (32.16)\\
ab^{-1} &= ba && (32.17)\\
ba^{-1} &= ab && (32.18)\\
aba^{-1} &= aba && (32.19)
\end{aligned}$$

The homomorphisms from $\mathbb{Z}$ to D_6 are given by the maps from the unique generator of $\mathbb{Z}$ to D_6,

$$\begin{aligned}
\phi_1(a) &= e && (32.20)\\
\phi_2(a) &= a && (32.21)\\
\phi_3(a) &= b && (32.22)\\
\phi_4(a) &= ab && (32.23)\\
\phi_5(a) &= ba && (32.24)\\
\phi_6(a) &= aba && (32.25)\\
& && (32.26)
\end{aligned}$$

Its commutator subgroup is given by (outside of trivial results)

$$\begin{aligned}
[a, b] &= a = [a, ba] = [a, aba] && (32.27)\\
[b, ab] &= b = [b, ba] = [ab, aba] && (32.28)\\
[b, aba] &= aba && (32.29)\\
[ba, aba] &= ba && (32.30)
\end{aligned}$$

This is the alternating group A_3, a normal subgroup of D_6.

Cosets :

its abelianization is $\mathbb{Z}_3$, with elements $0, 1, 2$, which is given as $\langle a \mid a^3 = e \rangle$, with the Abelian inclusion of

$$\iota_{\text{Ab}}(0) = e \quad (32.31)$$
$$\iota_{\text{Ab}}(1) = e \quad (32.32)$$
$$\iota_{\text{Ab}}(2) = e \quad (32.33)$$

Conjugacy, same coset?

Theorem 32.1.1. A relation $R \hookrightarrow 1 \times X$ is simply a predicate of belonging to R

Proof. As we have $1 \times X \cong X$, this reduces to the case of a unary relation. □

Theorem 32.1.2. A relation $R \hookrightarrow 0 \times X$ is the empty relation.

Proof. As $0 \times X \cong 0$, and □

From this definition of the relation between two generalized element, we can build the "graph" of the relation as a subcategory of the category of subobjects $\text{Sub}(X \times Y)$, where we require that all arrows of $\text{Sub}(X \times Y)$ factor through R. [slice category of $\text{Im}(r)$ in $\text{Sub}(X \times Y)$?]

$$\text{Graph}(R) = \quad (32.34)$$

Definition 32.1.2. For a relation $r : R \to X \times Y$, we say that it is a left (resp. right) *total relation* if for any $x \in X$ (resp. $y \in Y$), there exists at least one $y \in Y$ (resp. $x \in X$) for which xRy.

This definition is translated in categorical terms by

right uniqueness?

Normal subobject?

Theorem 32.1.3. If two relations $r_1, r_2 : R \to X \times Y$ share the same monomorphism in their epi-mono factorization, they are the same relationship, in the sense that for any two generalized elements $a : A \to X$, $b : B \to Y$, we have $ar_1b \leftrightarrow ar_2b$

Proof. □

As relations are classified specifically by the monomorphism part of their epi-mono factorization, we can consider

This means that as for any subobject category, the category of relations forms a poset. And in the case of a topos, this category is a Heyting algebra.

If we have

Lattice for relations [35]

Bottom : diagonal relation (equality)

Top : total relation (all elements related)

Meet, join

In set theory, we also have that functions are a specific type of relation, where for a function $f : X \to Y$, with the evaluation $f(x) = y$, if we consider it as some relation xFy with $F \hookrightarrow X \times Y$, we have the constraint that every element $x \in X$ is present exactly one time, or in other words,

$$\forall x \in X,\ \exists! r \in R,\ \mathrm{pr}_1(r) = x \tag{32.35}$$

Definition 32.1.3. The relations on an object X, denoted $\mathrm{Rel}(X)$, are the subobject category of the product $X \times X$:

$$\mathrm{Rel}(X) = \mathrm{Sub}(X \times X) \tag{32.36}$$

Birkhoff's representation theorem?

Of particular interest for relations are the relations on X itself, ie the relations

$$R \hookrightarrow X \times X \tag{32.37}$$

Theorem 32.1.4. The coequalizer of parallel morphisms $X \rightrightarrows Y$ defines a functor from the category of relations on X to the category of quotients on X :

$$\mathrm{coeq} : \mathrm{Rel}(X) \quad \to \quad \mathrm{Quot}(X) \tag{32.38}$$

Proof. Given a relation $R \in \mathrm{Rel}(X)$, we can form the parallel pair by going through the projections of the product,

$$R \xhookrightarrow{\ \iota\ } X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$$

for which, applying the coequalizer of $\mathrm{pr}_i \circ \iota$, we obtain the limit

$$\mathrm{coeq}(\mathrm{pr}_1 \circ \iota, \mathrm{pr}_2 \circ \iota) = X/R \tag{32.39}$$

with the diagram

$$X \xrightarrow{q} X/R$$

Up to equivalence of epimorphisms, this yields a quotient of X. □

This is the formalization of the intuition that a coequalizer is the quotient of a set given a relation.

Example 32.1.5. In **Top**, consider the unit interval $[0,1]$, with the open sets defined by

$$\mathrm{Open}([0,1]) = [[0,1], \Sigma] \tag{32.40}$$

the maps to the Sierpinski space.

if we take the relation

$$R = \{(0,1)\} \tag{32.41}$$

with the maps

$$\mathrm{pr}_1 \circ \iota((0,1)) = 0 \tag{32.42}$$
$$\mathrm{pr}_2 \circ \iota((0,1)) = 1 \tag{32.43}$$

its coequalizer gives the quotient map of the circle :

$$q : [0,1] \twoheadrightarrow S^1 \tag{32.44}$$

Sinc

The coeq functor has a right adjoint in the form of the kernel functor.

Theorem 32.1.5. The kernel of a morphism defines a functor from the quotients on X to relations on X.

Proof. Given a quotient $q : X \twoheadrightarrow Q$, its kernel

$$\begin{array}{ccc} \mathrm{Fib}_p(f) & \xrightarrow{!} & 1 \\ {\scriptstyle p^*f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

□

$$\mathrm{coeq} : \mathrm{Rel}(X) \to \mathrm{Quot}(X) \tag{32.45}$$
$$\ker : \mathrm{Quot(X)} \to \mathrm{Rel}(X) \tag{32.46}$$

(coeq ⊣ ker)

”a binary relation R is a function if it transmits elements and reflects distinctions.”

”an injective function is a function that transmits distinctions, and a surjective function is a function that reflects elements.”

32.2 Span categories

To put this notion of a relation into more categorical terms, let’s look at how categories interact with spans. If we have a category **C** with pullbacks, we can easily enough define a category of spans for **C** via the obvious process of simply taking the functor category of the span category,

$$\Lambda : -1 \xleftarrow{s} 0 \xrightarrow{t} +1 \tag{32.47}$$

ie $\mathbf{C}^{\Lambda}$, which is composed of all triples of objects $F(-1, 0, +1) = (A, B, C)$ with two morphisms

$$F(s) : B \quad \to \quad A \tag{32.48}$$
$$F(t) : B \quad \to \quad C \tag{32.49}$$

and whose morphisms are natural transformations η on those categories, which map central objects to each other and boundary objects to each other, obeying the naturality condition

$\Lambda_{A,B,C,f,g}$

$$\begin{array}{ccc} F(\bullet_i) & \xrightarrow{F(\sigma)} & F(\bullet_j) \\ \downarrow{\scriptstyle \eta_{\bullet_i}} & & \downarrow{\scriptstyle \eta_{\bullet_j}} \\ G(\bullet_i) & \xrightarrow{G(\sigma)} & G(\bullet_j) \end{array}$$

which means, as we have only two possible non-trivial morphisms in Λ, s and t,

$$\begin{array}{ccc} F(0) & \xrightarrow{F(s)} & F(-1) \\ \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_0} \\ G(0) & \xrightarrow{G(s)} & G(-1) \end{array}$$

$$\begin{array}{ccc} F(0) & \xrightarrow{F(t)} & F(+1) \\ \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_0} \\ G(0) & \xrightarrow{G(t)} & G(+1) \end{array}$$

in other words, the left and right legs of one span are mapped to the left and right leg of the other.

The evaluation map of those functors give us the functor that select the outer objects of the span

$$(\mathrm{ev}_{-1}, \mathrm{ev}_{+1}) : \mathbf{C}^{\Lambda} \to \mathbf{C} \times \mathbf{C} \tag{32.50}$$
$$(A \leftarrow B \rightarrow C) \mapsto (A, C) \tag{32.51}$$

From this, we can construct another category from spans, called the span category, based on **C** and whose morphisms are given by the fibers of the evaluation functor

$$\mathrm{Hom}_{\mathrm{Span}(\mathbf{C})}(X, Y) = (\mathrm{ev}_{-1}, \mathrm{ev}_{+1})^{-1}(X \times Y) \tag{32.52}$$

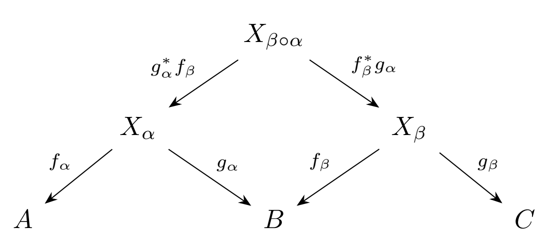
[36]

Definition 32.2.1. For a category **C** with pullbacks, the *span category* Span(**C**) is the category whose objects are the same as **C**, the morphisms $A \to B$ are given by spans $A \leftarrow X \to B$ (there is one morphism $A \to B$ per object X and per morphism f, g)

$$m_{X,f,g} = (f : X \to A,\ g : X \to B) \tag{32.53}$$

so that $s(m_{X,f,g}) = t(f)$ and $t(m_{X,f,g}) = t(g)$

composition $\beta \circ \alpha$ for $s(\beta) = t(\alpha)$ is given by the pullback of the common leg for two morphisms



so that $\beta \circ \alpha$ is the span

$$X_\alpha \overset{f_\alpha \circ g_\alpha^* f_\beta}{\longleftarrow} X_\alpha \times_B X_\beta \overset{g_\beta \circ f_\beta^* g_\alpha}{\longrightarrow} X_\beta \tag{32.54}$$

and the identity is given by the span of two identity morphisms,

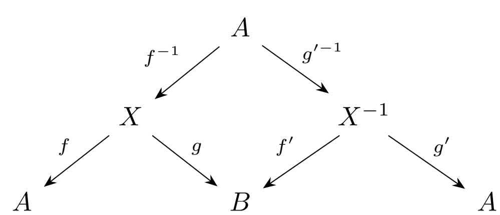
$$\mathrm{Id}_A = (A \leftarrow A \rightarrow A) \tag{32.55}$$

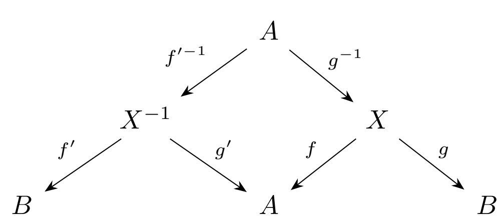
which can be checked to obey the identity properties by the pullback of identity morphisms.

As pullbacks are only defined up to isomorphism, this means that this is not strictly speaking a category, since the composition of two morphisms is not strictly defined, and this also entails that composition is not strictly associative. To construct the actual category, we need to consider the skeletonized version of **C** instead, which we will assume implicit from now on.

Theorem 32.2.1. In the span category of a category, isomorphisms are given by spans whose legs are both isomorphisms, meaning that assuming the underlying category skeletonized, this restricts them to identity spans.

Proof. From the composition law in span categories, the span corresponding to the inverse of a span $A \leftarrow X \rightarrow B$ will be given by another span $B \leftarrow X^{-1} \rightarrow A$ such that their pullback along either common legs will be such that the pullback morphisms are the inverse of the original, ie





□

Dagger category where the dagger exchanges the legs of the span

Embedding of the category in Span($\mathbf{C}$) is identity on objects, and for morphisms,

$$\begin{aligned} \iota : \mathbf{C} &\hookrightarrow \mathrm{Span}(\mathbf{C}) && (32.56) \\ (f : X \to Y) &\mapsto (X \xleftarrow{\mathrm{Id}_X} X \xrightarrow{f} Y) && (32.57) \end{aligned}$$

which obeys the appropriate composition law functorially. We can also have the (co?)reflection

$$\begin{aligned} T : \mathrm{Span}(\mathbf{C}) &\twoheadrightarrow \mathbf{C} && (32.58) \\ (A \xleftarrow{f} X \xrightarrow{g} B) &\mapsto g : (X \to B) && (32.59) \end{aligned}$$

Span category as the category of relations?

Theorem 32.2.2. For any span $S = (A \leftarrow X \to B)$, there exists a relation $R \hookrightarrow X \times Y$

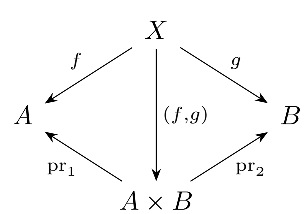
Proof. Given the span S with legs $f : X \to A$ and $g : X \to B$, we can construct an additional span from the product of A and B using the universal property of the product,

$$A \xleftarrow{\mathrm{pr}_1} A \times B \xrightarrow{\mathrm{pr}_2} B \tag{32.60}$$

and define the span morphism between those two spans given by the pair morphism (f, g) via the universal property :

$$(f, g) : X \to A \times B \tag{32.61}$$

with the diagram



and then, using the (epi, mono) factorization in the topos,

$$X \twoheadrightarrow \mathrm{Im}(f, g) \hookrightarrow A \times B$$

giving us the relation

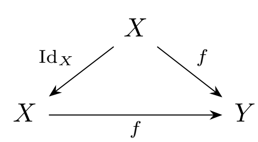
$$R = \mathrm{Im}(f, g) \hookrightarrow A \times B \tag{32.62}$$

□

The categorical equivalent of this relation giving a function if the relation has a unique pair for every element of A is that our left leg $f : X \to A$ should be such that its (epi, mono) factorization has a monomorphism which is also an isomorphism, ie

$$X \twoheadrightarrow \mathrm{Im}(f) \xhookrightarrow{\cong} A$$

in which case the corresponding morphism is simply (isomorphic to) the right leg of the span, which we can rewrite in canonical form as the span



In other words, the morphisms of a category correspond indeed to the spans of that form as we assumed earlier in our definition of the span category.

Example 32.2.1. In **Set**, the span category is given by

$$\mathrm{Span}(\mathbf{Set}) \cong \mathbf{Rel} \tag{32.63}$$

Proof. Given a span of two sets, $A \leftarrow X \to B$, □

Correspondence of a span and a morphism, is that the diagonal filler between the legs?

Theorem 32.2.3. A relation R from a span is left total if its left leg is an epimorphism. It is right total if its left leg is a monomorphism.

Proof. □

Beck-Chevalley condition

32.3 Partitions

As we will see in the next section, the dual of a relation is the notion of a partition, so first let's look briefly at what a partition is on a set.

The simple definition of a partition

Definition 32.3.1. A *partition* P of sets X is a collection of subsets of X, $\{S_i\}_{i\in I}$, $S_i \in \mathcal{P}(X)$, such that they are pairwise disjoint,

$$\forall i, j \in I,\ S_i \cap S_j \neq \varnothing \leftrightarrow i = j \tag{32.64}$$

and which form a cover of X :

$$X = \bigcup_{i\in I} S_i \tag{32.65}$$

or equivalently, as they are disjoint,

$$X = \coprod_{i\in I} S_i \tag{32.66}$$

The subsets S_i of the partition P are called the *blocks* of the partition.

Theorem 32.3.1. The partition of a set X with indexing set I can be expressed by the existence of a surjection

$$P : X \twoheadrightarrow Y \tag{32.67}$$

Proof. For every pair of elements $y_1, y_2 \in Y$, we have that their preimage are disjoint if $y_1 \neq y_2$ (by the property of functions having a unique pair $(x, f(x))$ for any given x)

$$P^{-1}(y_1) \cap P^{-1}(y_2) = \varnothing \tag{32.68}$$

and since every element of X belongs to the preimage of some element of Y,

$$X = \bigcup_{y\in Y} P^{-1}(y) = \coprod_{y\in Y} P^{-1}(y) \tag{32.69}$$

so that we can define the partition

$$P = \{P^{-1}(y)\}_{y\in Y} \tag{32.70}$$

□

As different functions could partition the set into the same partitions, simply by relabeling them with different elements of Y, we will consider partitions to be in fact equivalence classes to such surjections, so that the partition functions $P_i : X \to Y_i$ are equivalent if they are related by an isomorphism $Y_i \to Y_j$. In particular, if we consider the partition function over the same set Y, this is equivalence up to permutation of Y, S_Y.

Definition 32.3.2. Given two partitions P_1, P_2 of the same set X, we say that P_2 is a *refinement* of P_1 if for any block S_i of P_1, there exists a collection of blocks of P_2 which are a partition of S_i

Theorem 32.3.2. The notion of refinement for two partition functions $P_1 : X \to Y_1$, $P_2 : X \to Y_1$,

Theorem 32.3.3. Refinement defines a poset on the set of all partitions.

Proof. Any partition is a refinement of itself, where the partition of each block is simply the identity $S_i \to \{\bullet\}$. If P_1 is a refinement of P_2, and P_2 is a refinement of P_1, for each block $B_{1,i}$ we have a family of blocks of P_2,

$$B_{1,i} = \coprod_{j \in J_i} B_{2,j} \tag{32.71}$$

and likewise,

$$B_{2,n} = \coprod_{m \in K_n} B_{1,m} \tag{32.72}$$

meaning that, combining those relations, we have

$$B_{1,i} = \coprod_{j \in J_i} (\coprod_{m \in K_j} B_{1,m}) \tag{32.73}$$

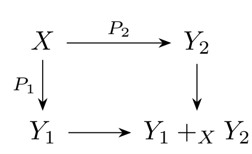
Since all the blocks are mutually disjoint, Antisymmetry Transitivity ☐

A logical predicate we can define on a partition is whether two elements of X are part of the same partition, that is,

$$P(x_1, x_2) \leftrightarrow P(x_1) = P(x_2) \tag{32.74}$$

Refinement of partitions

Pushout of partitions?



There is a partition of X by $Y_1 +_X Y_2$, the quotient of $Y_1 + Y_2$ by the equivalence relation $P_1(x) \sim P_2(x)$, ie given two indices i, j of the old partitions, those are the same index in the new partition if they overlap.

Lattice of partitions of a finite set [35] :

Bottom element : identity partition function (one block per element)

Top element : terminal morphism $X \to 1$ (partition of a single block)

Join : pushout

Meet :

Pushout is join

Theorem 32.3.4. The join of a partition P by a refinement of it is equal to the original partition.

Proof. □

As expected from a distributive lattice, the bottom element (the maximal partition), being a refinement for all partitions, is the identity of the join.

Example 32.3.1. Given the set $\{a, b, c, d, e, f\}$, consider the partition

$$B_1 = \{a, b, c\},\ B_2 = \{d, e, f\} \tag{32.75}$$

and the partition

$$B_1' = \{a, b\},\ B_2' = \{c\},\ B_3' = \{d, e, f\} \tag{32.76}$$

Those partitions overlap in such a way that

$$B_1 \sim B_1' \tag{32.77}$$

$$B_1 \sim B_2' \tag{32.78}$$

So that the new partition is simply that of B_1, B_2.

equivalence between the partial order of partitions ordered by refinement and the preorder of surjections ordered by factorization.

32.4 Corelations

As we have relations as the spans of a category, dually, we can also define the corelations as built from cospans [37].

"In fact, just as relations can be considered as subobjects of the Cartesian product, so corelations can be considered quotients of the coproduct (esp. when P is a regular epimorphism)."

If we consider the cospan diagram,

$$\mathbf{V} : -1 \xrightarrow{s} 0 \xleftarrow{t} +1 \tag{32.79}$$

and the associated functor category $\mathbf{C}^{\mathbf{V}}$

The cospan category will be likewise the category

$$\mathrm{Hom}_{\mathrm{CoSpan}(\mathbf{C})}(X, Y) = \tag{32.80}$$

In terms of sets, the relation, being a subset of a product, has as its dual, the corelation, the dual notion of a subset on a coproduct, which is that of a partition. While a subset of a product is given by a monomorphism into the product, a partition of a set is given by a surjection from the coproduct,

$$P : A + B \rightarrow X \tag{32.81}$$

For instance in the case $X = 2$, a set of two elements, we have that every element in X or Y has an associated label given by

$$P_i = P \circ \iota_{A/B} \tag{32.82}$$

The partition elements are then $P^{-1}(\bullet_i)$, giving the elements of the i-th partition.

The associated logical statement of a partition is then given by the composition $P \circ \iota_{A/B}$, where we have the statements

$$(P \circ \iota_A)(a) = x \tag{32.83}$$
$$(P \circ \iota_B)(b) = x \tag{32.84}$$
$$\tag{32.85}$$

that the element a or b belongs to the partition x of $A + B$.

Definition 32.4.1. A *corelation* α on two sets X, Y is a partition of the coproduct $X + Y$

Cospan category as a way to partition coproducts in two?

Relation to the negation on the subobject classifier?

32.5 Orthogonality and Quillen negation

[38, 39, 40]

The orthogonality of two morphisms that we saw in [X] can be interpreted by ways of spans and relations. If we consider two morphisms $i : A \to B$, $p : X \to Y$, and try to determine if they have the lifting property with respect to each other,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \exists ? \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

this defines a family of spans $\{\Lambda_{i,f}\}_{f \in \mathrm{Hom}(A,X)}$ and of cospans $\{V_{p,g}\}_{g \in \mathrm{Hom}(B,Y)}$, both with the legs B, X.

The spans define a family of relations

$$R_f : A \xrightarrow{(i,f)} \mathrm{Im}(f,i) \twoheadrightarrow B \times X$$

Its filler exists if furthermore, this relation

Interpretation of Quillen negation in logical terms?

Interpretation of orthogonality of morphisms

if $p \boxslash q$,

$$p : A \quad \to \quad B \tag{32.86}$$
$$q : X \quad \to \quad Y \tag{32.87}$$

p and q are predicates with values [context?] in A, Y

New predicate as the diagonal filler : $r : B \to X$

Typing : $p[a]$, $q[y]$, $r[b]$, we have $r[p[a]]$

Quillen negation gives us collections of morphisms : quantification on predicates, ie

$$p \in M^{\boxslash l} \leftrightarrow \forall q \in M^{\boxslash \cdots},\ p \boxslash q \tag{32.88}$$

Splitting of functions in orthogonality relations as one, the other, neither, both? is both always an isomorphism (since orthogonal to itself)

33
Internal logic

1

[41, 42]

From the Lindenbaum-Tarski correspondence between logic and algebra, we also have the converse of being able to associate a logic to every algebra on an order structure, at least for a broad enough definition of ”logic”.

[proof?]

This means in particular that in a category, for any object $X \in \mathbf{C}$, we have some logic associated with the ”type” of X given by the poset of its subobjects.

Objects as types

These ”logics” will depend on the properties of the category. In particular, we will not go very far in terms of a logic without a notion of finite products [correspondence to what?]

Syntatic category, category of contexts?

Meaning of the implication wrt Heyting implication

34

Internal hom

One component of the definition of a topos regards the behaviour of its *internal homs*, a way to internalize the hom-set of the category in its objects. In other words, every space of morphisms between two objects of the topos is itself an object of the topos, allowing us to talk about such function spaces internally to the topos itself.

Definition 34.0.1. In a symmetric monoidal category $(\mathbf{C}, \otimes, I)$, an *internal hom* is a bifunctor

$$[-,-] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C} \tag{34.1}$$

such that for any object $X \in \mathbf{C}$, the functor $[X, -]$ is right adjoint to the functor $(-) \otimes X$:

$$((-) \otimes X \dashv [X, -]) : \mathbf{C} \to \mathbf{C} \tag{34.2}$$

The simplest way to see the content of internal homs from their definition is via the adjunction of hom sets :

$$\mathrm{Hom}_{\mathbf{C}}(Z, [X, Y]) \cong \mathrm{Hom}_{\mathbf{C}}(Z \otimes X, Y) \tag{34.3}$$

If we consider the case where $Z = I$, we have the equivalence

$$\mathrm{Hom}_{\mathbf{C}}(I, [X, Y]) \cong \mathrm{Hom}_{\mathbf{C}}(I \otimes X, Y) \cong \mathrm{Hom}_{\mathbf{C}}(X, Y) \tag{34.4}$$

In other words, if we look at $\mathrm{Hom}_{\mathbf{C}}(I, [X, Y])$, the set of generalized elements of the monoidal object, then it is isomorphic to the actual hom-set of the category.

This internal hom is isomorphic to the hom-set as a set, but also carry the structure of $\mathbf{C}$ objects. If we pick the case of the Cartesian monoidal category $(\mathbf{Set}, \times, \{\bullet\})$ in particular, we recover an exact definition of the hom-set.

Effect on associator, unitor and braiding? [43]

Currying

As an adjunction of two functors, we have two natural transformations

$$\begin{aligned} \eta :& \qquad \mathrm{Id}_{\mathbf{C}} \to [X, ((-) \otimes X)] && (34.5) \\ \epsilon :& ([X, -] \otimes X) \to \mathrm{Id}_{\mathbf{C}} && (34.6) \end{aligned}$$

The natural transformation ϵ is the notion of an *evaluation map.* If we look at its components at Y, we can define the map on objects

$$\mathrm{eval}_{X,Y} = \epsilon_Y : ([X, Y] \otimes X) \to Y \tag{34.7}$$

The meaning here is that given a function from X to Y and a value in X, we can obtain a value in Y, hence its name of evaluation map. This can be seen explicitely by taking some generalized element $x : I \to X$ and considering the diagram

$$I \xrightarrow{x} X$$

If we have a morphism $Y \otimes X \to Z$, there is equivalently some morphism $Z \to [X, Y]$

Example : take $Z = [X, Y]$, take the morphism $\mathrm{Id}_{[X,Y]} : [X, Y] \to [X, Y]$. Its adjunct is

$$Y \otimes X \to [X, Y] \tag{34.8}$$

Evaluation map :

The counit of the adjunction for $[X, -]$ is called (evaluated at a component Y) the *evaluation map*

$$\mathrm{eval}_{X,Y} : [X, Y] \otimes X \to Y \tag{34.9}$$

Internal hom bifunctor

Example 34.0.1. In the category of vector spaces **Vec**, where the monoidal structure is the tensor product, the internal hom $[V, W]$ is the space of linear maps from V to W, which is indeed itself a vector space. The space of dual vectors is for instance given by $[V, I]$, leading to the equivalence A map between two vector spaces can be considered as the tensor product of the target space and the dual space of the domain. $[X \otimes Y, Z] \cong [X, [Y, Z]]$

Example 34.0.2. There is a category of smooth spaces which is the generalization of the category of smooth manifolds, as we will see later [x]. Partly this a way to include the internal hom for smooth manifolds, so that maps between manifolds are themselves a type of generalized manifold, which can be thought of as diffeological spaces, infinite dimensional manifolds like Hilbert manifoldd, inverse limit manifolds, pro-manifolds, etc.

Definition 34.0.2. For three objects X, Y, Z, with internal homs $[X, Y]$ and $[Y, Z]$, the *composition* morphism

$$\circ_{X,Y,Z} : [Y, Z] \times [X, Y] \to [X, Z] \tag{34.10}$$

is the adjunct of the map

$$f_{X,Y,Z} = \mathrm{ev}_{Y,Z} \circ (\mathrm{Id}_{[Y,Z]} \times \mathrm{ev}_{X,Y}) \tag{34.11}$$

so that

$$\circ_{X,Y,Z} = \overline{f}_{X,Y,Z} \tag{34.12}$$

Why that is composition :

Consider the morphism

$$\mathrm{Hom}(1, \circ_{X,Y,Z}) : \mathrm{Hom}(1, [Y, Z] \times [X, Y]) \to \mathrm{Hom}(1, [X, Z]) \tag{34.13}$$

Mapping a pair of functions $X \to Y$ and $Y \to Z$, to a function $X \to Z$.

$$\begin{aligned} \mathrm{Hom}(1, \circ_{X,Y,Z}) &\cong \mathrm{Hom}(1, \epsilon_{[X,Z]} \circ Z \times (\mathrm{ev}_{Y,Z} \circ (\mathrm{Id}_{[Y,Z]} \times \mathrm{ev}_{X,Y}))) \\ &\cong \mathrm{Hom}(1,) \end{aligned}$$

Theorem 34.0.1. Commutation of the product something

Theorem 34.0.2. Given two subobjects $A, B \hookrightarrow X$, their internal hom $[A, B]$ is a subobject of $[X, X]$ [and $[A, X]$ or $[X, B]$?]

Proof. Use the monomorphism property of monoidal products? □

Example 34.0.3. The *state monad*

34.1 Cartesian closed categories

As the product is an example of a monoidal structure, it is quite common to define the internal hom for it.

Definition 34.1.1. In a category with finite products, defined as a symmetric monoidal category $(\mathbf{C}, \times, 1)$, the *exponential object* of two objects X, Y, denoted Y^X is the internal hom with respect to that monoidal product :

$$Y^X = [X, Y]_{(\mathbf{C}, \times, 1)} \tag{34.14}$$

Definition 34.1.2. A category is *Cartesian closed* if it is a closed monoidal category with respect to the product, $(\mathbf{C}, \times, 1)$.

The internal hom of the Cartesian product is generally called more specifically the *exponential object* and denoted by

$$[X, Y] = Y^X \tag{34.15}$$

By analogy with the notation for the set of functions between two sets, which is the prototypical example of a Cartesian closed category.

Adjunction

$$(- \times A \dashv (-)^A) \tag{34.16}$$

Theorem 34.1.1. As an adjunction, the monoidal product functor preserves colimits and the internal hom functor preserves limits.

Example : the set of all function from X to some object $[X, -]$ preserves the terminal object :

$$[X, 1] \cong 1 \tag{34.17}$$

which is just the basic property of the hom-set for terminal object. Similarly, for two morphisms $f : Y \to X_1$ and $g : Y \to X_2$, we have

$$[Y, X_1 \times X_2] \cong [Y, X_1] \times [Y, X_2] \tag{34.18}$$

where the functions to a product are the product of those functions to each, ie morphism of the form (f, g)

Pullback : $[X, Y_1 \times_Z Y_2] \cong [X, Y_1] \times_{[X,Z]} [X, Y_2]$

Equalizer : $[X, \mathrm{eq}(f, g)] = \mathrm{eq}([X, g], [X, g])$.

Similarly for $(-) \otimes X$, $0 \otimes X = 0$. This is true for the Cartesian product with the empty set in **Set**

Theorem 34.1.2. Isomorphism

$$[X \otimes Y, Z] \cong [X, [Y, Z]] \tag{34.19}$$

$$\mathrm{Hom}_{\mathbf{C}}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C)) \tag{34.20}$$

34.2 Internal automorphisms

Definition 34.2.1. In a Cartesian closed category $\mathbf{C}$, an object is an *internal automorphism group* of an object X if it is the largest subobject of $[X, X]$ for which the composition map $\circ$ is invertible.

35

Enriched categories

By default, we consider the hom-sets of a category $\text{Hom}_{\mathbf{C}}(X, Y)$ to be sets, but many categories may have additional structure on their hom-sets. For instance, if we consider the category $\mathbf{Vec}_k$ of vector spaces over the field k, its morphisms are k-linear maps, and its hom-sets are

$$\text{Hom}_{\mathbf{Vec}_k}(V, W) = L_k(V, W) \tag{35.1}$$

However, in addition to being a set, $L_k(V, W)$, the k-linear maps, also form themselves a vector space, as we can define the sum $f + g$ of two linear maps, and the scaling αf, $\alpha \in k$, of a linear map.

To generalize this notion, we define enriched categories

Definition 35.0.1. An *enriched category* $\mathbf{C}$ over $\mathbf{V}$ a monoidal category $(\mathbf{V}, \otimes, I)$ is a category such that each hom-set $\text{Hom}_{\mathbf{C}}(X, Y)$ is associated to a *hom-object* $C(X, Y) \in \mathbf{V}$, such that every hom-object in $\mathbf{V}$ obeys the same rules regarding composition and identity, which are

$$\circ_{X,Y,Z} : C(Y, Z) \otimes C(X, Y) \to C(X, Z) \tag{35.2}$$

$$j_X : I \to C(X, X) \tag{35.3}$$

with the following commutation diagrams :

[composition is associative]

[Composition is unital]

Example 35.0.1. A category enriched in **Set** is a locally small category.

Proof. □

Most of the categories we have seen so far are in fact

Definition 35.0.2. A *k-linear category* is enriched over Vec_k.

Example 35.0.2. As said, $\mathbf{Vec}_k$ itself is a k-linear category.

Proof. □

Definition 35.0.3. A *cosmos* is a complete and cocomplete closed symmetric monoidal category.

36

Mixed-variance functors

It is common to have functors which will depend on a given category several times, sometimes in a covariant fashion, sometimes in a contravariant fashion.

36.1 Profunctors

The basic case of mixed-variance functors is that of profunctors.

Definition 36.1.1. A *profunctor* is a functor

$$H_F : \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set} \tag{36.1}$$

denoted by $F : \mathbf{C} \nrightarrow \mathbf{D}$

Example 36.1.1. The hom functor is a profunctor $\mathrm{Hom} : \mathbf{C} \nrightarrow \mathbf{C}$.

Theorem 36.1.1. The identity profunctor $\mathrm{Id} : \mathbf{C} \nrightarrow \mathbf{C}$ is the hom bifunctor

Proof. □

Theorem 36.1.2. The category of spans $\mathrm{Span}(\mathbf{C})$ is equivalent to the profunctor

$$s : \mathbf{C}^{\mathrm{op}} \nrightarrow \mathbf{C} \tag{36.2}$$

Proof. □

”In general, any functor like this may be interpreted as establishing a relation between objects in a category. A relation may also involve two different categories C and D. A functor, which describes such a relation, has the following signature and is called a profunctor:

”A profunctor F between C and D can also be viewed as a span C 0←F 1→D 0 that is compatible with composition in C and D in an appropriate way. ”

For a more general case, we can also consider enriched categories

Definition 36.1.2. A V-enriched profunctor is given by two categories **C**, **D** enriched over a closed monoidal category V,

$$H_F : \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to V \tag{36.3}$$

36.2 Supernatural transformations

The use of functors of mixed variance may require the extension of the notion of natural transformation, with the notions of *extranatural transformation* and *dinatural transformation.*

As you remember from 18, for two functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation between those functors $\eta : F \Rightarrow G$ has the naturality condition

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

For an extranatural transformation, we will deform the naturality condition in a certain way.

The basic example of an extranatural transformation is done by considering the hom bifunctor on **Set**, which is clearly of mixed variance :

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Set}} : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} &\to \mathbf{Set} && (36.4) \\ (Y, X) &\mapsto X^Y && (36.5) \end{aligned}$$

with X^Y the set of all functions from Y to X (this is the exponential object of **Set** 34.1.2). A simple natural transformation we can construct with this is the identity natural transformation,

$$\mathrm{Id}_{\mathrm{Hom}} : \mathrm{Hom}_{\mathbf{Set}} \to \mathrm{Hom}_{\mathbf{Set}} \tag{36.6}$$

with components of the form

$$\mathrm{Id}_{X,Y} : X^Y \to X^Y \tag{36.7}$$

which simply associates any function $Y \to X$ to itself. The naturality condition in Y says that for any function $g : Y \to Y'$, if we consider the function

$$X^g = \mathrm{Hom}_{\mathbf{Set}}(X, g) : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, Y') \tag{36.8}$$

which maps functions $f : X \to Y$ to functions $f' : X \to Y'$ by post-composition $f' = g \circ f$, then we have that the natural transformation gives us

$$\mathrm{Id}_{X,Y} X^g = X^g \mathrm{Id}_{X,Y'} : X^{Y'} \to X^Y \tag{36.9}$$

$$\begin{array}{ccc} X^{Y'} & \xrightarrow{\mathrm{Id}_{X,Y'}} & X^{Y'} \\ {\scriptstyle X^g}\downarrow & & \downarrow{\scriptstyle X^g} \\ X^Y & \xrightarrow[\mathrm{Id}_{X,Y'}]{} & X^Y \end{array}$$

As those are identity maps, this condition is fairly trivial. The role of the extranaturality will then be to deform this condition into a less trivial one. Since the exponential object can be defined in terms of the internal hom of **Set**, we can take $\mathrm{Id}_{X,Y} : [Y', X] \to [Y, X]$ and consider its adjunct, which is given by the eval map of the identity

$$\mathrm{eval}_{X,Y} : [Y, X] \otimes Y \to X \tag{36.10}$$

which is the object corresponding to a couple of a function $f : Y \to X$ and an element of Y, which is mapped to an element of X via the evaluation $\mathrm{eval}(f, x) = f(x) = y$.

This transformation is natural in X, as for any function $h : X \to X'$, which induces the morphism h^Y by pre-composition, ie

$$h^Y \circ f = f \circ h \tag{36.11}$$

we have that the eval natural transformation does form the commutative diagram

$$\begin{array}{ccc} X^Y \otimes Y & \xrightarrow{\mathrm{eval}_{X,Y}} & X \\ {\scriptstyle h^Y \otimes \mathrm{Id}_Y}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ X'^Y \otimes Y & \xrightarrow[\mathrm{eval}_{X',Y}]{} & X' \end{array}$$

[prove]

which is just the condition that applying the function h to some pair (f, y) before of after the evaluation does not change the outcome, ie the evaluation of $(f \circ h, y)$ is equal to $f(h(y))$.

However, the eval map is not natural in y. If we consider again the map $g : Y \to Y'$ and its induced map X^g,

$$\mathrm{eval}_{X,Y}(X^g \otimes Y) = \mathrm{eval}_{X,Y'}(X^{Y'} \otimes g) : X^{Y'} \otimes Y \to X \tag{36.12}$$

$$\begin{array}{ccc} X^{Y'} \otimes Y & \xrightarrow{\mathrm{Id}_{X^{Y'}} \otimes g} & X^{Y'} \otimes Y' \\ {\scriptstyle X^g \otimes \mathrm{Id}_Y}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{eval}_{X,Y'}} \\ X^Y \otimes Y & \xrightarrow[\mathrm{eval}_{X,Y}]{} & X \end{array}$$

"Notice how the extranatural variable y in eval x,y appears once covariantly [in the tensor factor] and once contravariantly [in the exponent], but together on the same side of the arrow [here the domain]."

Equivalent example for coeval

Those examples are somewhat equivalent to their adjunct version, but however fail to be natural transformations. To account for such phenomenon, we can define a wider condition of naturality called extranaturality :

Definition 36.2.1. For two functors of the form

$$F : \mathbf{A} \times \mathbf{B} \times \mathbf{B}^{\mathrm{op}} \to \mathbf{D} \tag{36.13}$$

and

$$G : \mathbf{A} \times \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{D} \tag{36.14}$$

an *extranatural transformation* is a family of morphisms

$$\alpha_{a,b,c} : F(a, b, b) \to a, c, c \tag{36.15}$$

which obey the following conditions : For any morphism $f : a \to a'$ in $\mathbf{A}$, and any objects $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

$$\begin{array}{ccc} F(a,b,b) & \xrightarrow{F(f,\mathrm{Id}_b,\mathrm{Id}_b)} & F(a',b,b) \\ {\scriptstyle\alpha_{a,b,c}}\downarrow & & \downarrow{\scriptstyle\alpha_{a',b,c}} \\ G(a,c,c) & \xrightarrow[G(f,\mathrm{Id}_c,\mathrm{Id}_c)]{} & G(a',c,c) \end{array}$$

For any morphism $g : b \to b'$ in $\mathbf{B}$, and any objects $a \in \mathbf{A}$ and $c \in \mathbf{C}$,

$$\begin{array}{ccc} F(a,b,b') & \xrightarrow{F(\mathrm{Id}_a,\mathrm{Id}_b,g)} & F(a,b,b) \\ {\scriptstyle F(\mathrm{Id}_a,g,\mathrm{Id}_b)}\downarrow & & \downarrow{\scriptstyle\alpha_{a,b,c}} \\ F(a,b',b') & \xrightarrow[\alpha_{a,b',c}]{} & G(a,c,c) \end{array}$$

and for any morhpism $h : c \to c'$ in $\mathbf{C}$, and any objects $a \in \mathbf{A}$ and $b \in \mathbf{B}$,

$$\begin{array}{ccc} F(a,b,b) & \xrightarrow{\alpha_{a,b,c}} & G(a,c,c) \\ {\scriptstyle\alpha_{a,b,c'}}\downarrow & & \downarrow{\scriptstyle G(\mathrm{Id}_a,h,\mathrm{Id}_c)} \\ G(a,c',c') & \xrightarrow[G(\mathrm{Id}_a,\mathrm{Id}_c,h)]{} & G(a,c',c) \end{array}$$

In other words, an extranatural transformation is a trifunctor that is natural in its first argument, and extranatural in the sense that we saw earlier for the eval and coeval map.

These concepts can be generalized further with dinatural transformations.

Definition 36.2.2. Given two functors $F, G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ A *dinatural transformation* $\alpha : F \Rightarrow G$ is a family of morphisms indexed by $c \in \mathbf{C}$

$$\alpha_c : F(c,c) \to G(c,c) \tag{36.16}$$

such that for any morphism $f : c \to c'$ in $\mathbb{C}$, the following diagram commute :

$$\begin{array}{ccccc} F(c',c) & \xrightarrow{F(f,c)} & F(c,c) & \xrightarrow{\alpha_c} & G(c,c) \\ {\scriptstyle F(c',f)}\downarrow & & & & \downarrow{\scriptstyle G(c,f)} \\ F(c',c') & \xrightarrow[\alpha_{c'}]{} & G(c',c') & \xrightarrow[G(f,c')]{} & G(c,c') \end{array}$$

36.3 Wedges and cowedges

Definition 36.3.1. For a bifunctor

$$F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D} \tag{36.17}$$

a wedge over F at $d \in \mathbf{D}$ is a dinatural transformation

$$e : \Delta_d \Rightarrow F \tag{36.18}$$

from the constant functor at d to F.

Definition 36.3.2. For a bifunctor

$$F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D} \tag{36.19}$$

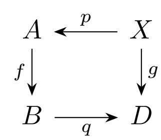
a cowedge over F at $d \in \mathbf{D}$ is a dinatural transformation

$$e : F \Rightarrow \Delta_d \tag{36.20}$$

from F to the constant functor at d.

Wedges and cowedges are in fact nothing but the version of cones and cocones in the mixed-variance case. To show this, let's first define the category of twisted arrows.

Definition 36.3.3. Given a category $\mathbf{C}$, its category of twisted arrows $\mathrm{Tw}(\mathbf{C})$ is the category whose objects are the morphisms of $\mathbf{C}$, and the morphisms are the twisted commutative diagrams, such that for two objects $f : A \to B$ and $g : X \to Y$, a morphism between them is given by two morphisms in $\mathbf{C}$, $p : X \to A$ and $q : B \to Y$, obeying $g = q \circ f \circ p$.



This category admits a mapping from our functors F to functors from the category of twisted arrows,

$$\overline{(-)} : [\mathbf{C}^{\mathrm{op}} \times \mathbf{C}, \mathbf{D}] \quad \to \quad [\mathrm{Tw}(\mathbf{C}), \mathbf{D}] \tag{36.21}$$

so that for any functor $F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$, we can associate the functor

$$\overline{F} : \mathrm{Tw}(\mathbf{C}) \quad \to \quad \mathbf{D} \tag{36.22}$$

$$f \quad \mapsto \quad F(s(f), t(f)) \tag{36.23}$$

Bifunctoriality for F corresponds to functoriality for $\overline{F}$.

Theorem 36.3.1. The category of twisted arrows is the category of elements of the hom functor :

$$\mathrm{Tw}(\mathbf{C}) \cong \mathrm{El}(\mathrm{Hom}_{\mathbf{C}}) \tag{36.24}$$

Proof. □

while a (co)cone over a diagram J is a morphism in the functor category $\mathbf{C}^J$,

36.4 Ends and coends

Definition 36.4.1. For a bifunctor

$$F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D} \tag{36.25}$$

the *end* of F is a terminal wedge

$$\omega : \underline{\mathrm{end}}(F) \Rightarrow F \tag{36.26}$$

with $\underline{\mathrm{end}}$ the end of the functor F.

Definition 36.4.2. For a bifunctor

$$F : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D} \tag{36.27}$$

the *coend* of F is an initial cowedge

$$\alpha : F \Rightarrow \underline{\mathrm{coend}}(F) \tag{36.28}$$

with $\underline{\mathrm{coend}}$ the coend of the functor F.

36.5 Grothendieck construction

An application of a coend is given by the *Grothendieck construction.* If we take a functor $F : \mathbf{C} \to \mathbf{Cat}$

$$BF = \mathbf{C}_{/(-)} \otimes \tag{36.29}$$

which is a coend in **Cat** of the mixed functor

Definition 36.5.1. Given a functor $F : \mathbf{C} \to \mathbf{Cat}$, the Grothendieck construction of F, denoted by $\int_{\mathbf{C}} F$, is the category with objects the pairs

$$(c, x) \in \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(F(c)) \tag{36.30}$$

and whose morphisms are given by pairs of morphisms (f, ϕ) acting on objects as

$$f : c \quad \to \quad c' \tag{36.31}$$

$$\phi : F(f)(a) \quad \to \quad a' \tag{36.32}$$

Example 36.5.1. For $X \in \mathbf{C}$, given a representable functor

$$\mathrm{Hom}_{\mathbf{C}}(-, X) : \mathbf{C} \to \mathbf{Set} \tag{36.33}$$

its Grothendieck construction is the slice category $\mathbf{C}_X$:

$$\mathbf{C}_X = \int_{Y \in \mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(Y, X) \tag{36.34}$$

Proof. The Grothendieck construction's objects are given by pairs of objects Y of $\mathbf{C}$ and elements of $\mathrm{Hom}_{\mathbf{C}}(Y, X)$, ie morphisms from Y to X :

$$(Y, f : Y \to X) \tag{36.35}$$

Which are isomorphic to the appropriate slice objects via

$$(Y, f : Y \to X) \rightleftarrows f : Y \to X \tag{36.36}$$

□

Example 36.5.2. If we consider some total orders such as $\mathbb{R} \times \mathbb{R}^{\mathrm{op}} \to \mathbb{R}$, a functor will here be an order-preserving function in the first variable and order-reversing in the second. The end of such a function will therefore be some value $x \in \mathbb{R}$. A natural transformation between two such functions $\eta : f \to g$ on this will have components obeying

$$\begin{array}{ccccc} (x_1, x_2) & & f(x_1, x_2) & \xrightarrow{\eta_x} & g(x_1, x_2) \\ \downarrow{\scriptstyle (\leq, \geq)} & & \downarrow{\scriptstyle f(\leq)} & & \downarrow{\scriptstyle g(\leq)} \\ (y_1, y_2) & & f(y_1, y_2) & \xrightarrow[\eta_y]{} & g(y_1, y_2) \end{array}$$

Example 36.5.3. Given the category of finite dimensional spaces over a field k, $\mathbf{FVect}_k$, consider the functor

$$F : \mathbf{FVect}_k \times \mathbf{FVect}_k^{\mathrm{op}} \to \mathbf{FVect}_k \tag{36.37}$$
$$(V, W) \mapsto V \otimes W* \tag{36.38}$$

$$\int_{V \in \mathbf{FVect}_k} F(V, V) \cong k \tag{36.39}$$

Structure map :

$$\epsilon_V : V \otimes V^* \to k \tag{36.40}$$
$$(v, \omega) \mapsto \omega(v) \tag{36.41}$$

Coend of $\mathrm{Hom}_{\mathbf{FVect}_k}(W, V) \cong V^* \otimes W$

Example 36.5.4. Given a metric space (X, d) considered as an $\mathbb{R}_+$-enriched category, where points $x \in X$ are objects and morphisms correspond to the distance between two points. Take some

37

Reflective subcategories

Definition 37.0.1. Given two categories $\mathbf{C}$, $\mathbf{D}$, we say that $\mathbf{C}$ is a reflective subcategory of $\mathbf{D}$ if it is a full subcategory, $\iota : \mathbf{C} \hookrightarrow \mathbf{D}$, and the inclusion functor has a left adjoint called the *reflector*

$$(T \dashv \iota) : \mathbf{C} \underset{\hookleftarrow \iota \to}{\overset{\leftarrow T -}{\rightleftarrows}} \mathbf{D}$$

In other words, for any object $X \in \mathbf{D}$, we can project it to the subcategory $\mathbf{C}$ so that its interactions with other objects $Y \in \mathbf{C}$ is given by the adjunction

$$\mathrm{Hom}_{\mathbf{C}}(T(X), Y) = \mathrm{Hom}_{\mathbf{C}}(X, \iota(Y)) \tag{37.1}$$

ie functions with X as source interacts the same way that it would in the larger category.

Likewise, we have its dual, the coreflective subcategory

Definition 37.0.2. Given two categories $\mathbf{C}$, $\mathbf{D}$, we say that $\mathbf{C}$ is a coreflective subcategory of $\mathbf{D}$ if it is a full subcategory, $\iota : \mathbf{C} \hookrightarrow \mathbf{D}$, and the inclusion functor has a right adjoint called the *coreflector*

$$(\iota \dashv T) : \mathbf{C} \underset{\leftarrow T -}{\overset{\hookleftarrow \iota \to}{\rightleftarrows}} \mathbf{D}$$

$$\mathrm{Hom}_{\mathbf{C}}(X, T(Y)) = \mathrm{Hom}_{\mathbf{C}}(\iota(X), Y) \tag{37.2}$$

"reflective v. coreflective : the subcategory is made of objects such that every object has a maximal "nice" quotient v. subobject, given by the reflector v. coreflector?"

[...]

Example 37.0.1. The category of Abelian groups **Ab** is a full subcategory of the category of groups,

$$\iota_{\mathbf{Ab}} : \mathbf{Ab} \hookrightarrow \mathbf{Grp} \tag{37.3}$$

ie there are no group homomorphisms between two Abelian groups which is not in **Ab**. This inclusion admits a left adjoint Ab, called the *Abelianization functor*.

Proof. In terms of hom-sets, we have that, if $\iota_{\mathbf{Ab}}$ admits a left-adjoint Ab,

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Ab}(G), H) = \mathrm{Hom}_{\mathbf{Grp}}(G, \iota_{\mathbf{Ab}}(H)) \tag{37.4}$$

Then for any group homomorphism $f : G \to \iota_{\mathbf{Ab}}(H)$, we have a corresponding Abelian group homomorphism $r : \mathrm{Ab}(G) \to H$

[...]

Look at group operations using the free group of one element **Z**? □

Example 37.0.2. The category of metric spaces **Met** with isometries as morphisms has as a full subcategory the category of complete metric spaces **CompMet**.

$$\iota_{\mathbf{CompMet}} : \mathbf{CompMet} \hookrightarrow \mathbf{Met} \tag{37.5}$$

The reflector associates the completion of the metric space to any metric space.

38

Monads

There are multiple definitions of what a monad is, depending on the context, from its use in computer science where it is understood as adding extra information to a function's return type, to a categorification of monoids.

The famed definition of it (from MacLane[44]) is that a monad is a monoid in the category of endofunctors.

Definition 38.0.1. A *monad* (T, η, μ) in a category $\mathbf{C}$ is composed of

- An endofunctor $T : \mathbf{C} \to \mathbf{C}$
- A natural transformation $\eta : \mathrm{Id}_{\mathbf{C}} \to T$, called the *unit*
- A natural transformation $\mu : T \circ T \to T$, called the *muliplication*

such that the multiplication is associative, $\mu \circ T\mu = \mu \circ \mu T$:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

and there exists an identity element : $\mu \circ T\eta = \mu \circ \eta T = \mathrm{Id}_T$

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & TT & \xleftarrow{T\eta} & T \\ & \searrow\!\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\!\swarrow & \\ & & T & & \end{array}$$

If we consider the category of endofunctors, in other words the functor category $\text{End}(\mathbf{C}) = [\mathbf{C}, \mathbf{C}]$, then a monad as defined here is indeed a monoid, in the sense that it is an algebra on an endofunctor 26.5. The category is $\text{End}(\mathbf{C})$, the object is T, and the functor is the component-wise map of composition with T :

$$\forall X \in \text{End}(\mathbf{C}),\ F_T(X) = X \circ T \tag{38.1}$$

From this the carrier of the algebra is the natural transformation $\mu : F_T(T) \cong T \circ T \to T$

In the context of computer science, the unit is also called the *return map*, while the multiplication is called the *apply map*.

Example 38.0.1. The simplest example of a monad is the identity functor which simply maps the category to itself, where the unit is the identity functor and the multiplication as well.

Example 38.0.2. The *maybe monad* Maybe on **Set** adds a single new element to a set,

$$\text{Maybe}(X) = X + \{\bullet\} \tag{38.2}$$

and transforms functions to

$$\text{Maybe}(f : X \to Y) = f' : (X + \{\bullet_X\}) \to (Y + \{\bullet_Y\}) \tag{38.3}$$

with the property that $f'(x \in X) = f(x)$ but $f'(\bullet_X) = \bullet_Y$.

The unit of the monad is given by, in component form,

$$\begin{aligned} \eta_X : X \quad &\to \quad X + \{\bullet\} && (38.4)\\ x \quad &\mapsto \quad x && (38.5) \end{aligned}$$

so that the original set is simply embedded into the new one via the obvious map, and the multiplication is given by

$$\mu_X : X + \{\bullet_1, \bullet_2\} \to X + \{\bullet\} \tag{38.6}$$

which is that upon another application of the maybe monad, the extra new element added is identified with the old.

The name of "maybe monad" stems from its use in programming to describe the formalization of error handling, where a function sends back a Maybe(Y)-typed value instead of M with the term $\bullet$ representing an error of the function, such as

$$a : \mathbb{R}, b : \mathbb{R} \dashv div(a, b) = \begin{cases} \bullet & b = 0 \\ a/b \end{cases} \tag{38.7}$$

in which case any function can be made to handle this new type Maybe($\mathbb{R}$). The unit [return] is simply that the usual type is embedded in the new one, while the multiplication (binding) is so that only one of error element is allowed to exist. A function of type $\mathbb{R}$ can therefore be extended to a function of type Maybe($\mathbb{R}$) via

$$\eta(f) : (f : \mathbb{R} \to Y) \to (f' : \text{Maybe}(\mathbb{R}) \to \text{Maybe}(Y)) \tag{38.8}$$

where a value of an error in the input sends back an error value in the output, which is usually the behaviour of NaN.

Furthermore we also have the avoidance of the multiplication of error values from the multiplication map, so that if we attempt divisions in a row, where the input might itself be an error

$$\eta(\text{div}) : (f : \mathbb{R} \to \mathbb{R} + \{\bullet\}) \to (f' : \mathbb{R} + \{\bullet\} \to \mathbb{R} + \{\bullet_1, \bullet_2\}) \tag{38.9}$$

The multiplication map will get rid of this extra error value :

Theorem 38.0.1. x

Monads from adjunctions

Theorem 38.0.2. Any adjunction of two functors

$$(L \dashv R) : \mathbf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{D}$$

defines a monad

$$T = R \circ L : \mathbf{C} \to \mathbf{C} \tag{38.10}$$

with unit the unit of the adjunction, and multiplication

$$\mu : \tag{38.11}$$

and a comonad

$$\overline{T} = L \circ R : \mathbf{D} \to \mathbf{D} \tag{38.12}$$

with counit the counit of the adjunction, and comultiplication

$$\delta : \tag{38.13}$$

Proof. The composition obviously defines an endofunctor on the appropriate category, so that we only need to prove the compatibility of the (co)unit and (co)multiplication map.

As the two functors are adjoint, we have the existence of a unit and counit,

$$\eta \ : \quad \mathrm{Id}_{\mathbf{C}} \to R \circ L \tag{38.14}$$
$$\epsilon \ : L \circ R \to \mathrm{Id}_D \tag{38.15}$$

Left whiskering by R the triangle identity give us

$$\begin{array}{ccc} RL & \xrightarrow{R \circ L \circ \eta} & RLRL \\ & \searrow^{\mathrm{Id}_{RL}} & \downarrow{\scriptstyle R \circ \eta \circ L} \\ & & RL \end{array}$$

Right whistering by L :

$$\begin{array}{ccc} RL & \xrightarrow{R \circ L \circ \eta} & RLRL \\ & \searrow^{\mathrm{Id}_{RL}} & \downarrow{\scriptstyle R \circ \eta \circ L} \\ & & RL \end{array}$$

□

38.1 Types

Monads on a category define a few different types of objects within that category.

Definition 38.1.1. Given a monad (T, η, μ) on $\mathbf{C}$, we say that an object X is T-*modal* if the unit on it is an isomorphism

$$\eta_X : X \xrightarrow{\cong} TX \tag{38.16}$$

If it is merely a monomorphism, it is *submodal*.

Example 38.1.1. In **Set**, with the maybe monad, any infinite set is a modal type, while all sets are submodal.

Proof. As the coprojection morphism in **Set** is an injection, $X \to X + 1$ is injective. If X is infinite, by definition, there is an injection $i : \mathbb{N} \to X$. Then we can define the function $X + 1 \to X$ by

$$f(x) = \begin{cases} i(0) & x = \bullet \\ i(n+1) & i^{-1}(x) = n \\ x & x \in X \setminus i(\mathbb{N}) \end{cases} \tag{38.17}$$

and we have

$$f(\eta_X(x)) \quad = \quad f(x) \tag{38.18}$$
$$= \tag{38.19}$$

□

Definition 38.1.2. Given a comonad (G, ϵ, δ) on $\mathbf{C}$, we say that an object X is T*-comodal* if the counit on it is an isomorphism

$$\epsilon_X : GX \xrightarrow{\cong} X \tag{38.20}$$

If it is merely a epimorphism, it is *supcomodal*.

Modal object, comodal object, submodal object, supcomodal object, anti-modal type

[45]

38.2 Eilenberg-Moore category

Just as we can define a (co)monad from a pair of adjoint functors 38.0.2, conversely, for any (co)monad, there is an associated subcategory equipped with two adjoint functors, so that for a monad (T, η, μ) on a category $\mathbf{C}$, we have the *Eilenberg-Moore category* $\mathbf{C}^T$, which has a pair of functors

38.3 Adjoint monads

Like any functor, monads can have adjoint functors, but in the case of monads, the term of adjoint typically has a more specific meaning.

Definition 38.3.1. Given a monad (T, μ, η) and a comonad (G, δ, ϵ), we say that they are left (resp. right) adjoint if T is left (resp. right) adjoint to G

Example 38.3.1. The basic adjoint modality example is the *even/odd modality* pair,

$$\text{Even} \dashv \text{Odd} \tag{38.21}$$

This is done on the category of integers as an ordered set, $(\mathbb{Z}, \leq)$, for which the morphisms are the order relations, and endofunctors are order-preserving functions.

The functor we consider here is the largest integer which is smaller to $n/2$:

$$\lfloor -/2 \rfloor : (\mathbb{Z}, \leq) \quad \to \quad (\mathbb{Z}, \leq) \tag{38.22}$$

$$n \quad \mapsto \quad \lfloor n/2 \rfloor \tag{38.23}$$

This functor has a left and right adjoint functor,

$$\text{even} : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{Z}, \leq) \quad (38.24)$$
$$n \mapsto 2n \quad (38.25)$$
$$\text{odd} : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{Z}, \leq) \quad (38.26)$$
$$n \mapsto 2n+1 \quad (38.27)$$
$$(38.28)$$

Proof :

$\lfloor -/2 \rfloor$ has as a domain the whole category

For a total order, the hom-set $\text{Hom}(X,Y)$ is simply empty if $X > Y$ and has a single element otherwise. For $\lfloor -/2 \rfloor$, the hom-set is then gonna be that $\text{Hom}_{\mathbb{Z}}(X, \lfloor Y/2 \rfloor)$ is empty if $2X > Y$ if Y is even, and $2X+1 > Y$ if Y is odd.

The left adjoint of $\lfloor -/2 \rfloor$ is a functor such that

$$\text{Hom}_{\mathbb{Z}}(L(-), -) \cong \text{Hom}_{\mathbb{Z}}(-, \lfloor -/2 \rfloor) \quad (38.29)$$

In the case of a total order, the isomorphism simply means that both sets have the same cardinality, ie they either have no elements (the two objects are not ordered) or one (the two objects are ordered). So

$$\text{Hom}_{\mathbb{Z}}(L(n), m) \cong \text{Hom}_{\mathbb{Z}}(n, \lfloor m/2 \rfloor) b \Leftrightarrow L(n) \leq m \leftrightarrow n \leq \lfloor m/2 \rfloor \quad (38.30)$$

If we have $L(n) = 2n$, we need to show this equivalence both ways.

If $2n \leq m$, then dividing by 2, we have $n \leq m/2$, which we can then apply the floor to both sides (it is monotonous), so $\lfloor n \rfloor \leq \lfloor m/2 \rfloor$. As n is an integer, $n \leq \lfloor m/2 \rfloor$

Converse : If $n \leq \lfloor m/2 \rfloor$: From properties of floor :

$$n \leq \lfloor \frac{m}{2} \rfloor \leftrightarrow 2 \leq \frac{\lceil m \rceil}{n} \quad (38.31)$$

As m is an integer, $2n \leq m$.

So the even function is indeed left adjoint.

Odd function is right adjoint :

From these three functions, we can define adjoint monads :

$$(\text{Even} \vdash \text{Odd}) \quad (38.32)$$

which send numbers to their half floor and then to their corresponding even and odd number :

$$\mathrm{Even}(n) = 2\lfloor n/2 \rfloor \tag{38.33}$$

$$\mathrm{Odd}(n) = 2\lfloor n/2 \rfloor + 1 \tag{38.34}$$

n	Even(n)	Odd(n)
-2	-2	-1
-1	-2	-1
0	0	1
1	0	1
2	2	3
3	2	3

Table 38.1: Caption

Monad and comonad, unit and counit, multiplication

Example 38.3.2. Integrality modality : Given the two total order categories $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$, the inclusion functor

$$\iota : (\mathbb{Z}, \leq) \hookrightarrow (\mathbb{R}, \leq) \tag{38.35}$$

$$n \mapsto n \text{ (as a real number)} \tag{38.36}$$

Left and right adjoints : $(L \dashv \iota \dashv R)$

$$\mathrm{Hom}_{\mathbb{Z}}(k, R(x)) \cong \mathrm{Hom}_{\mathbb{R}}(\iota(k), x) \tag{38.37}$$

$$R : \mathbb{R} \to \mathbb{Z} \tag{38.38}$$

meaning that the right adjoint is an integer which is superior to any integer k if

38.4 Monoidal monad

38.5 Algebra of a monad

Monads naturally form an algebra over each of the objects that they act upon.

Definition 38.5.1. For a monad (T, η, μ) on a category $\mathbf{C}$, a T-algebra is a pair (X, f) of an object $X \in \mathbf{C}$ and a morphism $\alpha : TX \to X$ making the following diagrams commute :

$$\begin{array}{ccc} X & \xrightarrow{\eta} & T(X) \\ & \searrow^{\mathrm{Id}_X} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array}$$

$$\begin{array}{ccc} T^2X & \xrightarrow{T(\alpha)} & TX \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \alpha} \\ TX & \xrightarrow{\alpha} & X \end{array}$$

The precise category in which this algebra is defined

Definition 38.5.2. The *Eilenberg-Moore category* of a monad is a category

Example 38.5.1. The free monoid monad, or list monad, is the composition of the free monoid functor $F : \mathbf{Set} \to \mathbf{Mon}$ with the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$. For some set S, we send it to the free monoid $F(S)$, which has as elements the n-tuples of elements of S of arbitrary size, with free monoidal operation their concatenation. For $S = (a, b, c, \dots)$,

$$(a, b, c, \dots) \cdot (\alpha, \beta, \gamma, \dots) = (a, b, c, \dots, \alpha, \beta, \gamma, \dots) \tag{38.39}$$

with underlying set all the tuples of finite length of S :

$$U(F(S)) = \coprod_{k \in \mathbb{N}} S^{\times k} \tag{38.40}$$

so that the list monad is given by

$$F : \mathbf{Set} \to \mathbf{Set} \tag{38.41}$$

$$S \mapsto \coprod_{k \in \mathbb{N}} S^{\times k} \tag{38.42}$$

Its unit is the morphism that associates to every element its singleton list,

$$\eta_S : S \to \mathrm{List}(S) \tag{38.43}$$

$$x \mapsto (x) \tag{38.44}$$

and its multiplication takes a list of list and bind them by concatenation,

$$\mu : \mathrm{List} \circ \mathrm{List} \to \mathrm{List} \tag{38.45}$$

$$\begin{pmatrix} (x_{11}, x_{12}, \dots x_{1n}) \\ \vdots \\ (x_{k1}, x_{k2}, \dots x_{kn}) \end{pmatrix} \mapsto (x_{11}, x_{12}, \dots x_{1n}, \dots, x_{k1}, x_{k2}, \dots x_{kn}) \tag{38.46}$$

Example 38.5.2. The maybe monad defines the algebra of the smash product over pointed sets. The category of objects with at least one element and morphisms preserving a specific element is the category of pointed sets, so that

$$\text{EM}(\text{Maybe}) = \mathbf{Set}_\bullet \tag{38.47}$$

If we pick some set X and a T-action $\alpha : \text{Maybe}(X) \to X$, this commutes if

$$\alpha(\text{Maybe}(x)) = x \tag{38.48}$$

So that a must map any element from the original set back to itself, leaving as the only freedom the element to which $\bullet$ is mapped. Given some chosen element $\overline{x}$, the T-action will be

$$\alpha_{\overline{x}}(\bullet) = \overline{x} \tag{38.49}$$

In other words, this defines a pointed set with a specific element $\overline{x}$ rather than our abstract pointed set.

[No algebra for $\varnothing$?]

Also :

$$\alpha() \tag{38.50}$$

The algebra defined on pointed sets is the one given by the smash product.

[...]

Free algebra : The free T-algebra of the maybe monad is given for some set X by the algebra over $X+\{\bullet\}$ with the multiplication map component $\mu : T^2X \to TX$ as its T-action. This means that we have some pointed set $X_\bullet \in \mathbf{Set}_\bullet$

Maybe monad is monoidal : the monad unit is given by

$$\epsilon : \{\bullet\} \tag{38.51}$$

composition law :

$$\mu_{X,Y} : \text{Maybe}(X) \times \text{Maybe}(Y) \to \text{Maybe}(X \times Y) \tag{38.52}$$

An important class of monads are the ones which are associated (in the internal logic of the category, cf. [X]) to the classical *modalities*, ie necessity and possibility.

Kripke semantics

Example 38.5.3. Necessity/Possibility modalities

38.6 Idempotent monads

One class of monads we will use extensively are the idempotent monads (and comonads), which are idempotent in the algebraic sense of having $T^1 \cong T$:

Definition 38.6.1. A monad (T, η, μ) is said to be an *idempotent monad* if

$$\mu : T^2 \to T \tag{38.53}$$

is an isomorphism.

In components, this means that the action of a monad is such that every component of the multiplication map is an isomorphism.

$$\eta_X : T^2 X \xrightarrow{\cong} TX \tag{38.54}$$

and has an inverse,

$$\eta_X^{-1} : TX \to T^2 X \tag{38.55}$$

So that the two are, up to isomorphism, the same object,

$$T^2 X \cong X \tag{38.56}$$

For simplicity, as this will be a very common operation, we will throughout this book mostly not write down the isomorphism μ_X for any operation involving $T^2 X$ on an idempotent monad, so that we will write morphisms $f : T^2 X \to Y$ equivalently as $f : TX \to Y$ rather than the more proper $f \circ \mu_X$, but it is implicit.

Example 38.6.1. The free-forgetful adjunction $(F \dashv U)$ between groups and Abelian groups is an idempotent monad.

Proof. □

Theorem 38.6.1. The Eilenberg-Moore category of an idempotent monad $\mathrm{EM}(T)$ is a reflective subcategory of $\mathbf{C}$.

Proof. □

With this we can consider a variety of monads easily simply by looking at various reflective subcategories we have seen in 37.

Example 38.6.2. From the reflector T of the inclusion functor $\iota : \text{CompMet} \hookrightarrow \text{Met}$, we can construct the monad of completion,

$$\text{Comp} : \textbf{Met} \to \textbf{Met} \tag{38.57}$$

which sends any metric space to its completion. [Moore closure?]

On functions :

$$\text{Comp}(f : X \to Y) = \overline{f} : \text{Comp}(X) \to \text{Comp}(f) \tag{38.58}$$

Sends functions of the underlying space to the one that's a reflection idk

Algebra?

We will see quite a lot more examples of idempotent monads and comonads in the chapter on objective logic IX, as this is the main tool by which this operates.

Theorem 38.6.2. Adjoint idempotent monads commute with limits and colimits

Proof. As we can decompose any idempotent adjoint monad into an adjoint triple of functors ($L \dashv C \dashv R$) on the Eilenberg-Moore category, we have that, as C is both left and right adjoint, L is left adjoint and R is right adjoint,

$$(C \circ L)\text{colim}F = \text{colim}(C \circ L)F \tag{38.59}$$

$$(C \circ R) \lim F = \lim(C \circ R)F \tag{38.60}$$

[check] □

39

Linear and distributive categories

Definition 39.0.1. A category $\mathbb{C}$ with finite products and coproducts is *distributive* if for any three objects $X, Y, Z \in \mathbb{C}$, the canonical distributivity morphism

$$(X \times Y) + (X \times Z) \to X \times (Y + Z) \tag{39.1}$$

is an isomorphism.

Example 39.0.1. Set is distributive.

Proof. For three sets X, Y, Z, using the Kuratowski definition of ordered pairs,

$$\begin{aligned} X \times (Y, a) &= X \times \{\{Y\}, \{Y, a\}\} && (39.2) \\ &= && (39.3) \end{aligned}$$

disjoint unions, we have

and their behavior with products,

$$\begin{aligned} X \times (Y \amalg Z) &= X \times ((Y, 0) \cup (Z, 1)) && (39.4) \\ &= (X \times (Y, 0)) \cup (X \times (Z, 1)) && (39.5) \\ &= && (39.6) \end{aligned}$$

since w ∎

Example 39.0.2. **Top** is distributive.

Proof. If we have three topological spaces X, Y, Z, with the topological product and coproduct, the topology of $Y + Z$ is given by

$$Y + Z = (Y + Z, \tau_Y + \tau_Z) \tag{39.7}$$

and its product with X is then

$$X \times (Y + Z) = (X \times (Y + Z), \tau) \tag{39.8}$$

with

$$\tau = \{U_i \times U_j \mid U_i \in \tau_X \wedge U_j \in (\tau_Y + \tau_Z)\} \tag{39.9}$$

□

Definition 39.0.2. A category **C** with finite products and coproducts is *linear*, or *semiadditive*, if it has a zero object, and its identity maps

$$\mathrm{Id}_{XY} : X + Y \to X \times Y \tag{39.10}$$

are isomorphisms for all X, Y.

Example 39.0.3. The category of vector spaces **Vec** is linear.

Proof. □

Example 39.0.4. The category of R-modules R**Mod** is linear.

Theorem 39.0.1. A semi-additive category **C** is enriched over the monoidal category of commutative monoids $(\mathbf{CMod}, \oplus, 0)$.

Proof. □

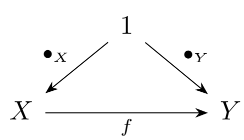
This is what is typically meant that the category is ”semi-additive”, in that it is possible to perform some basic addition on the morphisms. For the category to be upgraded to additive, we also require

40

Pointed objects

In a category $\mathbf{C}$ with a terminal object, a *pointed object* is an object X of a category along with a specific morphism $1 \to X$, called the *basepoint*. This can be described by an object of the coslice category $\mathbf{C}^{1/}$, which we will call the *category of pointed objects* of $\mathbf{C}$.

As usual for a coslice category, the morphisms induced from the original category are given by commuting triangles from the basepoint,



which means that the basepoint of X will always be mapped to the basepoint of Y, $f \circ \bullet_X = \bullet_Y$.

Example 40.0.1. The category of pointed sets is the category of pointed objects of $\mathbf{Set}$:

$$\mathbf{Set}_* = \mathbf{Set}^{1/} \tag{40.1}$$

where objects are the pointed sets $p : 1 \to X$ of sets X with basepoint p, denoted X_p, and the morphisms are the pointed functions preserving the basepoint, ie for any morphism $f : X \to Y$ in $\mathbf{Set}_*$, there's a corresponding function in $\mathbf{Set}$ obeying

$$f(x) = \begin{cases} f(x) & x \neq p \\ x & x = p \end{cases} \tag{40.2}$$

As with any coslice category, there is a forgetful functor

$$U : \mathbf{C}_* \to \mathbf{C} \tag{40.3}$$

which simply gives back the original object,

$$U(X_p) \cong X \tag{40.4}$$

Theorem 40.0.1. In the category of pointed objects $\mathbf{C}^{*/}$, the coslice object $1 \to 1$ is a zero object.

Proof. □

If the category has additionally finite coproducts, This functor has a left adjoint of a free functor which is nothing but the maybe monad we saw previously.

Theorem 40.0.2. The left adjoint of the forgetful functor $U : \mathbf{C}_* \to \mathbf{C}$ is a free functor $F : \mathbf{C} \to \mathbf{C}_*$ which sends any object to the coproduct of the object with the terminal object and, as a pointed object, is the coprojection of this coproduct : $\iota_1 : 1 \hookrightarrow X + 1$

Proof. □

Limits :

Theorem 40.0.3. Given a category $\mathbf{C}$ with all limits, the limit $\lim_{\mathbf{I}} F_*$ for a diagram $F_* : \mathbf{I} \to \mathbf{C}^{1/}$ corresponds to a limit in $\mathbf{C}$ via the forgetful functor U :

$$U(\lim_{\mathbf{I}}(F_*)) = (\lim_{\mathbf{I}}(U \circ F_*)) \tag{40.5}$$

Proof. □

Theorem 40.0.4. The product of a category of pointed objects $X_x \times Y_y$ is the pointed object $(X \times Y)_{(x,y)}$.

Theorem 40.0.5. The coproduct of a category of pointed objects $X_x + Y_y$, also called the *wedge sum* $X_x \vee Y_y$, is the pointed object given by the pushout

$$\begin{array}{ccc} 1 & \xrightarrow{[x,y]} & X_x + Y_y \\ {\scriptstyle \mathrm{Id}_1}\downarrow & & \downarrow \\ 1 & \xrightarrow{\bullet} & X_x \vee Y_y \end{array}$$

Examples of the wedge sum are common in topology where

Internal monoids in term of pointed objects

Definition 40.0.1. In a category of pointed objects over a monoidal category $(\mathbf{C}, \otimes, I)$, we can define the *smash product* $X \wedge Y$ as the pushout

$$X \wedge Y = 1 +_{(X \otimes 1) \vee (Y \otimes 1)} (X \otimes Y) \tag{40.6}$$

In particular, we can define a smash product with a product,

$$X \wedge Y = 1 +_{X \vee Y} (X \times Y) \tag{40.7}$$

which can be understood as the quotient of the image of $X \vee Y$ in $X \times Y$

The smash product is a symmetric monoidal product

Example 40.0.2. For two lines $\mathbb{R}$ in some appropriate geometric category (like say **Top**), their wedge sum $\mathbb{R} \vee \mathbb{R}$ is the cross, and their product the plane $\mathbb{R}^2$. The smash product is then the quotient of the cross (say the axis) in $\mathbb{R}^2$, giving four quadrants $\{(x, y) \mid \pm x > 0 \wedge \pm y > 0\}$ connected by the central point $(0, 0)$.

Example 40.0.3. For two circles S^1, their wedge sum $S^1 \vee S^1$ is a bouquet of two circles, and their smash product is given by the quotient of the torus $S^1 \times S^1$ by this bouquet of two circles representing a meridional and longitudinal circle on that torus, whose quotient is a sphere.

$$S^1 \wedge S^1 \cong S^2 \tag{40.8}$$

In some sense, pointed objects can be seen as a bridge between distributive categories (which represent spaces in some manner) with that of a linear category (representing some sort of algebra), by giving a space some natural notion of a preferred point and a few operations upon it.

Part III

Spaces

One of the main type of category we will use for objective logic are categories which are *spaces* or relate to spaces, in a broad sense, such as frames, sheaves and topoi.

[46, 47, 48]

2

41

General notions of a space

Before looking into how spaces work in category theory, let's first look at how spaces are treated both intuitively, in philosophical analysis, and the most common ways to treat spaces in mathematics.

41.1 Mereology

[49, 50]

The most basic aspect of a space in philosophical terms is that of *mereology*. The mereology of a space is the study of its parts, where we can decompose a space into regions with some specific properties. A space X is composed of a collection of regions $\{U_i\}$, which are ordered by a relation of inclusion $(\{U_i\}, \subseteq)$, called *parthood*, which obeys the usual partial order relations :

- Reflection :

$$U \subseteq U$$

- Symmetry :

$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_1 \to U_1 = U_2$$

- Transitivity :

$$U_1 \subseteq U_2 \wedge U_2 \subseteq U_3 \to U_1 \subseteq U_3$$

Those are the typical notion of a partial order : reflexivity (a region is part of itself), antisymmetry (if a region is part of another, and the other region is part of the first, they are the same region) and transitivity (if a region is part of another region, itself part of a third region, the first is part of the third). The antisymmetry allows us to define equality in terms of parthood, simply as $U_1 = U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_2 \subseteq U_1$.

As we will deal with posets extensively in this chapter, we will use Hasse diagrams to illustrate our mereologies here.

Definition 41.1.1. A *Hasse diagram* of a poset $(X, \leq)$ is a directed graph for which the nodes are elements of the set X, and the edges represent at least the minimal amount of order relations to generate the full order by transitivity.

For simplicity the direction of edges is usually implicit, simply being that the source of the edge is the one higher and the target is the one lower.

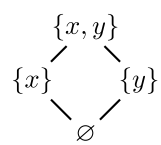


Figure 41.1: Hasse diagram of the power set of the set of two elements.

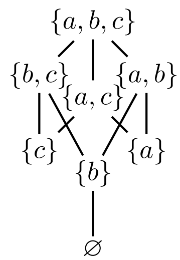
Let's look at a few basic examples of mereologies with Hasse diagrams. The simplest one is the *trivial mereology*, which is just a single node with no edges.

$$X$$

The interpretation of this diagram will depend slightly on what we impose as semantics, as it could either be an empty region (the mereology of a non-existing space) or a totally unified space with no subregions, including the empty region, à la Parmenides [Duality of nothingness and being?].

Another simple one is that of the *abstract point*. This is a single region whose only subobject is the empty region.

A general kind of mereology is the one given by discrete sets, where we consider the various subsets as subregions. For instance, in the case of a "space" of 3 regions, we have (assuming we also consider the empty region as a region) the following structure



As a space this could be interpreted as a discrete space with a finite number of points.

From the definition of a mereology, we can define some further operations. The most basic derived relation is that of *proper parthood* $\subset$, which is defined simply as parthood excluding equality :

$$U_1 \subset U_2 \leftrightarrow U_1 \subseteq U_2 \wedge U_1 \neq U_2 \tag{41.1}$$

The converse of proper parthood is *proper extension* : we say that U is a proper extension of U' if $U' \subset U$ and $U \neq U'$. In other words, U is a region that contains U' but is larger than it.

Those are roughly all the notions that can be expressed without quantifiers.

A first basic notion of mereology involving quantification is that of *overlap* : Two regions U_1, U_2 overlap if there exists a third region U_{12} which is part of both :

$$O(U_1, U_2) \leftrightarrow \exists U_{12},\ (U_{12} \subseteq U_1 \wedge U_{12} \subseteq U_2) \tag{41.2}$$

If we were to interpret this in set theoretical terms, this is equivalent to $U_1 \cap U_2 \neq \varnothing$.

The converse of overlap is *underlap*, where there exists a third region containing the first two : two regions U_1, U_2 underlap if there exists a third region U_{12} which contain them both :

$$U(U_1, U_2) \leftrightarrow \exists U_{12},\ (U_{12} \subseteq U_1 \wedge U_{12} \subseteq U_2) \tag{41.3}$$

If we were to interpret this in set theoretical terms, this is equivalent to $\exists U,\ U_1 \cup U_2 \subseteq U$.

We say that two regions are *disjoint* if they are not overlapping :

$$D(U_1, U_2) = \neg O(U_1, U_2) \tag{41.4}$$

Definition 41.1.2. The *overlap* of two regions is the existence of a third region which is a part of both :

$$U_1 \circ U_2 \leftrightarrow \exists U_3,\ [U_3 \subseteq U_1 \wedge U_3 \subseteq U_1] \tag{41.5}$$

Unless a mereological nihilist, we also typically define an operation to turn several regions into one, the *fusion* :

Definition 41.1.3. Given a set of regions $\{U_i\}_{i \in I}$, we say that U is the *fusion* of those region, $\sum(U, \{U_i\})$,

Definition 41.1.4. If a region does not have any proper part, we say that it is *atomic*

$$\mathrm{Atom}(U) \leftrightarrow \nexists U',\ U' \subset U \tag{41.6}$$

Atomic regions will, depending on the exact model, be either assimilable to points, or be the empty region. If there is an empty region, a point will then be a region whose only part is the empty region. We will generally denote points by the usual symbols $x, y, \ldots$.

Mereologies can vary quite a lot depending on what you wish to model or your own philosophical bent. *Mereological nihilism* will assume for instance that there are no objects with proper parts (so $U_1 \subseteq U_2$ implies $U_1 = U_2$), and we can only consider a collection of atomic points with no greater structure (in particular, there is no *space* itself which is the collection of all its regions), while on the other end of the spectrum is *monism* (such as espoused by Parmenides[1]), where the only region is the whole space itself, with no subregion.

Typically however, we tend to consider some specific base axioms for a mereology. Beyond the partial ordering axioms (which are referred to as M1 to M3), we also have

Axiom M4. Weak supplementation : if U_1 is a proper part of U_2, there's a third region U_3 which is part of U_2 but does not overlap with U_1 :

$$U_1 \subset U_2 \rightarrow \exists U_3,\ [U_3 \subseteq U_2 \wedge \neg U_3 \circ U_1] \tag{41.7}$$

[diagram]

Counterexample 41.1.1. A linear mereology $U_0 \subset U_1 \subset U_2 \subset \ldots \subset U_N$ is not weakly supplemented : for $U_{k-1} \subset U_k$, any other region $U \subseteq U_k$ will overlap with U_{k-1}

Axiom M5. Strong supplementation : If U' is not part of U, there exists a third region U'' which is part of U' but does not overlap with U.

$$\neg(U' \subseteq U) \rightarrow \exists U'',\ U'' \subseteq U' \wedge \neg(O(U', U)) \tag{41.8}$$

[diagram]

Axiom M5'. Atomistic supplementation : If U' is not a part of U, then there exists an atom x that is part of U' but does not overlap with

Axiom TOP. Top : There is a universal object W of which every region is a part of

$$\exists W,\ \forall U,\ U \subseteq W \tag{41.9}$$

Axiom BOT. Bottom : There is an atomic null object N which every region contains

$$\exists N,\ \forall U,\ N \subseteq U \tag{41.10}$$

[...]

Mereologies typically do not include all possible such axioms, but we have common systems that will include a variety of them[51, 52]. The smallest of these is just bare mereology, **M**, which is just given by M1, M2 and M3, and is therefore simply the theory of partial orders.

Minimal mereology **MM** is simply **M** with weak supplementation M4

extensional mereology **EM** is **M** with M5

classical extensional mereology **CEM** is **EM** with M6 and M7

general mereology **GM** is **M** with M8

general extensional mereology **GEM** is **EM** with M8

atomic general extensional mereology **AGEM** is **M** with M5' and M8

Set theory may be considered as some kind of mereology, as we can take the class of all sets and the subset relation as its parthood system. As such as system, it is [...]

A caveat however is that the mereology of set theory does not define it uniquely [53]

41.2 Topology

A common approach for space in mathematics is the notion of *topology*. We have already briefly defined the category **Top** of topological spaces, but as they form the basis for most of the common understanding of spaces in math, we should look into them more deeply : what their motivations are, what they are for, and how they may relate to other objects.

If we look at one of the common structuration of mathematics, popularized by Bourbaki[ref on structuralism], spaces are built first as sets, then as topological spaces, and they may afterward get further specified into other structures, such as metric spaces, etc.

This is only a convention, as there are many other structures one may choose, that can be easier, more general, more specific to a given property, etc. The point of topological spaces is that they are a good compromise between those constraints, being fairly easy to define and allowing to talk about quite a lot of properties.

First, let's look at the basic structure of sets. From the perspective of mereology, sets are a rather specific choice of structure, corresponding to an atomic unbounded relatively complemented distributive lattice 1, [see definition of points in philosophy too etc]

Historically, the notion that spaces are made of point is quite ancient [cf. Sextus Empiricus], but it has not had the modern popularity it now has until the works of [Riemann?] Cantor, Hausdorff, Poincaré, etc, and the notion was put into the modern mathematical canon with such works as Bourbaki, etc.

If we consider spaces only as sets however, the informations we can derive from them is rather limited. This is an observation from antiquity [Sextus again]

In modern terms, we can talk about subsets, cardinalities, overlap and unions, but we would be missing on quite a lot of intuitively important properties of a space. If we consider physical space as our example, as we've seen from mereology, some points seem to "belong together" more than other points, some may be "next to" a give subset even if they do not belong to it, two subsets may "touch" without any overlap, and so on.

To illustrate those notions, we can consider some subsets of the plane. If we compare let's say some kind of continuous shape, a disk and an uncountable set of points sprinkled in an area (for instance the two-dimensional Cantor dust),

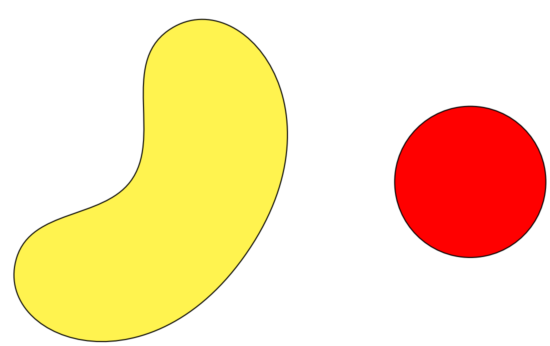


Figure 41.2: Three subsets of the plane of identical cardinality

We would expect the first two objects to have more in common than with the third, having what we will later call the same *shape*, but as sets, they are all isomorphic, simply by the virtue of containing the same amount of points.

The same goes for comparing one connected shape and one composed of two disconnected shapes

[...]

If we consider the set D of all points that are at a distance strictly inferior to a given value r from a central point o, we would like to say that a point at exactly a distance of r is somehow closer to D than other points, but as a set, this point is simply equivalent to every point outside of D, as $D \cap \{p\} = \varnothing$.

Convergence

quasitopological spaces, approach spaces, convergence spaces, uniformity spaces, nearness spaces, filter spaces, epitopological spaces, Kelley spaces, compact Hausdorff spaces, δ-generated spaces Cohesion, pretopology, proximity spaces, convergence spaces, cauchy spaces, frames, locales

Definition 41.2.1. A topological space (X, τ) is a set X and a set of subsets $\tau \subset \mathcal{P}(X)$, called *open sets*, such that for any collection of open sets $\{U_i\} \subseteq \tau$, we have

- The entire space X and the empty set $\varnothing$ are both open sets : $X, \varnothing \in \tau$
- For any collection of open sets $\{U_i\}_{i\in I}$, their union is an open set : $\bigcup_i U_i \in \tau$
- For any finite collection of open sets $\{U_i\}_{i\in I}$, $|I| \in \mathbb{N}$, their intersection is an open set : $\bigcap_i U_i \in \tau$.

Approaches via open/closed sets, closure operators, interior operators, exterior operators, boundary operators, derived sets

Definition 41.2.2. Given a set X, a *net* on X is a directed set $(A, \leq)$ and a map $\nu : A \to X$

In most of basic topology, nets are rarely used in their generality and we use instead the narrower notion of a sequence,

Example 41.2.1. A *sequence* is a net with the directed set $(\mathbb{N}, \leq)$.

A sequence would be in our case something like a family of nested open sets

$$\{U_i \mid i \in \mathbb{N},\ \forall n \in \mathbb{N},\ U_{n+1} \subseteq U_n\} \tag{41.11}$$

Example 41.2.2. Net of neighbourhoods

Definition 41.2.3. A net $x_\bullet$ on a directed set A is said to be eventually in $S \subseteq X$ if

Definition 41.2.4. A net is said to *converge* to an element $x \in X$ if

Definition 41.2.5. A *filter*

Theorem 41.2.1. A topological space (X, τ) defines a filter for every point x called the *neighbourhood filter*, which is generated by the set of open neighbourhood of x.

Proof. Given the point x, take the set of its open neighbourhood

$$N_o(x) = \{U \mid x \in U,\ U \in \tau\} \tag{41.12}$$

for which no element is non-empty by definition. If we look at its closure under supersets,

$$N(x) = \{U \mid \exists U' \in N_o(x),\ U' \subseteq U\} \tag{41.13}$$

this set will be closed under finite intersection, as for any two $U_1, U_2 \in N(x)$, we have that their intersection necessarily contains x as U_1 and U_2 both do, and therefore also contain an open neighborhood of x. □

Definition 41.2.6.

As categorical theories tend to work best when considering properties by their relations with morphisms, one useful thing to do here is to do so for open sets.

Theorem 41.2.2. For a topological space X, its set of open sets is isomorphic to the preimages of 1 for all continuous maps to the Sierpinski space.

Proof. If we consider the set of all functions $X \to \{0, 1\}$, which corresponds to the power set of X, via their interpretation as characteristic functions

$$\forall U \in \mathcal{P}(X),\ \chi_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases} \tag{41.14}$$

as a continuous function must have open preimages for open sets, and closed preimages for closed sets, χ_U is only continuous if U is an open set. □

Another definition of open sets

Theorem 41.2.3. A subset U is open if its inclusion map $\iota_U : U \hookrightarrow X$

41.3 Complexes

An additional notion of space common in mathematics, opposed to the "geometric" notion of space, is the so-called "combinatorial" notion of space, where instead of a continuous medium, we take some discrete set of spatial elements and connect them together.

There are many possible ways to define them, depending with what generality we want them or within which framework they would work best.

Definition 41.3.1. An *abstract simplicial complex* is a family of sets Δ closed under subsets, in that for any element $X \in \Delta$, any subset of X is also an element of Δ.

This can be understood by thinking of elements of X as being points in a space, and the sets in Δ as being structures built from those points. From this, we can define its total set of points,

$$X_0 = \bigcup_{X_i \in \Delta} X_i \tag{41.15}$$

Example 41.3.1. The basic examples of an abstract simplicial complex is the *simplex*, which is the case where, given some finite set X_0, the associated simplex of dimension $n = |X_0| - 1$ is

$$X = \mathcal{P}(X_0) \tag{41.16}$$

The first few simplices are :

- The empty simplex, $|X_0| = 0$, of dimension -1
- The point, $|X_0| = 1$, of dimension 0
- The line, $|X_0| = 2$, of dimension 1, with the dimension 1 edge being the subset $\{\bullet_0, \bullet_1\}$
- The triangle, $|X_0| = 3$, of dimension 2, with the edges $\{\bullet_0, \bullet_1\}$, $\{\bullet_1, \bullet_2\}$ and $\{\bullet_0, \bullet_2\}$, and the face $\{\bullet_0, \bullet_1, \bullet_2\}$

Example :

- The -1-simplex based on the empty set is a singleton $\{\varnothing\}$, and corresponds to an empty space.
- The 0-simplex $\{\{0\}, \varnothing\}$ composed of a single point and an empty set, corresponds to a single point.
- The 1-simplex $\{\{0, 1\}, \{0\}, \{1\}, \varnothing\}$

[simplex diagrams]

Examples of more complex complexes

Example 41.3.2. A basic example of a non-trivial complex is the square,

Definition 41.3.2. The *geometric realization* of a complex is a mapping of an abstract simplicial complex to a topological space

Example 41.3.3. A common geometric realization of complexes is their embedding in $\mathbb{R}^n$, a common such embedding being the embedding of simplices by convex sets :

$$\Delta^k = \{(t_1, \dots, t_k) \in \mathbb{R}^n \mid \sum_{i=0}^{k} t_i = 1,\ \forall i,\ t_i \geq 0\} \tag{41.17}$$

Definition 41.3.3. A *simplicial set* X is given by a collection of sets $(X_i)_{i\in\mathbb{N}}$, along with a collection of *face maps*

$$d_i : X_n \to X_{n-1} \tag{41.18}$$

and *degeneracy maps*

$$s_i : X_n \to X_{n-1} \tag{41.19}$$

obeying (assuming all compositions are on composable maps)

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i && i < j \\ d_j \circ s_j &= s_{j-1} \circ d_1 && i < j \\ d_j \circ s_j &= d_{j+1} \circ s_j = \mathrm{Id} && \\ d_j \circ s_j &= s_j \circ d_{i-1} && i > j+1 \\ s_i \circ s_j &= s_{j+1} \circ s_i && i \leq j \end{aligned} \tag{41.20}$$

42

Frames and locales

42.1 Order theory

All those notions of mereology and topology can be formalized within the context of category theory using the notion of frames and locales.

As we've seen, any formalization of a space can be at least formalized as a poset ordered by inclusion, already a category. All further notions relating to spaces will therefore be extra structures on posets, typically relating to their limits.

First we need to define the notion of semilattices for joins and meets.

Definition 42.1.1. A meet-semilattice is a poset $(S, \leq)$ with a meet operation $\wedge$ corresponding to the greatest lower bound of two elements (which is assumed to always exist in a meet-semilattice) :

$$m = a \wedge b \leftrightarrow m \leq a \wedge m \leq b \wedge (\forall w \in S,\ w \leq a \wedge w \leq b \to w \leq m) \qquad (42.1)$$

Example 42.1.1. In $\mathbb{Z}$ and $\mathbb{R}$ (in fact for any total order), the meet of two numbers is the min function :

$$k_1 \wedge k_2 = \min(k_1, k_2) \qquad (42.2)$$

Theorem 42.1.1. In a poset category, the meet is the coproduct.

Proof. By the semantics of morphisms in a poset category, if we look at the universal property of the coproduct, for any two objects X, Y with morphisms to a third object Z (so $Z \leq X$ and $Z \leq Y$), then there exists an object $X+Y$ with morphisms from X and Y (so $X+Y \leq X$, $X+Y \leq Y$) such that there is a unique morphism $Z \to X+Y$ (so that $Z \leq X+Y$). in other words, $X+Y$ is a lower bound, and if Z is also a lower bound, it is inferior to it, making $X+Y$ the greatest lower bound, ie the meet. □

Example 42.1.2. In the partial order defined by the power set of a set, the meet is the intersection of two sets.

Proof. The meet of two sets A, B is the largest set C that is a subset of both A and B. The intersection $A \cap B$ is by definition such a set, $A \cap B \subseteq A, B$. If $A \cap B$ is a strict subset of another lower bound C,

$$A \cap B \subset C \subseteq A, B \tag{42.3}$$

This means that □

Definition 42.1.2. A join-semilattice is a poset $(S, \leq)$ with a join operation $\vee$ corresponding to the least upper bound of two elements (which is assumed to always exist in a join-semilattice) :

$$m = a \vee b \leftrightarrow a \leq m \wedge b \leq m \wedge (\forall w \in S,\ a \leq w \wedge b \leq w \to m \leq w) \tag{42.4}$$

Theorem 42.1.2. In a poset category, the join is the product.

Proof. As with the proof for the meet, the universal property tells us that for any two morphisms $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$ (so that $Z \leq X_1, X_2$), we have a unique morphism $(f_1, f_2) : Z \to X_1 \times X_2$

$$x \tag{42.5}$$

Natural transformation (by components) :

$$\eta_X : \prod_i a_i \to a_i \tag{42.6}$$

There is one morphism from the join to each element, therefore $\prod_i a_i \leq a_i$

Upper bound is least : for any other b such that $b \leq a_i$ (ie the natural transformation $\alpha_b : b \to a_i$ for some a_i), then the unique morphism $f : b \to \prod a_i$ (b smaller than a_i)

□

Properties :

Proposition 42.1.1. The meet is commutative : $a \wedge b = b \wedge a$.

Proof. As the roles of a and b in the definition of the meet are entirely symmetrical, due to the commutativity of the logical conjunction, this is true. Alternatively, this is simply the commutativity of the coproduct. □

Proposition 42.1.2. The meet is associative : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Proof. If $m = b \wedge c$, then $a \wedge (b \wedge c) = a \wedge m$, meaning that the meet can be defined by some element m' such that

$$(m' \leq a) \wedge (m' \leq m) \wedge (m \leq b) \wedge (m \leq c) \tag{42.7}$$

$$\wedge (\forall w \in S,\ w \leq b \wedge w \leq c \to w \leq m) \tag{42.8}$$

$$\wedge (\forall w' \in S,\ w' \leq a \wedge w \leq m \to w \leq m') \tag{42.9}$$

as $(m' \leq m) \wedge (m \leq b)$ □

Definition 42.1.3. A *lattice* is a poset that is both a meet and join semilattice, such that $\wedge$ and $\vee$ obey the *absorption law*

$$a \vee (a \wedge b) = a \tag{42.10}$$

$$a \wedge (a \vee b) = a \tag{42.11}$$

Theorem 42.1.3. In a lattice, the join and meet are idempotent :

$$a \vee a = a \tag{42.12}$$

$$a \wedge a = a \tag{42.13}$$

Proof. By the second absorption law,

$$a \wedge (a \vee a) = a \tag{42.14}$$

then we can write by the first absorption law

$$a \vee a = a \vee (a \wedge (a \vee a)) = a \tag{42.15}$$

Likewise for $a \wedge a$,

$$a \vee (a \wedge a) = a \tag{42.16}$$

and

$$a \wedge a = a \wedge (a \vee (a \wedge a)) = a \tag{42.17}$$

□

Example 42.1.3. For sets, we have $A \cap A = A$ and $A \cup A = A$

Definition 42.1.4. A lattice is *distributive* if it obeys the distributy laws

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{42.18}$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{42.19}$$

Theorem 42.1.4. If a lattice obeys any of the two distributivity laws, it obeys both.

Proof. □

Example 42.1.4. For sets, the distributivity law is the distributivity of intersection and union :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (42.20)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (42.21)$$

Definition 42.1.5. A *Heyting algebra* is a bounded lattice, such that there exists both a bottom element 0 such that $0 \leq x$ for all x, and a top element 1 such that $x \leq 1$ for all x.

This top and bottom element will correspond to the equivalent top and bottom element of a mereology, generally interpreted as the ”total space” and the empty space.

Theorem 42.1.5. Heyting algebras are always distributive.

Proof. □

Example 42.1.5. The algebra of open sets on a topological space is a Heyting algebra.

Proof. □

A common poset structure that we will use is the algebra generated by a family of subsets. If we have a set X, and a family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, $\mathcal{B}$ forms a poset by the inclusion relation ordering, $(\mathcal{B}, \subseteq)$. Some common families of subsets of interest are the power set $\mathcal{P}(X)$, and more generally the set of opens $\mathrm{Op}(X)$ for a given topology.

It is a directed set with X the top element

In this context, the meet is the intersection

Theorem 42.1.6. The intersection of two sets is their meet.

Proof. As the intersection of A and B is defined via

$$A \cap B = \{x | x \in A \wedge x \in B\} \quad (42.22)$$

We can

We already know that $A \cap B \subseteq A, B$. If we assume a set $C \neq A \cap B$ such that $A \cap B \subseteq C$ and $C \subseteq A, B$, this means that C contains all the same elements as $A \cap B$ with some additional elements (since $A \cap B$ is a subset, we have

$C = (A \cap B) \cup (A \cap B)^C$, and as they are different, $(A \cap B)^C \neq \varnothing$). However, as C is a subset of A and B, that complement can only contains elements of A and B □

Theorem 42.1.7. The union of two sets is their join.

Proof. As the union of A and B is defined via

$$A \cup B = \{x | x \in A \vee x \in B\} \tag{42.23}$$

□

A family of sets is therefore a meet semilattice if it is closed under intersection, and a join semilattice if it is closed under union. If it is both, it is automatically a lattice as the absorption laws are obeyed by union and intersection.

[proof]

If the empty set is furthermore included, it is a bounded lattice, with $1 = X$, $0 = \varnothing$

Semi-lattice, lattice, Heyting algebra, frame (complete Heyting algebra)

Definition 42.1.6. A *Heyting algebra* H is a bounded lattice for which any pair of elements $a, b \in H$ has a greatest element x, denoted $a \to b$, such that

$$a \wedge x \leq b \tag{42.24}$$

Definition 42.1.7. The *pseudo-complement* of an element a of a Heyting algebra is

$$\neg a = (a \to 0) \tag{42.25}$$

Example 42.1.6. A bounded total order $0 \to 1 \to \ldots \to n$ is a Heyting algebra given by

$$a \to b = \begin{cases} n & a \leq b \\ b & a > b \end{cases} \tag{42.26}$$

The pseudo-complement is therefore just $\neg a = 0$.

Example 42.1.7. For the power set $\mathcal{P}(X)$ poset, the relative pseudo-complement of two sets A, B is

$$C = (X \setminus A) \cup B \tag{42.27}$$

This follows the property as $A \cap C = A \cap (B \setminus A)^C$

(eq. to the discrete topology)

Definition 42.1.8. A Heyting algebra is *complete* if it

Definition 42.1.9. A *boolean algebra* is a Heyting algebra satisfying the law of excluded middle,

$$a \wedge \neg a = 0 \tag{42.28}$$

Definition 42.1.10. A *frame* $\mathcal{O}$ is a poset that has all small coproducts (called joints $\vee$) and all finite limits (called meets $\wedge$), and satisfied the distribution law

$$x \wedge (\bigvee_i y_i) \leq \bigvee (x \wedge y_i) \tag{42.29}$$

Definition 42.1.11. The category of frames **Frm** is the category whose objects are frames and morphisms are frame homomorphisms

In terms of objects, frames are the same as complete Heyting algebras, but categorically this is not true, as frame homomorphisms are not the same as complete Heyting algebra homomorphisms.

Frames define a mereology by considering its objects as regions, its poset structure by the parthood relation, and joins and meets by

Mereological axiom for distribution law?

Definition 42.1.12. A *locale* is an object in the dual category of frames, the category of locales **Loc** :

$$\mathbf{Frm}^{\mathrm{op}} = \mathbf{Loc} \tag{42.30}$$

Locales are therefore formally frames, but locale homomorphisms are not

Example 42.1.8. A power set is a boolean algebra

Proof. The power set $\mathcal{P}(X)$ is as we've seen a Heyting algebra, and furthermore, we have

$$A \cap A^C = A \cap (X \setminus A) = \tag{42.31}$$

□

The basic example of a frame in math is that of the frame of opens for a topological space (X, τ).

Example 42.1.9. The category of open sets of a topological space X, $\mathrm{Op}(X)$, is a frame.

Proof. If we consider the poset of opens, as a union and intersection of open sets is itself an open set, we have a lattice, which is bounded by X itself and the empty set $\varnothing$. □

The frame of open is not boolean typically, as the negation $\neg$ can be defined as $\neg a \to 0$, and the implication

$$\begin{aligned} U \to V &= \bigcup \{W \in \mathrm{Op}(X) | U \cap W \subseteq V\} && (42.32) \\ &= (U^c \cup V)^\circ && (42.33) \end{aligned}$$

$$\begin{aligned} \neg U &= (U^c \cup \varnothing)^\circ && (42.34) \\ &= (U^c)^\circ && (42.35) \\ &= X \setminus \mathrm{cl}(U \cap X) && (42.36) \\ &= X \setminus \mathrm{cl}(U) && (42.37) \end{aligned}$$

The interior of the complement

$$\begin{aligned} U \cup (X \setminus U)^\circ &= (X \cup U) \setminus (\mathrm{cl}(U) \setminus U) && (42.38) \\ &= X \setminus \partial U && (42.39) \\ & && (42.40) \end{aligned}$$

Therefore a frame of open is boolean if open sets never have a boundary, which is that every open set is a clopen set.

Stone theorem [54]

Theorem 42.1.8. The category **Sob** of sober topological spaces with continuous functions and the category **SFrm** of spatial frames are dual to each other.

Examples :

Example 42.1.10. For a given set X, the partial order defined by inclusion of the power set $\mathcal{P}(X)$, is a complete atomic Boolean algebra.

Definition 42.1.13. A sober topological space

Theorem 42.1.9. Stone duality : The category of sober topological spaces **Sob** is dual to the category of spatial frames **SFrm**

In terms of categories, the various formalizations of mereology are expressed by different types of algebraic structures on posets. **M** is simply a poset with no extra structure.

The TOP axiom corresponds to the existence of a greatest element in this partial order (if we consider this applying to spaces, this is the object X of the space itself), BOTTOM to a least element (the empty set).

Most axiomatizations of mereology do not include the bottom element, but we will keep it for a better analogy with spaces in terms of a category, as they typically include one.

42.2 Subobjects of lattices

Definition 42.2.1. A *sublocale* of a locale $L \in \mathbf{Loc}$ is a regular subobject of L.

Example 42.2.1. For any object U in a locale L, the down set (the slice category L_U) is a sublocale

Proof. □

Moore closure

Theorem 42.2.1. The

double negation sublocale

Consider the map

$$\neg\neg : L \to L \quad (42.41)$$

$$U \mapsto \neg\neg U \quad (42.42)$$

A nucleus on L (a frame) is a function $j : L \to L$ which is monotone ($j(a \wedge b) = j(a) \wedge j(b)$), inflationary ($a \leq j(a)$) and $j(j(a)) \leq j(a)$

A meet-preserving monad.

Properties :

- $j(\top) = \top$
- $j(a) \leq j(b)$ if $a \leq b$
- $j(j(a)) = j(a)$

Quotient frames : L/j is the subset of L of j-closed elements of L (such that $j(a) = a$).

42.3 Lattice of subobjects

If we are to consider some category or object of a category as representing a space in some sense, a useful method to model it is to consider the structure given by its subobjects, as this is the best analogue that we have to a subregion of a space.

Given a category $\mathbf{C}$, we can look at the *poset of subobjects* $\mathrm{Sub}(X)$ for a given object X, with the following definition :

Definition 42.3.1. The poset of subobjects of X is the skeletal subcategory of the slice category $\mathbf{C}_{/X}$ from which we take every object of $\mathbf{C}_{/X}$ which is a monomorphism in $\mathbf{C}$, and identify every isomorphism to the identity. 22.2.2

Theorem 42.3.1. Every morphism in $\mathrm{Sub}(X)$ is a monomorphism in $\mathbf{C}$

Proof. As our objects are only monomorphisms, and any morhpism in $\mathrm{Sub}(X)$ will be a slice morphism g between two monomorphisms f, f', so that $f' \circ g = f$, we can just use the property that of $f \circ g$ is a monomorphism, then so is g. Therefore all morphisms in $\mathrm{Sub}(X)$ are monomorphisms in $\mathbf{C}$. □

They are in fact monomorphisms between the subobjects of X themselves.

Theorem 42.3.2. $\mathrm{Sub}(X)$ is a poset category

Proof. As the category of subobjects includes the identity morphism, it is reflexive. As we are assuming the equivalence class for everything here, if we have two morphisms f, g between two objects $\alpha, \beta \in \mathrm{Sub}(X)$, this means that we have two subobjects A, B of X with inclusion $\alpha : A \hookrightarrow X$ and $\beta : B \hookrightarrow X$ □

It is best to try to keep in mind when a statement about a category regards the category itself versus when it is about the category of subobjects, as those are typically categories of spaces versus poset categories, in which the different terms have vastly different meanings, and it is common for textbooks to not be terribly clear on this point.

As a poset category, we can say a few things about the limits of $\mathrm{Sub}(X)$, if they exist.

As we've seen in the chapter on limits, the initial and terminal object in a poset category correspond to the bottom and top element of the poset. We always have a top element of a poset of subobjects, the object itself, and if the category has an strict initial object, it will always be the bottom, as there is no monomorphism to 0 in this case. Notationally, we have

$$(0_X : 0 \hookrightarrow X) \quad \cong \quad 0 \in \mathrm{Sub}(X) \tag{42.43}$$

$$(\mathrm{Id}_X : X \hookrightarrow X) \quad \cong \quad 1 \in \mathrm{Sub}(X) \tag{42.44}$$

The product and coproduct in the poset of subobjects correspond to the join and meet, ie the greatest lower bound and least upper bound. In terms of limits in $\mathbf{C}$, we have that the product in $\mathrm{Sub}(X)$ corresponds to a

If our subobject poset is a join semilattice, this means that it is equipped with a product and a terminal object (X itself), meaning that we have in fact a monoidal category.

In this case, we can define the functor $(-) \times S$, which is simply a function on the poset mapping every object to their meet with S, which is a map from the subobject poset $\mathrm{Sub}(X)$ to the subobject poset $\mathrm{Sub}(S)$ by intersection.

If the class of functors $(-) \times S$ admits a right-adjoint for every S, we will also have an internal hom. The adjunction gives us

$$\mathrm{Hom}_{\mathrm{Sub}(X)}(S, [X, Y]) \cong \mathrm{Hom}_{\mathrm{Sub}(X)}(S \times X, Y) \tag{42.45}$$

Meaning that

$$S \leq [X, Y] \leftrightarrow S \wedge X \leq Y \tag{42.46}$$

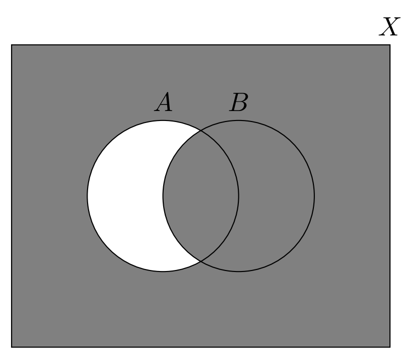
The internal hom $[X, Y]$ is therefore some subobject for which every subobjects is such that their meet with X is a subobject of Y. As we will see, this is what is defined as an implication in a Heyting algebra, denoted by

$$Y^X = X \to Y \tag{42.47}$$

Example 42.3.1. In **Set**, the implication between subsets $A \to B$ is the union of B and the complement $X \setminus A$,

$$A \to B = A^c \cup B \tag{42.48}$$

with the Venn diagram



Properties :

Proposition 42.3.1. X belongs to $\mathrm{Sub}(X)$ and is the top element.

Proposition 42.3.2. If the category has an initial object 0, it is in $\mathrm{Sub}(X)$ and is the bottom element.

Theorem 42.3.3. If **C** admits all finite limits, $\mathrm{Sub}(X)$ is a meet-semilattice.

Proof. As we've seen before, the intersection of two subobjects is defined by the pullback of their inclusion map. If a category has all finite limits, this is in particular true of the pullback, so that any two subobjects of X also admit an intersection, itself a subobject, such that $A \cap B$ is indeed a lower bound of both A and B. This lower bound is indeed the greatest by the universal property of the pullback : if we have any other lower bound C of A and B (monomorphisms to A and B, then there is a unique morphism $\iota_{C,A\cap B} : C \to A \cap B$ which is such that

$$\iota_{A\cap B,A} \circ \iota_{C,A\cap B} = \iota_{C,A} \tag{42.49}$$

which must be a monomorhpism itself since its composition here is also a monomorphism, so that we indeed have $C \subseteq A \cap B$ for any other lower bound. □

By the same reasoning, admitting all colimits will also make it a join-semilattice, with the join the union [?]

Definition 42.3.2. If the poset of subobjects $\mathrm{Sub}(X)$ admits

Theorem 42.3.4. The implication $A \to B$ in a poset of subobject is given by the exponential object

Proof. As an implication, we have that the subobject $A \to B$ is a subobject of X for which all subobjects C of $A \to B$ are such that

$$(C \wedge A) \leq B \tag{42.50}$$

□

Definition 42.3.3. If the poset of subobjects has a bottom element 0 (the inclusion $0 \hookrightarrow X$ in $\mathbf{C}$) and an implication, the negation $\neg_X A$ of a subobject A is the operation

$$\neg_X A = A \to 0 \tag{42.51}$$

This negation (what is called the pseudo-complement) means that for any subobject of the pseudocomplement,

$$S \subseteq \neg_X A \tag{42.52}$$

we have that

$$S \wedge A \subseteq 0 \tag{42.53}$$

As the bottom element, this simply means that any subobject of $\neg_X A$ is disjoint from A, hence its name of pseudocomplement. The pseudo being here due to the fact that we are however not guaranteed that this pseudocomplement contains *all* subobjects not in A, which is only true for the boolean case

Definition 42.3.4. If the poset of subobjects is a Heyting algebra and has the boolean property,

$$a \to b = \neg a \vee b \tag{42.54}$$

then it is a boolean category idk

In a boolean category we have indeed that $\neg_X A = A \to 0 = \not{A} \vee 0$

[...]

One type of property that we might want to impose on the poset of subobjects is to have some notion of connectedness. What we would like generally is that, given a coproduct $A + B$, it is in some sense "disconnected".

The basic notion for this is that of a disjoint coproduct,

Definition 42.3.5. A coproduct is *disjoint* if the intersection of its components is the initial object,

$$X \times_{X+Y} Y = 0 \tag{42.55}$$

Definition 42.3.6. A category $\mathbf{C}$ is *finitely extensive* if its slice categories behave as

$$\mathbf{C}_{/X} \times \mathbf{C}_{/Y} \cong \mathbf{C}_{/(X+Y)} \tag{42.56}$$

43

Coverage and sieves

To define a space in categorical terms, we need to have some formalization of an equivalent notion to mereology, open sets, frames or such that we saw earlier. The notion of *coverage* that we will see will be more general than that (in particular not necessarily be about subregions) but contain those as a special case.

Definition 43.0.1. A *cover* of an obhect X is given by a morphism $\pi : U \to X$. For a collection of covers, we speak of a *covering family*,

$$\{\pi_i : U_i \to X\}_{i\in I} \tag{43.1}$$

where I is some indexing set.

The raw definition does not give much properties to a covering family, but it is common to consider them to be

[define cover/covering family first?]

Definition 43.0.2. Given an object X in a category $\mathbf{C}$, a *coverage* J of X is a covering family for X

$$J = \{U_i \to X\}_{i\in I} \tag{43.2}$$

such that morphisms between two objects of $\mathbf{C}$ induce a coverage. For $g : Y \to X$, there exists a covering family $\{h_j : V_j \to Y\}_{j\in J}$ such that gh_j factors through f_i for some i :

$$\begin{array}{ccc} V_j & \xrightarrow{k} & U_i \\ \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow{g} & X \end{array}$$

If we take the case of topology that we've seen as an example, we define the standard coverage of a space X to be the collection of all families of open subsets that cover it, ie

$$J(X) = \{\{U_i \hookrightarrow X\} \mid U_i \subseteq X, \bigcup_i U_i = X\} \tag{43.3}$$

Its stability under pullback corresponds to the fact that for any continuous function $f : Y \to X$, as the pre-image of any open set is itself an open set, we can define a family

$$\{f^{-1}(U_i) \to Y\} \tag{43.4}$$

and as any point in X is covered by some U_i, any point in Y will similarly be covered by $f^{-1}(U_i)$, obeying the properties of a coverage.

"Another perspective on a coverage is that the covering families are "postulated well-behaved quotients." That is, saying that $\{f_i : U_i \to U\}_{i \in I}$ is a covering family means that we want to think of U as a well-behaved quotient (i.e. colimit) of the U_i. Here "well-behaved" means primarily "stable under pullback." In general, U may or may not actually be a colimit of the U_i; if it always is we call the site subcanonical. "

To define spaces in the mathematical sense of the word, we need to have some sort of equivalent definition of a *topology*.

If $\mathbf{C}$ has pullback : the family of pullbacks $\{g^*(f_i) : g^*U_i \to V\}$ is a covering family of V.

Grothendieck topology :

An important class of coverage is the *Grothendieck topology*

Čech nerve

Sieve

Definition 43.0.3. For a covering family $\{f_i : U_i \to U\}$ in a coverage J, its *sieve* is the coequalizer

$$\coprod_{j,k} \text{よ}(U_j) \times_{\text{よ}(U)} \text{よ}(U_k) \rightrightarrows \coprod_i \text{よ}(U_i) \to S(\{U_i\}) \tag{43.5}$$

with よ the Yoneda embedding $\text{よ} : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C})$

Example 43.0.1. For an open cover of a topological space, Other definition : A sieve $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ on $X \in \mathbf{C}$ is a subfunctor of $\mathrm{Hom}_{\mathbf{C}}(-, X)$

Objects $S(Y)$ are a collection of morphisms $Y \to X$, and for any morphism $f : Y \to Z$, $S(f)$

Pullback by a sieve :

Ordering : $S \subseteq S'$ if $\forall X$, $S(X) \subseteq S'(X)$

Category of sieves is a partial order, with intersection and union, it is a complete lattice

Definition 43.0.4. For a category $\mathbf{C}$ and an object $X \in \mathbf{C}$, a presieve P on X is a collection of arrows with codomain X.

Definition 43.0.5. Sieve is a collection of arrows with codomain X such that $f \in S \to f \circ g \in S$

Definition 43.0.6. A sieve S is *generated* by a presieve P on X if it is the smallest sieve containing it, that is, it is the collection of arrows to X which fector through an arrow in P.

"The Grothendieck topology generated by a coverage is the smallest collection of sieves containing it which is closed under maximality and transitivity."

43.1 Grothendieck topology

For a sheaf meant to emulate the notion of a space, we would like to have an analogous notion to open sets, as some class of preferred subobjects. This is given by the notion of a Grothendieck topology.

[covering sieves]

Definition 43.1.1. For a category $\mathbf{C}$, a *Grothendieck topology* is a collection of coverings

$$T = \{\phi_i : U_i \to U\}_{i \in I} \tag{43.6}$$

such that

- Any isomorphism is in T : $\mathrm{Iso}(\mathbf{C}) \subseteq T$
- T is stable under composition : if $\{U_i \to U\}$ is a covering, and for each i, $\{V_{i,j} \to U_i\}$ are coverings, then $\{V_{i,j} \to U\}$ is also a covering.
- T is stable under pullback : for a covering $\{U_i \to U\}$ and a morphism $V \to U$, the pullback $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}$ is a covering in T.

[Grothendieck pretopology?]

Trivial topology : only maximal sieves

Chaotic topology : Only one sieve, endomorphisms of the object. (”multiplicative N seen as a category with one object, we get the arithmetic site”)

discrete topology : All sieves

Atomic topology : all non-empty sieves

43.2 Čech nerves

Definition 43.2.1. the *Čech nerve* of a morphism $f : U \to X$ is the simplicial object where the k-simplices are given by the k-fold pushout of U with itself over X,

$$C(U) = \left(\ldots \longrightarrow U \times_X U \times_X U \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} U \times_X U \rightrightarrows U \right) \tag{43.7}$$

Given a covering sieve $\{U_i \to X\}$ with respect to a coverage,

44

Subobject classifier

In a category **C** with finite limits, a subobject classifier is given by an object Ω (the object of truth values) and a monomorphism

$$\top : 1 \to \Omega \tag{44.1}$$

from the terminal object 1, such that for every monomorphism [inclusion map] $\iota : U \hookrightarrow X$, there is a unique morphism $\chi_U : X \to \Omega$ such that U is the pullback of $* \to \Omega \leftarrow X$

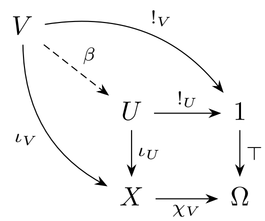
$$\begin{array}{ccc} U & \xrightarrow{!_U} & 1 \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\top} \\ X & \xrightarrow[\chi_U]{} & \Omega \end{array}$$

so that $U \cong X \times_\Omega 1$, or, if we look at it via the equalizer of a product,

$$U \cong X \times_\Omega 1 \to X \times 1 \underset{\chi_U \circ \mathrm{pr}_1}{\overset{\top \circ \mathrm{pr}_2}{\rightrightarrows}} \Omega$$

From our rough intuition of the pullback, we can see imagine that this means something like the object for which χ_U is equal to true, which is the usual sense of what a characteristic function is.

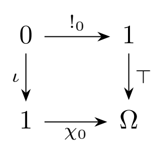
This diagram is furthermore universal, in the sense that for any other subobject V of X, with $\iota_V : V \to X$, the following diagram only commutes if V is itself a subobject of U :



ie that V has the same type of valuation in Ω as U through the characteristic function χ_U.

Theorem 44.0.1. In a locally finite category **C** with a terminal object, **C** has a subobject classifier if and only if there is some object Ω for which the hom functor is naturally isomorphic to the subobject functor.

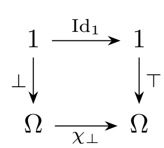
A particularly important case of the pullback is the subobject defined by the monomorphism $0 \hookrightarrow 1$ (if the category admits an initial object), in which case we get the following diagram



The morphism $\chi_0 : 1 \to \Omega$ is another truth value of Ω, which is the *false* truth value, denoted as

$$\bot : 1 \to \Omega \tag{44.2}$$

As this defines a subobject for Ω itself, we also have the pullback



$\chi_\bot$ is then some endomorphism on Ω, which we call the *negation*, $\neg$, as it can be understood to map the false value $\bot$ in Ω to the true value $\top$, being the characteristic function of the false value. From this diagram, we have the first equality on Ω,

$$\neg \circ \bot = \top \tag{44.3}$$

the negation of falsity is truth. Furthermore, if we apply the negation to the truth, we get

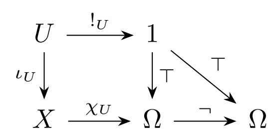
Theorem 44.0.2. The negation of truth $\top$ is falsity $\bot$.

Proof. If we take □

Theorem 44.0.3. If the category has no zero object $1 \cong 0$, the composition of the negation with any characteristic morphism $\chi_U : X \to \Omega$ causes the failure of U or any of its subobject to be subobjects of X.

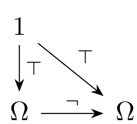
Proof. If we have the span

$$X \xrightarrow{\neg \chi_U} \Omega \longleftarrow 1 \tag{44.4}$$



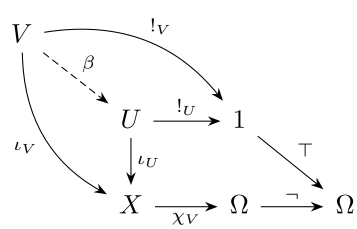
For U to be a subobject of X, we need the lower composition to form a pullback with

However, the triangle here does not commute



as from the definition, we have $\neg \circ \bot = \top$ [proof?]

Furthermore, if we call the pullback of this span $\neg U$, and we try to look at any subobject of U itself, none of those subobjects are subobjects of $\neg U$.



□

So that the negation of a characteristic function contains none of the original subobjects. We will call this the *pseudo-complement* of a subobject.

Definition 44.0.1. For a characteristic function $\chi_U : X \to \Omega$, its pseudocomplement is the negation

$$\neg_X \chi_U = \neg \circ \chi_U \tag{44.5}$$

and its pullback defines the pseudocomplemented subobject

$$\neg_X U = X \times_{\Omega, \neg\chi_U} 1 \tag{44.6}$$

While the pseudocomplement depends on the containing object X, it is typically obvious with respect to which object we are taking it, so that we will usually write it as $\neg$ unless there is some ambiguity.

The exact nature of this pseudo-complement will depend on the exact category that we are working on. As we will see, it is not necessarily true that the pseudo-complement is to be understood as "everything in X not in U" (that is, $U \cup \neg U = X$), but it does have a few of the characteristics we would expect from the complement.

Theorem 44.0.4. A subobject and its pseudo-complement are disjoint :

$$U \cap \neg U = 0 \tag{44.7}$$

or in other words, the pullback $U \times_X \neg U$ is the initial object.

but one thing that is true is that it does split the points of X into either. To show this, we need to consider a few things.

Proof that set is boolean

Proof that the hom set commutes with negation?

Theorem 44.0.5. In a category with a strict initial object and a two-valued subobject classifier, given the hom-functor h^X, the negation morphism on $\mathbf{C}$ is mapped to a negation morphism on $\mathbf{Set}$(???)

Proof. First we have to show that the falsity morphism is preserved. By the strictness of the initial object, we have that $\mathrm{Hom}_{\mathbf{C}}(1, 0) = 0$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{C}}(X, 0) & \xrightarrow{!_0} & 1 \\
\downarrow \iota & & \downarrow \top \\
1 & \xrightarrow{\chi_0} & \Omega
\end{array}$$

By preservation of limits, we simply map the pullback

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\scriptstyle \bot} & & \downarrow{\scriptstyle \top} \\ \Omega & \xrightarrow{\neg} & \Omega \end{array}$$

to

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\scriptstyle \bot} & & \downarrow{\scriptstyle \top} \\ \Omega & \xrightarrow{\chi_\bot} & \Omega \end{array}$$

(using $\mathrm{Hom}_{\mathbf{C}}(X, 1) \cong 1$) □

Theorem 44.0.6. Any point of X, $x : 1 \to X$ in $\mathrm{Hom}_{\mathbf{C}}(1, X)$, is either in U or $\neg U$.

Proof. As the pullback preserves limits, if we take the hom functor h^1, this leads to the following pullback in $^{\mathbf{Set}}$:

$$\begin{array}{ccc} \mathrm{Hom}(1, U) & \xrightarrow{\mathrm{Id}_1} & 1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \top} \\ \mathrm{Hom}(1, X) & \xrightarrow{\chi} & \Omega \end{array}$$

As **Set** is a boolean category □

Theorem 44.0.7. Conjunction

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{Id}_1} & 1 \\ {\scriptstyle (\top,\top)}\downarrow & & \downarrow{\scriptstyle \top} \\ \Omega \times \Omega & \xrightarrow{\cap} & \Omega \end{array}$$

Proof. □

Theorem 44.0.8. For any object X, the initial object 0 is always a subobject.

Proof. □

This is best exemplified by the simple case for sets :

Example 44.0.1. In **Set**, Ω is the set containing the initial object, $\Omega = \{\varnothing, \{\bullet\}\}$, also noted as $2 = \{0,1\}$.

For a subset $S \subseteq X$ with an inclusion map $\iota : S \hookrightarrow X$, the characteristic function $\chi_S : X \to 2$ is the function defined by $\chi_S(x) = 1$ for $x \in S$ and $\chi_S(x) = 0$ otherwise. The truth function simply maps $*$ to 1 in Ω.

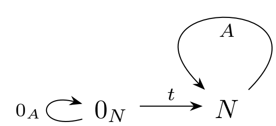
$$\forall x \in U,\ \chi_U(\iota_U(x)) = 1 \tag{44.8}$$

And conversely, if we look at another subobject $V \subseteq X$, the pullback works out

$$\forall x \in V,\ \chi_U(\iota_U(x)) = 1 \tag{44.9}$$

only if $V \subseteq U$, ie there exists a monomorphism from $V \to U$

Subobject classifiers can be more complex than the simple boolean domain true/false. A good illustration of this is the subobject classifier in the category of graphs[55]. A graph is composed by two sets, those of nodes N and of arrows A, with two functions s, t for the source and target of each arrow.



For a subgraph $\iota : S \hookrightarrow G$, the classifying map χ_S has the following behaviour :

- If a node in G is not in S, it is mapped to 0_N.
- If a node in G is in S, it is mapped to N.

Subobject classifier for a topological space

Negation complement :

Definition 44.0.2. Given the classifying arrow $\bot : 1 \to \Omega$ of the initial object $!_0 : 0 \hookrightarrow 1$, with the associated negation morphism $\neg : \Omega \to \Omega$ the classifying arrow of $\bot$, the pseudocomplement $\neg_X U$ of a subobject $U \hookrightarrow X$ is the pullback of $\neg\chi_U$ by $\top$:

$$\neg_X U = \mathrm{Fib}_\top(\neg\chi_U) \tag{44.10}$$

Theorem 44.0.9. A subobject and its pseudocomplement have no overlap $U \cap \neg_X U \cong 0$

Proof. □

To preserve negation : preserve the pullback and terminal object and initial object?

quillen negation for the true and false morphism?

Theorem 44.0.10. Quillen negation of the true and false morphism

Proof. Right Quillen negation :

$$\begin{array}{ccc} 1 & \xrightarrow{x} & X \\ {\scriptstyle\top}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ \Omega & \xrightarrow[g]{} & Y \end{array}$$

Left Quillen negation :

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ {\scriptstyle i}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle\top} \\ B & \xrightarrow[g]{} & \Omega \end{array}$$

If we have a diagonal filler, which will be the unique map $!_B : B \to 1$, this is in fact equivalent to the subobject diagram

$$\begin{array}{ccc} B & \xrightarrow{!_U} & 1 \\ \downarrow{\scriptstyle \mathrm{Id}_B} & & \downarrow{\scriptstyle\top} \\ B & \xrightarrow{\chi_B} & \Omega \end{array}$$

as $\chi_B \circ \mathrm{Id}_B = \chi_B = \top \circ !_B$. Therefore, the only morphisms in the left Quillen negation are of the form $\mathrm{Id}_B : B \to B$. The negation of the prototypical subobject $1 \to \Omega$ is the full inclusion of the entire object into itself. [Is that the only possible morphism? What if it's not a pullback] Left Quillen negation for the false map :

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ {\scriptstyle i}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle\bot} \\ B & \xrightarrow[g]{} & \Omega \end{array}$$

□

44.1 Algebra on the subobject classifier

Just as we have an algebra on the poset of subobjects by its use of limits and colimits, we will likewise have an equivalent algebra with the subobject classifier, being an internalization of the category of subobjects. As there is the isomorphism $\mathrm{Hom}_{\mathbf{C}}(X, \Omega) \cong \mathrm{Sub}(X)$

Theorem 44.1.1. The meet of two subobjects corresponds to the logical conjunction $\wedge$.

Proof. □

44.2 In a sheaf topos

Given the sheaf topos $\mathbf{H} = \mathrm{Sh}(\mathbf{C}, J)$, there is a natural subobject classifier

Example 44.2.1. A trivial case of a subobject classifier in a sheaf topos is the initial topos $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$. As the only sheaf is the empty sheaf, it is also trivially the sheaf corresponding to the subobject classifier of a Grothendieck topos. This means the only truth value, as given by the map $1 \to \Omega$, is the truth, or falsity, as we have that the terminal object, initial object and subobject classifier are the same object.

Example 44.2.2. The subobject classifier of **Set** is, as we've seen, 2. In terms of a presheaf category $\mathbf{Set} = \mathrm{Sh}(\mathbf{1})$, this is given by the presheaf to sieves of the probe. As there is only one such object, this is the set of sieves on the terminal category, ie the empty sieve and the maximal sieve.

Example 44.2.3. Given a spatial topos $\mathrm{Sh}(X)$ on a topological space X, the subobject classifier of that topos is the sheaf that associates to every open set U its open subsets :

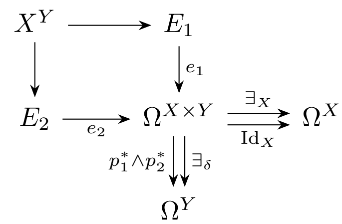
$$\Omega : \mathrm{Op}(X) \rightarrow \mathrm{Set} \tag{44.11}$$
$$U \mapsto \Omega(U) = \mathrm{Op}(U) \tag{44.12}$$

Proof. □

44.3 Exponential object

In a category with a subobject classifier as well as a Cartesian closed one, ie for which the Cartesian product defines a monoidal product with an associated internal hom,

One important use of the subobject classifier is the case where we consider the monoidal category to be that generated by the product, $(\mathbf{C}, \times, 1)$,



with the projections $p_i : X \times Y \times Y \to X \times Y$

44.4 Power object

Definition 44.4.1. Given an object X in a category $\mathbf{C}$ with a subobject classifier Ω, a power object of X is an object Ω^X along with a monomorphism

Theorem 44.4.1.

45
Points

As we have seen, we can represent a general notion of space as that of a frame, but a more contentious issue is how to define *points* in a space. This is an issue that goes back all the way to the foundation of geometry [56], and to this day is not an uncontentious one, as the assumption of point-like structures in space is still a thorny issue.

Categories for which we have fairly simple notions of discrete elements such as finite sets do not have too much trouble defining what a point could be, corresponding in some sense to the notion of discrete objects that were used uncontroversially in antiquity, but given a frame like that of continuous physical space, this becomes a more complex notion to define, as there is no internal notion of what a point is in the context of the frame of opens $\mathcal{O}(X)$.

In the abstract, we can define a point just as we would define an element for another category, simply as morphisms from some terminal object to the regions of space, but we are neither guaranteed the existence of such an object nor that the space is in some sense *composed* by those points rather than just those merely inhabiting it.

The intuitive notion, going back at least as far as [56], would be to consider a point as the limit of a shrinking family of open sets, but we could have for instance a family of regions $\{U_i\}$ which converge to another (non-point like) region, such as a family of disks of radius $r_n = 1 + 2^{-n}$. Furthermore, two different such sequences can converge to the same point so that we also need to be able to define the equivalence of such sequences.

To represent the notion of several sequences of regions converging to the same result, we need to use the notion of *filter*

A subset F of a poset L is called a filter if it is upward-closed and downward-directed; that is:

If $A \leq B$ in L and $A \in F$, then $B \in F$; for some A in L, $A \in F$; if $A \in F$ and $B \in F$, then for some $C \in F$, $C \leq A$ and $C \leq B$.

Points given a locale?

Given a locale X, a concrete point of X is a completely prime filter on $O(X)$. [Show equivalence with a continuous map $f : 1 \to X$: treat $f^* : O(X) \to O(1)$ as a characteristic function]

Completely prime filter :

A filter F is prime if $\perp \notin F$ and if $x \vee y \in F$, then $x \in F$ and $y \in F$. For every finite index set I, $x_k \in F$ for some k whenever $\bigvee_{i \in I} x_i \in F$.

[Some descent of open sets for a topological space?]

Example 45.0.1. Take the frame defined by the poset $0 \leq A \leq 1$, the simplest frame which is not boolean. Its joins and meets can all be deduced from the properties of the join and meet with respect to the top and bottom element and idempotency. Let's attempt to find a frame homomorphism ϕ to the terminal(?) frame $\{0 \leq 1\}$. As it must preserve joins and meets and bounds, we should have $\phi(0) = 0$, $\phi(1) = 1$, and therefore $\phi(A)$ must be mapped to either 1 or 0. We should therefore have that

$$\phi(A \to 0) = \phi(0) \quad (45.1)$$
$$\phi(A) \to 0 = 0 \quad (45.2)$$

and

$$\phi(A \to 1) = \phi(1) \quad (45.3)$$
$$\phi(A) \to 1 = 1 \quad (45.4)$$

The first implies that $\phi(A) = 0$, while the second implies that $\phi(A) = 1$. There is therefore no such map from this frame to the terminal frame, and therefore no points.

Example 45.0.2. The classic example of a pointless locale is the locale of surjections from the discrete space $\mathbb{N}$ to the continuous space $\mathbb{R}$ with its standard topology[57]. This locale is defined by the

As $|\mathbb{N}| < |\mathbb{R}|$, there is no such surjection.

Stone theorem

46
Presheaves

Definition 46.0.1. A *presheaf* on a small category $\mathbf{C}$ is a functor F

$$F : \mathbf{C}^{\text{op}} \to \mathbf{Set} \tag{46.1}$$

This definition also generalizes to any category. If we replace **Set** with any category $\mathbf{S}$, we speak of an S-valued presheaf, defined as

$$F : \mathbf{C}^{\text{op}} \to \mathbf{S} \tag{46.2}$$

In a similar manner, we have the dual of presheaves, called *copresheaves*, and defined as sheaves on the opposite category :

$$F : \mathbf{C} \to \mathbf{Set} \tag{46.3}$$

And similarly, for an S-valued copresheaf,

$$F : \mathbf{C} \to \mathbf{S} \tag{46.4}$$

Fundamentally, any functor can be described as a (co)presheaf, as any functor from a category $\mathbf{C}$ (or its opposite) fits the definition, but a presheaf is typically gonna be studied with more specific goals in mind, usually to turn them into sheaves or topos.

Theorem 46.0.1. The presheaf limit of a functor $F : D^{\mathrm{op}} \to C$ is the presheaf defined by

$$(\lim F)(X) = \mathrm{Hom}_{\mathbf{Set}^{\mathbf{D}^{\mathrm{op}}}}(\mathrm{pt}, \mathrm{Hom}_{\mathbf{C}}(X, F(-))) \tag{46.5}$$

and if this presheaf is representable, the object associated is the limit.

Theorem 46.0.2. A presheaf category $\mathrm{PSh}(\mathbf{C})$ has all limits and colimits.

Proof. By 19.0.1, as $\mathrm{PSh}(\mathbf{C})$ is a functor category, it has all limits and colimits of **Set**, which has all limits. □

46.1 Presheaf on a topological space

An example for the motivation of presheaves is to consider a topological space (X, τ). The category of interest here is the frame of opens $\mathrm{Op}(X)$. A sheaf on the frame of open is some functor associating a set to every open set :

$$\forall U \in \mathrm{Op}(X),\ F(U) = A \in \mathbf{Set} \tag{46.6}$$

or, in the case of an **S**-presheaf, some other object, typically something like a ring or Abelian group. This association is done in a way that preserves the functions contravariantly. In particular, if we have an inclusion $\iota : U \hookrightarrow U'$, its opposite is $\iota^{\mathrm{op}} : U' \to U$, and the functor maps it to

$$F(\iota^{\mathrm{op}}) : F(U') \to F(U) \tag{46.7}$$

Therefore for any inclusion, there exists some morphism sending the object of the larger open set to the smaller open set.

A common example of presheaves for the topological case is that of *structure presheaves*, which map those open sets to some function set (or ring or Abelian group). Common examples of this would be the set of continuous functions to some specific codomain, like $\mathbb{R}$

$$F(U) = C(U, \mathbb{R}) \tag{46.8}$$

or more restricted functions like smooth or analytic functions, $C^\infty(U, \mathbb{R})$ or $C^\omega(U, \mathbb{R})$. In those cases, the functor map $F(\iota^{\mathrm{op}})$ corresponds to the *restriction function.*

$$\forall f \in F(U'),\ \mathrm{res}_{U',U}(f) = f\big|_U \tag{46.9}$$

where the function f in the set of functions $F(U')$ is mapped to a function $f\big|_U$ in $F(U)$.

Properties : the restriction of an open set to itself is the identity :

$$\mathrm{res}_{U,U} = F(\mathrm{Id}_U) = \mathrm{Id}_{F(U)} \tag{46.10}$$

$$\mathrm{res}_{U'',U} \circ \mathrm{res}_{U',U} = \mathrm{res}_{U'',U} \tag{46.11}$$

We can show that this function

We will see in the section on sheaves the meaning of this construction.

Example 46.1.1. An S-valued presheaf on $\mathbf{C}$ is a *constant presheaf* if it is a constant functor, ie for some element $X \in \mathbf{S}$, the presheaf is just

$$\Delta_X : \mathbf{C}^{\mathrm{op}} \to \mathbf{S} \tag{46.12}$$

As presheaves are merely functors, there is a category of presheaves simply defined by the appropriate functor category, so that an $\mathbf{S}$-presheaf category on $\mathbf{C}$ is

$$\mathrm{PSh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{S}] \tag{46.13}$$

and likewise for copresheaves. This means in particular that morphisms of presheaves in this context are given by natural transformations between two presheaves.

injectivity, surjectivity, etc.

Definition 46.1.1. A *subpresheaf* $S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ of a sheaf $C : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf for which we have for any element of the site $U \in \mathbf{C}$

$$S(U) \subseteq X(U) \tag{46.14}$$

This notion generalizes to presheaves valued in other categories using monomorphisms.

Theorem 46.1.1. Subpresheaves define monomorphisms in the category of presheaves.

Proof. As presheaves are objects of a functor category, we need to look at the behaviour of monic natural transformations.

$$\iota : S \hookrightarrow X \tag{46.15}$$

So that for any presheaf Y, and any two natural transformations $g_1, g_2 : Y \to S$, we have

$$\iota \circ g_1 = \iota \circ g_2 \to g_1 = g_2 \tag{46.16}$$

If we look at the components of the natural transformation, we have that for any $U \in \mathbf{C}$,

$$\iota_U : S(U) \to X(U) \tag{46.17}$$

as we defined our subpresheaves to be subsets for every element of the site, this means that ι_U is a monomorphism for all U

$$\iota_U : S(U) \hookrightarrow X(U) \tag{46.18}$$

for which the monomorphism condition means that for any other natural transformations $g_1, g_2 : Y \to S$, in terms of components, we have

$$(\iota \circ g_1)(y) = (\iota \circ g_2)(y) \to g_1(y) = g_2(y) \tag{46.19}$$

□

Example 46.1.2. For the structure presheaf of continuous real functions on a topological space, $C(X, \mathbb{R})$, we have

$$C^\infty(X, \mathbb{R}) \hookrightarrow \ldots \hookrightarrow C^1(X, \mathbb{R}) \hookrightarrow C(X, \mathbb{R}) \tag{46.20}$$

46.2 Presheaf category

Theorem 46.2.1. The intersection of two subpresheaves $X, Y \in \mathrm{PSh}(\mathbf{C})$ is the subpresheaf mapping elements of $\mathbf{C}$ to the intersection of their images :

$$[X \cap Y](U) = X(U) \cap Y(U) \tag{46.21}$$

Proof. given two subpresheaves $\iota_1 : S_1 \hookrightarrow X$, $\iota_2 : S_2 \hookrightarrow X$, their intersection is the pullback $S_1 \times_X S_2$, meaning that, as limits are computed component-wise, we have

$$[X \cap Y](U) = X(U) \cap Y(U) \tag{46.22}$$

□

Theorem 46.2.2. The union of

$$[X \cup Y](U) = X(U) \cup Y(U) \tag{46.23}$$

46.3 The Yoneda embedding

As sheaves are fundamentally functors, this means that we can treat them within the context of the Yoneda lemma.

interpretation of presheaves $X(U)$ as a function U to X via Yoneda

Definition 46.3.1. A *representable presheaf* $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf that has a natural isomorphism to the hom-functor h_X fr some object $X \in \mathbf{C}$:

$$\eta : F \xrightarrow{\cong} h_X = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{46.24}$$

In other words a representable presheaf sends every object $Y \in \mathbf{C}$ to the set $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ of all morphisms from Y to X, and every morphism $f : Y \to Z$ to the function sending any morphism $g : Y \to X$ to its composite $f \circ g$.

As we will see later on, representable presheaves are often use to carry the notion that the objects of a category $\mathbf{C}$ correspond in some sense to some of the objects of a presheaf category. This is used for instance in the notion of constructing spaces from simpler spaces as presheaves, in which case the basic spaces are also included. For instance, we can consider manifolds as constructed by atlases to be presheaves on the category of open sets of $\mathbb{R}^n$, in which case the representable presheaves in this category will correspond to the manifolds which are simply raw open sets of $\mathbb{R}^n$.

Theorem 46.3.1. A representable presheaf is determined by a unique object $X \in \mathbf{C}$.

Proof. By the Yoneda lemma, □

Theorem 46.3.2. Representable functors preserve all limits.

Proof. □

This embedding in fact allows us to interpret presheaves as some sort of exten sion of a category, which has the benefit of being better behaved than the original category. This is the *Yoneda embedding*.

Definition 46.3.2. The *Yoneda embedding* $よ : \mathbf{C} \hookrightarrow [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is the embedding of a category $\mathbf{C}$ into its category of presheaves via its representable presheaves, so that

$$よ(X) = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{46.25}$$

Theorem 46.3.3. The Yoneda embedding is fully faithful.

Definition 46.3.3. The *Yoneda extension* of a functor $F : \mathbf{C} \to \mathbf{D}$ is the left Kan extension by the Yoneda embedding,

$$\tilde{F} = \mathrm{Lan}_{よ}(F) : [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{D} \tag{46.26}$$

Theorem 46.3.4. Given a functor $F : \mathbf{C} \to \mathbf{D}$, if the induced functor Lan_F : $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]$ preserves limits and all those limits are representable, then F

Does the Yoneda embedding preserve monomorphisms,

Theorem 46.3.5. The Yoneda embedding preserves monomorphisms and epimorphisms.

Proof. Given a morphism $f : X \to Y$, the Yoneda embedding applied to it gives us

$$よ(f) : よ(X) \to よ(Y) \tag{46.27}$$

□

46.4 Simplices

[58]

A basic example of presheaves is given by simplicial sets, which are presheaves over the simplex category $\mathbf{\Delta}$:

$$X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set} \tag{46.28}$$

By the Yoneda embedding [representable presheaves etc], any object in the simplex category is a simplicial set

Example 46.4.1. To start with we can look at a few representable presheaves. If we look at the zero case, of a single element

$$I_0 = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{0}) \tag{46.29}$$

As a terminal object in the simplex category, there is only a single morphism for every object $X \in \mathbf{\Delta}$,

$$\mathrm{Hom}_{\mathbf{\Delta}}(X, \mathbf{0}) \cong \{\bullet\} \tag{46.30}$$

and every k-simplex morphism gets mapped to the identity

For the 1-simplex,

$$I_1 = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{1}) \tag{46.31}$$

$I_1(0)$: two maps injecting the point to either object $I_1(1)$: A single identity functor

For the 2-simplex :

$$I_2 = \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{2}) \tag{46.32}$$

$I_2(0)$: three maps $I_2(1)$: Two maps

Furthermore, we can consider simplexes which are constructed from the combination of different simplexes

Example 46.4.2. Take the basic simplex of the triangle. Let's call this sheaf Δ_3. It is composed of a 2-simplex, three 1-simplices, and three 0-simplices. We should therefore expect those elements (at least for the non-degenerate ones). Take the three simplexes $\vec{\mathbf{2}}$, $\vec{\mathbf{1}}$ and $\vec{\mathbf{0}}$. We want to identify the edges of each to form a triangle.

$$S_3(\vec{\mathbf{0}}) = \varnothing \tag{46.33}$$

$$S_3(\vec{\mathbf{1}}) = \{\bullet\} \tag{46.34}$$

$$S_3(\vec{\mathbf{2}}) = \{\bullet\} \tag{46.35}$$

and this maps the morphism (functor) which maps each extremity of 1 and 2 together :

$$S_3(f : \vec{\mathbf{1}} \to \vec{\mathbf{2}}) = f'(S_3(\vec{\mathbf{1}}) \to S_3(\vec{\mathbf{2}})) \tag{46.36}$$

[...]

As an equalizer? $1 + 2/f$

47 Site

A site is roughly speaking the elements from which a (Grothendieck) topos is stitched together. This is done by considering some category of those elements, as well as some spatial structure on it, so as to be able to define those constructions properly using descent.

Definition 47.0.1. A site (C, J) is a category C equipped with a coverage J.

Sieve

Example 47.0.1. The terminal category 1 is a site. The covering family is simply the only function, $\{\mathrm{Id}_* : * \to *\}$. As there are no other objects in the category, we only need to check the induced coverage on itself. Given the morphism $\mathrm{Id}_* : * \to *$, the diagram commutes trivially by using the identity function everywhere.

$$\begin{array}{ccc} * & \xrightarrow{\mathrm{Id}_*} & * \\ {\scriptstyle \mathrm{Id}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_*} \\ * & \xrightarrow{\mathrm{Id}_*} & * \end{array}$$

Example 47.0.2. The category of opens of a topological space is a site

Example 47.0.3. The category of Cartesian spaces $\mathbf{CartSp}_{\text{smooth}}$ [with coverage?] is a site.

"Every frame is canonically a site, where U is covered by $\{U_i\}$ precisely if it is their join."

Is there some kind of relationship between the sheaves of a Grothendieck topos, and the elements of the site taken as (representable) sheaves + coproduct and equalizer

47.1 Site morphisms

Definition 47.1.1. Given two sites $(\mathbf{C}, J)$ and $(\mathbf{D}, K)$, a functor $F : \mathbf{C} \to \mathbf{D}$ is a morphism of sites if it is covering-flat and preserves covering families : for every covering $\{p_i : U_i \to U\}$ of $U \in C$, $\{f(p_i) : f(U_i) \to f(U)\}$ is a covering of $f(U) \in D$.

Covering-flat :

For a set-valued functor $F : C \to \mathbf{Set}$,

Filtered category : A filtered category is a category in which every diagram has a cocone.

For any finite category D and functor $F : D \to C$, there exists an object $X \in C$ and a nat. trans. $F \to \Delta_X$.

Simpler version :

- There exists an object of C (non-empty category)
- For any two objects $X, Y \in C$, there is an object Z and morphisms $X \to Z$, $Y \to Z$
- For any two parallel morphism, $f, g : X \to Y$, there exists a morphism $h : Y \to Z$ such that $hf = hg$.

Every category with a terminal object is filtered.

Every category which has finite colimits is filtered.

Interpretation : for any limit that the site has, they are preserved.

48

Descent

To deal with the notion of structures on spaces, we need to take into account the notion that spaces can be in some sense composed by its regions. If we wish to define a structure on a space, we need to figure out

- Is this structure also reflected on its constitutive regions
- Can we construct this structure by patching together the ones from its regions

The former is the type of notion that we saw with the presheaf's restriction map, and the latter will be that given by *descent*. This is the process by which we will consider our various spaces as a collection of subspaces and how the various structures on them are meant to work together.

First let's see how a space can come together as the union of its regions. For some topological space (X, τ), we can reconstruct X by the quotient of the coproduct of its open sets, ie

$$X \cong \coprod_i U_i / \cong \tag{48.1}$$

where we define the equivalence relation to be that if two open sets have a non-empty intersection, then the points in the intersection are identified. If we have for instance two open sets U_1, U_2 and their intersection $U_{12} = U_1 \cap U_2$,

$$\forall x, y \in U_1 \sqcup U_2,\ x \cong y \leftrightarrow x = y \vee x \in \iota(U_1) \wedge y \in \iota(U_2) \wedge \exists U_{12} \tag{48.2}$$

This notion can be used to derive a variety of global structures from local ones

Before this we need to talk about the notion of fibered categories, and first of Cartesian morphisms.

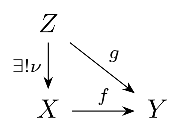
Definition 48.0.1. Given a functor $P : \mathbf{C} \to \mathbf{D}$, we say that a morphism $f : X \to Y$ in $\mathbf{C}$ is *P-Cartesian* if for every $Z \in \mathbf{C}$ and every morphism $g : Z \to Y \in \mathbf{C}$, and for every morphism $\alpha : P(Z) \to P(X)$ in $\mathbf{D}$ such that $P(g) = P(f) \circ \alpha$, there is a unique morphism $\nu : Z \to X$ in $\mathbf{C}$ such that

$$g = f \circ \nu \tag{48.3}$$

and

$$\alpha = P(\nu) \tag{48.4}$$

ie we have the following commuting diagram



Definition 48.0.2. For a functor $P : \mathbf{C} \to \mathbf{D}$, this is a *fibration* if for all $X \in \mathbf{C}$ and $f_0 : Y_0 \to P(X) \in \mathbf{D}$, there exists a Cartesian morphism $f : Y \to X \in \mathbf{C}$ such that $P(f) = f_0$.

Example 48.0.1. Category of arrows

49

Sheaves

The more important construction based on (co)presheaves is that of (co)sheaves, which are (co)presheaves with some additional conditions, meant to signify the spatial nature of the construction : the category corresponds in some sense to the piecing together of regions.

49.1 Sheaves on topological spaces

As for presheaves, the pedagogical model of sheaves is the one for structure sheaves on topological spaces, where we consider a functor from the frame of opens $\mathrm{Op}(X)$ to some ring, Abelian group or algebra of functions

$$F : \mathrm{Op}(X) \to \mathbf{CRing} \tag{49.1}$$

$$U \to F(U) = C(U) \tag{49.2}$$

Where typical cases are the rings of continuous, smooth or analytic real functions $U \to \mathbb{R}$.

In addition to the usual properties of a presheaf, simply stemming from its functoriality, we also have the additional properties

Definition 49.1.1. A sheaf on a topological space is a presheaf obeying the following properties :

- *Locality* : Given some open set $U \in \mathrm{Op}(X)$ and some open cover $\{U_i \hookrightarrow U\}_{i\in I}$, and two sections $s, t \in F(U)$. Then if we have $\mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t)$ for every i of its cover, $s = t$.
- Given some open set $U \in \mathrm{Op}(X)$ and some open cover $\{U_i \hookrightarrow U\}_i$, and a family of sections $\{s_i \in F(U_i)\}_{i\in I}$. If those sections agree pairwise on their overlaps, ie

$$\mathrm{res}_{U_i,U_i\cap U_j}(s_i) = \mathrm{res}_{U_j,U_i\cap U_j}(s_j) \tag{49.3}$$

then there exists a section $s \in F(U)$ which restricts to s_i on each U_i.

Counterexample 49.1.1. The presheaf of bounded functions does not form a sheaf.

Proof. While the presheaf of bounded function obeys locality [proof?], it is possible to construct a family of sections that does not have a global section. For instance if we pick $\{(x-1, x+1) \mid x \in \mathbb{R}\}$, which forms a cover of $\mathbb{R}$, we can form sections of bounded functions for each open set, for instance $f(x) = x$, with bound $(x-1, x+1)$, but its global section on $\mathbb{R}$ would be the identity function on $\mathbb{R}$, which is not bounded. □

Counterexample 49.1.2. The constant presheaf does not generally form a sheaf.

Proof. Given the topological space of two points with the discrete topology, Disc(2), with open sets

$$\varnothing, \{\bullet_1\}, \{\bullet_2\}, \{\bullet_1, \bullet_2\} \tag{49.4}$$

□

49.2 General sheaves

The more general notion of a sheaf is given by sieves [Grothendieck topology]

In the topological case,

[...]

Consider the Yoneda embedding of $\mathbf{C}$:

$$j : \mathbf{C} \hookrightarrow \mathrm{Psh}(\mathbf{C}) \tag{49.5}$$

Definition 49.2.1. Given a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, and given a coverage J of $\mathbf{C}$, F is a sheaf with respect to J if

- for every covering family $\{p_i : U_i \to U\}_{i\in I}$ in J

- for every compatible family of elements $(s_i \in F(U_i))_{i \in I}$,

there is a unique element $s \in F(U)$ such that $F(p_i)(s) = s_i$ for all $i \in I$.

If we consider the case where a covering family is composed of monomorphisms with subobjects (assuming the equivalence something something), then $p_i : U_i \hookrightarrow U$ can be considered [something], and the morphism generated by the sheaf is understood to be a *restriction* : $F(p_i)(s) = s_i$, we are restricting the section s on U to the subobject on U_i.

[The section of a sheaf is defined by its local elements]

The easiest example of this is to pick once again a presheaf on the frame of open $F \in \mathrm{Psh}(\mathrm{Op}(X))$, and as the coverage, pick the subcanonical coverage. For any open set $U \subseteq X$, a subcanonical coverage is a family of open sets $\{U_i\}$ such that

$$\bigcup_i U_i = U \tag{49.6}$$

[...]

A nice class of such sheaves are the sheaves given by function spaces over the appropriate sets, *sheaves of functions*. The prototypical example would be the sheaf of real valued continuous functions on a topological space, $C(-)$. For a given topological space X, take its frame of opens $\mathrm{Op}(X)$, and the sheaf associates to any open set $U \in \mathrm{Op}(X)$ the set of those functions $C(U)$.

As a presheaf, we have the notion that this assignment is a contravariant functor. In $\mathrm{Op}(X)$, any morphism $U \to X$ is an inclusion relation, $U \to X$ means that there is some inclusion map $\iota : U \hookrightarrow X$. The contravariant functorial character is then that, for the inclusion $U \hookrightarrow X$, with opposite category morphism $X \to U$, we have

$$C(X \to U) = C(X) \to C(U) \tag{49.7}$$

The corresponding operation is that of the restriction of a function. Given a function $f \in C(X)$, there is a corresponding function $C(\iota_U)(f)$, which is the restriction :

$$C(\iota_U)(f) = f\big|_U : U \to \mathbb{R} \tag{49.8}$$

Where the functorial rules furthermore say that $f\big|_X = f$ (since $F(\mathrm{Id}) = \mathrm{Id}$) and for $V \subset U \subset X$, $f\big|_U\big|_V = f\big|_V$.

The sheaf properties are usually expressed as a notion of *locality*,

is then that given a covering family $\{\iota_i : U_i \hookrightarrow X\}$

that for some function on X, $f \in C(U)$,

Example 49.2.1. For an S-valued sheaf on (C, J), the *constant sheaf*

Examples with sheaves on frames of opens

For $\mathrm{Op}(X)$ with the canonical coverage, a presheaf F is a sheaf if for every complete subcategory $\mathcal{U} \hookrightarrow \mathrm{Op}(X)$,

$$F(\varinjlim \mathcal{U}) \cong \varprojlim F(\mathcal{U}) \tag{49.9}$$

Proof. Complete full subcategory is a collection $\{\iota_i : U_i \hookrightarrow X\}$ closed under intersection. The colimit

$$\varinjlim(\mathcal{U} \hookrightarrow \mathrm{Op}(X)) \cong \bigcup_i U_i \tag{49.10}$$

is the union of these open subsets. By construction,

□

Example 49.2.2. The presheaf mapping all objects to the empty set $\Delta_\varnothing$ is a sheaf, called the *empty sheaf* (or *initial sheaf* as it is the initial object in the category of sheaves). As every covering family is mapped to the compatible family of elements of the empty set,

Example 49.2.3. The presheaf mapping all objects to the singleton $\{\bullet\}$ is a sheaf, called the *terminal sheaf.*

Definition 49.2.2. A *sheaf morphism* $\phi : F \to G$ for two sheaves on a site $\mathbf{C}$ is a natural transformation on the functors F, G, in particular preserving restriction maps via the commutative diagram

$$\begin{array}{ccc} F(V) & \xrightarrow{\rho_{V,U}} & F(U) \\ \downarrow{\scriptstyle \phi_V} & & \downarrow{\scriptstyle \phi_U} \\ G(V) & \xrightarrow{\rho'_{V,U}} & G(U) \end{array}$$

As **Set** has a partial order relation on it by the subset relation, we similarly have

Definition 49.2.3. A sheaf $S \in [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is a *subsheaf* of F if for all objects $U \in \mathbf{C}$, we have

$$S(U) \subseteq F(U) \tag{49.11}$$

Theorem 49.2.1. The inclusion functor of the category of sheaves to the category of presheaves has a left adjoint, called the *sheafification functor*.

$$(L \dashv \iota) : \mathrm{Sh}(\mathbf{C}) \underset{\iota}{\overset{L}{\rightleftarrows}} \mathrm{PSh}(\mathbf{C})$$

Definition 49.2.4. J-sheaves

Definition 49.2.5. The coverage of a sheaf is *subcanonical* if every representable presheaf is also a representable sheaf.

Trivial topology : "Every presheaf is a sheaf for this coverage (and in particular, it is subcanonical). The corresponding Grothendieck coverage consists of all sieves that contain a split epimorphism."

"On any regular category there is a coverage, called the regular coverage, whose covering families are the singletons $\{f : V \to U\}$ where f is a regular epimorphism. It is subcanonical."

"On any coherent category there is a a coverage, called the coherent coverage, whose covering families are the finite families $\{f_i : U_i \to U\}_{1 \leq i \leq n}$ the union of whose images is all of U. It is subcanonical. Likewise there is a geometric coverage on any infinitary-coherent category."

"On any extensive category there is a coverage, called the extensive coverage, whose covering families are the inclusions into a (finite) coproduct. It is subcanonical. The coherent coverage on an extensive coherent category is generated by the union of the regular coverage and the extensive one."

"Any category has a canonical coverage, defined to be the largest subcanonical one. (Hence the name "subcanonical" = "contained in the canonical coverage.") The covering sieves for the canonical coverage are precisely those which are universally effective-epimorphic, meaning that their target is their colimit and this colimit is preserved by pullback"

"The canonical coverage on a Grothendieck topos coincides with its geometric coverage, and moreover every sheaf for this coverage is representable. That is, a Grothendieck topos is a (large) site which is equivalent to its own category of sheaves."

49.3 Subsheaves

Similarly to subpresheaves, we have that a sheaf S will be the subsheaf of another sheaf X

Theorem 49.3.1. Subsheaves are subobjects in the category of sheaves on that site.

Proof. □

Subobject category for sheaves

Theorem 49.3.2. The empty sheaf is the bottom element of the poset of subsheaves.

Proof. As the empty sheaf simply maps every site object to the empty set, this simply stems from $\varnothing \subseteq A$ for any set A. ☐

49.4 Concrete sheaves

While we have seen that it is possible to define generalized spaces as sheaves, the notion of space involved may be too general in some cases, in that the "topology" of the space may not be spatial : the poset of subobjects may not be such that it could be define by relations on sets of points as we do in topological spaces.

Example 49.4.1. From the non-spatial frame that we've defined 45.0.1, we could define a sheaf over it. Take our frame

$$\mathbf{F} = \{1 \to A \to 0\} \tag{49.12}$$

with the natural frame site structure, ie for any $U \in \mathbf{F}$, we consider the families of morphism $\{U_i \to U_i\}$ given by joins :

$$\bigvee_i U_i = U \tag{49.13}$$

For this frame, there are no such coverages outside of the trivial one, $\{U \to U\}$.

define the sheaf $\mathrm{Sh}(\mathbf{F})$

A simple enough presheaf to consider is just the representable presheaf of one of those element,

$$X_A = \mathrm{Hom}_{\mathbf{F}}(-, A) \tag{49.14}$$

which has the following probes :

$$X_A(0) = \{\leq_{0,A}\} \tag{49.15}$$
$$X_A(A) = \{\leq_{A,A}\} \tag{49.16}$$
$$X_A(1) = \varnothing \tag{49.17}$$

To try to recover our more usual notion of spaces as collections of points, we need to look into the notion of concrete sheaves.

The intuition behind a concrete sheaf is that its spatial behavior is entirely determined by set of points, similarly to a topological space. That is, to a concrete sheaf X representing a space, we associate some set of points $|X|$, and to all subsheaves $S \subseteq X$ their own set of points, in a way that the interaction of those subsheaves behaves in a way consistent with them being sets of points.

First, we need to define the notion of a concrete site, so that the elements from which the sheaf is constructed itself has points :

Definition 49.4.1. A *concrete site* is a site with a terminal object 1 for which

- The hom-functor $h^1 = \mathrm{Hom}_{\mathbf{C}}(1, -)$ is faithful
- For a covering family $\{f_i : U_i \to U\}$,

$$\coprod_i \mathrm{Hom}_{\mathbf{C}}(1, f_i) : \coprod_i \mathrm{Hom}_{\mathbf{C}}(1, U_i) \to \mathrm{Hom}_{\mathbf{C}}(1, U)$$

  is surjective.

For a concrete site $\mathbf{C}$, there is an *underlying set* to any object, which is given by the h^1 functor. For brevity, it will be denoted by

$$\forall U \in \mathbf{C},\ |U| \cong \mathrm{Hom}_{\mathbf{C}}(1, U)$$

This definition means that if we look at U's points, they can be entirely covered by the points of its covering family, ie for any point $x \in |U|$, there exists corresponding points in $\sqcup|U_i|$ which will map onto it. In this sense, the covering family actually "covers" U pointwise, as opposed to say the case of an object with some number of point whose covering family contains no points.

[example of the pointless frame?]

To have a concrete presheaf, in addition to being constructed from concrete sites, we will also need those concrete patches to mesh together in a way that behaves properly for a set.

Definition 49.4.2. A presheaf $X \in \mathrm{PSh}(\mathbf{C}, J)$ is a *concrete presheaf* if for any $U \in \mathbf{C}$, the function

$$\tilde{X}_U : X(U) \to \mathrm{Hom}_{\mathbf{Set}}(|U|, X(1)) \tag{49.18}$$

is injective.

Just as for the site, a concrete sheaf also has an underlying set, given by its probes from the terminal object, $X(1)$. We can also write

$$|X| \cong X(1) \tag{49.19}$$

This map is meant to parallel the case of the Yoneda lemma, where for this same presheaf, we also have

$$X(U) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathbf{C})}(よ(U), X) \tag{49.20}$$

So that we can in fact rewrite the condition as

$$\mathrm{Hom}_{\mathrm{PSh}(\mathbf{C})}(よ(U), X) \to \mathrm{Hom}_{\mathbf{Set}}(|U|, X(1)) \tag{49.21}$$

In the interpretation of the Yoneda lemma as the plots of a space, this means that every plot of the sheaf gives rise to a different function on its underlying sets, ie there is no case where a plot acts the same on points but differ in a purely "spatial" manner.

"A concrete presheaf is a subobject of the presheaf

$$U \mapsto \mathrm{Hom}_{\mathbf{Set}}(|U|, |X|)$$

"

"First, for $U \in C$, write $|U|$ for the underlying set $\mathrm{Hom}_C(*, U)$, and note that we can regard $\mathrm{Hom}_C(U, V) \subseteq \mathrm{Hom}_{Set}(|U|, |V|)$ since $\mathrm{Hom}_C(*, -)$ is faithful.

Then a concrete presheaf X is given by a set $|X|$ together with, for each $U \in C$ a $|U|$-ary relation $X(U) \subseteq |X|^{|U|}$, such that for any $f : U \to V$ in C, $g \in X(V)$ implies $gf \in X(U)$, and such that X(*)=—X—.

This data defines a concrete presheaf $X : C^{op} \to Set$, and every concrete presheaf is isomorphic to one of this form.

To give a natural transformation between concrete presheaves $X \to Y$ is to give a function $|X| \to |Y|$ that preserves the relations."

The hom-set $\mathrm{Hom}_{\mathbf{Set}}(|U|, X(1))$ is meant to represent the probes of the sheaf in terms of points.

Example : take a sheaf of two points, $\{\bullet_1, \bullet_2\}$, with the concrete site $\mathbf{1}$. There is exactly two such maps, Δ_1, Δ_2.

Is there a map from concrete sheaves to topological spaces?

Theorem 49.4.1. There is a functor from a concrete site $\mathbf{S}$ to the category of topological spaces $\mathbf{Top}$.

$$\mathrm{TopReal} : \mathbf{S} \to \mathbf{Top} \tag{49.22}$$

Proof. Given some object $U \in \mathbf{S}$, we take its set of underlying points as the points of the topological space

$$|U| = \mathrm{Hom}_{\mathbf{S}}(1, U) \tag{49.23}$$

and we define on it the final topology making

□

Theorem 49.4.2. Given the subcategory of concrete (pre?)sheaves in a (pre?)sheaf category, there exists a functor to the category of topological spaces. (Topological realization?)

Proof. Given a concrete sheaf X, we can consider its set of points to be $|X|$. The topology on this underlying set is then □

50
Topos

[59, 60, 61, 62, 63]

One important type of categories that will be the main focus of study here is that of a *topos* (plural *topoi*). There are many possible definitions and intuitions of what a topos is, many of them listed in [57], but for our purpose, a topos will mostly be about

- A universe of types in which to do mathematics
- A category of spaces
- A categorification of some types of logics

There are a few different nuances to what a topos can be, but the most general case we will look at for now (disregarding things such as higher topoi) is that of an elementary topos.

Definition 50.0.1. An *elementary topos* $\mathbf{H}$ is a category which has all finite limits, is Cartesian closed, and has a subobject classifier.

This definition of an elementary topos fits best in the first sense of the definitions, in that it is a universe in which to do mathematics. These properties are overall modeled over **Set**, and in some sense it is the generalization of a set. As we will see in details in 69, **Set** itself is a topos.

Its use as a mathematical universe is given by its closure under limits (and as we will show, colimits) and exponentiation. We can easily talk about any (finite) construction of objects in a topos, as well as any function between two elements of a topos, without leaving the topos itself, and the subobject classifier will give us some notion of subobjects in the category.

Theorem 50.0.1. An elementary topos has all finite colimits.

Proof. Contravariant power set functor :

$$\Omega^{(-)} : \mathbf{H}^{\mathrm{op}} \to \mathbf{H} \tag{50.1}$$

□

Properties : locally Cartesian closed, finitely cocomplete, Heyting category

Giraud-Rezk-Lurie axioms

Theorem 50.0.2. There are no finite topoi except for the initial topos.

Proof. If a topos is not the initial topos $\mathrm{Sh}(\mathbf{0})$, □

Theorem 50.0.3. For any topos $\mathbf{H}$, the slice category given by one of its object $\mathbf{H}_{/X}$ is itself a topos.

Proof. For two objects in the slice category, $f : A \to X$, $g : B \to X$, and two morphisms between those objects, $\alpha, \beta : f \to g$, □

Theorem 50.0.4. Any epimorphism in a topos is effective.

Proof. Given an epimorphism $f : X \to Y$, □

[64, 65]

50.1 Lattice of subobjects

From the existence of all limits and colimits, we can deduce that the poset of subobjects has itself all the limits and colimits deriving from those of the underlying category, since any (co)limit in $\mathrm{Sub}(X)$ is the (co)limit in $\mathbf{C}$ with a diagram of two extra arrows to X.

Theorem 50.1.1. In a topos, the subobjects $\mathrm{Hom}_{\mathbf{H}}(X, \Omega)$ of an object X form a Heyting algebra.

Proof. From its completeness and cocompleteness, the poset of subobject is equipped with meet and join, making it a lattice, and a strict initial object, making it a bounded lattice.

internal hom? □

Is Sub(X) also a closed cartesian category?

Heyting algebra of subobject : given two subobjects $A, B \in \mathbf{H}$

$$A \wedge \tag{50.2}$$

In addition to this, as the topos is guaranteed a subobject classifier, the subobject presheaf of X is isomorphic to the hom-set

$$\mathrm{Sub}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \Omega) \tag{50.3}$$

We have therefore that every (co)limit in $\mathbf{C}$ corresponds to [some algebra on Ω].

Theorem 50.1.2. Every object X of an elementary topos has a power object PX, given by

$$PX = \Omega^X \tag{50.4}$$

Proof. □

Inclusion map : $X \hookrightarrow PX$

Theorem 50.1.3. Pullback of a function to a power set :

$$\begin{array}{ccc} U & \xrightarrow{g} & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \{\}} \\ X & \xhookrightarrow{h} & PY \end{array}$$

[57]

Theorem 50.1.4.

Theorem 50.1.5. An elementary topos is finitary extensive.

Proof. To show this, let's show that coproducts are disjoint. Consider the pushout square given by the inclusion maps of the initial object, $X +_0 Y \cong X + Y$

$$\begin{array}{ccc} 0 & \xrightarrow{0_Y} & Y \\ {\scriptstyle 0_X}\big\downarrow & & \big\downarrow{\scriptstyle \iota_Y} \\ X & \xhookrightarrow{\iota_X} & X + Y \end{array}$$

By the previous theorem, this is also a pushout square of $X \hookrightarrow X + Y \hookleftarrow Y$, which is the intersection of X and Y, so that the coproduct is disjoint. As all colimits are stable under pullback in a topos, this means that the category is finitary extensive. □

Theorem 50.1.6. Any elementary topos is a distributive category, ie

$$X \times Y + X \times Z \cong X \times (Y + Z) \tag{50.5}$$

topos is always a distributive category

As a Heyting algebra, any algebra of subobjects of a topos admits the double negation property

$$U \to \neg_X \neg_X U \tag{50.6}$$

$$\neg\neg = [[-, \varnothing], \varnothing] \tag{50.7}$$

50.2 Grothendieck topos

The most common type of topos, and the one we will typically use, is the Grothendieck topos, based around the category of sheaves on a given site.

As most of the proofs related to this will be easier going first through presheaves as a topos, let's first look at the presheaf topos.

Definition 50.2.1. A *presheaf topos* on a category $\mathbf{C}$ is the functor category of all presheaves on $\mathbf{C}$,

$$\mathrm{PSh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$$

Theorem 50.2.1. A presheaf topos is an elementary topos.

Proof. By the property of limits on presheaves 46.0.2, $\mathrm{Psh}(\mathbf{C})$ has all limits. The □

Definition 50.2.2. A *Grothendieck topos* E on a site $\mathbf{C}$ with coverage J is the category of sheaves over the site $(\mathbf{C}, \mathcal{J})$

$$\mathcal{E} \cong \mathrm{Sh}(\mathbf{C}, \mathcal{J}) \tag{50.8}$$

Proposition 50.2.1. A Grothendieck topos is an elementary topos

Proof. □

Example 50.2.1. A trivial example of a Grothendieck topos is the initial topos, which is the sheaf topos over the empty category with the empty topology (which is the maximal topology on this category), $\mathrm{Sh}(\mathbf{0})$. The only element of this topos is the empty functor, with the identity natural transformation on it (as there is no possible components to differentiate them on the empty category, this is the only one). We therefore have

$$\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1} \tag{50.9}$$

Its only object $*$ is both the initial and terminal object (therefore a zero object), its product and coproduct are simply $* + * \cong *$ and $* \times * \cong *$

One important nuance in topos theory is that a topos can be considered alternatively as a space in itself, or as a category in which every object is a space. The former is referred to as a *petit topos*, while the latter is a *gros topos*. A typical example of this would be for instance the topos of smooth spaces **Smooth**, which contains (among other things) all manifolds, as a gros topos, while the topos of the site of opens on a topological space $\mathrm{Sh}(\mathrm{Op}(X))$ would be an example of a petit topos.

50.0.3

This construction allows us to bridge the gros and petit topos, in that given a space X in a gros topos $\mathbf{H}$, its corresponding petit topos will be $\mathbf{H}_{/X}$.

"Also in 1973 Grothendieck says the objects in any topos should be seen as espaces etales over the terminal object of the topos, in a generalized sense that includes saying any orbit of a group action lies "etale" over a fixed point. "

As a topos, a Grothendieck topos has a subobject classifier Ω, which, as an object of a sheaf category, will be itself a sheaf. From the properties of a sheaf, we can also look at its various properties.

Any sheaf topos has an object with the property of a subobject classifier, given by the sheaf of principal sieves. That is, for any object U of the site,

$$\Omega(U) = \{S | S \text{ is a sieve on } U\} \tag{50.10}$$

$$\Omega(f) : \Omega(U) \to \Omega(V),\ \Omega(g)(S) = S\big|_g = \{h | g \circ h \in S\} \tag{50.11}$$

To show the subobject classifier, we first need to show the existence of a sheaf morphism from the terminal sheaf to this sheaf, $\top : 1 \to \Omega$. This is simply given by the components

$$\begin{aligned} \top_X : 1(X) &\to \Omega(X) && (50.12)\\ * &\mapsto t(X) && (50.13) \end{aligned}$$

Example 50.2.2. In the sheaf topos $\mathbf{Set} \cong \mathrm{Sh}(\mathbf{1})$

Sheaf topos is Cartesian closed, internal hom from this monoidal structure

Theorem 50.2.2. In a Grothendieck topos, all coproducts are disjoint, in the sense that for any two subobjects of the same object, $\iota_1, \iota_2 : X_1, X_2 \hookrightarrow Y$, their intersection (pullback) is empty (the initial object) if their union (pushout) is isomorphic to the coproduct. For the diagram $F = X_1 \hookrightarrow Y \hookleftarrow X_2$,

$$\lim_I F \cong 0 \to \mathrm{colim}_I(F) \cong X_1 + X_2 \tag{50.14}$$

Example 50.2.3. Given the site of the interval category $\tilde{\mathbf{2}} = \{0 \to 1\}$, the corresponding presheaf topos is the *Sierpinski topos*,

$$\mathrm{PSh}(1 \to 2) \tag{50.15}$$

Theorem 50.2.3. In a Grothendieck topos, the internal hom $[X, Y]$ is the sheaf

$$[X, Y] : U \mapsto \mathrm{Hom}(よ(U) \times X, Y) \tag{50.16}$$

Proof. □

50.3 Lawvere-Tierney topology

For a topos that is not necessarily a Grothendieck topos, an equivalent notion to the Grothendieck topology is that of the Lawvere-Tierney topology [66].

Definition 50.3.1. On a topos $\mathbf{H}$, a *Lawvere-Tierney topology* is given by a morphism j on the subobject classifier

$$j : \Omega \to \Omega \tag{50.17}$$

which obeys the following properties :

- $j \circ \top = \top$
- $j \circ j = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

As a morphism on Ω corresponds to a subobject of it, by

$$\mathrm{Sub}(\Omega) \cong \mathrm{Hom}_{\mathbf{H}}(\Omega, \Omega) \tag{50.18}$$

the Lawvere-Tierney topology can alternatively be defined as a subobject of Ω via its pullback along $\top$,

$$J = \Omega \times_{\Omega} 1 \hookrightarrow \Omega \tag{50.19}$$

This morphism defines a special class of subobjects, as we can define for any subobject $\chi_U : X \to \Omega$ a new subobject $j \circ \chi_U$, and the first property means that the subobject associated to the pullback of $j \circ \chi_U$ is

Definition 50.3.2. For a subobject U of X, the *closure operator* induced by j is a morphism

$$\overline{(-)}_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X) \tag{50.20}$$
$$U \mapsto \overline{U}_X \tag{50.21}$$

which obeys

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(X,\Omega) & \xrightarrow{\cong} & \mathrm{Sub}(X) \\ {\scriptstyle \mathrm{Hom}_{\mathbf{C}}(1,j)}\downarrow & & \downarrow{\scriptstyle \overline{(-)}_X} \\ \mathrm{Hom}_{\mathbf{C}}(X,\Omega) & \xrightarrow{\cong} & \mathrm{Sub}(X) \end{array}$$

Example 50.3.1. The basic example of a Lawvere-Tierney topology is in the topos of sets, where it is simply the identity map, obeying trivially all the properties of that operator. The associated subobject for this is the truth subobject $\top : 1 \hookrightarrow \Omega$.

The closure of any subobject is

Analog of Grothendieck topology for a topos? [66]

The term of a closure operator derives from its interpretation in the context of a sheaf on a topological space. If given a topological space (X, τ) we pick our sheaf on the site of its category of opens, the category $\mathrm{Op}(X)$ with the canonical coverage, and given a collection $C = \{U_i\}_{i\in I}$, the locality operator j maps C to the open sets covered by C.

$$j(C) = \{U \in \mathrm{Op}(X) | U \subseteq \bigcup_{i\in I} U_i\} \tag{50.22}$$

Properties :

- If $U \in C$, $U \in j(C)$
- $j(\mathrm{Op}(X)) = \mathrm{Op}(X)$
- $j(j(C)) = j(C)$
- $j(C_1 \cap C_2) \subseteq j(C_1) \cap j(C_2)$
- If C_1, C_2 are sieves, $j(C_1 \cap C_2) = j(C_1) \cap j(C_2)$
- If $C = S(U)$, $j(C)$ is also a sieve.

If C is a sieve, it is an element of $\Omega(U)$, the subobject classifier on the topos of presehaves on X.

Generalization : j is a map $j : \Omega \to \Omega$ on the subobject classifier of a topos, the *Lawvere-Tierney topology*, with properties

- $j \circ \top = \top$
- $j \circ j = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

j is a modal operator on the truth values Ω.

Example 50.3.2. In the Grothendieck topos $E = \mathrm{Set}^{\mathrm{Op}(X)^{\mathrm{op}}}$ of presheaves on X a topological space, Classifier object $U \mapsto \Omega(U)$, the set of all sieves S on U, a set of open subsets $V \subseteq U$ such that $W \subseteq V \in S$ implies $W \in S$.

Each open subset $V \subseteq U$ determines a principal sieve $\hat{V}$ consisting of all opens $W \subseteq V$

The map $\top_U : 1 \to \Omega(U)$ is the map that picks the maximal sieve $\hat{U}$ on U.

$$J(U) = \{S|\ S \text{ is a sieve on } U \text{ and } S \text{ covers } U\ \} \tag{50.23}$$

S covers U means

Adjoint of closure : interior?

50.4 Open subobjects

Earlier on, we saw that open sets in topology could be defined in terms of their continuous functions via the Sierpinski space 41.2.2. This type of definition can be easily extended to the categorical case, where the equivalent of the Sierpinski space is what are called *Sierpinski objects.*

While **Top** does not have a subobject classifier, it does possess a strong subobject classifier, which is the Sierpinski space

Theorem 50.4.1. The Sierpinski space S is a strong subobject classifier.

Proof. □

Given any object X in a topos, with the terminal morphism

$$!_X : X \to 1 \tag{50.24}$$

we say that X is *subterminal* if $!_X$ is a monomorphism. As the terminal object is the terminal sheaf mapping every object of the site to $\{\bullet\}$, this means that a subterminal object is a sheaf mapping every object to a subsingleton set, ie either empty or a singleton.

For $g_1, g_2 : Y \to X$,

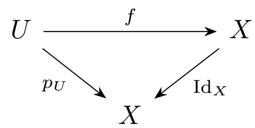
$$!_X \circ g_1 =!_X \circ g_2 \to g_1 = g_2 \tag{50.25}$$

Open sets : comodal objects of the interior comonad?

Gros topos / petit topos thing

Terminal object in $\mathbf{H}_X$: $\mathrm{Id}_X : X \to X$

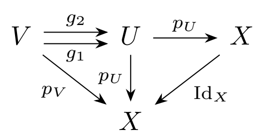
Map $!_U : p_U \to 1$, $p_U : U \to X$, a morphism $f : U \to X$ such that $\mathrm{Id}_X \circ f = p_U$, ie $f = p_U$



ie any terminal map $p_U \to 1$ is the map corresponding to $p_U : U \to X$

Monomorphism : for another object V, defining a bundle $p_V : V \to X$, with two morphisms $\overline{g}_1, \overline{g}_2 : p_V \to p_U$, corresponding to two morphisms $g_1, g_2 : V \to U$ where $p_V \circ g_i = p_U$. This is a monomorphism if

$$!_U \circ \overline{g}_1 =!_U \circ \overline{g}_2 \to \overline{g}_1 = \overline{g}_2 \tag{50.26}$$



or in terms of the original category [?]

$$p_U \circ g_1 = p_U \circ g_2 \to g_1 = g_2 \tag{50.27}$$

if p_V, p_U are monomorphisms,

Example 50.4.1. In a Grothendieck topos, every bundle from an object of the site is subterminal.

Proof. Given a topos $\mathbf{H} = \mathrm{Sh}(\mathbf{C})$, for some object of the site $U \in \mathbf{C}$, we can embed it in $\mathbf{H}$ via the Yoneda embedding よ(U). For any other object of the site U', we have

$$\text{よ}(U)(U') = \mathrm{Hom}_C(U', U) \tag{50.28}$$

If we have some morphism f : よ$(U) \to X$ for some object X, in the slice category $\mathbf{H}_{/X}$,

$$Z \rightrightarrows \mathrm{Hom}(-, U) \to X \tag{50.29}$$

□

Open subobjects : modal types of j?

Is there an equivalent of the compact open topology for internal hom objects?

50.5 Factorization system

In any topos, we have an orthogonal factorization system such that for any morphism $f : X \to Y$, this can be factorized uniquely as

$$f = e \circ m \tag{50.30}$$

For $e \in E$, the class of all epimorphisms, and $m \in M$, the class of all monomorphisms. This is the *(epi, mono)-factorization system.*

Theorem 50.5.1. Any elementary topos admits an (epi, mono)-factorization system.

Proof. □

Example 50.5.1. Sets and the restriction, corestriction system

50.5.1 Generating families

[67]

51

Stalks and étale space

[66]

51.1 In a topological context

The notion of a stalk is meant to encode local behaviours in a sheaf. For instance, if we consider a structure sheaf on a topological space, like say some set of continuous maps $X \to \mathbb{R}$ with some specific properties, the stalk of that sheaf is the behaviour of those maps at a single point. The stalk is to be understood as some equivalence class of those functions that "behave similarly" at the given point.

If we considered for instance the discrete space X, for which all functions (as functions on sets) $X \to \mathbb{R}$ are continuous, the stalk at $x \in X$ would simply be the value at that point, as there are no further constraints on those functions. Any function with that value at that point will be part of the stalk.

However, if we take a more complex case such as smooth functions on a manifold, the local behavior of a function cannot be only reduced to its mere value at that point. It has slightly further "reaching" behaviors, such as the values of its derivatives etc.

To get the proper definition of a stalk, we need to look at the frame of opens of our space $\mathrm{Op}(X)$, and

Definition 51.1.1. Given a sheaf of rings F on a topological space X, its stalk at x is given as the directed limit over the neighbourhood net of x

$$F_x = \lim_{U \in N(x)} F(U) \tag{51.1}$$

Example 51.1.1. As we discussed, the stalks for the structure sheaf of continuous functions of a discrete space $C(X, Y)$ for some space Y is Y itself.

Proof. In a discrete space, the neighbourhood net of a point is the poset given by every subset containing that point ordered by inclusion

$$N(x) = \{S \subseteq X \mid x \in X\} \tag{51.2}$$

The associated directed set is given by reverting the arrows and applying the functor

□

In **Top**, consider an object B (the base space), and take the slice category $\mathbf{Top_{/B}}$, the category of bundles $\pi : E \to B$ over B.

If π is a local homeomorphism, ie for every $e \in E$, there is an open neighborhood U_e such that $\pi(U_e)$ is open in B, and the restriction $\pi|_{U_e} : U_e \to \pi(U_e)$ is a homeomorphism, then we say that $\pi : E \to B$ is an *étale space*, with $E_x = \pi^{-1}(x)$ the *stalk* of π over x.

For $\mathrm{Sh}(\mathbf{C}, J)$ a topos, if F is a sheaf on $(\mathbf{C}, J)$, the slice topos $\mathrm{Sh}(\mathbf{C}, J)/F$ has a canonical étale projection $\pi : \mathrm{Sh}(\mathbf{C}, J)/F \to \mathrm{Sh}(\mathbf{C}, J)$, a local homeomorphism of topoi, the étale space of F.

For any object $X \in \mathbf{C}$, $y(X)$ the Yoneda embedded object,

$$U(X) = \mathrm{Sh}(\mathbf{C}, J)/y(X) \tag{51.3}$$

Sections of π_F over $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ are in bijection with elements of $F(X)$.

If $(\mathbf{C}, J)$ is the canonical site of a topological space, each slice $\mathrm{Sh}(\mathbf{C}, J)/F$ is equivalent to sheaves on the étale space of that sheaf. In particular, $U(X) \to \mathrm{Sh}(\mathbf{C}, J)$ corresponds to the inclusion of an open subset.

51.2 In a sheaf

Stalks can be generalized from the case of sheaves on a topological space to the general case.

Definition 51.2.1. Given a point of a topos $p : \mathbf{Set} \to \mathbf{H}$, the stalk of an object $X \in \mathbf{H}$ is the inverse image of that geometric morphism :

$$\mathrm{stalk}_p(X) = p^*(X) \tag{51.4}$$

If $\mathbf{H}$ is a Grothendieck topos, this is the inverse image generated by an actual point as a site morphism, $p : \mathbf{1} \to \mathbf{C}$

Topological case

52

Topological topoi

While the category of topological spaces **Top** is *not* a topos, I feel like I should bring up some comments on this, as the notion of spaces in mathematics as a topological space is historically very deeply anchored, for various reasons such as history, ease of use, broadness, etc.

In terms of category, we have the category of topological spaces **Top**, with objects the class of all topological spaces, and morphisms the class of all continuous functions. From this definition, we can tell that **Top** is a concrete category, where the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is simply the functor associating the underlying set X of a topological space (X, τ), and it is faithful as we have

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y) = C(X, Y) \tag{52.1}$$

The injectivity to $\mathrm{Hom}_{\mathbf{Set}}(U(X), U(Y))$ implies that two different continuous functions lead to two different functions, which is trivially true since we defined our continuous functions to merely be a subset of the functions here.

The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has a left and right adjoint, the first giving us the reflective subcategory of discrete spaces, the second giving us the coreflective subcategory of trivial spaces.

[proof]

Adjoints

Theorem 52.0.1. The initial and terminal objects of **Top** are the empty topological space $\varnothing$ and the singleton topological space $\{\bullet\}$, both of which have a unique topology :

$$\tau_\varnothing = \mathcal{P}(\varnothing) = \{\varnothing\} \quad (52.2)$$

$$\tau_{\{\bullet\}} = \mathcal{P}(\{\bullet\}) = \{\varnothing, \{\bullet\}\} \quad (52.3)$$

Proof. As the initial object is given by the empty set, for which there is only one possible function to any other function (the empty function), we only need to prove that this is always a continuous function, which is vacuously true since an empty function has an empty preimage.

The same goes for the terminal object, as any set has a single morphism to the singleton, so that we only need to show it to be a continuous function. This is □

Theorem 52.0.2. Top admits a product and coproduct, which are given by the product topology and coproduct topology.

Proof. The product topology of (X, τ_X) and (Y, τ_Y) is given by the set $X \times Y$ equipped with the topology

□

Theorem 52.0.3. Top admits equalizers and coequalizers, which are given by the subspace topology and the quotient topology.

Proof. Given two topological spaces X, Y and two parallel morphisms between them,

$$X \to Y \quad (52.4)$$

□

From this, we have that **Top** is both finitely complete and cocomplete. However, this is as far as we can go to make it a topos.

Theorem 52.0.4. Top is not a balanced category.

Proof. In **Top**, morphisms that are both mono and epi are continuous bijective functions, ie There are however continuous bijections which are not homeomorphisms. Take the topological spaces $\mathbb{R}_{\text{can}}$, the real line with the canonical topology, and $\mathbb{R}_{\text{disc}}$, the real line with the discrete topology. Take the continuous function which is the identity function on sets,

$$f : \mathbb{R}_{\text{disc}} \to \mathbb{R}_{\text{can}} \quad (52.5)$$

As any map from a discrete space to a topological space is continuous, this function is continuous, as well as trivially bijective. However, the image of an open set $\{x\}$ in $\mathbb{R}_{\text{disc}}$ is a closed set in $\mathbb{R}_{\text{can}}$, and therefore not a homeomorphism. □

From this we have that there cannot be a subobject classifier on **Top**.

The non-boolean structure of the poset of subobjects for **Top** is due to

The other obstruction to this is the lack of exponential object in **Top** :

Theorem 52.0.5. There is no exponential object in **Top**.

Proof. To prove this we only need to find an object of **Top** that is not exponentiable, which is given by the fact that a topological space X is only exponentiable if the functor $X \times -$ preserves coequalizers, ie if we have a quotient map $q : Y \to Z$, for some space Y, then the function

$$q \times \mathrm{Id}_X : Y \times X \to Z \times X \tag{52.6}$$

is also a quotient map.

"This functor always preserves coproducts, so this condition is equivalent to saying that $X \times -$ preserves all small colimits. This is then equivalent to exponentiability by the adjoint functor theorem."

A counter example to this is to take $\mathbb{Q}$ the space of rational numbers with the subspace topology from $\mathbb{R}$ and the quotient map from $\mathbb{Q}$ to the quotient space $\mathbb{Q}/\mathbb{Z}$, the rationals up to the equivalence

$$\forall k \in \mathbb{Z},\ q \sim q + k \tag{52.7}$$

with the appropriate quotient topology. This is roughly the space of all rational points on the circle S^1. Given this, we can show that the Cartesian product $\mathbb{Q} \times -$ does not preserve quotients, as we have that

$$\begin{aligned} q \times \mathrm{Id}_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} &\to (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q} && (52.8) \\ (q_1, q_2) &\mapsto (f(q_1), q_2) && (52.9) \end{aligned}$$

is not a quotient map. To prove this,

Take the sequence

$$r_n = \begin{cases} 1 & n = 0 \\ \sqrt{2}/|n| & n > 0 \end{cases} \tag{52.10}$$

which is irrational for $n > 0$ and converges to 0. Take A_n to be an open subset of $[n, n+1] \times \mathbb{R}$ such that the closure of A is given by the boundary points

$$x \tag{52.11}$$

□

Due to this, **Top** is not a topos, quasi-topos or even Cartesian closed category, and is therefore not a particularly good category to perform categorical processes in.

There are several possible ways to try to find a compromise to have a topos of topological spaces. One is to take some subcategory that does form a topos [sober spaces idk]

A better behaved subcategory of **Top** is the category of compact Hausdorff spaces **CHaus**, which, in addition to being complete and cocomplete [proof?],

As compact Hausdorff spaces are rather limiting, we will instead use the broader category of compactly generated topological spaces.

Definition 52.0.1. A topological space X is said to be *compactly generated* if it obeys one of the equivalent properties :

- For any space Y and function on the underlying sets $f : |X| \to |Y|$, then the lifts

Convenient category of topological spaces : subcategory of Top such that

Every CW complex is an object **C** is Cartesian closed **C** is complete and co-complete Optional : **C** is closed under closed subspaces in Top : if $X \in \mathbf{C}$ and $A \subseteq X$ is a closed subspace, then A belongs to **C**.

Theorem 52.0.6. Every first-countable topological space is compactly generated.

Proof. □

Another way is to find some other category that has a broad intersection with **Top** (ex smooth spaces).

Theorem 52.0.7. There's an embedding of Δ-generated topological spaces in the topos of smooth spaces.

Why **Top** isn't a topos : not Cartesian closed or localy Cartesian closed[68]

53

Geometry

The broad notion of ”geometry” in a topos involved the use of so-called *geometric morphisms* (although in terms of the topos itself, those are actually functors).

The underlying notion of geometric morphisms can be understood from the point of view of Grothendieck topoi. Take two such topoi, with sites $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$. We will consider some morphism of sites $f : X \to Y$, inducing a functor by precomposition :

$$(-) \circ f : \mathrm{Sh}(Y) \to \mathrm{Sh}(X) \tag{53.1}$$

$$(F : Y^{\mathrm{op}} \to \mathbf{Set}) \mapsto (G : X^{\mathrm{op}} \to \mathbf{Set}) \tag{53.2}$$

ie for some element $y \in Y$, and a presheaf F, we have the map

$$F \circ f : Y \to \mathrm{Set} \tag{53.3}$$

Upon restriction to act on sheaves, this is our inverse image functor f^*, with the right adjoint to this being the direct image functor.

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y) \tag{53.4}$$

$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X) \tag{53.5}$$

Example 53.0.1. The basic example which gives the morphisms their names is the case of a sheaf over a topological space X, where the site is the category of opens $\mathrm{Op}(X)$, and a common type of sheaf is the sheaf of functions to some space $A : \mathrm{Sh}(\mathrm{Op}(X)) = C(X, A)$. A presheaf F is then a map

$$F : \mathrm{Op}(X) \to \mathrm{Set} \tag{53.6}$$
$$U \mapsto C(U, A) \tag{53.7}$$

The site morphism is then given by some continuous function $f : X \to Y$, in which case the direct image functor is

and the inverse image functor is given by

$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X) \tag{53.8}$$
$$F \mapsto F \circ f \tag{53.9}$$

ie given some open set $U_Y \in \mathrm{Op}(Y)$, with image $f(U_Y) = U_X$, we have

$$(f^*F)(U_Y) = F \circ f(U_Y) = F(U_X) \tag{53.10}$$

A morphism of site in this case is a continuous function $f : X \to Y$, which induces a functor on $\mathrm{Op}(X)$ by restriction :

$$f(U \in \mathrm{Op}(X)) = \tag{53.11}$$

$$f_*F(U) = F(f^{-1}(U)) \tag{53.12}$$

For two toposes E, F, a geometric morphism $f : E \to F$ is a pair of adjoint functors (f^*, f_*)

$$f_* : E \to F \tag{53.13}$$
$$f^* : F \to E \tag{53.14}$$

such that the left adjoint f^* preserves finite limits. f_* is the *direct image functor*, while f^* is the *inverse image functor*.

Example 53.0.2. Another similar example to look at the basic functioning of the geometric morphisms is to consider two slice topos from the category of sets. Taking two sets X and Y, consider the slice topoi $\mathbf{Set}_{/X}$ and $\mathbf{Set}_{/Y}$.

53.1 Terminal geometric morphism

One of the most common type of geometric morphism on a (Grothendieck) topos is the case of global sections. The site morphism involved is from whichever site we decide on our topos X to the trivial site $*$, so that our geometric morphism is between our topos and the topos of sets, $\mathbf{Set} = \mathrm{Sh}(*)$. The only site morphism available here is the constant functor

$$p : X \to * \tag{53.15}$$

which is a site morphism as any covering family of X is sent to $\mathrm{Id}_* : * \to *$, which is the only covering of 1. As the terminal category does not have much in the way of limits, we will have to show that this functor is filtered.

The induced functor is therefore some functor from the category of sets to our topos, so that for any object $F : X^{\mathrm{op}} \to \mathbf{Set}$ in our topos, and any object $x \in X$ in the site, the precomposition becomes

$$(-) \circ p : \mathrm{Set} \quad \to \quad \mathrm{Sh}(X) \tag{53.16}$$

$$(A : * \to \mathrm{Set}) \quad \mapsto \quad (A \circ p : X^{\mathrm{op}} \to * \to \mathbf{Set}) \tag{53.17}$$

ie for any "sheaf" $* \to$ A (a set), we obtain a sheaf on X simply giving us back this set.

there is only one morphism between two sites, $\mathrm{Id}_* : * \to *$. The induced functor on the sheaf is

$$F \circ \mathrm{Id}_* : * \tag{53.18}$$

direct image functor is

$$x \tag{53.19}$$

If f^* has a left adjoint $f_! : E \to F$, f is an essential geometric morphism.

Direct image functor :

$$f_* F(U) = F(f^{-1}(U)) \tag{53.20}$$

Global section : if $p : X \to *$, $*$ the terminal object of the site

Inverse image functor :

$$f^{-1} G(U) = G(f(U)) \tag{53.21}$$

LConst functor

Theorem 53.1.1. The global section functor possesses a left adjoint called the locally constant sheaf functor LConst

$$(\mathrm{LConst} \dashv \Gamma) : \mathbf{E} \underset{\leftarrow \mathrm{LConst} -}{\overset{-\Gamma \rightarrow}{\rightleftarrows}} \mathbf{Set}$$

Proof. □

Over a different base topos?

53.2 Specific geometric morphisms

Definition 53.2.1. If a geometric morphism $(f^* \dashv f_*)$ has a further right adjoint $f^!$,

$$(f^* \dashv f_* \dashv f^!) : \mathbf{E} \begin{array}{c} \xrightarrow{f^*} \\ \xleftarrow{f_!} \\ \xrightarrow{f^*} \end{array} \mathbf{S}$$

such that $f^!$ is fully faithful, we say that it is a *local geometric morphism.*

Definition 53.2.2. If a geometric morphism $(f^* \dashv f_*)$ has a further left adjoint $f_!$,

we say that it is an *essential geometric morphism.*

Connection to image, orthogonal factorization

53.3 Category of topoi

Definition 53.3.1. The category Topos is the category with as objects all topoi and as morphisms the geometric morphisms between those topoi.

54
Subtopos

As the geometric morphism is the natural map between topoi, we need some kind of inclusion map to define the notion of a subtopos. This is given by the geometric embedding :

Definition 54.0.1. A *geometric embedding* $f : \mathbf{H}_S \hookrightarrow \mathbf{H}$ is a geometric morphism for which the direct image functor f_* is full and faithful (so that $\mathbf{H}_S$ is a full subcategory).

In particular, in a Grothendieck topos, a

Theorem 54.0.1. $\epsilon : f^* f_* \to \mathrm{Id}_{\mathbf{H}_S}$ is an isomorphism.

Proof. □

Reflective localization?

Example 54.0.1. The initial topos $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$ is always a subtopos of any topos.

Proof. As there is only one map from $\mathbf{H}$ to $\mathbf{1}$, we can take this as a baseline.

$$\Delta_* : \mathbf{H} \hookrightarrow \mathbf{1} \tag{54.1}$$

Then the left adjoint of this map is

$$\mathrm{Hom}_{\mathbf{H}}(L(*), X) \cong \mathrm{Hom}_{\mathbf{1}}(*, \Delta_*(X)) \tag{54.2}$$

so that there is only one map from $L(*)$ to any object of the topos, ie it is the constant map to the initial object, Δ_0. Dually if we try to find its right adjoint, it will be the constant map to the terminal functor Δ_1. □

Theorem 54.0.2. The slice category $\mathbf{H}_{/X}$ of a topos $\mathbf{H}$ is itself a topos and a subtopos of $\mathbf{H}$.

Proof. (Maclane thm 7.1) As the topos contains an initial object, any of its slice categories contains an initial and final object. The product in a slice category is simply the pullback in $\mathbf{H}$, which does always exist in a topos. [equalizer?] "the equalizer of two arrows X Y in E / B is clearly just the equalizer in E equipped with the evident map to B" The subobject classifier $\Omega \times X \to X$ Power object □

Definition 54.0.2. For subterminal object $U \in \mathbf{H}$, the map $o_U(V) = U \to V$ defines a Lawvere-Tierney topology

over topos, comma topos?

Definition 54.0.3. A subtopos $\iota : \mathbf{H}_j \hookrightarrow \mathbf{H}$ with Lawvere-Tierney topology j is a *dense subtopos*

Lawvere-Tierney topology j

Level of a topos : an essential subtopos $H_l \hookrightarrow H$ is a *level* of H.

"the essential subtoposes of a topos, or more generally the essential localizations of a suitably complete category, form a complete lattice"

"If for two levels $H_1 \hookrightarrow H_2$ the second one includes the modal types of the idempotent comonad of the first one, and if it is minimal with this property, then Lawvere speaks of "Aufhebung" (see there for details) of the unity of opposites exhibited by the first one."

55 Motivic yoga

[Keep?]

In addition to the geometric morphisms and their two induced functors, direct images f

[69, 70, 71] Six functor formalism

56

Localization

[72, 73, 74]

In some cases we wish to create a new category from an old one by "quotienting it" along some specific morphisms, at least up to isomorphisms. That is, we wish to declare that a given morphism is in fact an isomorphism, by adjoining an inverse morphism for all such morphisms.

56.1 Localization of a commutative ring

A non-categorical example of this is given by the notion of localizing a commutative ring.

If given some commutative ring $(R, +, \cdot)$, we speak of a *localization* at some subset $S \subseteq R$, denoted by $R[S^{-1}]$, if we give each of those elements a multiplicative inverse.

Example 56.1.1. For the ring $\mathbb{Z}$, its localization at $S = \mathbb{Z} \setminus \{0\}$ is the field of rational numbers, $\mathbb{Q}$.

As we are making every element of S invertible, we can also much more simply localize it at the value of its primes, since any rational numbers can be written out as

$$q = \frac{a}{b} = a \prod_i (\frac{1}{p_i})^{k_i} \tag{56.1}$$

So that in fact we can write it as $\mathbb{Q} = \mathbb{Z}[\mathbb{P}^{-1}]$. This is a general notion that can be applied,

Theorem 56.1.1. For any localization $R[S^{-1}]$, [something something prime ideal]

Example 56.1.2. The localization of the ring of polynomials $k[x]$

We can of course localize rings at narrower sets, such as for the case of $\mathbb{Z}[\frac{1}{2}^{-1}]$, the ring of half-integers, which is for instance the ring of values for particle spin in quantum mechanics.

More interesting for us is the case of smooth rings, such as $C^\infty(M)$

Theorem 56.1.2. The localization of $C^\infty(M)$ at the set S_x of functions that do not vanish on x is the germs of smooth functions at x :

$$S_x^{-1} C^\infty(M) \cong C^\infty_x(M) \tag{56.2}$$

Proof. □

56.2 Categorical localization

In the case of a category $\mathbf{C}$, we speak of a localization at a set $W \subseteq \mathrm{Mor}(\mathbf{C})$ of morphisms. The localized category $\mathbf{C}[W^{-1}]$ is then a larger category for which every morphism in W admits an inverse.

Definition 56.2.1. A localization of a category $\mathbf{C}$ by a set of morphisms $\mathbf{C}[W^{-1}]$ is a category $\mathbf{C}[W^{-1}]$ equipped with a functor

$$Q : \mathbf{C} \to \mathbf{C}[W^{-1}] \tag{56.3}$$

such that

$$\forall f : X \to Y \in W,\ Q(f) \in \mathrm{Iso}(X, Y) \tag{56.4}$$

A motivating example for this is algebraic cases. For instance, if we have a natural number object N, we can consider its automorphisms corresponding to additions by some number, ie

$$(+k) : N \to N \tag{56.5}$$

$$n \mapsto n + k \tag{56.6}$$

In terms of internal operations, this is given by all the maps $k : 1 \to N$ and the addition map $+ : N \times N \to N$, where the $(+k)$ map is then obtained by

$$N \xrightarrow{\langle \mathrm{Id}_N, k \circ !_N \rangle} N \times N \xrightarrow{+} N \tag{56.7}$$

$n \mapsto (n, k) \mapsto n + k$

These maps are not invertible, as there is (outside of +0) no inverse in N for each of these.

This does give us the definition on an *integers object* Z.

This process can be used for a wide variety of cases, identifying objects such as homotopy equivalent objects,

[75] For a category C and a collection of morphisms $S \subseteq \mathrm{Mor}(C)$, an object $c \in C$ is S-local if the hom-functor

$$C(-, c) : C^{\mathrm{op}} \to \mathrm{Set} \tag{56.8}$$

sends morphisms in S to isomorphisms in Set, so that for every $(s : a \to b) \in S$,

$$C(s, c) : C(b, c) \to C(a, c) \tag{56.9}$$

is a bijection

"localization of a category consists of adding to a category inverse morphisms for some collection of morphisms, constraining them to become isomorphisms"

Example 56.2.1. The basic example of a localization is that of a commutative ring. localizing with prime 2 : $\mathbb{Z}[1/2]$, localization away from all primes : $\mathbb{Q}$

Example 56.2.2. Localization of $\mathbb{R}[x]$ away from a : rational functions defined everywhere except at a

Localization at a class of morphisms W : reflective subcategory of W-local objects (reflective localization).

Localization of an internal hom : localization of the morphisms defined by $\prod_{X,Y}[X, Y]$?

Localization of a topos corresponds to a choice of Lawvere topology, localization of a Grothendieck topos to a Grothendieck topology.

Duality of a localization?

Example 56.2.3. The simplest example of a localization is the localization by the class of all morphisms,

$$\mathbf{C}[\mathrm{Mor}(\mathbf{C})] \tag{56.10}$$

and it is equivalent to the terminal category $\mathbf{1}$. The map $\mathbf{C}[\mathrm{Mor}(\mathbf{C})]$ is the unique such functor, while its inverse can be given by the choice of any object and morphism in the category of fraction, so the constant functor $\Delta_X^{\mathbf{1}}$ for some X. Their composition is then

$$! \circ \Delta_X = \mathrm{Id}_* \tag{56.11}$$
$$\Delta_X^{\mathbf{1}} \circ ! = \Delta_X^{\mathbf{C}[\mathrm{Mor}(\mathbf{C})]} \tag{56.12}$$

The first being obviously naturally isomorphic to the identity functor, while the other also is by the fact that its components are all isomorphisms.

Example 56.2.4.

56.3 Reflective localization

Definition 56.3.1. A localization $\mathbf{C}[W^{-1}]$ is *reflective* if the localization functor Q admits a fully faithful right adjoint :

$$(Q \dashv T) : \mathbf{C} \to \mathbf{C}[W^{-1}] \tag{56.13}$$

Theorem 56.3.1. Every reflective subcategory $(\iota, T) : \mathbf{C} \to \mathbf{D}$ is a reflective localization at the preimage of the isomorphism in $\mathbf{C}$, ie

$$W = T^{-1}(\mathrm{Iso}(\mathbf{C})) \tag{56.14}$$

so that

$$\mathbf{C} = \mathbf{D}[W^{-1}] \tag{56.15}$$

Proof. □

56.4 Colocalization

A further way to define the localization of a category $\mathbf{C}$ by a class of morphisms W is to take the category $\mathbf{C}[W^{-1}]$, with the same objects

$$\mathrm{Obj}(\mathbf{C}[W^{-1}]) = \mathrm{Obj}(\mathbf{C}) \tag{56.16}$$

and whose morphisms $A \to B$ are given by spans whose left leg is in W. In other words, it is a wide subcategory of $\mathrm{Span}(\mathbf{C})$.

Theorem 56.4.1. A localization in that sense is equivalent to the localization [etc]

Dual to the localization, we can also define the *colocalization*, which is defined in the same way except that we are only considering the right leg of the spans to be in W.

57

Moduli spaces and classifying spaces

As we've seen in 46.3, for categories of presheaves on some underlying category **C**, there exists special presheaves called the *representable presheaves*, which are such that they can be written as a hom-functor

$$F = \mathrm{Hom}_{\mathbf{C}}(-, X) \tag{57.1}$$

with X the *representing object.* Those are in some sense the equivalent of the objects of **C** in our topos.

Representation of a functor F :

$$\theta : \mathrm{Hom}_{\mathbf{C}}(-, X) \xrightarrow{\cong} F \tag{57.2}$$

"As above, the object c is called a representing object (or often, universal object) for F, and the element e is called a universal element for F. Again, it follows from the Yoneda lemma that the pair (c,e) is determined uniquely up to unique isomorphism."

Fine v. coarse moduli space

Definition 57.0.1. An object $X \in [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is a *coarse moduli space*

Example 57.0.1. In the category of sheaves on manifolds, $[\mathrm{SmoothMan}^{\mathrm{op}}, \mathbf{Set}]$, The moduli space of k-differential forms Ω^k associates to every manifold M its set of k-differential forms $\Omega^k(M)$ Correspondence between the actual space of k-differential form $\Omega^k(M)$ as a sheaf itself and the internal hom $[X, \Omega^k]$

Moduli space : space that modulates a property on another space

Classifying morphisms,

beep

[Connection between localization, moduli spaces, lifting?]

58

Number objects

One benefit of topoi as a category is the guaranteed existence of a natural number object under broad circumstances [76], which is an internalization of the notion of positive integers. As we wish for our topoi to be a universe for mathematics, this is a fairly fundamental object.

Definition 58.0.1. A *natural number object* for a topos is an object denoted $\mathbb{N}$ such that there exists the morphisms

- The *zero morphism* $z : 1 \to \mathbb{N}$
- The *successor morphism* $s : \mathbb{N} \to \mathbb{N}$

such that for any diagram $q : 1 \to X$, $f : A \to A$, there is a unique morphism u

$$\begin{array}{ccccc} 1 & \xrightarrow{z} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow^{q} & \downarrow u & & \downarrow u \\ & & A & \xrightarrow{f} & A \end{array}$$

The map $z : 1 \to N$ is the zero element of N, while $s : N \to N$ is the successor morphism, analogous to the operation $s(n) = n + 1$, so that any integer can be understood as the n-fold composition of s :

$$n = s \circ \ldots \circ s \circ z \tag{58.1}$$

This number object is in fact (isomorphic to) the algebra generated by the functor underlying the maybe monad Maybe(n), which in this context we will call the *successor functor* $S(n)$ 26.5.2

$$S(X) = X + 1 \tag{58.2}$$

Theorem 58.0.1. The natural number object is an initial algebra for the successor functor.

f defines a sequence, such that $a_0 = q$ and $a_{n+1} = f(a_n)$

Show that a morphism $\mathbb{N} \to A$ induces a diagram $A \to A \to A \to \ldots$, which induces a limit

$$\lim_f A \tag{58.3}$$

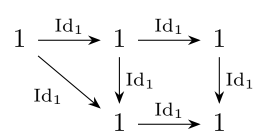
Theorem 58.0.2. The natural number object is stable under coproduct with the terminal object :

$$N + 1 \cong N \tag{58.4}$$

Theorem 58.0.3. Every Grothendieck topos admits a natural number object.

Proof. □

Remark 58.0.4. Beware that a "natural number object" may be a poor model of natural numbers in some extreme corner cases. A good example of this is the initial topos $\mathbf{1} = \mathrm{S}h(\mathbf{0})$, for which the natural number object is



which does obey all appropriate properties for it, but is only really the trivial algebra $(\{0\}, +)$.

Theorem 58.0.5. The natural number object is a rig object.

Proof. □

Counterexample 58.0.1. The topos of finite sets **FinSet** has no natural number object.

Proof. As for a finite set, there is no injection $X + 1 \hookrightarrow X$ (by the pigeonhole principle), an object N cannot exist due to 58.0.2. □

Definition 58.0.2. The addition morphism is defined as

$$+ : \tag{58.5}$$

1

Theorem 58.0.6. A natural number object defines a total order relation by

$$R_{\leq} : N_{\leq} \hookrightarrow N \times N \tag{58.6}$$

Example in set : $N_{<}$ is the set of pairs (n_1, n_2), $n_1 < n_2$

Definition 58.0.3. A finite cardinal $[n]$ is the pullback of $\mathrm{pr}_2 \circ R_{<}$ along the integer $n : 1 \to N$.

This is the object given by the condition of being inferior to the number n

Example 58.0.1. In **Set**, a finite cardinal as defined is the set of pairs $(n_1, n_2) \in R_{<}$ for $n_2 = n$, which is a set of n elements.

Definition 58.0.4. An object in a topos is a *finite object*

Are finite objects isomorphic to coproducts of 1?

Definition 58.0.5. An *integers object* Z is an object equipped with the morphisms

$$z \; : \; 1 \to Z \tag{58.7}$$

$$s \; : Z \to Z \tag{58.8}$$

Likewise, topoi also induce a *real number object*

Definition 58.0.6. A *real number object* R is a commutative ring object

$$0 \; : \; 1 \to R \tag{58.9}$$

$$1 \; : \; 1 \to R \tag{58.10}$$

$$+ \; : R \times R \to R \tag{58.11}$$

$$\cdot \; : R \times R \to R \tag{58.12}$$

with an *apartness relation*

59

Pointed spaces

For each object X of a topos, we can define a family of associated pointed spaces given by a pair of a given point $p : 1 \to X$ and the space X. This is defined by the notion of pointed objects :

Definition 59.0.1. A *pointed object* of a category with a terminal object 1 is an element of the coslice category $\mathbf{C}^{1/}$

For brevity, the pointed objects $p : 1 \to X$ will be denoted as X_p. The point p is the *base point* of the pointed space X_p.

Theorem 59.0.1. The morphisms of the category of pointed objects $\mathbf{C}^{1/}$ map every base point to another base point,

$$\forall f : X_p \quad \to \quad Y_q,\ f \circ p = q \tag{59.1}$$

Proof. By definition of morphisms in a coslice category. □

Theorem 59.0.2. The Eilenberg-Moore category of the maybe monad on a category is isomorphic to its category of pointed objects.

$$\mathrm{EM}(\mathrm{Maybe}) \cong \mathbf{C}^{1/} \tag{59.2}$$

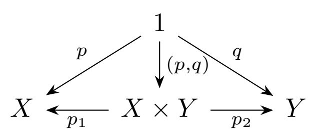
Hom-sets are pointed sets, pointed by the zero morphism.

Theorem 59.0.3. Wedge sum

Proof. □

Theorem 59.0.4. If **C** has a product, the pointed category $\mathbf{C}^{1/}$ has a canonical tensor product, the smash product.

Proof. Given two objects X, Y with product $X \times Y$, and two pointed objects X_p, Y_q, the universal property of the product gives us



□

Theorem 59.0.5. The pointed category of a topos has a zero object.

Proof. From the universal property of the terminal object in the topos, there is only one pointed object $1 \to 1$. This object is terminal in $\mathbf{H}^{1/}$ as any morphism to it will correspond to the morphism $f : X \to 1$, which factors as

$$\mathrm{Id}_1 = f \circ !_X \tag{59.3}$$

as there is only one such morphism, the object is terminal. It is also initial as any morphism from it will correspond to the morphism $g : 1 \to X$, which factors as

$$g \circ \mathrm{Id}_1 = !_X \tag{59.4}$$

meaning that $g = !_X$, so that there is also only one such morphism. □

60
Ringed topos

As with sheaves in general, we do not have to consider our topos to be exclusively set-valued, and we can give it a variety of other types. One of the most commonly used is that of a ringed topos.

Definition 60.0.1. A *ringed topos* $(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$ is a topos $\mathbf{H}$ with a distinguished (unital) ring object $\mathcal{O}_{\mathbf{H}}$

Example 60.0.1.

61

Integration

61.1 Integration on Banach spaces

61.2 Integration on topos

A generic notion of integration (or more generally, distributions) on a topos can be done via the use of coends [77, 78]. To find some equivalent of a distribution

Definition 61.2.1. For a topos $\mathbf{H}$ over a base topos $\mathbf{B}$, a *Lawvere distribution* on $\mathbf{H}$ with values in a topos $\mathbf{H}'$ over $\mathbf{B}$ is given by a $\mathbf{B}$-indexed adjoint pair $(\mu \dashv \mu^*)$

$$(\mu \dashv \mu^*) : \mathbf{H} \underset{\mu^*}{\overset{\mu}{\rightleftarrows}} \mathbf{H}'$$

μ is a cosheaf on $\mathbf{H}$ with values in $\mathbf{H}'$.

The underlying base topos $\mathbf{B}$ is generaly $\mathbf{Set}$, where we will often look at the real line object $\mathbb{R}$.

Category

$$\mathrm{Dist}_{\mathbf{B}}(\mathbf{H}, \mathbf{H}') \tag{61.1}$$

Example 61.2.1. Dirac measure

[79, 80]

If we take some appropriate monoidal category $(V, \otimes, I)$, and two functors from our topos to it,

$$F : \mathbf{H} \to V \tag{61.2}$$

$$G : \mathbf{H} \to V \tag{61.3}$$

Consider the functor

$$F \otimes G : \mathbf{C}^{\text{op}} \times \mathbf{C} \to V \tag{61.4}$$

Example of "dual" things to consider : current theory, homology of simplices, cohomology of n-forms

Basic idea : function is given by $[X, R]$ for some ring object R, functions between them are $[[X, R], [Y, R]]$

those form a ring-enriched subcategory

Opposite subcategory : given by dual $[X, R]^*$, which is apparently $[[X, R], R]_{R\text{Mod}}$?

$$\int_X : [X, R] \times [X, R]^* \to R \tag{61.5}$$

$$R \cong \int^{X \in R\text{-mod}} [X, R]_R \otimes_R X \tag{61.6}$$

For a commutative ring object in $\mathbf{H}$

[77]

Part IV

Schemes, algebras and dualities

In our analysis of topoi, a very common way to define them will be via duality, where rather than define our categories intrinsically, we define them as the duals of simpler categories. This is due to the duality of spaces as they are defined by their probes (as we can describe manifolds as the charts $\mathbb{R}^n \to M$), and spaces as they are defined by their values (as we can describe manifolds by the function spaces living on them, such as $C^\infty(M)$). While for manifolds it is commonly more useful to describe them with the former, ie as (pre)sheaves, we will deal also with many spaces for which the latter is more practical, ie as copresheaves.

For this, we will need to define what a dual is exactly in category theory.

[81]

62
Algebras

The notion of algebra has come about quite a few time throughout but we need now to figure out some rather general description of it.

Definition 62.0.1. An *algebraic structure*

Algebraic structures, rings, algebras, etc

linear categories? 39.0.2

One example we saw early on regarding the duality between spaces and algebras is that of **FinSet**, whose opposite category we saw was that of finite boolean algebras.

Copresheaf on a point?

For our case of looking at physical categories, one instructive example is that of C^∞-rings, which are roughly speaking analogues of the standard algebra of smooth functions on $\mathbb{R}^n$, $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Definition 62.0.2. A C^∞-algebra (or smooth algebra, also called C^∞-ring) is a copresheaf on **CartSp** that preserves finite products.

Definition 62.0.3. The category of C^∞ algebras is the category of all such copresheaves and the natural transformations between them, ie

$$C^\infty\mathbf{Alg} \subset [\mathbf{CartSp}, \mathbf{Set}] \tag{62.1}$$

Example 62.0.1. As a basic example, we can look at the case of the assignment given by the representable functor

$$h^X : \mathbf{CartSp} \to \mathbf{Set} \tag{62.2}$$
$$\mathbb{R}^n \mapsto \mathrm{Hom}_{\mathbf{CartSp}}(X, \mathbb{R}^n) \cong C^\infty(X, \mathbb{R}^n) \tag{62.3}$$

By definition this is the set of every smooth function from the Cartesian space X to $\mathbb{R}^n$, and a copresheaf. This is a C^∞ algebra since the hom functor preserves limits,

$$C^\infty(X, \mathbb{R}^n \times \mathbb{R}^m) \cong C^\infty(X, \mathbb{R}^n) \times C^\infty(X, \mathbb{R}^m) \tag{62.4}$$

where smooth maps to a product of Cartesian spaces are isomorphic to a pair of smooth maps to those Cartesian spaces.

$$f(x) = \mathrm{pr}_1(f(x)) \times \mathrm{pr}_2(f(x)) \tag{62.5}$$

In particular, this property means that we can merely consider the sets of smooth real functions on a given Cartesian space, as any representable C^∞-algebra can then simply be written as

$$C^\infty(X, \mathbb{R}^n) \cong (C^\infty(X, \mathbb{R}))^{\times n} \tag{62.6}$$

Theorem 62.0.1. The hom-set of algebra homomorphisms to $\mathbb{R}$, $C^\infty(\mathbb{R}^n) \to \mathbb{R}$, are isomorphic to the underlying set of $\mathbb{R}^n$,

$$\mathrm{Hom}_{\mathbf{CartSp}}(\mathbb{R}^0, \mathbb{R}^n) \cong \mathrm{Hom}_{\mathbf{Alg}}(C^\infty(\mathbb{R}^n), \mathbb{R}) \tag{62.7}$$

via the evaluation map

$$\mathrm{ev} : \tag{62.8}$$

Proof. Given some function $f : \mathbb{R}^n \to \mathbb{R}$, and the i-th projection function $x_i : \mathbb{R}^n \to \mathbb{R}$

$$x_i = \mathrm{pr}_i(x) \tag{62.9}$$

□

We will see later on exactly how this

63

Affine varieties

Definition 63.0.1. Given the affine space A^n, and a set of polynomials valued in A^n, an *affine variety* is given by the intersection of the sets of zeros of those polynomials.

$$V(f_1, \ldots, f_n) = \{x \in A^n \mid \forall i,\ f_i(x) = 0\} \tag{63.1}$$

64

Schemes

[82, 83]

Definition 64.0.1. The spectrum of a ring R is a topological space $\mathrm{Spec}(R) = (P, Z)$, with P the set of the prime ideals of R

$$P = \{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal of } R\} \tag{64.1}$$

and Z is the *Zariski topology* on P, defined by the closed sets that, for any ideal I, its equivalent closed set is given by

$$V(I) = \{\mathfrak{p} \in P \mid I \subseteq P\} \tag{64.2}$$

along with a sheaf of rings $\mathcal{O}_R$

Example 64.0.1. For the ring of real numbers, $\mathbb{R}$, its only ideals are the trivial ideal 0, which is a prime ideal, and itself. There is then only two closed sets defined by the Zarinski topology, $V(\mathbb{R}) = \varnothing$ and $V(\{0\}) = \{0\}$, giving us the terminal topology.

Example 64.0.2. Take the ring of smooth functions on an underlying manifold, $C^\infty(M)$. An example of ideals is the set of functions vanishing on a region U, ie $f\big|_U = 0$:

$$\forall g \in C^\infty(M),\ (fg)\big|_U = 0 \tag{64.3}$$

Let's note those ideals by $\mathfrak{m}_U$. Those ideals are related by $U' \subseteq U$ implying $\mathfrak{m}_U \subseteq \mathfrak{m}_{U'}$, since a function vanishes on U' if it vanishes on a larger set, so that those ideals cannot be maximal unless U is a single point $\{p\}$, which we will denote $\mathfrak{m}_p$.

We can show that those ideals are maximal as they are

the prime ideals are Whitney

Example 64.0.3. Taking the ring of integers $\mathbb{Z}$, its ideals are given by the sets of multiples,

$$k\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}\ n = km\} \tag{64.4}$$

such as the set of even numbers $2\mathbb{Z}$. The prime ideals are given by the multiples of the prime numbers, $p\mathbb{Z}$ for $p \in \mathbb{P}$ (in addition to the zero ideal), so that its spectrum is given by [Multiples or primes themselves?]

$$\mathfrak{p} = \mathbb{P} \cup \{0\} \tag{64.5}$$

65

Duality

In its most generality, a duality is simply an equivalence of categories between one category and the dual of another (in the sense of the opposite category) [84, 85], which is the notion of dual adjunction we saw in 30.1 for the case of an equivalence.

Definition 65.0.1. A duality is an equivalence between a category $\mathbf{C}$ and the dual of a category $\mathbf{D}$, that is, we have two functors going between each as

$$S : \mathbf{C} \to \mathbf{D} \tag{65.1}$$

$$T : \mathbf{D} \to \mathbf{C} \tag{65.2}$$

where their composition in either order is naturally isomorphic to the identity functor on each category :

$$\eta : \mathrm{Id}_{\mathbf{C}} \to TS \tag{65.3}$$

$$\epsilon : \mathrm{Id}_{\mathbf{D}} \to ST \tag{65.4}$$

obeying that for every $X \in \mathbf{C}$ and $Y \in \mathbf{D}$

$$T\epsilon_X \circ \eta_{TX} = \mathrm{Id}_{TX} \tag{65.5}$$

$$S\eta_Y \circ \epsilon_{SY} = \mathrm{Id}_{SY} \tag{65.6}$$

but for the most part, we will be interested in more specific cases of dualities via certain methods.

Definition 65.0.2. If two dual categories **C**, **D** are concrete, ie with forgetful functors

$$U : \mathbf{C} \to \mathbf{Set} \quad (65.7)$$
$$V : \mathbf{D} \to \mathbf{Set} \quad (65.8)$$

that are faithful and representable,

$$U \cong \mathrm{Hom}_{\mathbf{C}}(E_{\mathbf{C}}, -) \quad (65.9)$$
$$V \cong \mathrm{Hom}_{\mathbf{D}}(E_{\mathbf{D}}, -) \quad (65.10)$$

[is concreteness and representability needed]

Definition 65.0.3. A *concrete duality*

[86]

Dualizing object :

Example 65.0.1. The standard example of duality is that of dual vectors, with dualizable object $I \cong k \in \mathbf{Vec}_k$. The two functors are simply

$$U = V = h_I = \mathrm{Hom}_{\mathrm{Vec}_k}(I, -) \quad (65.11)$$

, mapping vector spaces to their set of points. Then a dualizable object is a vector space X

$$T : \mathrm{Vec}_k^{\mathrm{op}} \xrightarrow{h_X} \mathrm{Vec}_k \quad (65.12)$$
$$S : \mathrm{Vec}_k^{\mathrm{op}} \xrightarrow{h_X} \mathrm{Vec}_k \quad (65.13)$$

$$US : \mathbf{Vec}_k \to \mathbf{Set} \quad (65.14)$$
$$VT : \mathbf{Vec}_k \to \mathbf{Set} \quad (65.15)$$

representable : $\exists X, Y \in \mathbf{Vec}_k$ such that

$$US = \mathrm{Hom}_{\mathbf{Vec}_k}(-, X) \quad (65.16)$$
$$VT = \mathrm{Hom}_{\mathbf{Vec}_k}(-, Y) \quad (65.17)$$

Representing elements : $\phi = \mathrm{Hom}_{\mathbf{Vec}_k}(X, X)$, $\psi = \mathrm{Hom}_{\mathbf{Vec}_k}(Y, Y)$, space of square matrices

nat :

$$\eta : \mathrm{Id}_{\mathbf{Vec}_k} \to TS \quad (65.18)$$
$$\epsilon : \mathrm{Id}_{\mathbf{Vec}_k} \to ST \quad (65.19)$$

Those are respectively the resolution of the identity and [trace?]

Contravariant representation?

Canonical isomorphism $\omega : |X| \to |Y|$ via

$$|X| \xrightarrow{X} USTX \tag{65.20}$$

Adjunction

$$(-) \otimes A \dashv (-) \otimes A^* \cong [A, -] \tag{65.21}$$

$$X^* \cong k \otimes X^* \cong [A, k] \tag{65.22}$$

[...]

In the context of presheaves, this situation is simply the case where

As a presheaf can be considered as the generalization of a space that can be probed by specific objects, a copresheaf can be considered as the generalization of a quantity on that space with values in the same objects.

The basic example for this is to consider the algebra of smooth functions on a Cartesian space, via the functor associating R-algebras to Cartesian spaces :

$$C^\infty : \mathbf{SmoothMan} \to \text{R-}\mathbf{Alg} \tag{65.23}$$

Associating the algebra $C^\infty(M)$ to any manifold M. Adding the forgetful functor $U : R{-}\mathbf{Alg} \to \mathbf{Set}$, this gives us some copresheaf on the category of Cartesian spaces.

Theorem 65.0.1. The opposite category of **CartSp** is the category of smooth real functions.

$$\mathbf{CartSp}^{\text{op}} \cong C^\infty \tag{65.24}$$

Proof. Given the category of smooth algebras on Cartesian spaces, with objects the smooth algebras $C^\infty(\mathbb{R}^n)$ and morphisms the algebra homomorphisms, take the functor

$$C^\infty(-) : \mathbf{CartSp} \to \mathbf{C}^\infty \tag{65.25}$$

We can define this functor via the contravariant hom-functor $\text{Hom}_{\text{CartSp}}(-, \mathbb{R})$. On objects, this clearly sends any Cartesian space to their set of smooth functions, by definition of the morphisms in **CartSp**. As a contravariant functor, any morphism is sent to

$$\text{Hom}(f, \mathbb{R}) : \text{Hom}(Y, \mathbb{R}) \to \text{Hom}(X, \mathbb{R}) \tag{65.26}$$

So that we have a morphism $C^\infty(Y) \to C^\infty(X)$ defined by precomposition, ie for any $g \in C^\infty(Y)$, $C^\infty(f)(g) = g \circ f$.

If we consider the standard smooth algebra on that set, those transformations are always algebra homomorphisms, as the algebraic operations commute with function composition

$$C^\infty(f)(\alpha g_1 + \beta g_2) = (\alpha g_1 + \beta g_2) \circ f \quad (65.27)$$
$$= (\alpha g_1 \circ f + \beta g_2 \circ f) \quad (65.28)$$

$$C^\infty(f)(g_1 g_2) = (g_1 g_2) \circ f \quad (65.29)$$
$$= (g_1 \circ f)(g_2 \circ f) \quad (65.30)$$

Conversely, any algebra isomorphism on smooth algebras □

[Smooth algebras on $\mathbb{R}$ is enough since we can define products of functions and diagonal for $\mathbb{R}^n$]

Theorem 65.0.2. Milnor duality :

$$\mathrm{Hom}_{\mathrm{SmoothMan}}(X, Y) \cong \mathrm{Hom}_{\mathbf{CAlg}_{\mathbb{R}}}(C^\infty(Y), C^\infty(X)) \quad (65.31)$$

This copresheaf is furthermore a cosheaf. If we consider the restriction of our manifold to some submanifold $\iota : S \hookrightarrow M$,

Copresheaf : functor $F : \mathbf{C} \to \mathbf{Set}$

Cosheaf : copresheaf has for a covering family $\{U_i \to U\}$

$$F(U) \cong \varinjlim \left(\coprod_{ij} F(U_i \times_U U_j) \rightrightarrows \coprod_i F(U_i) \right) \quad (65.32)$$

Equivalence via Yoneda :

$$\mathrm{CoSh}(\mathbf{C}) \cong \quad (65.33)$$

The copresheaf assigns to each test space $U \in \mathbf{C}$ the set of allowed maps from A to U - U-valued functions on A

Example 65.0.2. There is a duality between the algebra $\mathbb{R}$ and the single point space $\{0 < 1\}$

Proof. As $\mathbb{R}$ has only a single ideal, 0,

$$\mathrm{Spec}(\mathbb{R}) \cong \{\bullet\} \quad (65.34)$$

Its Zariski topology is simply given by the □

Duality for functors and monads and comonads? Given functors between two presheaves, what is the associated functor on the two dual copresheaves?

Theorem 65.0.3. Given a functor between two presheaves in a presheaf category,

$$F : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D}) \tag{65.35}$$

there is a dual functor between their copresheaves

Gelfand duality

Ringed topos example

Isbell duality [87]

Theorem 65.0.4. Given a cosmos $\mathbf{V}$ 35.0.3, and a small $\mathbf{V}$-enriched category, with $[\mathbf{C}^{\mathrm{op}}, \mathbf{V}]$ and $[\mathbf{C}, \mathbf{V}]^{\mathrm{op}}$ its enriched functor categories, then there is an adjunction

$$(\mathcal{O} \dashv \mathrm{Spec}) : [\mathbf{C}, \mathbf{V}]^{\mathrm{op}} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} [\mathbf{C}^{\mathrm{op}}, \mathbf{V}]$$

Such that for a $\mathbf{V}$-sheaf $F \in [\mathbf{C}^{\mathrm{op}}, \mathbf{V}]$, $\mathcal{O}$ gives a $\mathbf{V}$-cosheaf acting on $\mathbf{C}$ as

$$\mathcal{O}(F) : X \mapsto \mathrm{Hom}_{[\mathbf{C}^{\mathrm{op}}, \mathbf{V}]}(F, \mathrm{Hom}_{\mathbf{C}}(-, X)) \tag{65.36}$$

and for a cosheaf $G \in [\mathbf{C}, \mathbf{V}]^{\mathrm{op}}$, Spec gives a $\mathbf{V}$-sheaf acting on $\mathbf{C}^{\mathrm{op}}$ as

$$\mathrm{Spec}(G) : X \mapsto \mathrm{Hom}_{[\mathbf{C}, \mathbf{V}]^{\mathrm{op}}}(\mathrm{Hom}_{\mathbf{C}}(X, -), G) \tag{65.37}$$

Proof. □

Example 65.0.3. Case of $\mathbf{V} = \mathbf{Set}$

66
Q-categories

The duality between spaces and quantities is, when it comes to algebras, generally only well defined for the commutative cases. But in a modern context there are many cases where we want a non-commutative geometry, which is the dual of a non-commutative algebra.

Unlike the commutative case, there is no sheaf dual to that kind of algebra, but there is a more general construction

Definition 66.0.1. A *Q-category* is given by a pair of adjoint functor

$$(u^* \dashv u_*) : \overline{\mathbf{A}} \underset{u_*}{\overset{u^*}{\leftrightarrows}} \mathbf{A}$$

such that u^* is a fully faithful functor.

Theorem 66.0.1. For a field k and the category of k-commutative algebras $\mathbf{CAlg}_k$, and

[88]

67
Weil algebras and infinitesimal spaces

One of the duality we will exploit the most here is that given by the Weil algebras.

Definition 67.0.1. A *Weil algebra* (or *Artinian ring*) is a commutative unital ring of the form

$$A = R \oplus W \tag{67.1}$$

composed of a ring R, and a module of finite rank over R consisting of nilpotent elements, ie

$$\forall x \in W,\ \exists n \in \mathbb{N},\ x^n = 0 \tag{67.2}$$

The nilpotent elements are in the context of infinitesimal geometry interpreted to be "infinitesimal elements", where their nilpotency indicates their "smallness" in the fact that their product is formally zero, an algebraic version of the cutting off of higher order terms like $(dx)^2$.

In this manner, every ring is trivially a Weil algebra via

$$R \oplus \{0\} \tag{67.3}$$

Example 67.0.1. The simplest non-trivial example of a Weil algebra is the ring of *dual numbers*, defined as the quotient of the real polynomial ring by the square

$$\mathbb{R}[\varepsilon^2] = \mathbb{R}[\varepsilon]/\langle \varepsilon^2 \rangle \tag{67.4}$$

which has the graded structure

$$\mathbb{R}[\varepsilon] = \mathbb{R} \oplus W \tag{67.5}$$

where W contains the monomials of degree 1, for which $x^2 = 0$ by definition.

The relationship between this Weil algebra and infinitesimals can be seen by considering its behaviour on polynomial functions,

$$(x+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\varepsilon^k = x^n + nx^{n-1}\varepsilon + 0 \tag{67.6}$$

So that the polynomial of a dual number is equivalent to the pairing of the polynomial of the actual function on $\mathbb{R}$ and of its derivative,

$$\forall f \in \mathbb{R}[x],\ f(x+\varepsilon) = f(x) + \varepsilon f'(x) \tag{67.7}$$

Prime ideals of dual numbers : For a product of two dual numbers,

$$(x_1 + y_1\varepsilon)(x_2 + y_2\varepsilon) = x_1x_2 + (x_1y_2 + x_2y_1)\varepsilon \tag{67.8}$$

Similar arguments can be used for higher nilpotent elements, such as for the quotient

$$\mathbb{R}[\varepsilon^k] = \mathbb{R}[X]/\langle X^k \rangle \tag{67.9}$$

which has the nilpotent elements $X^k = 0$ for $k > 1$.

Category of smooth loci

Definition 67.0.2. The category of formal Cartesian spaces is the category for which objects

[...]

The copresheaf on the category of formal Cartesian spaces can then be interpreted as some space with structures upon it with values in a formal Cartesian space.

$$[X, \mathbb{R}^n \times W] \tag{67.10}$$

Infinitesimal objects as left adjoints of the internal hom?

The basic example of a Weil algebra is the algebra of *dual numbers*, a vector space composed by a pair of real numbers, $\mathbb{R}[\varepsilon]/\varepsilon^2$

$$(x + \epsilon y)^2 = x^2 + xy\varepsilon \tag{67.11}$$

As a copresheaf?

Its formal dual is the spectrum

$$D = \mathrm{Spec}(\mathbb{R}[\varepsilon]/\varepsilon^2) \tag{67.12}$$

which is the first order infinitesimally thickened point in one dimension (also sometimes called a ”linelet”, an infinitesimally small line).

Part V

Example categories

[89]

For the consideration of the methods to be studied, we need to look at a few good examples of appropriate categories. We will mostly look at topos (the main focus of this), more specifically Grothendieck topos, as well as a few categories of some physical interest relating to quantum theory, which may differ from the topos structure, hopefully illuminating the differences with the more classical logic associated with classical mechanics.

Quantales? Topos of the sheaves of commutative algebras on Hilbert space? Effectus categories?

68
Spatial topoi

As the classic example of a sheaf is that of a sheaf on the set of opens of a topological space, let's first look at its topos, the full category of sheaves on that same set of opens, the sheaf category $\mathrm{Sh}(\mathrm{Op}(X))$ with the subcanonical coverage, simply written as $\mathrm{Sh}(X)$ for short. This is a *spatial topos*. This topos will contain for instance the structure sheaves of continuous functions to some topological space Y.

[...]where the restriction maps $\rho_{U,V}$ are simply given by the domain restriction

Restriction maps, gluing, locality

Theorem 68.0.1. The initial object of a spatial topos is the empty set, $よ(\varnothing)$.

Theorem 68.0.2. The terminal object of a spatial topos is the whole space, $よ(X)$

Interpretation as a structure sheaf from the computation of stalks?

Theorem 68.0.3. Given a sheaf on a spatial topos,

"The problem with excluded middle in topological models is that it may not hold continuously: e.g. a subspace $U \subseteq X$ and its complement $X \setminus U$ will together contain all the points of X, but the map from their coproduct $U + (X \setminus U)$ to X does not have a continuous section because the topology on the domain is different. Thus, we cannot expect to have the full law of excluded middle in a topological model"

69
Category of sets

The most basic topos (outside of the initial topos $\mathrm{Sh}(\mathbf{0}) \cong \mathbf{1}$) is the category of sets **Set** [90, 91], with objects made from the class of all sets and morphisms the class of all functions.

In terms of a sheaf topos, sets can be defined as the sheaf on the terminal category : $\mathbf{Set} = \mathrm{Sh}(\mathbf{1})$, which is the functor from $\mathbf{1}^{\mathrm{op}} \cong \mathbf{1}$ to Set. As the set of all functions from $\{\bullet\}$ to any set X is isomorphic to X itself, the presheaves on **1** are easily seen to be isomorphic to **Set**. We only need to consider sieves on its unique object $*$, and as there is only one possible morphism there, there are only two possible sieves : the empty sieve $S_\varnothing$ which maps $*$ to the empty set, and the maximal sieve S_* which maps it to the singleton containing the identity map.

This allows us two possible topologies, the *chaotic topology* for the empty sieve and the *standard topology* for the maximal sieve.

For the chaotic topology, if we consider the category of presheaves on **1** (isomorphic to **Set**), the empty sieve on $*$ generates the subfunctor

$$F_\varnothing \hookrightarrow よ(*) \cong \mathrm{Hom}_{\mathbf{1}}(-, *) \tag{69.1}$$

which assigns its unique object $*$ to the empty set :

$$F_\varnothing(*) = \varnothing \tag{69.2}$$

our sheaves are then the local objects of PSh(**1**) on which this subfunctor

$$\mathrm{Hom}_{\mathrm{PSh}(\mathbf{1})}(よ(*), F) \cong \mathrm{Hom}_{\mathbf{Set}}(\varnothing, F) \tag{69.3}$$

[Find something here : 1]

[...]

The empty sieve is therefore equivalent to the initial topos, so that we will only consider here the

The choice of the terminal category with the maximal sieve is in fact equivalent to **Set** being a spatial sheaf on the topological space of one point. Sets can therefore also be seen as functions to sets on a single point, which is simply the identity $[1, X] \cong X$.

Having a Grothendieck topology equivalent to the trivial topology, the category is subcanonical, with its unique representable presheaf

$$よ(*) = \mathrm{Hom}(-, *) \tag{69.4}$$

which maps any element of the site (so simply $*$ itself) to the set

$$\mathrm{Hom}(*, *) = \{\mathrm{Id}_*\} \tag{69.5}$$

As the site only has a trivial coverage, there is only a fairly limited amount of assembly that we can do from it.

A variation of **Set** is **FinSet**, the category of finite sets. Trivially a full subcategory of **Set** via the obvious inclusion functor.

Theorem 69.0.1. The category FinSet is a subtopos of **Set**.

Proof. As we are only concerned with finite limits here, all basic arguments for **Set** being a topos apply here. All finite limits of finite sets in **Set** are also finite sets, as are all internal homs and the subobject classifier. □

For any set, sheaf on that set equivalent to Set? (with the standard coverage)

Theorem 69.0.2. For any discrete category with the maximal coverage, the sheaf on that site is equivalent to **Set**.

Proof. For some discrete category **n**, if we consider its sheaf

$$\mathrm{Sh}(\mathbf{n}) \tag{69.6}$$

for any such sheaf $F : \mathbf{n}^{\mathrm{op}} \to \mathbf{Set}$, we have an equivalent sheaf which has empty image for any $n \in \mathbf{n}$ other than one :

$$F'(\bullet_0) = \coprod_{i=0}^{n} F(\bullet_i) \tag{69.7}$$

□

69.1 Limits and colimits

As the prototype for the very notion of limits and colimits, **Set** has all the finite limits and colimits. In fact, it has all small limits and colimits, ie when the diagram itself is small enough to be a set.

Theorem 69.1.1. The empty set $\varnothing$ is the initial object of **Set**.

Proof. We need to show that for any set $X \in \mathrm{Obj}(\mathbf{Set})$, there is a unique function $f : \varnothing \to X$. A function $f : A \to B$ is a subset of $A \times B$ obeying some properties, therefore we need to look at the set of subsets of $\varnothing \times A$. By properties of the Cartesian product,

$$\varnothing \times A = \{\varnothing\} \tag{69.8}$$

There is therefore only one element to choose from, $\varnothing$, which is indeed a function since it obeys (vacuously) the constraints on functions. □

Theorem 69.1.2. Any singleton set $\{\bullet\}$ is a terminal object of **Set**, all isomorphic.

Proof. We need to show that for any set $X \in \mathrm{Obj}(\mathbf{Set})$, there is a unique function $f : X \to \{\bullet\}$. We can simply pick the Cartesian product

$$X \times \{\bullet\} \tag{69.9}$$

which, as a graph, satisfies the condition of a function. This is the only possible function as any subset of this would simply have the domain of the function be smaller than X. □

[formal proof 1]

The terminal object is also the unique representable presheaf on **1**, simply given by $\mathrm{Hom}_{\mathbf{1}}(*, *)$, mapping the unique element of **1** to the set of the unique identity morphism of $*$, as we would expect.

As a category of sets, which are fundamentally defined by $\in$, **Set** has global elements $x : I \to X$. Those global elements are separators

Well-pointed topos

Theorem 69.1.3. The product on **Set** is isomorphic to the Cartesian product.

Proof. The Cartesian product has by construction two projection operators, pr_1 and pr_2. Given two functions $f_i : X \to Y_i$, there is a natural function $f = f_1 \times f_2$ given by

$$f(x) = (f_1(x), f_2(x)) \tag{69.10}$$

which obeys the universal property of the product.

□

Theorem 69.1.4. The coproduct on **Set** is isomorphic to the disjoint union.

Proof. □

Theorem 69.1.5. Given two functions $f, g : A \to B$, the equalizer in **Set** is the subset $C \subseteq A$ on which those functions coincide,

$$\mathrm{eq}(f, g) = \{c \in A | f(c) = g(c)\} \tag{69.11}$$

Theorem 69.1.6. The equalizer of two functions $f, g : A \rightrightarrows B$ is the set of elements of A whose image agree :

$$\mathrm{eq}(f, g) = \{x \in A | f(x) = g(x)\} \tag{69.12}$$

Theorem 69.1.7. The coequalizer of two functions $f, g : A \rightrightarrows B$ is the quotient set on A by the equivalence relation

$$x \sim y \leftrightarrow f(x) = g(y) \tag{69.13}$$

Proof.

$$A \to B \to C \tag{69.14}$$

□

Theorem 69.1.8. The pullback of the cospan $A \to C \leftarrow B$ can be viewed as the indexed set

$$X \times_Z Y \cong \coprod_{x \in X} g^{-1}(\{f(x)\}) \tag{69.15}$$

or symmetrically,

$$X \times_Z Y \cong \coprod_{y \in Y} f^{-1}(\{g(y)\}) \tag{69.16}$$

That is, for any element $x \in X$, the

Proof. □

Theorem 69.1.9. The pushout of the span $A \leftarrow C \to B$ is the

Theorem 69.1.10. Directed limit

Given these, we can see that **Set** has all small limits and colimits.

69.2 Elements

Fairly obviously, given its status as the model for it, **Set** has global elements : $x : \{\bullet\} \to X$, corresponding to the functions

$$\forall x \in X,\ x(\bullet) = x \tag{69.17}$$

so that explicitly, $x = \{(\bullet, x)\}$ (this set can be shown to exist with the axiom of pairing). Those points are furthermore *separators*, in that the morphisms on **Set** are entirely defined by them

$$f = g \leftrightarrow \forall x : \{\bullet\} \to X,\ f \circ x = g \circ x \tag{69.18}$$

As it is also a non-degenerate topos, in that $\{\bullet\} \neq \varnothing$, this makes **Set** a well-pointed topos.

Set is boolean, two-valued, has split support

69.3 Subobject classifier

For **Set**, the subobject classifier is the set $\{\varnothing, \{\bullet\}\}$, also denoted by $\{0, 1\}$ or $\{\bot, \top\}$, corresponding to the two boolean valuations of a subobject : either being a subset or not being a subset. While we could define it as such, this is also simply the subobject classifier of the terminal topos, as this is also the closed sieve on the terminal category **1**, so that the subobject classifier is the functor

$$\Omega : \mathbf{1} \to \{\varnothing, \{\bullet\}\} \tag{69.19}$$

which, in the equivalence of sheaves $\mathbf{1} \to \mathbf{Set}$ with sets, is simply a set of two elements.

In terms of set theory, a subobject is a subset (up to isomorphism), that is

$$A \subseteq X \leftrightarrow \exists \chi_A : X \to \Omega \tag{69.20}$$

For a subset $\iota_S : S \hookrightarrow X$

$$\begin{array}{ccc} S & \xrightarrow{!_S} & 1 \\ \downarrow{\scriptstyle \iota_S} & & \downarrow{\scriptstyle \text{true}} \\ X & \xrightarrow{\chi_S} & \Omega \end{array}$$

The function χ_S is more typically called the characteristic function and uses the notation $\chi_S : U \to \mathbb{B}$

$$\chi_U(x) = \begin{cases} 0 \\ 1 \end{cases} \tag{69.21}$$

Internal hom : The set of functions

$$[X, Y] = \{f | f : X \to Y\} \tag{69.22}$$
$$= \{f \subseteq X \times Y | \forall x \in X,\ \exists y \in Y,\ (x, y) \in f \wedge ((x, y) \in f \wedge (x, z) \in f \to y = z)\} \tag{69.23}$$

69.4 Natural number object

Sets have as their natural number objects any of the standard constructions for the set of natural numbers (up to isomorphism). We can look for instance at the von Neumann construction.

The von Neumann construction is given by starting with the empty set $\varnothing$, and defining the successor function recursively, where for any element $n \in \mathbb{N}$,

$$\mathrm{succ}(n) = n \cup \{n\} \tag{69.24}$$

and then our set of integers is given by the usual axiom of infinity

[...]

This set obeys the natural number object properties, with $z : 1 \to N$ being the zero element, $0 = \varnothing \in \mathbb{N}$, and s being the successor function. If we then have some sequence of sets, given by a morphism $u : \mathbb{N} \to A$, or in other words, $\{A_i\}_{i \in \mathbb{N}}$ for $A_i \in A$, q will simply be the first element of that sequence, A_0, while the function f is the mapping giving us the next element,

$$f(A_i) = A_{i+1} \tag{69.25}$$

This construction is equivalent to that seen previously. Up to isomorphism for instance, the successor function is simply the maybe monad, as we have $\{n\} \cong \{\bullet\}$

The maybe monad is simply

$$\mathrm{Maybe}(X) = X \sqcup \{\bullet\} \tag{69.26}$$

Coproduct for natural number :

$$0 + 0 = \{\} \tag{69.27}$$

Lawvere-Tierney topology : some morphism $j : \mathbf{2} \to \mathbf{2}$

Properties : $j(\{\bullet\}) = \{\bullet\}$, $j(j(x)) = j(x)$, $j(a \wedge b) =$

$j(\{\bullet\}) = \{\bullet\}$ reduces the choice to $j = \mathrm{Id}_\Omega$ and $j = \{\bullet\}$, the constant map.

For the identity map : Given a subset $\iota : S \hookrightarrow X$, with classifier $\chi_S : A \to \Omega$, the composition $j \circ \chi_S$ defines another subobject $\bar{\iota} : \overline{S} \hookrightarrow A$ such that s is a subobject of $\bar{\iota}$, $\bar{s}$ is the j-closure of s

Identity map closure : every object is its own closure. This is the *discrete topology.*

Constant map $j(x) = \{\bullet\}$: the composition $j \circ \chi_S$ is the "always true" characteristic function, which is just χ_A. The closure of a set S in A is the entire set A. This is the *trivial* or *codiscrete* topology.

Those are the only two allowed topologies in **Set**.

Relation to $\mathrm{loc}_{\neg\neg}$: $\neg : \Omega \to \Omega$ is

$$\neg(\{\bullet\}) = \varnothing \tag{69.28}$$

$$\neg(\varnothing) = \{\bullet\} \tag{69.29}$$

$\neg\neg$ is simply the identity on **Set**. The j-closure associated to it is the identity map.

Localization?

Theorem 69.4.1. The power object Ω^X of a set X is the power set $\mathcal{P}(X)$.

Proof. As every subobject of X is isomorphic to a morphism $X \to \Omega$, the hom-set $\mathrm{Hom}_{\mathbf{Set}}(X, \Omega) \cong \Omega^X$ is the set of all subsets of X. □

In more intuitive terms, the functions of Ω^X send every element of X to a truth value, signifying whether or not it belongs to it.

69.5 Closed Cartesian

That the category of sets is closed Cartesian simply stems from the usual construction of functions on sets in basic set theory. As functions from A to B are defined as a subset of the Cartesian product $A \times B$, this is guaranteed by the existence of the Cartesian product of sets and the axiom schema of specification.

Definition 69.5.1. For any two sets A, B, the hom-set $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ is itself a set, by the traditional set definition of functions

$$f : A \to B \leftrightarrow f = \{(a, b) \subseteq A \times B | f(a) = b\} \tag{69.30}$$

so that []

The evaluation map of a function in **Set** is given by the traditional formulation of function evaluations in set theory. For a function $f : A \to B$, its evaluation by an element $x \in A$ is the unique element y in Y for which $(x, y) \in R_f$. In set theoretical terms, this can be defined Russell's iota operator,

$$\mathrm{ev}(x, f) = \iota y,\ xR_f y \tag{69.31}$$

$$= \bigcup \{z \mid \{y \mid xR_f y\} \ = \{z\}\} \tag{69.32}$$

Counit of the tensor product/internal hom adjunction?

$$S \times (-) \dashv [S, -] \tag{69.33}$$

currying

69.6 Internal objects

The category of **Set** contains just about all the typical internal objects that we would expect, as most basic structures in mathematics are defined using set theory. Any group in the traditional definition of the sense (as sets with some extra structure) have an equivalent internal group in **Set**, with the expected underlying set, and the same is true for internal rings, internal modules, etc etc.

We have in particular our various number objects with their associated internalized structures. The real number object $\mathbb{R}$ can be given an internal group structure with the addition morphism, it can be given an internal ring structure, etc.

69.7 Integration

Set is trivially a concrete category, with free-forgetful adjunction ($\mathrm{Id} \dashv \mathrm{Id}$). Its dualizing object is simply the terminal object $\{\bullet\}$, such that we have

$$\mathrm{Hom}_{\mathbf{Set}}(\{\bullet\}, -) \tag{69.34}$$

for any set X, its dual (the complete atomic boolean algebra) is given by

$$X^* = \mathrm{Hom}_{\mathbf{Set}}(\{\bullet\}, X) \tag{69.35}$$

So that on objects, this is a self duality : the complete atomic boolean algebra based on X is done on the same set as X.

Example of Lawvere distribution : use $\mathbb{N}$ for an enriched category?

The trivial case is to simply pick the usual case of **Set** enriched over itself, which is just the usual categorical case, then define the endofunctors

$$F : \mathbf{Set} \to \mathbf{Set} \tag{69.36}$$
$$G : \mathbf{Set} \to \mathbf{Set} \tag{69.37}$$

then define the mixed variance functor

$$F \otimes G : \mathrm{Set}^{\mathrm{op}} \times \mathbf{Set} \to \mathbf{Set} \tag{69.38}$$
$$(X, Y) \mapsto F(X) \otimes G(Y) \tag{69.39}$$

Case of a geometric map $\mu : \mathbf{Set} \rightleftarrows \mathbf{Set}$, point of a topos, "dirac" distribution?

Distribution : case $X = \{\bullet\}$:

70

The Sierpinski topos

The simplest non-trivial site is the one given by the interval category $\Delta[\vec{1}]$ in the simplex category, with a distinct initial object and terminal object. The Sierpinski topos[89, 92] is the presheaf category over that site :

$$\mathbf{SierpTop} = \mathrm{PSh}(\Delta[\vec{1}]) \tag{70.1}$$

As a presheaf topos, any object of the topos F is defined by two sets, $F(0)$ and $F(1)$, along with the restriction map

$$r : F(1) \to F(0) \tag{70.2}$$

so that any presheaf is fundamentally defined by a function between two sets.

Theorem 70.0.1. The Sierpinski topos is equivalent to the sheaf topos over the Sierpinski topological space.

Proof. The Sierpinski site is given by the set of two elements $\{0, 1\}$ with the topology $\{\varnothing, \{1\}, \{0, 1\}\}$ and the poset structure : The coverage of each element

$$\begin{array}{c} \{0,1\} \\ | \\ \{1\} \\ | \\ \varnothing \end{array}$$

is given by

$$\begin{array}{rrcl} \varnothing : & \varnothing & \hookrightarrow & \varnothing \\ \{1\} : & \varnothing & \hookrightarrow & \{1\} \\ & \{1\} & \hookrightarrow & \{1\} \\ \{0,1\} : & \varnothing & \hookrightarrow & \{0,1\} \\ & \{1\} & \hookrightarrow & \{0,1\} \\ & \{0,1\} & \hookrightarrow & \{0,1\} \end{array}$$

Then any sheaf in the sheaf topos is given by the three sets $F(\varnothing)$, $F(\{1\})$, and $F(\{0,1\})$, along with the restriction maps

$$r_{\{0,1\},\{1\}} : F(\{0,1\}) \to F(\{1\}) \tag{70.3}$$
$$r_{\{1\},\varnothing} : F(\{1\}) \to F(\varnothing) \tag{70.4}$$
$$r_{\{0,1\},\varnothing} : F(\{0,1\}) \to F(\varnothing) \tag{70.5}$$

such that

$$r_{\{0,1\},\varnothing} = r_{\{1\},\varnothing} \circ r_{\{0,1\},\{1\}} \tag{70.6}$$

Locality : there are no open covers here (except trivially), so no matter Gluing : Same

Isomorphism of topos :

$$\Phi : \mathrm{PSh}(\Delta[\vec{1}]) \to \mathrm{Sh}(\mathrm{Sierp}) \tag{70.7}$$
$$F \mapsto \Phi(F) \tag{70.8}$$

obeying

$$\Phi(F)(\{0,1\}) = F(1) \tag{70.9}$$
$$\Phi(F)(\{1\}) = F(0) \tag{70.10}$$
$$\Phi(F)(\varnothing) = 1 \tag{70.11}$$

□

The fact that the Sierpinski topos can be entirely defined by functions is also reflected by its isomorphism with the arrow category of sets :

Theorem 70.0.2. The Sierpinski topos is isomorphic to the arrow category of **Set** :

$$\mathbf{SierpTop} \cong \mathrm{Arr}(\mathbf{Set}) \tag{70.12}$$

Proof.

$$\Phi : \mathbf{SierpTop} \to \mathrm{Arr}(\mathbf{Set}) \tag{70.13}$$

$$F \to f \tag{70.14}$$

obeying

$$\Phi(F(0)) = s(f) \tag{70.15}$$

$$\Phi(F(1)) = t(f) \tag{70.16}$$

□

70.1 Limits and colimits

Being the sheaf topos of a topological space, its initial and terminal object are given by the initial and terminal sheaf.

Theorem 70.1.1. The initial object of **SierpTop** is the initial sheaf mapping all objects to the empty set, or equivalently, the empty function

$$f_\varnothing : \varnothing \to \varnothing \tag{70.17}$$

Theorem 70.1.2. The terminal object of **SierpTop** is the terminal sheaf mapping all objects to $\{\bullet\}$, or equivalently, the identity function on the singleton :

$$f_1 : \{\bullet\} \to \{\bullet\} \tag{70.18}$$

We can

70.2 Integration

Being a functor category, and having $\Delta[\vec{1}] \cong \Delta^{\mathrm{op}}[\vec{1}]$, we have that the Sierpinski cotopos

71

The topos of G-sets

For a given group G, the category of G-sets is the category whose objects are sets equipped with a G-group action, (X, ρ), where

$$\rho : G \times X \to X \tag{71.1}$$

and whose morphisms are the equivariant maps on those sets, ie for any morphism $f : (X_1, \rho_1) \to (X_2, \rho_2)$, f acts as a function on X_1 with the property that it is left equivariant :

$$f(\rho_1(g, X)) = \rho_2(g, f(X)) \tag{71.2}$$

This category forms a topos, and as it will furnish us a few useful counterexamples later on, let's look into it for a bit.

Definition 71.0.1. Given a group G as a monoid category, the topos of G-sets is the presheaf topos

$$G\mathbf{Set} = \mathrm{PSh}(G) \tag{71.3}$$

Being a site of a single object, the associated sheaves are simply sets, but there are non-trivial automorphisms on this object, which are mapped functorially onto those sets.

Theorem 71.0.1. The two definitions of the category of G-sets are isomorphic.

Proof. Given a presheaf F on G, we have that $F(\bullet)$ is any set, and each arrow of G gets mapped to some set automorphism of $F(\bullet)$.

Morphisms are natural transformations between presheaves $\square$

72

Topos on the simplex category

[93]

As one of our example of a presheaf, the simplicial sets, given by the presheaves on the simplex category

$$X : \mathbf{\Delta} \to \mathbf{Set} \tag{72.1}$$

form a topos, as they are simply the presheaf category over the simplicial category

$$\mathbf{sSet} = \mathrm{PSh}(\mathbf{\Delta}) \tag{72.2}$$

For each simplicial complex $\Delta[\vec{\mathbf{n}}]$, let's define the sets of n-simplices as

$$X_n = \Delta[\vec{\mathbf{n}}] \tag{72.3}$$

Forgetful functor to the various simplices?

Definition 72.0.1. A *simplicial set* X is composed of a sequence of sets $(X_n)_{n\in\mathbb{N}}$, its set of n-simplices, represented by a totally ordered set (ordered simplices)

For every injective map $\delta_i^n : [n-1] \to n$, there's a map $d_i^n : X_n \to X_{n1}$, the i-th face map on n-simplices

Theorem 72.0.1. Simplicial sets as presheaves on the simplex category and as collections of sets are isomorphic

Proof. □

Theorem 72.0.2. The simplicial category $\mathbf{\Delta}$ is a concrete site.

Proof. $\mathbf{\Delta}$ has a terminal object, which is $\Delta\left[\tilde{\mathbf{1}}\right]$. Its functor $\mathrm{Hom}_{\mathbf{\Delta}}(1, -)$ maps $\Delta\left[\tilde{\mathbf{n}}\right]$ to the set of n elements, and as we've seen,

$$\mathrm{Hom}_{\mathbf{\Delta}} = \tag{72.4}$$

□

Unlike more complex cases like sheaves over Cartesian spaces73, we will not have objects that do not "locally look like" the underlying category, as the category is locally finitely presentable : every object in it can simply be represented as colimits of [is that true]

[In case of non-injective map, pullback or something idk]

[Locally representable ergo locally presentable, means that we don't have any funny business?]

Initial and terminal object :

Theorem 72.0.3. The initial simplicial set 0 is the simplicial set of no points.

Theorem 72.0.4. The terminal simplicial set 1 is the constant simplicial set mapping every simplicial category to the singleton, equivalent to a point and degenerate higher simplices.

Proof. As the terminal sheaf, it is simply the sheaf

$$X(\Delta[\vec{\mathbf{n}}]) = \{\bullet\} \tag{72.5}$$

So that this is the simplicial set of a single point, for which the (degeneracy map?) □

Theorem 72.0.5. The product on **sSet** is the product of their underlying sets

$$(X \times Y)_k = X_k \times Y_k \tag{72.6}$$

for which the degeneracy maps and face maps are the products of those morphisms :

Proof. □

example :

Functors of **sSet** : to top, to simplicial complexes, etc

Concrete simplicial sets are simplicial complexes?

Non-concrete example : single point with multiple edges to itself?

72.1 Limits and colimits

Theorem 72.1.1. The initial object for simplicial sets is the empty simplicial set.

Proof. As usual, the initial sheaf maps every object of the site to the empty set, meaning that the underlying set 0_0 is the empty set, and likewise for any higher simplex 0_n. □

Theorem 72.1.2. The terminal object for simplicial sets is the points with every degenerate higher simplex mapped onto it.

Proof. The terminal object is as usual the map to the singleton $\{\bullet\}$ □

Theorem 72.1.3. The product of two simplicial sets is given by the simplicial set where every k-simplex is given by

$$(X \times Y)_k = X_k \times Y_k \tag{72.7}$$

and with face and degeneracy maps

$$d_i^{X\times Y}(x, y) = (d_i^X(x), d_i^Y(y)) \tag{72.8}$$

$$s_i^{X\times Y}(x, y) = (s_i^X(x), s_i^Y(y)) \tag{72.9}$$

Example 72.1.1. The product of two intervals $I \times I$ is a square.

Proof. The non-degenerate components of an interval are two points, $\{\bullet_0, \bullet_1\}$, and a line $\{\ell\}$, with the face map

$$s \tag{72.10}$$

So that the points are given by

$$\{\bullet_0 \times \bullet_0, \bullet_0 \times \bullet_1, \bullet_1 \times \bullet_0, \bullet_1 \times \bullet_1\} \tag{72.11}$$

or for short,

$$\{\bullet_0, \bullet_1, \bullet_2, \bullet_3\} \tag{72.12}$$

□

Theorem 72.1.4. The coproduct of two simplicial sets

72.2 Subobject classifier

The subobject classifier of the category of simplicial sets

73

Category of smooth spaces

[94, 95, 96, 97, 98, 99]

A more geometric category for a topos is the topos of smooth spaces **Smooth**, which is defined as the sheaf over the category of smooth Cartesian spaces,

$$\mathbf{Smooth} = \mathrm{Sh}(\mathbf{CartSp}_{\mathrm{Smooth}}) \tag{73.1}$$

The category of smooth spaces can be given a variety of sites equivalently, such as the site of smooth manifolds SmoothMan, the site of open subsets of $\mathbb{R}^n$, or the site of $\mathbb{R}^n$, all with morphisms being the smooth maps between such objects. As we will see, all those possible sites end up producing the same topos, and we will typically just use the category of Cartesian spaces **CartSp** where the only objects are $\mathbb{R}^n$, as this will make for the simpler site. This stems from the fact that the other sites can be constructed by limits and colimits of each other that we do not have to worry too much. While the objects are $\mathbb{R}^n$, this is of course only up to diffeomorphism so that this will include all manners of contractible domains like open balls.

The coverage of this site is slightly tricky. The most obvious cover is simply the coverage by open sets (diffeomorphic to $\mathbb{R}^n$ in our case). While we can construct a sheaf over this coverage (and it will in fact lead to an equivalent topos), there are coverages with better properties.

Definition 73.0.1. A *good open cover* is an open cover for which any finite intersection of open sets is contractible, ie a good open cover $\{f_i : U_i \to X\}$ of

X

$$\int \prod_{i\in I,\ X} U_i \cong \star \tag{73.2}$$

Example 73.0.1. The typical cover of the circle, S^1, is usually composed by two lines, U_N, U_S, where each line relates to the circle via the stereographic projection at the north and south. This cover is however not a good open cover as the overlap of those open sets is the coproduct of two intervals,

$$U_N \cap U_S \cong I + I \tag{73.3}$$

a good open cover of the circle will be for instance given by three intervals, each overlapping with each other in exactly one connected region,

$$U_i \cap U_j \cong I \tag{73.4}$$

This is taken typically taken as the cover as, from algebraic topology, we know that such covers have better properties, as the Čech cohomology of the cover will be identical to that of the space itself.

Theorem 73.0.1. Any smooth manifold admits a good open cover.

This implies a homeomorphism to the open ball

Definition 73.0.2. A good open cover $\{f_i : U_i \to X\}$ is a *differentially good open cover* if finite intersections of the cover are all diffeomorphic to the open ball.

$$\prod_{i\in I,\ X} U_i \cong B^k \tag{73.5}$$

Theorem 73.0.2. All three coverage of $\mathbf{CartSp}_{\text{smooth}}$ lead to isomorphic sheaves :

$$\text{Sh}(\mathbf{CartSp}_{\text{smooth}}, \mathcal{J}_{\text{open}}) \cong \text{Sh}(\mathbf{CartSp}_{\text{smooth}}, \mathcal{J}_{\text{good}}) \cong \text{Sh}(\mathbf{CartSp}_{\text{smooth}}, \mathcal{J}_{\text{diff}})$$

Proof. □

Therefore for our purpose we can pick the best behaved coverage.

Sheaves on the category of Cartesian spaces is best understood, in the context of geometry, as being *plots*, the more general version of what would be an atlas in the case of manifolds.

Definition 73.0.3. A *plot* is a map between an open set of a Cartesian space $\mathcal{O} \subseteq \mathbb{R}^n$ and a topological space X

From the Yoneda lemma, we have that for any sheaf $X \in \mathbf{Smooth}$ and Cartesian space U, we have the isomorphism

$$X(U) \cong \mathrm{Hom}_{[\mathbf{CartSp},\mathbf{Set}]}(よ(U), X) \tag{73.6}$$

$$よ(U) : \mathbf{CartSp}^{\mathrm{op}} \to \mathbf{Set} \tag{73.7}$$

$$O \mapsto \tag{73.8}$$

While this is a good intuitive way to understand the spaces probed by plots, it can be useful to know that in fact the topological space X itself is not necessary as a data to define a space cattaneo.

Theorem 73.0.3. Given the set of transition functions on a manifold, the topological space can be reconstructed as

$$M = \coprod O_i / \sim \tag{73.9}$$

where two points in $O_i \sqcup O_j$ are equivalent if $\tau_{ij}(x_i) = x_j$

This is what we do with the smooth sets topos, as we are only considering the existence of those maps (as the set $F(\mathbf{Cartsp})$), and the behavior of those plots over overlapping regions.

Example 73.0.2. Take the circle S^1, which we will defined a bit simplistically as a plot over $\mathbb{R}$. To avoid any issue of bad covers, let's take three different overlapping plots so that all overlaps are pairwise contractible. Just considering those plots, we get

$$S^1(\mathbb{R}) = \{\varphi_1, \varphi_2, \varphi_3\} \tag{73.10}$$

In terms of an atlas, if we considered our circle as the interval $[0, 1]$ with ends identified, we could cover it via the three intervals $(0, 3/4)$, $(1/2, 1)$ and $(3/4, 1/2)$ (simply pick any map $\mathbb{R} \to (0, 1)$ to appropriately rescale everything such as the arctan map)

[diagram]

with overlaps

$$1;2 \quad : \quad (1/2, 3/4) \tag{73.11}$$

$$1;3 \quad : \quad (0, 1/2) \tag{73.12}$$

$$2;3 \quad : \quad (3/4, 1) \tag{73.13}$$

with the transition maps

$$\tau_{12} = \tag{73.14}$$
$$\tau_{13} = \tag{73.15}$$
$$\tau_{23} = \tag{73.16}$$

In terms of overlap, we have $U^+ \cap U^- \cong I \sqcup I$, so that we need to consider additionally the plot of that Cartesian space (slightly complicated by the non-connected aspect of it, but we can consider the open set of the line $(-1,0) \cup (0,1)$. While we can do it this is partly why we generally consider good open covers)

$$S^1(I \sqcup I) = \{\varphi^\pm\} \tag{73.17}$$

which maps this overlap region onto S^1. The inclusion of this overlap area is done as

$$\iota_+(x \in I \sqcup I) = x \tag{73.18}$$
$$\iota_-(x \in I \sqcup I) = 2 - x \tag{73.19}$$

The overlap in terms of the plot is that we map $(-1,0) \cup (0,1)$ to the interval I as

Those morphisms on **CartSp** are mapped onto opposite mappings on **Set** :

$$S^1(\iota_+) : \{\varphi^+, \varphi^-\} \to \{\varphi^\pm\} \tag{73.20}$$

If we take the less abstract case of a concrete sheaf to look at smooth spaces, considering CartSp is a concrete site, the concrete presheaf of Sh(CartSp) is the category of *diffeological spaces* DiffeoSp, where each global element $X : 1 \to$ DiffeoSp is a diffeological space.

[Diff is a quasitopos]

An important subcategory is also the category of smooth manifolds **SmoothMan**.

$$\mathbf{SmoothMan} \subseteq \mathbf{DiffeoSp} \subseteq \mathbf{Smooth} \tag{73.21}$$

SmoothMan is not itself a topos, as it lacks an exponential object (Hom-sets between manifolds such as $C^\infty(M,N)$ are not themselves manifolds, although they are close to it [100]), and the quotients or equalizers of manifolds are not themselves manifolds [examples]

Smooth manifolds are locally representable objects of **Smooth**. If $X : 1 \to$ **Smooth** is a concrete smooth space (diffeological space), it is locally representable if there xists $\{U_i \hookrightarrow X\}$, $U_i \in$ **Smooth** such that the canonical morphism out of the coproduct

$$\coprod_i U_i \to X \tag{73.22}$$

Is an effective epimorphism in **Smooth**.

$$\coprod_i U_i \times_X \coprod_j U_j \rightrightarrows \coprod_i U_i \to X \tag{73.23}$$

By commutativity of coproduct and pullback [prove it]

$$\coprod_{i,j} (U_i \times_X U_j) \rightrightarrows \coprod_i U_i \to X \tag{73.24}$$

Theorem 73.0.4.

$$\mathbf{Smooth} \cong \mathrm{Sh}(\mathbf{SmoothMan}) \tag{73.25}$$

An important property of **Smooth** is that it contains a large proportion of **Top**, more specifically the category of

Status wrt top, delta generated top, etc

Due to this wide variety of physically important objects in **Smooth**, it will typically be (or at least some wider categories that we will define later) the topos serving as the setting for physics in general.

Subobject classifier : for any $U \in$ **CartSp**, $\Omega(U)$ is the set of subsheaves of h_U.

Finer diffeology same as subobject relation of sheaves?

Definition 73.0.4. For a smooth space X, given a set F of parametrizations of X, the *diffeology generated by* F is the finest diffeology containing F.

$$\langle F \rangle = \bigcap \mathcal{D} \tag{73.26}$$

Example 73.0.3. For any diffeological space, the diffeology generated by the empty family $F = \varnothing$ is the discrete diffeology.

Dimension [101]

Definition 73.0.5. For a smooth space X, the dimension of X is the infimum of the dimension of its generating family.

$$\dim(X) = \inf_{\langle F \rangle = D} \dim(F) \tag{73.27}$$

[67]

Definition of dimension via the use of a subtopos of sieves on sites of specific dimension for the factoring through?

Note that the topos of smooth sets is much larger than that of manifolds, including infinite dimensional manifolds, and contains quite a lot of spaces which do not have any obvious interpretation in those terms.

Example 73.0.4. The *wire diffeology* is given by taking the standard diffeology on $\mathbb{R}^2$ [representative sheaf?] and only keeping the plots for which the parametrization factors through $\mathbb{R}$, ie for any parametrization $p : U \to \mathbb{R}^2$, for any $u \in U$, there exists an open neighbourhood $u \in V \subseteq U$ with a smooth map $F : V \to \mathbb{R}$ and a smooth curve $q : \mathbb{R} \to \mathbb{R}^2$ such that $p\big|_V = q \circ F$ [Same differential structure as $\mathbb{R}^2$ 1] The identity map is not a plot

73.1 Limits and colimits

Proposition 73.1.1. The initial object of **Smooth** is the constant functor

$$\Delta_\varnothing : \mathbf{CartSp}_{\mathrm{Smooth}} \to \mathbf{Set} \tag{73.28}$$

which maps every cartesian space to the empty set $\varnothing$.

The interpretation of this is that the initial object can be seen as the empty manifold, with the empty atlas.

Theorem 73.1.1. The terminal object of **Smooth** is the constant functor

$$\Delta_{\{\bullet\}} : \mathbf{CartSp}_{\mathrm{Smooth}} \to \mathbf{Set} \tag{73.29}$$

which maps every cartesian space to the singleton $\{\bullet\}$.

The interpretation of this is that the terminal object of **Smooth** is a *point*. This can be shown as the space obviously has a single point, since $p : \mathbb{R}^0 \to \{\bullet\}$, but every other plot factors through $\mathbb{R}^0$, since for any map $!_n : \mathbb{R}^n \to \mathbb{R}^0$, unique for every Cartesian space, we have

20.3.2

as in particular the plot of points $p : \mathbb{R}^0 \to \{\bullet\}$ is the only one which is an isomorphism. This can be seen by the fact that this smooth space is generated by the diffeology $\mathbb{R}^0 \to \{\bullet\}$, by taking the pullbacks of every map through $\mathbb{R}^0$. ie for our further maps $\mathbb{R}^k \to \{\bullet\}$, those can all be generated through the unique map $\mathbb{R}^k \to \mathbb{R}^0$,

$$\mathbb{R}^k \xrightarrow{!_{\mathbb{R}^k}} \mathbb{R}^0 \xrightarrow{p} \{\bullet\} \tag{73.30}$$

Theorem 73.1.2.

Important functors :

Definition 73.1.1. The forgetful functor

$$U_{\mathbf{Set}} : \mathbf{Smooth} \to \mathbf{Set} \tag{73.31}$$

is the functor mapping every smooth space to its plot of points.

73.2 Subobject classifier

As a sheaf topos, the subobject classifier of **Smooth** is the sheaf associating any Cartesian space $\mathbb{R}^n$ to the set of sieves closed under the good open coverage on that object

Collection of subopens of X that are closed under union?

$$\Omega(X) = \{\} \tag{73.32}$$

acting on morphisms as

$$\Omega(f : X \to Y) = f^* \tag{73.33}$$

(pullback of sieves)

Weak subobject classifier? [95]

73.3 Subcategories of smooth sets

As a sheaf topos, we have that all objects of the site correspond themselves to objects of the topos via the representable presheaves. Any Cartesian space $\mathbb{R}^k$ therefore has a corresponding smooth set via

$$X_{\mathbb{R}^k} = \mathrm{Hom}_{\mathbf{CartSp}}(-, \mathbb{R}^k) \tag{73.34}$$

such that their plots are given by

$$\mathrm{Plot}_{X_{\mathbb{R}^n}}(\mathbb{R}^l) = \mathrm{Hom}_{\mathbf{CartSp}}(\mathbb{R}^l, \mathbb{R}^k) = C^\infty(\mathbb{R}^l, \mathbb{R}^k) \tag{73.35}$$

and their transition functions are

The subcategory of concrete sheaves in **Smooth** is of particular importance as it represents the diffeological spaces.

Definition 73.3.1. A *diffeological space* is a pair $(X, \mathcal{D})$ of a set X and a *diffeology* $\mathcal{D}$, which is a collection of plots such that

- covering axiom : every constant map is a plot
- : locality axiom : Given a map $\phi : U \to X$, if for every point $p \in U$, there is a neighborgood $U_p \subseteq U$ such that ϕ_{U_p} is a plot, then ϕ itself is a plot.
- smooth compatibility axiom : if $\phi : U \to X$ is a plot and f is a smooth map $V \subseteq \mathbb{R}^n \to U$, then $\phi \circ f$ is a plot.

Theorem 73.3.1. The category of diffeological spaces and of concrete smooth spaces are isomorphic.

Proof. □

Example 73.3.1.

Smooth manifolds : locally representable sheaves

One important aspect of smooth sets is that it is possible to embed a wide variety of topological spaces into them (on a practical standpoint, pretty much any such space that we may wish to use in physics), making it an appropriate category for the notion of a "topological topos" that we discussed in 52.

First, we can map between **Top** and **Smooth**, via the following adjunction :

Theorem 73.3.2. There is an adjunction between the category of topological spaces and diffeological spaces,

$$(\mathrm{Dtplg} \dashv \mathrm{Cdfflg}) : \mathbf{Top} \underset{\mathrm{Cdfflg}}{\overset{\mathrm{Dtplg}}{\rightleftarrows}} \mathbf{DifflgSp}$$

Proof. Given a topological space X, we can construct a diffeological space $\overline{X}$ via the *continuous diffeology*. Given the same underlying set,

$$\overline{X}(\mathbb{R}^0) = |X| \tag{73.36}$$

its diffeology is given by the set of all continuous functions from $\mathbb{R}^n$ (taken as a topological space with the standard topology) to X. This gives us an appropriate diffeology as all constant continuous functions are continuous (leading to all constant functions being plots),

Functoriality

Conversely, given a diffeological space X, we can define a topological space $\overline{X}$, called the *D-topology*, for which we have likewise the same underlying set

$$X(\mathbb{R}^0) = |\overline{X}| \tag{73.37}$$

□

Furthermore, if we consider the subcategory of Δ-topological spaces, this mapping leads to a fully faithful embedding into **Smooth**.

Theorem 73.3.3. Δ-topological spaces are a reflective subcategory of diffeological spaces.

Proof. □

Those functors will help us get the appropriate underlying topologies of our spaces [Are they logical functors]

Example 73.3.2. The topology of $よ(\mathbb{R}^n)$ is the standard topology on $\mathbb{R}^n$.

Proof. □

73.4 Stalks

Stalks via family of open balls?

73.5 Non-concrete objects

As we've seen, the concrete sheaves in **Smooth** do not form the entire topos, leaving non-concrete sheaves.

[Coarse moduli spaces?]

The archetypal example of this is the smooth set of differential k-forms,

$$\Omega^k : \mathbf{CartSp} \rightarrow \mathbf{Set} \quad (73.38)$$
$$U \mapsto \Omega^k(U) \quad (73.39)$$

which associates to every Cartesian space the set of k-forms over that space. The value of objects is simply the set of all possible k-forms on that Cartesian space, while the morphisms $f : U_1 \to U_2$ induce a pullback

Theorem 73.5.1. The universal smooth set of differential k-forms is a smooth set.

Proof. Let's take the function associating to any Cartesian space its set of differential k-forms (where we consider here only the objects of the category)

$$\Omega^k : \mathbf{CartSp} \to \mathbf{Set} \quad (73.40)$$

To rise to a functor, the smooth maps between Cartesian space must obey functorial rules, ie for some smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$, we have a corresponding function (in opposite order for a presheaf)

$$\Omega^k(f) : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^m) \tag{73.41}$$

which is given by the pullback of differential form : any k-form of $\mathbb{R}^n$ is mapped to a k-form of $\mathbb{R}^m$ via f :

$$\Omega^k(f)(\omega) = f_* \omega \tag{73.42}$$

and we indeed have $\mathrm{Id}_* = \mathrm{Id}_{\mathbf{Set}}$, with the composition law [...]

Sheaf properties :

$\square$

If we attempt to look at the set of "points" of this space, if that term can be applied here, that would be the plot of the terminal object in the site, $\mathbb{R}^0$. But of course, in the sense of the sheaf as described here, this will just be the set of all k-forms over the point $\Omega(\mathbb{R}^0)$, which will just include the zero section, so that if we try to consider this plot as the "point content" of the space, there is but a single point :

$$\Omega(\mathbb{R}^0) = \{0\} \tag{73.43}$$

As we would not really consider the elements of this space to be that single section, it is therefore important to be mindful of what the plots of the sheaf represent.

Global sections :

$$\Gamma(\Omega^k) = \mathrm{Hom}_{\mathbf{Smooth}}(1, \Omega) \tag{73.44}$$

$$\Omega^k(X) \cong [X, \Omega^k] \tag{73.45}$$

Specific k-form on X : $1 \to [X, \Omega^k]$

Value of the k-form at a point :

Example 73.5.1. Another non-concrete object is the moduli space of Riemannian structures

Example 73.5.2. Moduli space of symplectic structures

73.6 Important objects

The category of smooth spaces contains most of the objects of importance in physics and other fields, so that it is useful to look at the various types of objects within it.

First, as a topos, it has a terminal object as we've seen (the constant sheaf 1 which maps all probes to a single element, the constant plot). From this and the coproduct, we can construct objects similar to sets as we wish (this is in fact what the discrete functor will be later on), and as with any topos, a natural number object N in particular.

As the coverage is subcanonical, the Yoneda embedding makes any Cartesian space a smooth space via its representable presheaf,

$$よ(\mathbb{R}^n) : U \mapsto \mathrm{Hom}_{\mathbf{CartSp}}(U, \mathbb{R}^n) \tag{73.46}$$

As we have seen, any diffeological space is a smooth space, in fact every concrete smooth space is a diffeological space. Manifolds are in particular the locally representable concrete objects.

By the Cartesian closed character of the topos, for any pair of manifolds, the set of all smooth maps between them is itself a smooth space, ie

$$C^\infty(M, N) \in \mathbf{Smooth} \tag{73.47}$$

What is the underlying topology? Compact open?

$$\mathrm{Cdfflg}([M, N]) \tag{73.48}$$

We can also ask for various additional properties on those exponential objects. For instance, for two objects X, Y which are internal vector spaces, we can ask for the subset X^Y that respects

Theorem 73.6.1. Linear maps between two internal vector spaces are a smooth space.

Proof. □

Full category of real vector spaces internal to smooth?

Important classes of non-concrete sheaves are the *moduli spaces*, which are sheaves giving back appropriate function spaces on a Cartesian space. For instance the moduli space of Riemannian metrics Met is a sheaf

$$\begin{aligned} \mathrm{Met} : \mathbf{CartSp}^{\mathrm{op}} &\to \mathbf{Set} && (73.49)\\ U \subseteq \mathbb{R}^n &\mapsto && (73.50) \end{aligned}$$

where $\mathrm{Met}(U)$ is the set of all Riemannian metrics on U. For instance, as there is only one metric on a point (since the tangent bundle there is zero dimensional), we have

$$\mathrm{Met}(\mathbb{R}^0) = \{0\} \tag{73.51}$$

And there is only one component to the metric on the line which must also be positive, so its set of metric is that of the positive definite smooth functions.

Other moduli structures of importance are the moduli space of differential forms $\Omega^\bullet$,

$$[X, \Omega^\bullet] = \Omega^\bullet(X) \tag{73.52}$$

and for classical mechanics, the moduli space of symplectic forms omega

True for any section?

Theorem 73.6.2. The moduli spaces of sections is a smooth space

Moduli space of connections

Other moduli spaces of importance are gauge-related ones, such as the moduli space of principal bundles, but this will be looked at in a more general lens in VII.

73.7 Internal objects

Definition 73.7.1. An internal group of **Smooth** is a *Lie smooth space.*

A particular case of this is the common case of Lie groups, which is simply an internal group whose underlying object is a manifold.

A more general example of a Lie smooth space that is not a Lie group would be for instance the diffeomorphism group. Given the power object of a manifold M^M, the diffeomorphism group is the subobject of invertible endomorphisms

Theorem 73.7.1. The diffeomorphism group is a smooth space.

Proof. □

This gives Diff(M) a natural internal group structure by composition.

As with many categories, one of the fundamental internal ring of the topos is $\mathbb{R}$, the representable functor of $\mathbb{R}$ itself.

Theorem 73.7.2. The representable object $R = よ(\mathbb{R})$ is a ring object.

Proof. □

A particularly important class of internal objects for smooth spaces is that of internal R-modules. By default, our internal function spaces are of the form of internal homs, $[X, Y]$. If we wish to speak of some real-valued function, it is an object in $[X, R]$, but there is no notion of this space being a module. Maps $X \to R$ are not enriched as modules (there is no notion of adding or multiplying them), nor can we do so with elements of $[X, R]$.

This is where the notion of internal R-module comes in. An internal R-module is fairly clear from the other notions of internalization seen before, we simply have some object M which is an internal Abelian group $(M, +)$ along with some morphism $\rho : R \times M \to M$, the action of the ring object R on M.

Theorem 73.7.3. Any internal hom $[X, R]$ to a ring object R defines an R-module object.

Proof. First we must show the structure of $[X, R]$ as an Abelian group object. As a ring object, we have some smooth map $+ : R \times R \to R$ and zero map $0 : 1 \to R$ [...] We need to find some equivalent morphisms on $[X, R]$. For the addition, this is

$$+ : [X, R] \times [X, R] \to [X, R] \tag{73.53}$$

With some implicit braiding to move the product around, this is constructed using the adjunct of this morphism :

$$\tilde{+} : [X,R]\times[X,R]\times X \overset{\mathrm{Id}_{[X,R]\times[X,R]}\times\Delta_X}{\longrightarrow} [X,R]\times[X,R]\times X\times X \overset{\mathrm{ev}_{X,R}\times\mathrm{ev}_{X,R}}{\longrightarrow} R\times R \overset{+}{\longrightarrow} R$$

whose right adjunct is

$$+ : [X, R] \times [X, R] \to [X, R] \tag{73.54}$$

and similarly for the additive inverse, this is some morphism $[X, R] \to [X, R]$, given by

$$\tilde{-} : [X, R] \times X \overset{\mathrm{ev}_{X,R}}{\longrightarrow} R \overset{-}{\longrightarrow} R \tag{73.55}$$

with likewise its right adjoint. The zero morphism is given by

$$\tilde{-} : [X, R] \times X \xrightarrow{\mathrm{ev}_{X,R}} R \xrightarrow{-} R \tag{73.56}$$

proof of associativity

proof of unitality proof of invertibility □

Does $[X, R]$ also form an algebra?

Basic example : The real line itself is equivalent to the algebra over a point? $[1, R]$

73.8 Dualities

73.9 Integration

As a category of locally Euclidian spaces which contains its own functions, one notion that we would like to see expressed in **Smooth** is that of integration, and more generally of distributions. If we have our space of functions $[X, R]$, our real line object R and some subspace of X in its power object X^{Ω}, we would like to have some notion of associating those functions over those subspaces to a value.

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Category of classical mechanics

The exact category to give to classical mechanics is somewhat controversial, due to the difficulties of finding an appropriate notion of morphisms. If we pick the most obvious candidate (symplectic manifolds and symplectomorphisms), all morphisms have to preserve the symplectic form. For two symplectic spaces (P_1, ω_1) and (P_2, ω_2), the map $f : (P_1, \omega_1) \to (P_2, \omega_2)$ implies

$$f^* \omega_2 = \omega_1 \tag{74.1}$$

[...]

An alternative category is given by the Weinstein symplectic category, whose objects are symplectic manifolds

alternative : the *category of Poisson manifolds*?

Definition 74.0.1. A *Poisson manifold* (P, π) is a manifold P equipped with a Poisson bivector $\pi \in \Gamma(\bigwedge^2 P)$

Poisson bracket :

$$\{f, g\} = \langle df \otimes dg, P \rangle \tag{74.2}$$

Definition 74.0.2. An *ichtyomorphism* is a smooth map preserving the Poisson bivector : $f^* \pi = \pi$

From this, the category of Poisson manifolds **Poiss** is the category with Poisson manifolds as objects and ichtyomorphisms as morphisms. In terms of the smooth topos, we have to look at the moduli space of Poisson bivectors :

$$\Pi(-) : \mathbf{CartSp} \to \mathbf{Set} \tag{74.3}$$
$$\mathbb{R}^n \mapsto \tag{74.4}$$

which is the set of all Poisson bivectors over $\mathbb{R}^n$. A choice of a Poisson manifold is therefore given by the internal hom of our underlying manifold and the moduli space of Poisson bivectors :

$$P(X) = [X, P] \tag{74.5}$$

Theorem 74.0.1. The category of symplectic manifolds is isomorphic to the slice topos $\text{Smooth}_{/\Omega^2}$ over the moduli space of symplectic forms Ω^2.

Proof. □

Poisson manifold : locally representable concrete object?

What are the morphisms in $\text{Smooth}_{/\Omega^2}$

74.1 Limits and colimits

Theorem 74.1.1. The product of two Poisson manifolds

74.2 Logic

The logic of classical mechanics is tied to the logic of measurement of observables. If we have some classical theory, with a Poisson manifold

Example : phase space of a point particle in n dimensions $\mathbb{R}^{2n}$, with the Poisson bracket

If we have some observable

$$f_o : \mathbb{R}^{2n} \to \mathbb{R} \tag{74.6}$$

Inversely, f_o selects a subset of the Poisson manifold. The statement that the measurement m_o is in the Borel subset $\Delta_o \subseteq \mathbb{R}$ is equivalent to a subobject of **Poiss**

$$S_o = f_o^{-1}(\Delta_o) \tag{74.7}$$

Limits and colimits :

$$S_{o_1} \sqcup S_{o_2} \tag{74.8}$$

What is the topos

Logic and presheaf etc

[103, 104, 105]

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Categories for quantum theories

[106, 107, 108, 109, 110]

To contrast with the topos we have seen for classical mechanics, we need to also consider an appropriate category to look at the behaviour of quantum mechanics, to see if specific categorical properties have some important things to tell us about quantum systems.

There are for this quite a lot of different categories we can look at. These are :

- The category of Hilbert spaces
- The slice category of a Hilbert space
- The spectral presheaf of a Hilbert space
- The Bohr topos

Those categories can all be used for quantum mechanics with various results [etc]

[111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126]

75.1 Quantum mechanics as a symmetric monoidal category

The basic formulation of quantum mechanics in terms of category theory is to simply look at the categories of its main objects, which are Hilbert spaces and C^*-algebras.

Definition 75.1.1. The category **Hilb** of Hilbert spaces has as its objects Hilbert spaces and as morphisms bounded linear maps between two Hilbert spaces.

The condition of bounded linear maps is here to guarantee the existence of a dual on every operator, which will be of use for us, as otherwise linear maps are not guaranteed a dual in the infinite dimensional case, for instance by picking the linear map on $L^2(\mathbb{R}^n)$

$$f(\psi) = \psi(0) \tag{75.1}$$

which is indeed linear etc, but corresponds to a "dual vector" $\chi = \delta_0$, which is not an $L^2(\mathbb{R}^n)$ Hilbert space, but part of the rigged Hilbert space of the position operator, $\Phi^*_{\hat{x}}$.

[proof]

Hilb can be entirely defined in categorical terms[127], so that we will look at this in a bit more detail to get some feel of the categorical structures involved here.

If we were to say them all at once, then **Hilb**, the category of (complex) Hilbert spaces with bounded linear operators, is defined as :

- A dagger compact symmetric monoidal category with duals, $(\mathbf{Hilb}, \otimes, I, \dagger, {}^*)$
- The monoidal unit I is a separator
-

The monoidal symmetric part is simply enough the tensor product $\otimes$, which is symmetric by interchange, and the monoidal unit is $\mathbf{C}$. The dagger † corresponds to the adjoint on operators, and the compact closedness means that any object X has a dual object X^*, with the unit and counit

$$\eta : \tag{75.2}$$

[...]

Interpretations :

The monoidal unit I can be shown to be equivalent to the complex numbers. First we take the hom-set

$$\mathrm{End}(I) = \mathrm{Hom}_{\mathbf{Hilb}}(I, I) \tag{75.3}$$

and show that it has some scalar structure.

In terms of quantum mechanics, we have that for a given object $\mathcal{H}$, the Hilbert space vectors are given by morphisms $\psi : \mathbb{C} \to \mathcal{H}$, as the underlying field is the free Hilbert space here and therefore the appropriate object for generalized elements. Dually, maps of the form $\omega : \mathcal{H} \to \mathbb{C}$ form dual vectors on the Hilbert space

Dagger

In addition to all we've seen, we can also consider the enrichment of the category **Hilb** over Banach spaces, turning it into a C^**-category*. This enrichment means that for any three Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$,

[...]

Zero map, sum of maps, scalar product of maps, involution

Definition 75.1.2. Two subobjects $\iota_1 : \mathcal{H}_1 \hookrightarrow \mathcal{H}$, $\iota_2 ! \mathcal{H}_2 \hookrightarrow \mathcal{H}$ are said to be *orthogonal* if $\iota_1^* \iota_2 = 0$.

Theorem 75.1.1. The duality of an operator defines a canonical splitting of monomorphisms in that for any inclusion map $\iota : \mathcal{H}' \hookrightarrow \mathcal{H}$,

$$\iota : \tag{75.4}$$

Proof. This is the equivalent of requiring orthogonal projectors in Hilbert spaces, where a projection is orthogonal if for $\psi \in \mathcal{H}$,

$$\langle P\psi, (1 - P)\psi' \rangle = 0 \tag{75.5}$$

meaning in categorical terms that the adjoint map $P\psi$ □

Those properties are in fact those of a C^*-algebra. As the notion of a C^*-algebra will be quite important later on for our other quantum categories, it is good to keep in mind.

Theorem 75.1.2. The C^* algebra of an object $\mathcal{H} \in$ **Hilb** is its hom-object with the Banach space enrichment.

In addition to C^*-algebras, we will also need to look at the von Neumann algebras of our Hilbert spaces

Definition 75.1.3. A von Neumann algebra (or W^*-algebra) is a C^*-algebra A that admits a predual, a complex Banach space A_* with an isomorphism of complex Banach spaces

$$* : A \to (A_*)^* \tag{75.6}$$

75.2 Slice category of Hilbert spaces

For a short detour in terms of interpreting quantum theory in a categorical manner, we will need to look at slice categories of **Hilb**. If we pick some Hilbert space $\mathcal{H}$, the slice category $\mathbf{Hilb}_{/\mathcal{H}}$ will be the appropriate setting to talk about this specific Hilbert space.

The objects of $\mathbf{Hilb}_{/\mathcal{H}}$ are the (bounded) linear maps from any Hilbert space to $\mathcal{H}$. For our interest later on, this contains in particular all the subspace maps from all subspaces of $\mathcal{H}$. As the slice category of an enriched category, it is furthermore an enriched slice category,

Hilb is a category in which every monomorphism is split, so every object of our slice category can be give by an appropriate retract, a projection.

[unique by orthogonal projections?]

75.3 Daseinisation

While monoidal categories are a perfectly serviceable setting for dealing with quantum mechanics, it has a few issues making it unsuitable for this analysis. In some sense it corresponds to the construction of an actual "quantum object" with an existence independent of measurement, giving it fairly problematic properties from a logical perspective (this is the content of the Kochen-Specker theorem). Due to this it also famously fails to be a topos, which is the main object we are concerned with here.

To deal with those problems, we have to deal with the *Daseinisation*[120] of the category, where rather than deal with some quantum object directly, we only consider its measurements in some context.

The simplest way to consider a measurement in quantum mechanics is to look at the projectors P of the theory. If we ignore the wider case of positive operator-valued measure and only look at projection-valued measure (we will assume no additional source of uncertainty beyond quantum theory), every measurement in a quantum theory can be modelled by this. If a measurement is associated with an observable A with spectrum $\sigma(A)$, and of projection-valued measure

$$E : \Sigma(\sigma(A)) \to \mathrm{Proj}(\mathcal{H}) \tag{75.7}$$

$$\Delta \mapsto E(\Delta) \tag{75.8}$$

The Born rules is that the probability of the measurement lying in some measurable subset of the spectrum Δ is

$$P(X \in \Delta|\psi) = \langle \psi, E(\Delta)\psi \rangle \tag{75.9}$$

After said measurement the system will collapse to the state $E(\Delta)\psi$. Our logic is that a system is indeed such that $X \in \Delta$ if it was last measured to be so. The creation of a *context* from there is to consider the set of all measurements composed from commutative operators so that they can be said to be both true at the same time in a manner consistent with classical logic. If we have another measurement derived from an observable A' with a projection-valued measure E', the two PVM commute, in the sense that for any two measurable subsets of their spectra, $\Delta \subset \sigma(A), \Delta' \subset \sigma(A')$, we have

$$E(\Delta)E(\Delta') = E(\Delta')E(\Delta) \tag{75.10}$$

Meaning that if we have done a first measure $E(\Delta)$ (meaning $x \in \Delta$), and a second measure $E'(\Delta')$ ($x' \in \Delta'$), a third measure of the original quantity will yield the same result :

First measurement : Collapse

$$\psi \to \frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \tag{75.11}$$

Second measurement : Collapse $E(\Delta)\psi$ to $E'(\Delta')E(\Delta)\psi$

$$\frac{E(\Delta)\psi}{\|E(\Delta)\psi\|} \to \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|} \tag{75.12}$$

Third measurement :

$$\begin{aligned}
P(X \in \Delta | \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}) &= \langle \frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}, E(\Delta)\frac{E'(\Delta')E(\Delta)\psi}{\|E'(\Delta')E(\Delta)\psi\|}\rangle \\
&= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2}\langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)E(\Delta)\psi\rangle \\
&= \frac{1}{\|E'(\Delta')E(\Delta)\psi\|^2}\langle E'(\Delta')E(\Delta)\psi, E'(\Delta')E(\Delta)\psi\rangle \\
&= 1
\end{aligned} \tag{75.13}$$

Therefore in a context, we can say that the measured values are "real" in that they do not depend on the measurement.

As the identity and the zero projector both commute with every operator, they are a part of every context.

We will furthermore need the notion of ordering of projectors, which corresponds to the ordering of the lattice in quantum logic, ie we say that two projectors P_1, P_2 are ordered if

$$P_1 \leq P_2 \leftrightarrow \mathrm{im}(P_1) \subseteq \mathrm{im}(P_2) \tag{75.14}$$

or equivalently, $P_1 P_2 = P_2 P_1 P = P_1$. This means

Example : Given a projection-valued measure P and a measurable set of its spectrum Δ, with some subset $\Delta' \subseteq \Delta$, by the rules

$$E(\Delta') = E(\Delta' \cap \Delta) = E(\Delta')E(\Delta) \tag{75.15}$$

We therefore have $E(\Delta') \leq E(\Delta)$.

In terms of interpretation, this means that for $P \leq P'$, P' is *weaker* : we only know that our state is in some subspace larger than for P. This can be seen in the case of projection-valued measures on some interval, where the weaker statement is $x \in [a-\varepsilon_1, b+\varepsilon_2]$ compared to the more precise statement $x \in [a, b]$. The best one could find is in fact the 1-dimensional projector, as no projector is smaller than that (except for the zero projector which cannot provide any information), and corresponds to the measurement of the exact state. Due to this, two 1-dimensional projectors are never ordered, unless they are the same

$$\forall P, P',\ \dim(\mathrm{im}(P)) = \dim(\mathrm{im}(P')) = 1 \to (P \leq P' \leftrightarrow P = P') \tag{75.16}$$

Properties :

$$\forall P,\ 0 \leq P \tag{75.17}$$

$$\forall P,\ P \leq \mathrm{Id} \tag{75.18}$$

The point of daseinisation is to consider measurements in general not as projectors in the category of Hilbert spaces, but spread onto all possible contexts that a system may have by considering the closest approximation of that measurement in a given context. This approximation is given by the narrowest projection that is superior to our projector, ie for all the projectors P' in the context, we wish to find the one such that $P \leq P'$, and for any other projector P'' which also obeys $P \leq P''$, $P' \leq P''$. This projector is denoted by, for a context V, $\delta(P)_V$, the V-support of P. In terms of lattice notation, this is given by

$$\delta(P)_V = \bigwedge \{P' \in \mathrm{proj}(V) \mid P \leq P'\} \tag{75.19}$$

As Id is always part of every context and the supremum of any context, we are always guaranteed to have such a projector more precise or equal to the identity projector, which merely informs us that the state is in the Hilbert space at all and nothing more. If $P \in \mathrm{proj}(V)$, $\delta(P)_V = P$.

Example of a subset again

We will need to consider the approximation of $E(\Delta)$ in every possible contexts

$$P \to \{\delta(E(\Delta))_V | V \in \mathcal{V}(\mathcal{H})\} \tag{75.20}$$

Why is this a sheaf? Contexts are ordered

(in the usual category of compact symmetric monoidal objects etc of quantum logic) is transformed to a (clopen) sub-object $\delta(P)$ of the spectral presheaf in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$

Kochen-Specker theorem : equivalent to the presheaf on the category of self-adjoint operator has no global element

Take a C^*-algebra (von Neumann?) A.

Subcategory of commutative subalgebras $\mathrm{ComSub}(A)$ is the poset wrt inclusion maps

for any operator (self-adjoint?) A, let W_A be the spectral algebra. W_A is the boolean algebra of projectors $E(A \in \Delta)$ that projects onto the eigenspaces associated with the Borel subset Δ of the spectrum $\sigma(A)$. $E[A \in \Delta]$ represents the proposition $A \in \Delta$

Spectral theorem : for all Borel subsets J of the spectrum of $f(A)$, the spectral projector $E[f(A) \in J]$ for $f(A)$ is equal to the spectral projector $E[A \in f^{-1}(J)]$ for A. In particular, if $f(\Delta)$ is a Borel subset of $\sigma(f(A))$, since $\Delta \subseteq f^{-1}(f(\Delta))$,

$$E[A \in \Delta] \leq E[A \in f^{-1}(f(\Delta))] \tag{75.21}$$

$$E[A \in \Delta] \leq E[f(A) \in f(\Delta)] \tag{75.22}$$

This means $f(A) \in f(\Delta)$ is weaker than $A \in \Delta$. $f(A) \in f(\Delta)$ is a *coarse graining* of $A \in \Delta$.

If $A \in \Delta$ has no truth value defined, $f(A) \in f(\Delta)$ may have for some f

Relations between two logical systems here :

First, any proposition corresponding to the zero element of the Heyting algebra should be valued as false, $\nu(0_L) = 0_{T(L)}$.

If $\alpha, \beta \in L$, $\alpha \leq \beta$, then α implies β. Ex : $A \in \Delta_1$, $A \in \Delta_2$, $\Delta_1 \subseteq \Delta_2$. Valuation should be $\nu(\alpha) \leq \nu(\beta)$ (monotonicity).

If $\alpha \leq \alpha \vee \beta$, $\beta \leq \alpha \vee \beta$, then $\nu(\alpha) \leq \nu(\alpha \vee \beta)$ and $\nu(\beta) \leq \nu(\alpha \vee \beta)$, and therefore

$$\nu(\alpha) \vee \nu(\beta) \leq \nu(\alpha \vee \beta) \tag{75.23}$$

Not as strong as $\nu(\alpha) \vee \nu(\beta) = \nu(\alpha \vee \beta)$. For instance for $A = a_1$, $A = a_2$, the projection operator for both of these proposition projects on the 2D span of the eigenvectors, not their union.

Similarly,

$$\nu(\alpha \wedge \beta) \leq \nu(\alpha) \wedge \nu(\beta) \tag{75.24}$$

Exclusivity : a condition and its complementation cannot both be totally true :

$$\alpha \wedge \beta = 0_L \wedge \nu(\alpha) = 1_{T(L)} \to \nu(\beta) \leq 1_{T(L)} \tag{75.25}$$

Unity condition : $\nu(1_L) = 1_{T(L)}$

Take the boolean subalgebra W of the lattice $P(H)$ of projection operators. Forms a poset under subalgebra inclusion. W is a poset category.

Take the set $\mathcal{O}$ of all bounded, self-adjoint operators on $\mathcal{H}$. Spectral representation :

$$A = \int_{\sigma(A)} \lambda dE^A_\lambda \tag{75.26}$$

$\sigma(A) \subseteq \mathbb{R}$ the spectrum of A, $\{E^A_\lambda | \lambda \in \sigma(A)\}$ a spectral family of A.

$$E[A \in \Delta] = \int_\Delta dE^A_\lambda \tag{75.27}$$

for Δ a borel subset of $\sigma(A)$. If a belongs to the discrete spectrum of A, the projector ontop the eigenspace with eigenvalue a is

$$E[A = a] := E[A \in \{a\}] \tag{75.28}$$

for $f : \mathbb{R} \to \mathbb{R}$ any bounded Borel function,

$$f(A) = \int_{\sigma(A)} f(\lambda) dE^A_\lambda \tag{75.29}$$

Categorification of $\mathcal{O}$: Objects are elements of $\mathcal{O}$, morphisms from B top A if an equivalence class of Borel functions $f : \sigma(A) \to \mathbb{R}$ exists such that $B = f(A)$, ie

$$B = \int_{\sigma(A)} f(\lambda) dE^A_\lambda \tag{75.30}$$

Definition 75.3.1. The spectral algebra functor $W : \mathcal{O} \to W$ is

- Objects mapped $W(A) = W_A$, W_A is the spectral algebra of A
- Morphisms : if $f : B \to A$, then $W(f) : W_B \to W_A$ is the subset inclusion of algebras $i_{W_B W_A} : W_B \to W_A$.

Spectral algebra for $B = f(A)$ is naturally embedded in the spectral algebra for A since $E[f(A) \in J] = E[A \in f^{-1}(J)]$ for all Borel subsets $J \subseteq \sigma(B)$

$$i_{W_{f(A)}W}(E[f(A) \in J]) = E[A \in f^{-1}(J)] \tag{75.31}$$

[...]

Category $\mathcal{O}_d$ of discrete spectra self-adjoint operators

Definition 75.3.2. The spectral presheaf on $\mathcal{O}_d$ is the contravariant functor $\Sigma : \mathcal{O}_d \to \mathbf{Set}$

- $\Sigma(A) = \sigma(A)$ (spectrum of A)
- if $f_{\mathcal{O}_d} : B \to A$, so that $B = f(A)$, then $\Sigma(f_{O_d}) : \sigma(A) \to \sigma(B)$ is defined by $\Sigma(f_{\mathcal{O}_d})(\lambda) = f(\lambda)$ for all $\lambda \in \sigma(A)$

Works because on discrete spectrum $\sigma(f(A)) = f(\sigma(A))$.

$$\Sigma(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d}) = \Sigma(f_{\mathcal{O}_d}) \circ \Sigma(g_{\mathcal{O}_d}) \tag{75.32}$$

global section : function γ that assigns for every object of the site an element γ_A of the topos, such that if $f : B \to A$, then $H(f)(\gamma_A) = \gamma_b$.

For the spectral functor, a global section / element is a function that assigns to each self-adjoint operator A with a discrete spectrum a real number $\gamma_A \in \sigma(A)$, such that if $B = f(A)$, then $f(\gamma_A) = \gamma_B$.

Kochen-Specker theorem : if $\mathrm{Dim}(\mathcal{H}) > 2$, there are no global sections of the spectral presheaf.

Continuous case

By Gelfand duality, the presheaf topos PSh(ComSub(A)) contains a canonical object, the presheaf

$$\Sigma : C \mapsto \Sigma_C \tag{75.33}$$

which maps a commutative C^*-algebra $C \hookrightarrow A$ to (the point set underlying) its Gelfand spectrum Σ_C.

[128]

75.3.1 von Neumann algebras

projection

To formalize this idea, we will need to use the notion of von Neumann algebra. While we could merely use C^* -algebras, there will be a difficulty if we do so : the projectors of C^*-algebras do not form a complete lattice, ie there may be subsets $S \subseteq \mathrm{proj}(A)$ which lack a lower or upper bound.

An example for this would be the algebra of compact operators $K(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$

Every projection in $K(\mathcal{H})$ has finite rank :

$$\dim(\mathrm{im}(P)) < \infty \tag{75.34}$$

If we consider an infinite dimensional Hilbert space, like $L^2(\mathbb{R})$, consider this subset : $\{P_i\}_{i\in\mathbb{N}}$, such that P_i maps to an i-dimensional subspace, and we have

$$\mathrm{im}(P_i) \subset \mathrm{im}(P_{i+1}) \tag{75.35}$$

Union of these is dense in $\mathcal{H}$?

$$\overline{\bigcup_{i\in\mathbb{N}} \mathrm{im}(P_i)} = \mathcal{H} \tag{75.36}$$

Supremum : $\sup(\{P_i\}) = I$, but I is not a compact operator.

[Why it happens? Relation to topology]

This possible lack of supremum and infimum would lead to the absence of disjunctions and conjunctions in our category [cf logic chapter]. While not tragic (this would only affect infinite conjunctions of propositions), we will try to keep things complete.

To insure the completeness of the lattice, we will use instead von Neumann algebras

Definition 75.3.3. A *von Neumann algebra* A is a C^*-algebra with a predual A_*, a Banach space dual to A.

Weak operator topology : The basis of neighbourgoods of 0 given by sets of the form

$$U(x,f) = \{A \in L(V,W) | f(A(x)) < 1\} \tag{75.37}$$

for $x \in V$, $f \in W^* = \mathrm{Hom}_{\mathrm{TVS}}(W,k)$. A sequence of operators (A_n) converges to A iff $(A_n(x))$ in the weak topology on W.

von Neumann algebras are closed in weak operator topology : any limit of net converges.

[...]

Definition 75.3.4. An Abelian von Neumann algebra

Definition 75.3.5. For any Abelian von Neumann algebra over $\mathcal{B}(\mathcal{H})$, there exists a self-adjoint operator generating it as [...]

75.4 The Bohr topos

[129, 130]

A broader notion of categorical quantum theory is given by the *Bohr topos*. Rather than only consider the spectral presheaf of spectra associated with each projector,

As with the case of the spectral presheaf, we start out with the [Partial?] C^*-algebra A of our quantum system, and define its poset of commutative subalgebras $\mathrm{ComSub}(A)$. This can be given as the functor

$$\mathcal{C} : C^*\mathbf{Alg} \to \mathbf{Poset} \tag{75.38}$$

[Image of morphisms]

Furthermore, we consider the functor sending any poset to its Alexandrov topology

$$\mathrm{Alex} : \mathbf{Poset} \to \mathbf{Top} \tag{75.39}$$

Definition 75.4.1. Given a quantum system defined by its C^*-algebra A, its *Bohr site* is

$$B(A) = \mathrm{Alex}(\mathcal{C}(A)) \tag{75.40}$$

Definition 75.4.2. The *Bohr topos* of a C^*-algebra A is the ringed Grothendieck topos on the Bohr site,

$$\mathrm{Bohr}(A) = (\mathrm{Sh}(\mathrm{Alex}(\mathcal{C}(A))), \underline{A}) \tag{75.41}$$

with $\underline{A}$ the tautological copresheaf

$$\mathrm{Bohr}_{\neg\neg}(A) = (\mathrm{Sh}_{\neg\neg}(\mathrm{Alex}(\mathcal{C}(A))), \underline{A}) \tag{75.42}$$

Definition 75.4.3. The Bohrification of A is the tautological sheaf

$$\underline{A} \in \mathrm{Sh}(\mathrm{ComSub}(A)) \tag{75.43}$$

which maps any commutative subalgebra C to its underlying set

$$\underline{A}(C) = |C| \tag{75.44}$$

and morphisms of the inclusions $C \subseteq D$ to the injection of sets

$$\underline{A}(C \subseteq D) = |C| \hookrightarrow |D| \tag{75.45}$$

Theorem 75.4.1. The Bohrification of A is an internal C^*-algebra of the topos, with morphisms

$$0 : \underline{1} \quad \to \quad \underline{A} \tag{75.46}$$
$$+ : \underline{A} \times \underline{A} \quad \to \quad \underline{A} \tag{75.47}$$
$$\cdot : \tag{75.48}$$

Relationship between Bohr topos and the spectral presheaf? Is it a specific sheaf in the dual topos?

[131]

75.5 The finite dimensional case

Take the case $\mathbb{C}^n$ of the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$, which is for instance used in quantum computing. The C^*-algebra is just

$$C^*(\mathcal{H}) = L(\mathcal{H}, \mathcal{H}) \tag{75.49}$$

(denoted $L(\mathcal{H})$ for short), with operator composition as its algebraic operation and complex conjugate as involution, as all finite-dimensional linear maps are bounded. If we pick a specific basis, this is the algebra of $n \times n$ matrices on $\mathbb{C}^n$ with matrix multiplication.

The algebra $L(\mathcal{H})$ is also a von Neumann algebra [proof]

any subalgebra is a von Neumann subalgebra

The projections of this algebra are formed by the orthogonal projections, as any oblique projection would not be self-adjoint, classified by the Grassmannians of the space,

$$\bigoplus_{i=0}^{n} \mathrm{Grass}(i, \mathcal{H}) \tag{75.50}$$

$$P = I_r \oplus 0_{d-r} \tag{75.51}$$

An operator A will simply be one of the linear map $A \in L(\mathcal{H})$

A context here is an Abelian von Neumann subalgebra of $L(\mathcal{H})$. The category of contexts $\mathcal{V}(L(\mathcal{H}))$, equivalently a set of commuting matrices

"An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator."

Theorem 75.5.1. Any abelian von Neumann algebra on a separable Hilbert space is $*$-isomorphic to either

- $\ell^\infty(\{1, 2, \ldots, n\})$
- $\ell^\infty(\mathbb{N})$
- $L^\infty([0, 1])$
- $L^\infty([0, 1] \cup \{1, 2, \ldots, n\})$
- $L^\infty([0, 1] \cup \mathbb{N})$

There is therefore some surjection from the self-adjoint operators to commutative von Neumann algebras :

$$f : \mathcal{B}_{\text{sa}}(\mathcal{H}) \to \mathcal{V}(W^*(\mathcal{H})) \tag{75.52}$$

Spectral theorem :

Theorem 75.5.2. For a bounded self-adjoint operator, there is a measure space (X, Σ, μ) and a real-valued essentially bounded measurable function f on X and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that

$$U^\dagger T U = A \tag{75.53}$$

$$[T\varphi](x) = f(x)\varphi(x) \tag{75.54}$$

and $\|T\| = \|f\|_\infty$

Finite dimensional :

Theorem 75.5.3. There exists eigenvalues $\{\lambda_i\}$ (ordered by value by i) of A and eigen subspaces $V_i = \{\psi \in \mathcal{H} | A\psi = \lambda_i \psi\}$ such that

$$\mathcal{H} = \bigoplus_{i=1}^{n} V_j \tag{75.55}$$

Theorem 75.5.4. For self-adjoint A, there exists an orthonormal basis of eigenvectors of A.

Theorem 75.5.5. For a self-adjoint operator A with respect to an orthogonal matrix, there exists an orthogonal matrix T such that $T^{-1}AT$ is diagonal.

Theorem 75.5.6. For a self-adjoint operator A, there exists different eigenvalues $\{\lambda_i\}$, $i \leq j \to \lambda_i \leq \lambda_j$, and eigen subspaces,

$$W_i = \{\psi \in \mathcal{H} \mid A\psi = \lambda_i \psi\} \tag{75.56}$$

Let P_i be the orthogonal projection of $\mathcal{H}$ onto W_i, then

- $\mathcal{H}$ is an orthogonal direct sum of W_i : $\mathcal{H} = \bigoplus_{i=1}^n W_i$, and $W_i \perp W_j$ for $i \neq j$
- $P_i P_j = \delta_{ij} P_i$ and $\mathrm{Id}_{\mathcal{H}} = \sum_i P_i$
- $A = \sum_i \lambda_i P_i$

Theorem 75.5.7. For a normal operator A (ie, commutes with its adjoint), there is a spectral resolution of A.

Spectrum in finite dimension :

$$\sigma(A) = \tag{75.57}$$

For our observable A,

$$A = \sum_i \lambda_i^m P_i \tag{75.58}$$

The commutative algebra generated is that which is spanned by those projective operators, ie

$$\forall B \in \mathrm{ComSub}(A),\ \exists \{c_i\} \in \mathbb{C}^k,\ B = \sum_i^m c_i P_i \tag{75.59}$$

All those operators are commutative, simply by the commutativity of the projectors between themselves.

Example of two operators with the same commutative subalgebra : any two operators with the same projectors but different eigenvalues

Alternatively : define them entirely by sets of projectors (up to a scale?), ie some subset of commutative projector (between 0 and n)

The Gelfand spectrum of a von Neumann algebra is the unique measurable space we define

"The predual of the von Neumann algebra B(H) of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm ——A——= Tr(—A—). The Banach space of trace class operators is itself the dual of the C*-algebra of compact operators (which is not a von Neumann algebra)."

Self-duality in finite dimension due to every operator being trace-class

Spectral measure [132] (1) :

For (X, Ω) a Borel space, a spectral measure is

$$\Phi : \Omega \to \mathcal{B}(\mathcal{H}) \tag{75.60}$$

- $\Phi(U)$ is an orthogonal projection for all U, $\Phi(U)^2 = \Phi(U) = \Phi(U)^*$
- $\Phi(\varnothing) = 0$ and $\Phi(X) = \mathrm{Id}$
- $\Phi(U \cap V) = \Phi(U)\Phi(V)$
- For a sequence (U_i) of pairwise disjoint Borel subsets,

$$\Phi(\bigcup_i U_i) = \sum_i \Phi(U_i)$$

(convergence wrt strong operator topology)

[...]

Spectral measure for finite dimensional case : Given the Abelian von Neumann algebra generated by

$$A = \sum_i \lambda_i P_i \tag{75.61}$$

with functions

$$f(A) = \sum f(\lambda_i) P_i \tag{75.62}$$

$$\begin{aligned}
\int f d\mu_\psi &= \langle \psi, f(A)\psi \rangle && (75.63)\\
&= \sum_i f(\lambda_i) \langle \psi, P_i \psi \rangle && (75.64)\\
&= \sum_i f(\lambda_i) \| P_i \psi \|^2 && (75.65)
\end{aligned}$$

measure is the counting measure

$$\mu_\psi = \sum_i \|P_i \psi\| \delta_{\lambda_i} \tag{75.66}$$

Gelfand dual :

- The space is the discrete space $\sigma(A)$
- The sigma-algebra is the discrete sigma algebra given by $\mathcal{P}(\sigma(A))$
- The measure is the counting measure

The spectral presheaf is then the presheaf

$$\underline{\Sigma} : \mathcal{V}(\mathrm{VNA}(\mathcal{H}))^{\mathrm{op}} \to \mathbf{Set} \tag{75.67}$$

which maps

Decomposition of operators : Given a set of n 1-dimensional orthogonal projectors, $\{P_i\}$, $P_i P_j = 0$,

75.5.1 The two-dimensional case

The simplest case we can use is the one-dimensional case, $\mathbb{C}$, but having only a single state in its projective Hilbert space, is a bit too trivial (its only projection is the identity, and therefore its underlying category is the terminal category), so let's look at $\mathbb{C}^2$.

To classify its orthogonal projectors, let's look at the Grassmannians of various dimensions for $\mathbb{C}^2$:

- $\mathrm{Gr}_0(\mathbb{C}^2) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^2) \cong \mathbb{C}P^1$
- $\mathrm{Gr}_2(\mathbb{C}^2) = \{\mathbb{C}^2\}$

The zero and two dimensional cases are simple enough, the zero-dimensional projection operator being the zero operator 0, with Abelian von Neumann algebra the trivial algebra $\{0\}$, and the two-dimensional projection operator is the identity map $\mathrm{Id}_{\mathbb{C}^2}$, with Abelian von Neumann algebra the scaling matrices, $c\mathrm{Id}_{\mathbb{C}^2}$

The one-dimensional case will contain most of the cases of interest. for some point $p \in \mathbb{C}P^1$, ie a point on the Riemann sphere $p \in S^2$, $p = (\theta, \phi)$, there is a projector to that line in the complex plane.

Given any self-adjoint operator $\mathcal{B}_{\text{sa}}(\mathbb{C}^2)$, the finite-dimensional spectral theorem tells us that the Hilbert space can be decomposed into orthogonal subspaces $\{W_i\}$ which each contain one or more of the eigenvectors of the operator. As there can only be as many orthogonal spaces as the sum of their dimension being inferior or equal to the total dimension, this will only allow the trivial case (Just the 0-dimensional subspace), a single 1-dimensional subspace, two 1-dimensional subspace, or a single 2-dimensional subspace. The first case is simply the projector 0, corresponding only to the 0 operator. The second case is, for the choice of a point (θ, ϕ) on the Riemann sphere,

$$A = \lambda P_{(\theta,\phi)} \tag{75.68}$$

The third case is

$$A = \lambda_1 P_{(\theta_1,\phi_1)} + \lambda_2 P_{(\theta_1,\phi_2)} \tag{75.69}$$

And the last case is a diagonal operator,

$$A = \lambda \mathrm{Id}_{\mathbb{C}^2} \tag{75.70}$$

The Abelian von Neumann algebras are therefore classified by those two points on the Riemann sphere,

$$((\theta_1, \phi_1), (\theta_2, \phi_2)) \to \mathrm{VNA}(\mathbb{C}^2) \tag{75.71}$$

The category of contexts is therefore such that

- The trivial von Neumann algebra is included in all algebras
- The scaling von Neumann algebra is not included in any other algebra?
- The von Neumann algebra constructed from a single one dimensional projection $P_{(\theta,\phi)}$ is included in any von Neumann algebra constructed from two one-dimensional projections, as long as they share that projection.

Diagram of the category

$\mathrm{VNA}(P_{(\theta,\phi)}) \leq \mathrm{VNA}(P_{(\theta,\phi)}, P_{(\theta',\phi')})$

Approximation of a projection : For any projection P, there is only two possible cases :

- The projection is 0, and the V-

Kochen-Specker : $\mathbb{C}^2$ is not concerned by this.

75.5.2 The three-dimensional case

To have a case that is actually covered by the big quantum theorems properly, we will have to consider the case of the Hilbert space $\mathbb{C}^3$. This is for instance the case given by massive spin 1 particles.

The classification of projectors is much the same as previously, thanks to the duality of Grassmannians,

- $\mathrm{Gr}_0(\mathbb{C}^3) = \{0\}$
- $\mathrm{Gr}_1(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_2(\mathbb{C}^3) = \mathrm{Gr}_{3-2}(\mathbb{C}^3) = \mathbb{C}P^2$
- $\mathrm{Gr}_3(\mathbb{C}^3) = \{\mathbb{C}^3\}$

The orthogonal subspaces of an operator will be

1. The empty subspace 0
2. One 1-dimensional subspace
3. Two 1-dimensional subspace
4. Three 1-dimensional subspace
5. One 2-dimensional subspace
6. One 1-dimensional subspace and one 2-dimensional subspace
7. One 3-dimensional subspace

As before, the first and last case are trivial, consisting of the trivial subspace and the whole subspace.

75.6 The infinite-dimensional case

For a Bohr topos with a more interesting structure, such as a differential cohesive structure that we will need later on IX, let's consider instead a simple infinite dimensional case, of the theory of a quantum particle in one dimension, with the traditional Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}, \ell) \tag{75.72}$$

with ℓ the Lebesgue measure and the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^\dagger \psi_2 \mu_\ell \tag{75.73}$$

where two functions ψ, ψ' are identified if they have the same inner product with all other functions, being identical up to differences on a set of measure zero.

Due to its much more complex nature, the full classification of projection operators, and therefore contexts, is not gonna be attempted here, so that only a few representative examples will be look at here.

The basic operator we will investigate is the position operator, which is a multiplication operator of the constant function, $\hat{x} = \hat{M}_x$, acting on wavefunctions as

$$\hat{x}\psi(x) = x\psi(x) \tag{75.74}$$

Theorem 75.6.1. The operator $\hat{x}$ is unbounded.

Proof. The norm of the operator is given by

$$\|\hat{x}\| = \sup_{\|\psi\|=1} |\langle \psi, \hat{x}\psi \rangle| \tag{75.75}$$

The supremum of the expectation value. If we pick a rather boring L^2 state such as the Heaviside function of width and height 1 centered at a given parameter x_0, we have a collection of wavefunctions of expectation value x_0 for any value of $x_0 \in \mathbb{R}$, so that

$$\|\hat{x}\| = \infty \tag{75.76}$$

□

Theorem 75.6.2. The domain of $\hat{x}$ is the space of bounded functions $L^\infty(\mathbb{R})$

$$D(\hat{x}) = L^\infty(\mathbb{R}) \tag{75.77}$$

which is dense in $L^2(\mathbb{R})$.

Proof. □

Theorem 75.6.3. The spectrum of $\hat{x}$ is $\mathbb{R}$.

Proof. If we take the resolvent of the position operator at the point λ,

$$\hat{x} - \lambda \hat{I} \tag{75.78}$$

We can define another multiplicative operator

$$\hat{x}_\lambda^{-1}\psi = \frac{1}{x - \lambda}\psi \tag{75.79}$$

Domain of $\hat{x}_\lambda^{-1}$? □

Theorem 75.6.4. The position operator $\hat{x}$ is self-adjoint.

Proof. □

Theorem 75.6.5. The projection-valued measure for the position operator is, for some subset $S \subseteq \mathbb{R}$ of the spectrum $\sigma(\hat{x})$, given by

$$\hat{P}_S \psi = \hat{M}_{\chi_S} \psi \tag{75.80}$$

Proof. □

From this we can see that the projection (and therefore measurement) of the position on a subset of $\mathbb{R}$ is identical up to a set of measure 0.

1-dimensional classification : every projector is part of the set of all 1-dimensional subspaces of $\mathcal{H}$, is it the projective limit $\mathcal{C}P^\infty$? The Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, classifier of U(1) bundles

Kuiper's theorem?

Examples of operators [projectors?] with continuous spectrum

As a continuous operator, $\hat{x}$ does not have an eigenbasis[133] (outside of the more general case of the Gelf'and triple/rigged Hilbert space), but we can instead compute its projection-valued measure.

One hierarchy of projections we can do for this is to consider the projections for the localization of the particle (we are assuming the non-relativistic case here where the particle *can* be localized). If the particle ψ is entirely localized in some interval $[a, b]$, that is,

$$\int_a^b \psi^\dagger(x) x \psi(x) dx = 1 \tag{75.81}$$

then we will denote this by the projector operator $P_{[a,b]}$. This selects wavefunctions of support $[a, b]$ (almost everywhere). Given the projection-valued measure of the position operator μ_x, this projection is given by

$$P_{[a,b]} = \int_a^b \hat{\mu}_x \approx \int_a^b dx\ |x\rangle\langle x| \tag{75.82}$$

with $|x\rangle\langle x|$ the standard physicist notation for the projection on the rigged Hilbert space basis.

Those projectors obey the following properties. Being the multiplication operator M_x, for some Borel subset $I \subseteq \mathbb{R}$, its spectral measure is given by the multiplication operator

$$P_I = M_{\chi_I} \tag{75.83}$$

Therefore, for a set of measure zero like the singleton, we have

$$P_{[a,a]} = 0 \tag{75.84}$$

Since wavefunctions are only defined modulo sets of measure 0.

For some subinterval $[a,b] \subset [a',b']$, we have

$$P_{[a,b]}P_{[a',b']}\psi = \chi_{[a,b]}\chi_{[a',b']}\psi \tag{75.85}$$
$$= \chi_{[a,b]\cap[a',b']}\psi \tag{75.86}$$
$$= \chi_{[a,b]}\psi \tag{75.87}$$

Hence $P_{[a,b]}P_{[a',b']} = P_{[a',b']}P_{[a,b]} = P_{[a,b]}$

$$\neg P_{[a,b]} = P_{[0,1]\setminus[a,b]} \tag{75.88}$$

Given two intervals, $[a,b]$, $[c,d]$, we have the following operations on $P_{[a,b]}$ and $P_{[c,d]}$.

Heyting algebra of the projectors as a homomorphism of algebra from the interval algebra of $\mathbb{R}$?

Power set boolean algebra for $\mathbb{R}$?

For any two subsets of $\mathbb{R}$, A, B, we have the standard boolean algebra, ie

$$A \leq B \quad \rightarrow \quad A \subseteq B \tag{75.89}$$
$$A \wedge B = A \cap B \tag{75.90}$$
$$A \vee B = A \cup B \tag{75.91}$$
$$\neg A = \mathbb{R} \setminus A \tag{75.92}$$
$$A \rightarrow B = A^c \cup B \tag{75.93}$$
$$\top = \mathbb{R} \tag{75.94}$$
$$\bot = \varnothing \tag{75.95}$$

From the properties of the indicator function χ_A, we have the following algebra homomorphism

$$P_{A\wedge B}\psi = \chi_{A\cap B}\psi \tag{75.96}$$
$$= \chi_A\chi_B\psi \tag{75.97}$$
$$= P_A P_B\psi \tag{75.98}$$
$$= P_B P_A\psi \tag{75.99}$$

$$\begin{aligned} P_{A\vee B}\psi &= \chi_{A\cup B}\psi & (75.100)\\ &= (\chi_A + \chi_B - \chi_A\chi_B)\psi & (75.101)\\ &= (P_A + P_B - P_A P_B)\psi & (75.102)\\ & & (75.103) \end{aligned}$$

$$\begin{aligned} P_{\neg A}\psi &= \chi_{\mathbb{R}\setminus A}\psi & (75.104)\\ &= (\chi_{\mathbb{R}} - \chi_A)\psi & (75.105)\\ &= (P_{\mathbb{R}} - P_A)\psi & (75.106)\\ &= P_A^{\perp}\psi & (75.107) \end{aligned}$$

$$\begin{aligned} P_{A\to B}\psi &= \chi_{A^c\cup B}\psi & (75.108)\\ &= (\chi_{A^c} + \chi_B - \chi_{A^c}\chi_B)\psi & (75.109)\\ &= (P_{\mathbb{R}} - P_A + P_B - (P_{\mathbb{R}} - P_A)P_B)\psi & (75.110)\\ &= (P_A^{\perp} + P_B - P_A^{\perp}P_B)\psi & (75.111) \end{aligned}$$

For a fuller picture of the daseinisation of the theory, we also need some non-commutative operator with the position, the natural choice being the momentum operator.

$$\hat{p} = -i\hbar\frac{d}{dx} \tag{75.112}$$

Domain [133]

$$\text{Dom}(\hat{p}) = H^{0,1} = \{\psi \in L^2([0,1]) \mid \psi' \in L^2([0,1])\} \tag{75.113}$$

Projection valued measure?

Interaction in the Heyting algebra

If we consider our position operator $\hat{x}$ as the generator of an Abelian von Neumann algebra,

We now have two commutative von Neumann algebras, V_x and V_p. For any projector in either of those, their support is simply themselves

$$\delta_{V_x}(P_x) = \bigwedge\{P \in V_x \mid P_x \leq P\} \tag{75.114}$$

as $P_x \in V_x$ and P_x is its own greatest lower bound.

Now if we wish to compare a momentum projector in a position context, we need to figure out the interaction of the two measurements. By the Paley-Wiener theorem, if we perform any measurement localizing the wavefunction to some compact subset, giving a wavefunction of compact support after collapse, then the resulting momentum space wavefunction is holomorphic, and therefore analytic. If it is equal to zero on any open set, then it is identically zero, meaning that the spectrum will always be the full momentum space.

This result does not generalize to the full subset algebra of $\mathbb{R}$, however, as it is possible to have wavefunctions where the support is not the entire real line nor a dense subset of it, but neither does the momentum space wavefunction[134]. For simplicity here we will only consider regions of compact support.

As the momentum is merely a multiplicative operator on the Fourier transform of the wavefunction, given some initial measurement on the subset I, so that our wavefunction is some L^2 function with essential support in I

[...]

$$P_{[p_1,p_2]}P_{[x_1,x_2]} \tag{75.115}$$

$$\delta_{V_x}(P_p) = \bigwedge\{P \in V_x \mid P_p \leq P\} \tag{75.116}$$

Part VI

Logic

[135, 136, 137]

One element of interest of topoi as a good foundation for math is that there is a connection between topos and logical theories. Given some category $\mathbf{C}$, there exists some equivalence with a type theory $L(\mathbf{C})$, which, given some appropriate constraints on the category, can be interpreted as a logic on this category.

76

Logic and order structures

To look at the connection between logics and categories, first let's look at how logics, in the broad sense, are composed. First, let's look at the "bare" logic of propositions. This is composed by a set of symbols for propositional variables

$$P = \{p_1, p_2, \dots\} \tag{76.1}$$

and the standard n-ary functions on propositions, ie $\neg$, $\wedge$, $\vee$ and $\rightarrow$, and parenthesis for bracketing, with the syntax

- Any propositional variable p_i is a proposition.
- For any proposition α, there is a proposition $\neg\alpha$.
- For any two propositions α, β, there is a proposition for any binary function, ie $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$

with eventually the constant symbols $\top, \bot$ added to the list of propositions.

Sequent calculus

An example of application of the calculus can be done with some basic application of logical operators with a single hypothesis,

Theorem 76.0.1. Modus ponens : Given the proposition $p \rightarrow q$ and p, we have q :

$$\frac{\Gamma_1 \vdash \Delta_1, p \rightarrow q \qquad \Gamma_2 \vdash \Delta_2, p}{\Gamma_1, \Gamma_2 \ \vdash \Delta_1, \Delta_2, q}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{I})}{\Gamma_1, p \vdash p}\,(\mathrm{WL})}{p, \Gamma_1 \vdash p}\,(\mathrm{PL})}{p, \Gamma_1 \vdash p, \Delta_1}\,(\mathrm{WR})}{p, \Gamma_1 \vdash \Delta_1, p}\,(\mathrm{PR}) \quad \dfrac{x}{x}\,(\mathrm{I})}{p \to q, \Gamma_1 \vdash \Delta_1, p}\,(\to\mathrm{L})$$

□

Theorem 76.0.2. Given the equivence $p \leftrightarrow q$, this entails $p \to q$ and $q \to p$

Proof. Simply by □

Theorem 76.0.3. If two propositions are equivalent, $p \leftrightarrow q$, they can prove the same proposition, ie we have

$$\frac{\Gamma, p \vdash \Delta, A \qquad (p \leftrightarrow q) \vdash}{\Gamma, q \vdash \Delta, A}$$

Proof.

$$\frac{\Gamma, p \vdash \Delta, A \qquad (p \leftrightarrow q) \vdash}{\Gamma, q \vdash \Delta, A}$$

□

The correspondence between a logical theory and an algebra on an order structure is known as the *Lindenbaum-Tarski algebra* of the theory.

Theorem 76.0.4. For any type of logical system with an associated sequent calculus including at least [sequent rules], where we identify equivalent propositions, $\alpha \sim \beta$ meaning that $\alpha \to \beta \wedge \beta \to \alpha$ is true, then there is a corresponding [order?]

Proof. The mapping f is done using the translation that any proposition α corresponds to an element of the poset, any sequent of the form

$$\pi \vdash \alpha \tag{76.2}$$

corresponds to an order relation $f(\pi) \leq f(\alpha)$

[...]

For this to be the appropriate [order],

The mapping is reflexive : comes from the axiom of identity, $A \vdash A$ [Add extra context] The mapping is antisymmetric : if $A \vdash B$ and $B \vdash A$, $\vdash A \cong B$ [what does it mean on a lattice level]

Transitive : $A \vdash B$, $B \vdash C$, then $A \vdash C$ (cut rule?)

□

In addition to this bare logic, we will also involved typed entities. For instance, in the case of sets, every

Definition 76.0.1. In a logical theory, a (first-order) signature is composed of

- A set Σ_0 of *sorts*
- A set Σ_1 of *function*
- A set Σ_2 of *relations*

Example 76.0.1. For classical logic, its signature is given by the single sort of propositions

$$\Sigma_0 = \{\text{Prop}\} \tag{76.3}$$

some set of logical functions, for instance

$$\Sigma_1 = \{\} \tag{76.4}$$

Definition 76.0.2. For a category $\mathbf{C}$ with finite products, and a signature Σ, a Σ-*structure* M in $\mathbf{C}$ defines :

- A function between sorts in Σ_0 and objects in $\mathbf{C}$
- A function between functions in Σ_1 and morphisms in $\mathbf{C}$
- A function between relations and subobjects

Connection between logic and order structures, relation to subobject order structures

[internal v. external logic]

Support object, h-propositions

$$\text{isProp}(A) = \prod_{x:A} \prod_{y:A} (x = y) \tag{76.5}$$

difference between h-propositions and propositions as types?

[138]

Logic from types, logic from topos, Heyting algebra [139] The subobjects of objects X in a topos $\mathbf{H}$ form a Heyting algebra, with operations $\cap, \cup, \rightarrow$ the partial ordering $\subseteq$ and the greatest and smallest elements 1_A, 0_A.

The language $L(\mathbf{H})$ of a topos is a many-sorted first-order language having the objects $X \in H$ as types for the terms of $L(\mathbf{H})$, there is a type operator τ which assigns to any term of $L(\mathbf{H})$ an object $\tau(p)$ of $\mathbf{H}$ called the type of p.

- 0_H is a constant term of type 1.
- For any object A of E, there is a countable number of variables of type A
- For any map $f : A \rightarrow B$, there is an "evaluation operator" $f(-)$ for terms of type A to terms of type B : p of type $A \Rightarrow f(p)$ of type B
- For any ordered pair (A, B) of $\mathbf{H}$, there is an ordered pair operator $\langle -, - \rangle$
- For any subobject $M : A \rightarrow \Omega$, there is a unary "membership-predicate $(-) \in M$ for elements of A. $x \in M$ is an atomic formula provided $x \in A$.
- The propositional connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$ are allowed for new formulas
- For any object A and variable $x \in A$, the quantifier $\exists x \in A$ and $\forall x \in A$ are allowed

Definition 76.0.3. Two objects $x, y \in A$ are equal if

$$x = y \leftrightarrow \langle x, y \rangle \in \Delta_A \tag{76.6}$$

$\Delta_A : A \times A \rightarrow \Omega$ the diagonal operator

Unique equality :

$$(\exists! x \in A)\phi(x) \leftrightarrow \exists x \in A, \forall y \in A, (\phi(y) \leftrightarrow x = y) \tag{76.7}$$

Membership :

$$x \in y \leftrightarrow \langle y, x \rangle \in (\mathrm{ev} : PA \times A \rightarrow \Omega) \tag{76.8}$$

For $x \in A$ and $F \in B^A$,

$$F(x) = (\mathrm{ev} : B^A \times A \rightarrow B)\langle F, x \rangle \tag{76.9}$$

For any map $f : A \rightarrow B$ with exponential adjoint $\overline{f} : 1 \rightarrow B^A$, we define an element $f_e = \overline{f}(0_e) \in B^A$ which represents f internally.

Example 76.0.2. Some of the most barebones "logics" one can have for a category are the initial category **0** and the terminal category **1**. The initial category has no types and therefore no terms or propositions, corresponding to the empty logic. The terminal category has a single type (the unit type), and only one term given by the identity, and a single proposition $* \to *$, which corresponds to the truth statement (or alternatively, the false statement, since we have $\top \leftrightarrow \bot$ in this system). As all the limits in the terminal category are its single object, this transfers to its category of subobjects, meaning that every logical construction there is also $1 \to 1$, so that this is the trivial logic, for which any proposition is true (this can be seen as stemming from the initial and terminal object being the same here, and therefore not having a notion of falsehood different from truth).

Example 76.0.3. A standard example of an internal logic is given by a group category, in the sense of a groupoid of one element with endomorphisms isomorphic to G. Having only one object, there is only one associated type, the group type G, and only one subobject relation, which is $\mathrm{Id}_G : G \hookrightarrow G$.

Lawvere-Tierney topology as locality modality

77

Logical structures in a category

77.1 Proposition as types

In the type theory section, we saw that a common method to treat logic in type theory is the notion of propositions as types, where we say that a proposition is true if its associated type is inhabited. Constructively speaking, the inhabitant $x : X$ of the type is a witness of the truth of the proposition, ie a value of the predicate for which it is true.

We will use a similar notion for categories, where given a predicate defined by some morphism $p : S \to X$, we say that p is true if the image of S via p is inhabited.

Definition 77.1.1. For an object X in a regular category $\mathbf{C}$, its *support* $[X]$ is the image factorization of the terminal morphism $X \to 1$:

$$X \twoheadrightarrow [X] \hookrightarrow 1 \tag{77.1}$$

in other words,

$$[X] = \mathrm{im}(!_X) \tag{77.2}$$

from this, we can see that $[X]$ is always a subterminal object.

the support of an object is also called its (-1)-*truncation* (or *truncation* for short, see 84.2.4 for more details), as this can be understood to the the truncation of the object to the level of a (-1)-category [21], which is simply given by two truth values $\{\top, \bot\}$. The truncation is given by projecting the objects down to the truth values by the support. It is also called the *bracket type* or the *squash type* of X.

For the interpretation of the support, let's consider a few cases.

If $[X] \cong 0$, the object has an epimorphism onto 0, and is therefore isomorphic to it.

If $[X] \cong 1$, $!_X$ is an epimorphism. The type theoretic interpretation here is that X contains some element, which we can write as

$$x : X \dashv \top \tag{77.3}$$

but there is a subtlety here, in that this element is only explicitly contained in X if $!_X$ is furthermore a split epimorphism, ie there exists a section $p : 1 \to X$. In logical terms, this is the difference between the mere stated existence of such an element versus the constructive proof of its existence.

If we choose to forego the constructivity of such elements in favor of allowing the existence of those mere propositions (so that we allow the possibility of an epimorphic terminal morphism that does not split), those correspond in the internal logic to *principles of omniscience* [Foundations of Constructive Analysis], where we assume the existence of elements without a construction process

The classic example of this in classical logic is the proof by contradiction, stemming from the law of excluded middle,

$$\vdash p \wedge \neg p \tag{77.4}$$

If we have a predicate $p[x]$, for which there exists

77.2 Lawvere theory

[140, 141, 142]

An important part of the logical structure of a category is given by its algebraic structure. In a logical theory, given some set X, we define the n-tuples of that set X^n, with $X^0 = \{\bullet\}$. An n-ary function is then some function $f : X^n \to X$.

As functions $X^0 = \{\bullet\} \to X$ are isomorphic to X, the nullary functions simply represent constant values.

Universal algebra :

Once a set of such operations has been defined, an algebra is defined by a set of *equational laws*

example : commutativity

$$\forall x, y \in X,\ x * y = y * x \tag{77.5}$$

specific relations :

$$1() + 1() = 2() \tag{77.6}$$

Example 77.2.1. A group is a algebra given by a set G and three functions, the binary multiplication $\cdot$, the unary inverse $(-)^{-1}$, and the nullary neutral element e

Generally the smallest amount of structure we can ask of a theory to give something deserving of the name logic is a finite product. Without this, we would not even be able to define [equivalence, multiple propositions in sequents?]

A logical theory associated with an object for which finite products are defined with its subobjects is called a *Lawvere theory*

Definition 77.2.1. Given the skeletal category of finite sets, denoted by

$$\aleph_0 = \mathrm{sk}(\mathbf{FinSet}) \tag{77.7}$$

equipped with the finite coproduct defined by disjoint union, we define its opposite category $\aleph_0^{\mathrm{op}}$, with an associated finite product.

As every element of $\aleph_0$ is a coproduct of 1, we have that

$$\mathrm{Hom}_{\aleph_0}(n, m) \cong \mathrm{Hom}_{\aleph_0}(1, m)^n \tag{77.8}$$

so that every morphism in $\aleph_0$ is an n-fold product of coproduct injections. Its opposite category is then such that every object is an n-fold product of 1

As we've seen, the opposite category for finite sets are finite boolean algebras.

Definition 77.2.2. For a small category $\mathbf{L}$ with strictly associative finite products, we say that it is a *Lawvere theory* if there exists an identity on objects functor $I : \aleph_0^{\mathrm{op}} \to \mathbf{L}$ that preserves finite products strictly.

As the functor is an identity on objects, the interpretation of a Lawvere theory is that there exists some distinguished object X of the category (corresponding to $I(1)$), called the *generic object* (or *generating object*), and every other object of L is simply a power of L, via

$$I((\coprod 1)^{\mathrm{op}}) = \prod I(1) \tag{77.9}$$

so that we have that any object of $\mathbf{L}$ is simply X^n. In particular, X^0 is the initial object.

L is *one-sorted*.

Example 77.2.2. The simplest such Lawvere theory is

Example 77.2.3. For a given group G considered as a category of one object with morphisms $\mathrm{Mor}(G) = G$, Lawvere theory of groups

Definition 77.2.3. A model of a Lawvere theory is a finite product preserving functor

$$M : \mathbf{L} \to \mathbf{C} \tag{77.10}$$

for some category $\mathbf{C}$ with finite products.

77.3 Relations

As we saw in 32, there is a natural definition of relations in categories, and even two of them : we can either define them as subobjects of product or as spans. This gives us a natural definition of relations for the internal logic in categorical terms, where the predicate of relation

$$a : A,\ b : B \vdash R[a, b] \tag{77.11}$$

77.4 Internal logic

From this, we assign a type theory, and hopefully a logic, to a given category.

Definition 77.4.1. In a category $\mathbf{C}$ with finite products, and a logical signature Σ, a Σ-*structure* on $\mathbf{C}$ is given by the following three functions :

- A function $M : \Sigma \to \mathbf{C}$ which assigns for every finite list of Σ-sorts an object of $\mathbf{C}$, such that

  $$M(S_1, \ldots, S_n) = M(S_1) \times \ldots \times M(S_n) \tag{77.12}$$

  and for the nullary version,

  $$M() = 1 \tag{77.13}$$

- A function assigning to every function symbol $f : S_1 \times \ldots \times S_n \to T$ an arrow

  $$M(f) : M(S_1, \ldots, S_n) \to M(T) \tag{77.14}$$

- A function assigning to every relation $R(S_1, \ldots, S_n)$ a subobject

  $$M(R) \hookrightarrow M(S_1, \ldots, S_n) \tag{77.15}$$

Correspondence with the topos being boolean

For the product $X \times X$, with projectors $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$

$$\mathrm{isProp}(X) = \mathrm{coeq}(\mathrm{pr}_1, \mathrm{pr}_2) \tag{77.16}$$

The notion of a the *propositional axiom of choice*, PAC [143]

locally v. globally inhabited

An application of this can be constructed using the pointless locale from 45.0.2 of surjections $\mathbb{N} \to \mathbb{R}$.

Difference between intuitionist and classical logic, propositional axiom of choice v. axiom of choice.

Truncation

Mere propositions

Explanation if $[X]$ is in between the two?

77.5 Internal logic of a topos

As categories with fairly rich structures, topoi typically have fairly well-behaved associated logical theories. From its properties, we can already guess that it will be a Lawvere theory, contain function types, etc

Theorem 77.5.1. Any internal logic of an elementary topos $\mathbf{H}$ is an intuitionistic higher order logic.

Proof. Left and right "and" elimination : $(p \wedge q) \to p$, $(p \wedge q) \to q$: projection functions

Axiom of simplification : $p \to (q \to p)$. Distributivity :

$$\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r)) \tag{77.17}$$

□

[Since every morphism has a pullback in a topos, every morphism is a display morphism and therefore corresponds to a predicate]

$$[\![X]\!] = X \tag{77.18}$$

78

Mitchell-Benabou language

In the case of a category with a subobject classifier, there is a simpler way of expressing the internal logic of a category, as the subobjects of a given object can be simply given by morphisms $X \to \Omega$. This isomorphism is the *Mitchell-Benabou language* of the category.

Theorem 78.0.1. In the internal logic of a category, a proposition can equivalently be described by a characteristic morphism.

Example 78.0.1. If we are given the natural number object $\mathbb{N}$ of a topos, any property proposition

79

Modal logic

[144]

The internal logic of a category will translate monadic and comonadic operators in terms of modalities.

[...]

It has been historically a point of contention between analytic and continental philosophers as to whether or not there was a point to modalities. Part of this contention relies on the reducibility of modal logic (at least the main type of modal logic of use back then, S4/S5 modal logic) to standard predicate logic via the Kripke semantics.

Our goal here will not be to judge of its *necessity*, but rather of its utility.

80

The internal logic of **Set**

The internal language $L(\mathbf{Set})$ will roughly correspond to classical logic (as applied to sets). More precisely, in terms of a set theory, it corresponds to *bounded Zermelo set theory*, which is similar to ZFC set theory, but with bounded comprehension and choice [145].

- For any predicate ϕ with only bounded quantifiers ($\forall x \in A$, $\exists x \in B$), and a set B, then $\{x \in B \mid \phi(x)\}$ is a set
- Bounded choice?

This is due to the lack of tools to talk of unbounded quantification over all objects internally [but see stack semantics].

In terms of its "fundamental" operations, this means that any atomic predicate will be of the form $x \in A$, including the equality operation :

$$A = B \leftrightarrow \forall x \in A,\ x \in B \land \forall x \in B,\ x \in A \tag{80.1}$$

So that any proposition will fundamentally depend on the morphisms $x : 1 \to A$ (this simply stems from the category being well-pointed).

Many-sorted first order language having the objects of the topos as types

Boolean algebra of subsets : for $X \in \mathbf{Set}$, we consider the boolean algebra of subobjects $\mathrm{Sub}(X)$ with the correspondences

$\to$: every element except the elements of A that aren't also elements of B.

Boolean algebra	Set operator
$a, b, c, \ldots$	$A, B, C \subseteq X$
$\wedge$	$\cap$
$\vee$	$\cup$
$\leq$	$\subseteq$
0	$\varnothing$
1	X
$\neg A$	$X \setminus A = A^c$
$A \to B$	$(X \setminus A) \cup B = A^c \cup B$

Table 80.1: Correspondence of algebra and logic

Boolean algebra identities :

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{80.2}$$
$$A \cup B = B \cup A \tag{80.3}$$
$$A \tag{80.4}$$

[...]

A basic example of statement in our topos is given by the truth morphism $\top : 1 \to \Omega$, which trivially factors through itself,

$$1 \xrightarrow{\mathrm{Id}_1} 1 \xrightarrow{\top} \Omega \tag{80.5}$$

which corresponds to the trivial statement

$$\vdash \top \tag{80.6}$$

Simply stating that truth is always internally valid. Conversely, the falsity morphism $\bot : 1 \to \Omega$, representing falsehood, should not have any such factoring, ie

$$1 \xrightarrow{f} 1 \xrightarrow{\top} \Omega \tag{80.7}$$

such that $\top \circ f = \bot$. As there is only one endomorphism on 1, this can only be the identity map, and therefore is only true if $\top = \bot$. This would however mean that, as $\top$ is the classifying map of 1 and $\bot$ that of 0 as subobjects of 1, by pullback this would mean that both 0 and 1 are the same object, which is not the case in **Set**.

Therefore, we can write that falsity is indeed false,

$$\nvdash \bot \tag{80.8}$$

but on the other hand, the negation of falsity, $\neg\bot$, is true, as

[...]

More generally, we can derive this

Theorem 80.0.1. If the negation of a morphism is true, ie

$$A \xrightarrow{f} \Omega \xrightarrow{\neg} \Omega \tag{80.9}$$

factors through 1 :

$$A \xrightarrow{f'} 1 \xrightarrow{\top} \Omega \tag{80.10}$$

such that $\neg \circ f = \top \circ f'$, then this morphism is false, ie there is no such factoring of f through $\top$.

Proof. □

Axioms :

Negation : for $p : A \hookrightarrow X$, ie

$$A \overset{p}{\hookrightarrow} X \xrightarrow{\chi_A} \Omega \tag{80.11}$$

The negation of the set $\neg A$ is such that $\neg p : \neg A \hookrightarrow X$ factors through the negation and p,

$$\chi_{\neg A} = \chi_A \circ \neg \tag{80.12}$$

Statement with context : $p, p \to q \dashv q$: slice category $\mathbf{Set}_{/(p:A\to X)\times[]}$.

Localization modality : $\bigcirc_j$ for $j = \mathrm{Id}_\Omega$:

[...]

An example of internal logic we can derive in **Set** is by looking at the natural number object **Set** as an internal ring. If we just look at $\mathbb{N}$ and $\{\bullet\}$, we have the integer type $\mathbb{N}$ and the unit type 1, so that

The successor function $s : \mathbb{N} \to \mathbb{N}$ is an integer term with a free integer variable, ie we have

$$n : \mathbb{N} \dashv s(n) : \mathbb{N} \tag{80.13}$$

We also have the subobjects

Proof of induction?

Theorem 80.0.2. Given a sequence given by $1 \xrightarrow{q} A \overset{f}{A}$, with

$$\begin{aligned} a_0 &= q \quad (80.14) \\ a_{n+1} &= f(a_n) \quad (80.15) \end{aligned}$$

If a property ϕ on an element of A is true for a_0 and if $\phi(a_n) \to \phi(a_{n+1})$, then ϕ is true for a subobject of A such that $P \cong N$ (or something) 1

81
The internal logic of a spatial topos

Logic of Sh($\mathbf{X}$), Sh($\mathbf{CartSp_{Smooth}}$)
Locality modality j

82

The internal logic of smooth spaces

As a logical system, the topos of smooth sets has as types the various smooth sets

Proposition : subspaces $S \hookrightarrow X$

Subobject classifier : Ω is the sheaf associating to any $U \subseteq \mathbb{R}^n$ such that

$$\Omega(U) = \{S | S \text{ is a } J\text{-closed sieve on } U\} \tag{82.1}$$

$$\Omega(f) = f^* \tag{82.2}$$

$\top : 1 \to \Omega$: maximal sieve on each object

Points in Ω : all the maps $1 \to \Omega$, all the J-closed sieves on $\mathbf{R}^0$.

83

The internal logic of classical mechanics

Internal logic for Poisson manifolds

Symmetric monoidal category with projection

In this context, a configuration is a subobject $f : 1 \to \text{Phase}$ for a given phase space.

let's consider the maps from configurations to the real numbers, [1]

A particular Poisson structure we can give is the trivial Poisson structure,

$$\{f, g\} = 0 \tag{83.1}$$

If we consider the map $\mathbb{R} \to P$ corresponding to this trivial structure on the real line, we will get our local example of a real line object. Statements about the measurements of a classical system can therefore be understood as morphisms from

Are the measurement given by $\mathbb{R}$ with the trivial Poisson structure

types given by morphisms to R?

A basic type of proposition on R is the notion of a measurement being in some interval. If we want to associate some interval $I \hookrightarrow R$ to a configuration in Phase, this is given by a morphism to the power object,

$$\text{Phase} \to \Omega^R \tag{83.2}$$

such that for a given configuration, we have the following diagram commute

Example 83.0.1. If we only look at observables that give out a well-defined value in R, this is the case where the observable only gives us singleton values, so that in fact

For many practical circumstances, we will want to know more specifically propositions about values. If we have a phase space Phase and some point in that phase space $(x, p) : 1 \to \text{Phase}$

84

The internal logic of quantum mechanics

There are three possible internal logics that we can consider for quantum mechanics here. If we consider it as a symmetric monoidal category, this is a form of *linear logic*, what we will call *quantum linear logic*. If we consider a given Hilbert space $\mathcal{H}$, the logic of the slice category $\mathbf{Hilb}_{\mathcal{H}}$ is von Neumann's *quantum logic*. And finally, we will look at the internal logic of the Bohr topos that we have constructed.

beep

[146]

84.1 Linear logic

[147]

The category of Hilbert spaces and linear logic are not quite like the other ones that we have looked into so far, not forming a topos. As we do not have a subobject classifier here, we will not be able to use a Mitchell-Benabou language. But we can still perform that translation using the basic translation as a type theory.

If we pick a given Hilbert space $\mathcal{H}$ as a reference, propositions are given by monomorphisms $\iota : W \hookrightarrow \mathcal{H}$ (up to isomorphisms). In the category of Hilbert spaces, all monomorphisms are split, meaning that there exists a retraction P_W

$$W \overset{\iota_W}{\hookrightarrow} \mathcal{H} \xrightarrow{P_W} W \tag{84.1}$$

such that $P_W \circ \iota_W = \mathrm{Id}_W$. This is the notion we've seen before for the state of a system depending on a projection of the Hilbert space via some measurement operator.

The category of Hilbert spaces, like the category of vector spaces, has an initial and terminal object that are the same, the *zero object* 0, corresponding to the Hilbert space $\mathbb{C}^0$. Being the subobject of any Hilbert space, the unique map $0 : 0 \to \mathcal{H}$ has an interpretation as a proposition for any Hilbert space,

Interpretation of daggers logically

Interpretation as resource logic (Petri nets), dynamic logic

[148]

Part VII

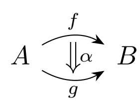
Higher categories

84.2 n-categories

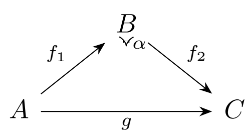
As we defined the notion of 2-categories 18.2, we can generalize this notion to an arbitrarily higher degree by introducing the concept of *higher categories*, in which we introduce the possibility of transformations between any two objects of the same type, called k-morphisms.

Definition 84.2.1. A k-morphism is defined inductively as an arrow for which the source and target is a $(k-1)$-morphism, and an object is a 0-morphism.

Therefore the morphisms we saw so far are 1-morphisms (arrows between 0-morphisms or objects), 2-morphisms are morphisms between two 1-morphisms, and so forth. In diagrammatical terms, we will express higher order morphisms by arrows between arrows, such as the 2-morphism $\alpha : f \to g$



While all 2-morphisms can be expressed this way, called a *globular* 2*-morphism*, for brevity we will also define 2-morphisms on slightly more complex diagrams.



This is called a *simplicial* 2*-morphism*, and it is simply the globular 2-morphism for $\alpha : f_2 \circ f_1 \to g$. Similarly we have the *cubical* 2*-morphism*

[diagram]

which is the globular 2-morphism for $f_2 \circ f_1 \to g_2 \circ g_1$.

A basic example of this is that just as functors can be considered as a morphism in the category of categories **Cat**, so can natural transformations be considered as 2-morphisms in the 2-category **Cat**. Let's show it explicitly :

Theorem 84.2.1. Considered as a 2-category, 2-morphisms in **Cat** are natural transformations.

Proof. As a 2-morphism is formally a function between two morphisms, which are functors in **Cat**, all we need to do is to show naturality of those functions. An object of a category in **Cat** is given by a functor from the terminal category $\mathbf{1}$, $\Delta_X : \mathbf{1} \to \mathbf{C}$, so that a mapping from an object in $\mathbf{C}$ to one in $\mathbf{D}$ is given by the commutative triangle

$$F \circ \Delta_X = \Delta_Y \tag{84.2}$$

Morphisms between two components in that sense are given by 2-morphisms between those morphisms, ie for $f : X \to Y$ in $\mathbf{C}$, we have some 2-morphism

$$\eta_f : \Delta_X \to \Delta_Y \tag{84.3}$$

and composition of those morphisms are given by vertical composition. For $f : X \to Y$ and $g : Y \to Z$, we have

$$\eta_{g \circ f} = \eta_g \circ \eta_f \tag{84.4}$$

The action of a functor on such morphisms, $F(f) : F(X) \to F(Y)$, is that of whiskering the 2-morphism by that functor :

$$F\eta_f : (F \circ \Delta_X) \to (F \circ \Delta_Y) = F(X) \to F(Y) \tag{84.5}$$

The components of a natural transformation are given by whiskering with that constant functor,

$$\eta_X = \Delta_X \eta \tag{84.6}$$

We therefore just need to show the naturality condition, $\eta_Y \circ F(f) = G(f) \circ \eta_X$:

$$\eta_Y \circ F(f) \quad = \quad \eta \tag{84.7}$$

□

”The objects in the hom-category C(x,y) are the 1-morphisms in C from x to y, while the morphisms in the hom-category C(x,y) are the 2-morphisms of C that are horizontally between x and y.”

Arrow category? Over category?

Example 84.2.1. In the category **Top**, homotopies between two continuous functions are 2-morphisms.

From this we can define the notion of n-category to be categories with morphisms up to level n, with the appropriate rules regarding composition and identity.

Definition 84.2.2. An n-category $\mathbf{C}$ is given by a sequence of n classes of k-morphisms $(\mathrm{Mor}_k(\mathbf{C}))_{0 \leq k \leq n}$, such that for every $k > 0$, we have two functions

$$s_k, t_k : \mathrm{Mor}_k(\mathbf{C}) \to \mathrm{Mor}_{k-1}(\mathbf{C}) \tag{84.8}$$

and a function of composition of k-morphisms,

$$\circ_k : \mathrm{Mor}_k(\mathbf{C}) \times \mathrm{Mor}_k(\mathbf{C}) \to \mathrm{Mor}_k(\mathbf{C}) \tag{84.9}$$

which have to obey the condition that and such that for every $k > 0$, for every $k-1$-morphism α, there exists a k-morphism Id_α for which $s_k(\mathrm{Id}_\alpha) = t_k(\mathrm{Id}_\alpha) = \alpha$

We can technically extend this definition to include cases for $k = 0$, in which case the only remaining class is $\mathrm{Mor}_0 = \mathrm{Obj}$, ie 0-categories are equivalent to sets.

The concept of an ∞-category is for the case where the set of classes of k-morphism is of infinite cardinality (typically countable).

We will also define more specifically the concept of (n,k)-categories :

Definition 84.2.3. An (n,k)-category for $n,k \in \mathbb{N} \cup \{\infty\}$ is an n-category for which every l-morphism for $k < l \leq n$ possesses an inverse, ie for any k-morphism $\alpha \in \mathrm{Mor}_k(\mathbf{C})$, $k > 1$, then there exists another k-morphism α^{-1} which obeys

$$\alpha \circ \alpha^{-1} = \mathrm{Id}_{t(\alpha)} \tag{84.10}$$

$$\alpha^{-1} \circ \alpha = \mathrm{Id}_{s(\alpha)} \tag{84.11}$$

Example 84.2.2. A group G interpreted as a category $\mathbf{G}$ ($\mathrm{Mor}(\mathbf{G}) \cong G$) is a $(1,0)$-category.

Example 84.2.3. The category of topological spaces with continuous maps as morphisms and homotopie equivalences between any two continuous map is a $(2,1)$-category.

Gauge example?

"An (n,r)-category is an r-directed homotopy n-type." Ex : a $(0,0)$-category is isomorphic to a set (the set of all objects), a $(1,0)$-category is a groupoid, a $(1,1)$-category is a category

$(\infty,0)$: ∞-groupoid (∞,∞) :

Descent to negative degrees : $(-1,0)$-category : truth values $(-2,0)$-category : Point

n-truncation : a category is n-truncated if it is an n-groupoid

Truncation

Definition 84.2.4. The n-truncation functor τ_n is an endofunctor on an ∞-category mapping all morphisms of order $k \geq n$ to the identity k-morphism.

An n-truncated category therefore acts as the equivalent n-category. For the case of a 0-truncation, this is simply a basic category.

85

∞-Groupoids

[149, 150]

The prototypical ∞-category is $\infty\mathbf{Grpd}$, the ∞-category of ∞-groupoids, along with its various truncations, the k-groupoids,

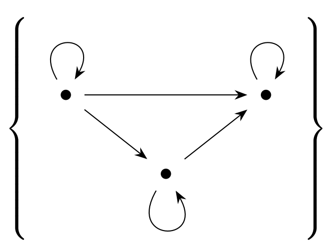
$$k\mathbf{Grpd} \cong \tau_k \infty\mathbf{Grpd} \tag{85.1}$$

where in particular, the 0-groupoids, simply being given by 0-morphisms, are just sets, and the 1-groupoids are the usual notion of groupoids.

There are a number of different ways that we can consider the various levels of groupoids

First, any ∞-category with all invertible k-morphisms (an $(\infty, 0)$-category) can be embedded in $\infty\mathbf{Grpd}$. Therefore any such diagram can be considered as such, which we will write either as some category $\mathbb{C}$, or by their pure diagram (neglecting any trivial arrows)

$\mathcal{G} =$



(85.2)

Another method is the traditional groupoid notation $\mathcal{G} : G_1 \rightrightarrows G_0$, where G_0 is the set of objects, G_1 the set of morphisms, and s, t are the source and target.

This method can be extended to the ∞-groupoid case by having an infinite sequence of k-morphisms

$$G_1 \equiv\rightrightarrows G_0$$

Given an ∞-group,

Definition 85.0.1. The category of ∞-groupoids is the category whose objects are ∞-groupoids and morphisms are ∞-groupoid homomorphisms.

Delooping

Theorem 85.0.1. For any group G, the G-sets of G can be expressed as a groupoid.

Proof. For a group G

$$\mathcal{G} : G \times X \to X \tag{85.3}$$

□

Equivariant maps in this context?

85.1 Truncation

The k-truncation of an n-groupoid is the equivalent k-groupoid

Theorem 85.1.1. The k-truncation of an ∞-category is a functor.

Proof. As the k-truncation sends every k'-morphism, $k < k'$, to the identity k'-morphism, □

Theorem 85.1.2. The n-truncation monad has an adjoint modality of the n-connected monad.

Definition 85.1.1. If we have some left adjoint of the n-truncation functor,

$$\mathrm{Hom}_{\mathbf{C}}() \tag{85.4}$$

The n-truncation monad is given by the

85.2 Cech ∞-groupoid

86
Homotopy categories

[151, 152, 153]

There exists a weaker notion than that of a higher category which is that of a category with weak equivalences, where rather than any higher morphisms, we define classes of morphisms which are weak equivalences, a special class of morphisms which are not strictly speaking isomorphisms (they do not admit an inverse such that $f \circ f^{-1} = \mathrm{Id}$), but we consider them to have a weak inverse which we will consider to be something akin to an equivalence.

Definition 86.0.1. A category $\mathbf{C}$ with weak equivalences is given by some subcategory $\mathbf{W} \hookrightarrow \mathbf{C}$ which contains all isomorphisms and which obeys the two out of three property : for any two composable morphisms f, g in $\mathbf{C}$, if two out of $\{f, g, g \circ f\}$ are in $\mathbf{W}$, then so is the third.

In particular, if we have composable morphisms of the types $f : X \to Y$ and $g : Y \to X$ in $\mathbf{W}$, then we can say that X and Y are weakly equivalent

[...]

In terms of n-categories, weak equivalences are meant to represent the notion of 2-morphisms to the identity. That is, for any morphism $f \in \mathbf{W}$, we have equivalently some 2-morphism $\alpha : \mathrm{Id}_{\mathbf{C}} \to f$

[...]

Definition 86.0.2. A category $\mathbf{C}$ is *homotopical* if in addition to being a category with weak equivalences, the subcategory $\mathbf{W}$ also obeys the two out of six property : for any sequence of composable morphisms,

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \tag{86.1}$$

if $h \circ g$ and $g \circ f$ are in $\mathbf{W}$, then so are f, g, h and $h \circ g \circ f$.

The homotopy category $\mathrm{Ho}(\mathbf{C}, \mathbf{W})$ of a category with weak equivalences $(\mathbf{C}, \mathbf{W})$ is the localization of that category along the weak equivalences, ie for any morphism $f \in \mathbf{W}$, we adjoin an actual inverse f^{-1} to transform the weak equivalence into an equivalence.

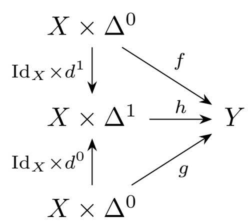
Example 86.0.1. For the category of topological spaces **Top**, we take the subcategory of weak equivalences to be that of continuous maps which are bijective on the path components, ie for $f : X \to Y$, $f \in \mathbf{W}$ if

$$f_* : \pi_0(X) \to \pi_0(Y) \tag{86.2}$$

[Higher homotopy groups?]

Another common example of a homotopy category is given by the simplicial sets, which are in some sense a "skeletal" version of topological spaces. This requires us to look at homotopies in simplicial sets first :

Definition 86.0.3. A simplicial homotopy between two simplicial morphisms $f, g : X \to Y$ means that the following diagram exists (using the isomorphism $X \times \Delta^0 \cong X$) :



Keeping the skeletal topological space interpretation, the simplicial homotopy is given by a morphism "parametrized" by Δ^1, for which

Interpreting simplicial sets as skeletal topological spaces, we can see for instance that the interval Δ_1

Example 86.0.2. The category of simplicial sets **sSet** can be turned into a homotopy category by taking its simplicial homotopy equivalence morphisms. For a simplicial morphism $f : X \to Y$, we say that it is a simplicial homotopy equivalence if there is a weak inverge $g : Y \to X$

In addition to weak equivalences to reflect the notion of homotopy equivalence, we can also look at two additional notions from homotopy theory, which are fibrations and cofibrations.

As a reminder, those notions in topology are

Definition 86.0.4. A fibration is a continuous map $p : E \to B$ such that every space X satisfies the homotopy lifting property.

Definition 86.0.5. A cofibration is a continuous map $i : A \to X$ such that for every space Y

"good subspace embedding". Example of use : given a subspace $\iota : S \hookrightarrow X$, and a map $f : S \to R$, is there an extension of f to X?

Example 86.0.3. $0 \hookrightarrow X$ is a cofibration

Example 86.0.4. A homeomorphism is a cofibration

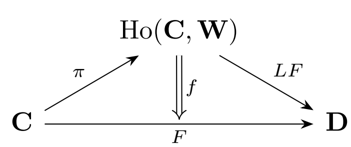
Definition 86.0.6. A fibration in a category with weak equivalences is a

Definition 86.0.7. A *model category* is a category $\mathbf{C}$ with three classes of morphisms $\mathbf{W}, \mathbf{Fib}, \mathbf{Cofib}$ where $(\mathbf{C}, \mathbf{W})$ is a category with weak equivalences, and

Similarly to how any category can be considered an ∞-category by simply considering all higher morphisms to be identities, we can also consider any category to be a category with weakly equivalences by simply picking W to be only isomorphisms, or a homotopy category by

in which case the homotopy projection is simply the skeletal category $\mathrm{Ho}(\mathbf{C}, \mathbf{W}) \cong \mathrm{Sk}(\mathbf{C})$

Definition 86.0.8. Given a category with weak equivalences $(\mathbf{C}, \mathbf{W})$ and some functor $F : \mathbf{C} \to \mathbf{D}$, its *left derived functor* LF is the right Kan extension of F along the projection $p : \mathbf{C} \to \mathrm{Ho}(\mathbf{C}, \mathbf{W})$



In a category with weak equivalences, the notions of limits and colimits do not typically preserve weak equivalences. ie if two diagram functors F, F' are weakly equivalent ($F(X) \cong_w F'(X)$ for all X), it is however not guaranteed that the limits from such a diagram are.

Example : pushout for $1 \leftarrow S^n \to D^{n+1}$ and $1 \leftarrow S^n \to 1$. The first colimit is S^{n+1} while the other is 1, which are not weakly equivalent in homotopy top

Definition 86.0.9. A *homotopy limit* of a functor $F : I \to \mathbf{C}$ with diagram of shape I is the right derived functor of the limit functor $\lim_I : [I, C] \to C$

$$\mathrm{holim_I} F = (R \lim_I) F \tag{86.3}$$

Theorem 86.0.1. Any discrete limit is homotopy invariant under [some condition] :

$$\lim_{\mathbf{n}} F = \mathrm{holim}_{\mathbf{n}} F \tag{86.4}$$

Definition 86.0.10.

Homotopy limit

Homotopy categories can be described by ∞-categories (2-categories, in fact), via

Theorem 86.0.2. Given a homotopy category ($\mathbf{C}$)

87

Intervals, loops and paths

The interaction of categories of spaces like topoi and ∞-categories or homotopy categories is what gives rise to concepts of homotopy. To talk of a notion of homotopy in our spaces (at least the type of homotopy we typically associate to spaces), we need to be able to talk about some notion of a path. Obviously our main idea behind such notions is that of the standard path in geometry, a map from the unit interval to a space

$$\gamma : [0,1] \to X \tag{87.1}$$

But we will need to keep things somewhat general.

Definition 87.0.1. An interval object I is some object containing two points, the boundary points

$$1 \sqcup 1 \underset{\iota_0}{\overset{\iota_1}{\rightrightarrows}} I \tag{87.2}$$

This is a pretty broad definition but we also typically depend that it be contractible.

Definition 87.0.2. A *contractible interval object* is an interval object

Typical examples of interval objects are the 1-simplex in the category of simplicial sets, where its boundary points are the maps $\vec{1} :\to \Delta[1]$, the unit interval $[0,1]$ for a variety of spatial categories, or the standard interval object $0 \to 1$ in the category of categories.

A path space object is then to be interpreted as the maps from this interval object to another space, where a path will be some element of the exponential object X^I

Definition 87.0.3.

$$X \xrightarrow{S} \text{Path}_X \overset{(d_0,d_1)}{\longrightarrow} X \times X \tag{87.3}$$

Loop space object and suspension object

Theorem 87.0.1. The localization of a category by its

Definition 87.0.4. Path ∞-groupoid

Definition 87.0.5. Loop space ΩX is the homotopy pullback $1 \to X \leftarrow 1$

Definition 87.0.6. In an ∞-category with a terminal object 1, a *suspension object* ΣX of X is the homotopy pushout

$$\Sigma X = X +^h_1 1 \tag{87.4}$$

Theorem 87.0.2. Fiber and cofiber of loop and suspension

Long (co)fiber sequences (Puppe sequence)

[154]

homotopy circle

Boring homotopy theory : interval object is terminal?

Example 87.0.1. If we pick as our line object the terminal object 1, the path space object X^I is isomorphic to X itself, with

$$X \to X^I \to X \times X \tag{87.5}$$

[...]

Trivial model category?

88 Line object

Given a Lawvere theory T, if it contains the theory of Abelian groups,

Example 88.0.1. In the topos of smooth spaces, the line object is the Yoneda embedding of the real line, $R = よ(\mathbb{R})$, is a line object.

Proof. $\square$

89

Stable homotopy theory

Definition 89.0.1. Given a pointed $(\infty, 1)$-category $\mathbf{C}$ with finite limits, it is *stable* if every morphism admits a kernel and a cokernel, and every exact triangle is coexact and vice versa.

Every morphism is the cokernel of its kernel and the kernel of its cokernel.

[155]

90

Postnikov etc

Example 90.0.1. To find the Postnikov tower for the sphere S^2, let's look at some of its homotopy sequence :

$$\pi_0(S^2) = 0 \quad (90.1)$$
$$\pi_1(S^2) = 0 \quad (90.2)$$
$$\pi_2(S^2) = \mathbb{Z} \quad (90.3)$$
$$\pi_3(S^2) = \mathbb{Z} \quad (90.4)$$
$$\pi_4(S^2) = \mathbb{Z}_2 \quad (90.5)$$
$$\pi_5(S^2) = \mathbb{Z}_2 \quad (90.6)$$

Whitehead tower

A useful example of this in physics is given by the delooping of the orthogonal group, $\mathbf{B}\mathrm{O}(n)$. As the orthogonal group has an exactly known periodic homotopy group,

$$\pi_{8k+0}(\mathrm{O}(n)) = \mathbb{Z}_2 \quad (90.7)$$
$$\pi_{8k+1}(\mathrm{O}(n)) = \mathbb{Z}_2 \quad (90.8)$$
$$\pi_{8k+2}(\mathrm{O}(n)) = 0 \quad (90.9)$$
$$\pi_{8k+3}(\mathrm{O}(n)) = \mathbb{Z} \quad (90.10)$$
$$\pi_{8k+4}(\mathrm{O}(n)) = 0 \quad (90.11)$$
$$\pi_{8k+5}(\mathrm{O}(n)) = 0 \quad (90.12)$$
$$\pi_{8k+6}(\mathrm{O}(n)) = 0 \quad (90.13)$$
$$\pi_{8k+7}(\mathrm{O}(n)) = \mathbb{Z} \quad (90.14)$$

First Stiefel-Whitney class, second Stiefel-Whitney, first fractional Pontryagin class, second frac. Pontr. class, etc

orthogonal group, special orthogonal group, spin group, string group, etc

Orthogonal structure, orientation, spin structure, string structure, etc

91

Homotopy localization

In addition to the localization of every possible weak isomorphism to form the homotopy category $\mathrm{Ho}(\mathbf{C})$, another possibility is to perform this localization at a smaller set of such weak isomorphisms.

Definition 91.0.1. Given an object X in a model category $(\mathbf{C}, W, \mathrm{Fib}, \mathrm{CoFib})$, its *homotopy localization* is its localization at the set of morphisms W_X of the form

$$Y \times X \xrightarrow{\mathrm{pr}_1} Y \tag{91.1}$$

$$W_X = \tag{91.2}$$

"In homotopy theory, for example, there are many examples of mappings that are invertible up to homotopy; and so large classes of homotopy equivalent spaces"

Example 91.0.1. Category of homotopy spaces

92

Adjunctions and monads

From the definition of an n-category, we can rewrite definitions for adjunctions and monads internally to a category. In particular, we can think of the definition we've seen in terms of the category of categories, **Cat**.

Given this, we can not only define adjunctions and monads within that language, but in fact generalize it for any 2-category or higher n-category.

Definition 92.0.1. In a 2-category $\mathcal{K}$, an adjunction

$$\eta, \mu : L \vdash R : X \to Y \tag{92.1}$$

is a pair of 1-morphisms $L : B \to A$ and $R : A \to B$ and 2-morphisms $\eta : \mathrm{Id} \to RL$ and $\mu : LR \to \mathrm{Id}$ which obey the triangle identities :

Working in the 2-category **Cat**, we end up with the appropriate definition for adjunctions.

Example 92.0.1. A non-**Cat** example of this is to pick the homotopy category of topological spaces, in which case we can look at the *suspension-loop adjunction*,

We can furthermore also define monads and comonads in this context

Definition 92.0.2. A *monad* in a 2-category is a triple (T, η, μ) over an object X given by an endomorphism $T : X \to X$, a 2-morphism $\eta : \mathrm{Id}_X \to T$, and a 2-morphism $\mu : T^2 \to T$, such that [...]

Theorem 92.0.1. Given two adjunctions

$$\eta, \mu : L \vdash R : C \quad \to \quad D \tag{92.2}$$

$$\eta', \mu' : L' \vdash R' : C' \quad \to \quad D' \tag{92.3}$$

and two 1-morphisms $F : C \to C'$ and $G : D \to D'$, the following 2-morphisms are isomorphic :

$$\zeta : L'G \xrightarrow{L'G\eta} \tag{92.4}$$

Definition 92.0.3. Given two such 2-morphisms as ξ and ζ, we say that ξ and ζ are *mates* under adjunction, written $\xi \dashv \zeta$.

93

∞-sheaves

Just as categories have sheaves as functors to the category of sets, ∞-categories have ∞-sheaves as functors to the category $\infty\mathbf{Grpd}$.

Before we get into ∞-sheaves, however, let's first look at the simpler case of stacks, or groupoid stacks more specifically (throughout we will simply refer to groupoid stacks as stacks, as we will not consider any other category for valuation)

Definition 93.0.1. A *pre-stack* is a groupoid-valued presheaf, ie a functor

$$S : \mathbf{C}^{\mathrm{op}} \to \mathbf{Grpd} \tag{93.1}$$

Definition 93.0.2. A *stack* is a prestack

Example 93.0.1. A traditional example of a stack is given by the stack of similar triangles[156, 157]. Take the set of all possible triangles, with sides (a, b, c). To only consider similar triangles, ie up to equivalence $(a, b, c) \cong (\alpha a; \alpha b, \alpha c)$, we will take triangles of a constant perimeter :

$$a + b + c = 2 \tag{93.2}$$

The space of all such values is therefore some two-dimensional subspace of $[0, 2]^3 \subseteq \mathbb{R}^3$, and more specifically this is a simplex.

For each point of this simplex, we have an associated triangle, and we attach to each of those point the group of symmetries of those triangle. The structure thus formed is a groupoid where each point is a group. The groups involved are typically the symmetric groups S_2 (isoceles triangles), S_3 (equilateral triangles) as well as the reflection symmetry $\mathbb{Z}_2$

Example 93.0.2. The simplest non-trivial example of a smooth stack is that of an *orbifold*, which are spaces modeled locally by quotients of open subsets of $\mathbb{R}^n$ by some finite group action. The simplest example of this being a line acted on by $\mathbb{Z}_2$, which resembles the half-line $[0, \infty)$.

As in the context of a stack we are instead interested in mapping a Cartesian space to a groupoid, we will instead look at the action of the functor H (for half line) on $\mathbb{R}$.

We are

[158]

Definition 93.0.3. A ∞-presheaf F is given by a functor

$$F : \mathbf{C}^{\mathrm{op}} \to \infty\mathbf{Grpd} \tag{93.3}$$

Definition 93.0.4. A ∞-sheaf F is an ∞-presheaf

Example 93.0.3. If we consider a set X as an ∞-sheaf on the terminal site of 0-type, and we consider its quotient under group action by some group object G, via the action

$$\rho : G \times X \to X \tag{93.4}$$

94

∞-topos

Definition 94.0.1. An ∞-category $\mathbf{H}$ is an ∞-topos if

Example 94.0.1. As in the case of 1-topoi, two basic examples of ∞-topoi are given by the initial ∞-topos, $\mathrm{Sh}_\infty(\mathbf{0})$, which is the initial ∞-category, and the terminal topos $\mathrm{Sh}_\infty(\mathbf{1})$, which is $\infty\mathbf{Grpd}$.

94.1 Principal bundles

Definition 94.1.1. A *G-principal bundle* P over an object X is a pullback of a morphism $X \to \mathbf{B}G$ and (some point?) $1 \to \mathbf{B}G$.

The only non-trivial morphism of the pullback is the projection $\pi : P \to X$.

[150, 159]

95

Simplicial homotopy

A simple case of a model category is given by the model category of simplicial sets, **sSet**, the Quillen model category.

Given the category **sSet**, its model structure is defined

Cylinder functor

$$\begin{array}{ccccc} X \cong X \times \Delta[0] & \xrightarrow{\mathrm{Id}_X \times \delta^1} & X \times \Delta[1] & \xleftarrow{\mathrm{Id} \times \delta^0} & X \times \Delta[0] \cong X \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \eta} & \swarrow {\scriptstyle g} & \\ & & Y & & \end{array}$$

96

Smooth groupoids

The higher categorical equivalent of the category of smooth spaces is the ∞-topos of smooth ∞-groupoids over the site of Cartesian spaces, which are smooth sets along with their homotopy equivalences. This will allow us to treat the notion of groups without internalization in the context of smooth spaces, such as what we would need for gauge theories.

If we look at those spaces once more in the context of their analogies with diffeological spaces, rather than mapping our probes to some sets of atlases, we are mapping them to some ∞-groupoid. All of our previous diffeological spaces are simply sheaves that are non-empty only for the discrete objects of the ∞-groupoid (in the sense of being 0-truncated, ie all their higher order morphisms are just the identities). Equivalently this can be done via the composition of the smooth set and the embedding of **Set** into ∞**Grpd**.

Definition 96.0.1. A *pre-smooth groupoid* is a presheaf of groupoids on the site **CartSp**,

$$X : \mathbf{CartSp}^{\mathrm{op}} \to \mathbf{Grpd} \tag{96.1}$$

The most basic kind of non-0-truncated smooth groupoid is a 1-truncated smooth groupoid of a single point, which simply corresponds to a group, which as a sheaf is given by

$$G(\mathbb{R}^0) = G \tag{96.2}$$

Another example is the *classifying space* of a group. This will be in the context of a smooth groupoid a diffeological space for which the k-homotopy groups will be equal to G.

Example 96.0.1. The circle S^1 is the classifying space of the group $\mathbb{Z}$. As a smooth groupoid, this is given by considering the sheaves on $\mathbb{R}^1$ (for instance using the stereographic projection)

A less trivial example for this is the notion of a G-manifold. Just as the case of ∞**Grpd**, where G-sets form a subcategory, G-manifolds can likewise be expressed as ∞-smooth groupoids.

Definition 96.0.2. A G-manifold a manifold M equipped with a smooth group action by the Lie group G

$$\rho : M \times G \to M \tag{96.3}$$

Example 96.0.2. the pair of the line manifold $L \cong \mathbb{R}$ and the one-dimensional translation group $T \cong \mathbb{R}$ (as a group).

$$\rho : M \times G \to M \tag{96.4}$$

Action groupoid

Definition 96.0.3. An orbifold X is a topological space with an atlas of orbifold charts (U, G, ϕ), where U is an open set of X, ϕ is a continuous map from some connected open subset $\mathcal{O} \subseteq \mathbb{R}^n$ to U, and G is a finite group acting smoothly and effectively on $\mathcal{O}$, such that ϕ is a G-equivariant map.

Example 96.0.3. The simplest non-trivial orbifold is the real line reflected along $\mathbb{Z}_2$. Take the unique chart $(\mathbb{R}, \mathbb{Z}_2, \phi)$, with the action the typical $\mathbb{Z}_2$ involution :

$$g_{-1} \in \mathbb{Z}_2,\ g_{-1}(x) = -x \tag{96.5}$$

Topologically the reflected real line is simply $[0, \infty)$, and one of its orbifold charts is given by

$$\phi = \mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R} \tag{96.6}$$

As a Lie groupoid, we have

$$\mathcal{G} : \mathbb{Z}_2 \times \mathbb{R} \quad \rightrightarrows \quad \mathbb{R} \tag{96.7}$$

Theorem 96.0.1.

Example 96.0.4. The localization of the circle S^1 is homotopy equivalent to the simplicial circle $\Delta_1 / \partial \Delta_0$

Proof.

$$S^1 = R/\mathbb{Z} \tag{96.8}$$

Localization :

$$\mathrm{loc}_R(S^1) \cong \mathrm{loc}_R(R/\mathbb{Z}) \tag{96.9}$$

Left adjoint due to reflective localization, preserves colimits, therefore commutes with the coequalizer

$$\mathrm{loc}_R(S^1) \cong 1/\mathbb{Z} \tag{96.10}$$

Stack? □

96.1 Principal bundle

In the case of a smooth groupoid, the principal bundle is the usual notion of a G-principal bundle.

The basic example is that of a manifold M, ie some locally representable smooth 0-type, with G some Lie group of the same type. If we consider some open cover $\{U_i \to M\}$, and its Čech groupoid $C(\{U_i\})_\bullet$,

96.2 Connections

Having defined principal bundles, we are now able to define (gauge) connections in this formalism.

Connections have many equivalent definitions, but the one that will interest us here is that of a transport operator. A connection will give us an isomorphism between two bundles after transport along a curve. If we have a curve $\gamma : [0,1] \to M$ on some manifold, and a principal bundle $\pi : P \to M$ a G-principal bundle, the transport along γ is some homomorphism

$$\mathrm{tra}_\nabla(\gamma) : P_{\gamma(0)} \to P_{\gamma(1)} \tag{96.11}$$

In terms of categories, our curve is an object in the path space object $\gamma : 1 \to [I, M]$

Groupoidal property of the transport

In the context of connections, we will usually need more information than that given by the fundamental groupoid (we don't expect all homotopic paths between two points to have the same transport property, as this is prevented by curvature), but less than that of the path space object : the transport along a path and then its reverse should in fact be isomorphic to the null path. The groupoid we are interested in is the *path groupoid* [160]. To define this, we first have to define which paths are meant to be equivalent.

Definition 96.2.1. A curve $\gamma : 1 \to [I, M]$ has *sitting instants*

Definition 96.2.2. Thin homotopy

Definition 96.2.3. The *path groupoid* $P(M)$ of a manifold M is

Generalization to smooth ∞-groupoids [161]:

Definition 96.2.4. The *path ∞-groupoid* $P_\infty(X)$ of a smooth ∞-groupoid X

Gravitational connection?

tratritra

97 Cohomology

As ∞-categories will typically encode data relating to the homotopy of their objects, there is some natural notion of (co)homology emerging from it,

Definition 97.0.1. Given some ∞-topos $\mathbf{H}$, the cohomology of $X \in \mathbf{H}$ with values in $A \in \mathbf{H}$ is the set of connected components of the hom-∞-groupoid :

$$H(X; A) = \pi_0 \mathrm{Hom}_{\mathbf{H}}(X, A) \tag{97.1}$$

and its degree n cohomology is given by the n-fold delooping of A :

$$H^n(X; A) = \pi_0 \mathrm{Hom}_{\mathbf{H}}(X, A_n) \tag{97.2}$$

Definition 97.0.2. The Eilenberg-MacLane object

Example 97.0.1. In the $(\infty, 1)$-category of topological spaces Top with homotopies, the Eilenberg-MacLane object $K(\mathbb{Z}, 1)$ is the circle S^1.

Proof. As the circle is connected, we have $\pi_0(S^1) = \{\bullet\}$, and its higher homotopy groups

$$(0, \mathbb{Z}, 0, 0, \ldots) \tag{97.3}$$

□

Example 97.0.2. If we take the $(\infty, 1)$-category of topological spaces Top with homotopies, and the Eilenberg-MacLane object $K(\mathbb{Z}, 1)$ as its values, the singular cohomology is

Example 97.0.3. The cohomology of the simplicial circle in the category of simplicial sets, $\Delta_1/\partial\Delta_0$

Homology? Dold-Kan correspondence

98 Higher order logic

The associated internal logic of a higher category theory is that of a type theory with homotopy types

Part VIII

Subjective logic

[162, 163]

The lower level of Hegel's logic is what he calls the *subjective logic*, which is roughly comparable to propositional or predicate logic or type theory. The subjective logic is not about the concepts and notions themselves, but about their relations. That is, we can speak of propositions and objects without ever giving specific examples of those.

99
Moments

The basic objects in the objective logic are *concepts* and *moments* [is that the subjective logic actually].

The concepts are meant in Hegel to represent thoughts (as this is primarily an idealist philosophy), notions, etc, which can include those of actual physical objects. This would fit notions of mathematical objects well enough, although whether they fit perfectly in the framework of a category, who knows.

Immanence $\approx$ internal logic?

For any two concepts, we can look at morphisms as relations between those, following the semantics of morphisms as predicates

The general properties of an object in a category defined by the slice category of that object?

Relations between concepts as morphisms

The formalization of qualities of an object are given by the concept of *moments* of an object, given by some "projection operator" $\bigcirc : C \to C$, such that

$$\bigcirc \bigcirc X \cong \bigcirc X \tag{99.1}$$

If we reduce the object X to merely the qualities given by $\bigcirc$, there is nothing left to remove so that any subsequent projection will be isomorphic to it. In categorical term, we also demand that the projection given as $X \to \bigcirc X$ be, within the category of qualities $\bigcirc$, an equivalence :

$$\bigcirc(X \to \bigcirc X) \in \mathrm{core}(X) \tag{99.2}$$

In terms of types, this is an idempotent monad $(\bigcirc, \eta, \mu)$, where the projection $X \to \bigcirc X$ is the unit of the monad,

$$\eta^{\bigcirc} : \mathrm{Id}_{\mathbf{H}} \to \bigcirc \tag{99.3}$$

or in components,

$$\eta^{\bigcirc}_X : X \to \bigcirc X \tag{99.4}$$

and the isomorphism is given by the multiplication map

$$\mu^{\bigcirc} : \bigcirc \xrightarrow{\cong} \bigcirc\bigcirc \tag{99.5}$$

or in components,

$$\mu^{\bigcirc}_X : \bigcirc^2 X \xrightarrow{\cong} \bigcirc X \tag{99.6}$$

Dually, we can also define idempotent comonads $(\Box, \epsilon, \delta)$, where we have some counit

$$\epsilon^{\Box} : \Box \to \mathrm{Id}_{\mathbf{H}} \tag{99.7}$$

$$\epsilon^{\Box}_X : \Box X \to X \tag{99.8}$$

and the comultiplication map (an isomorphism, for an idempotent comonad)

$$\delta^{\Box} : \Box\Box \xrightarrow{\cong} \Box \tag{99.9}$$

$$\delta^{\Box}_X : \Box^2 X \xrightarrow{\cong} \Box X \tag{99.10}$$

Definition 99.0.1. A *moment* on a type system/topos $\mathbf{H}$ is either an idempotent monad $\bigcirc$ or comonad $\Box$.

If we keep the monad or comonad status of the moment ambiguous, we will denote it by the functor Δ or $\bigstar$.

We will denote by $\mathbf{H}_{\bigcirc}$ the Eilenberg-Moore category of a monad $\bigcirc$, and by $\mathbf{H}_{\Box}$ that of a comonad $\Box$. As an idempotent (co)monads, this Eilenberg-Moore category is a (co)reflective subcategory of the original topos, where we have for a monad $\bigcirc$

$$(T_{\bigcirc} \dashv \iota_{\bigcirc}) : \mathbf{H}_{\bigcirc} \underset{\iota_{\bigcirc}}{\overset{T_{\bigcirc}}{\leftrightarrows}} \mathbf{H}$$

with the monad unit being the unit of the adjunction

$$\eta : \mathrm{Id}_{\mathbf{H}} \to \iota_{\bigcirc} T_{\bigcirc} \tag{99.11}$$

and multiplication being

$$\mu^{\bigcirc} = \iota_{\bigcirc} \triangleleft (\epsilon \triangleright T_{\bigcirc}) \tag{99.12}$$

And for a comonad $\square$,

$$(\iota_{\square} \dashv T_{\square}) : \mathbf{H}_{\square} \underset{T_{\square}}{\overset{\iota_{\square}}{\rightleftarrows}} \mathbf{H}$$

with the comonad counit being the counit of the adjunction,

$$\epsilon^{\square} : \iota_{\square} T_{\square} \to \mathrm{Id}_{\mathbf{H}} \tag{99.13}$$

and the comultiplication

$$\delta = \iota_{\square} \triangleleft (\eta \triangleright T_{\square}) \tag{99.14}$$

where the adjoint of the inclusion and its (co)reflector, $(T_{\bigcirc} \dashv \iota_{\bigcirc})$ or $(\iota_{\square} \dashv T_{\square})$, gives the moment via the usual composition of monads and comonads from adjoint pairs,

$$\bigcirc = \iota_{\bigcirc} \circ T_{\bigcirc} \tag{99.15}$$

$$\square = \iota_{\square} \circ T_{\square} \tag{99.16}$$

Those Eilenberg-Moore categories are fully faithful subcategories of the original topos, by the idempotence of those (co)monads.

The unit of the monad $\eta_{\bigcirc}$ gives us a mapping from the original object to the object in this subcategory

[Free algebra?]

Therefore, for a monadic or comonadic moment, we have both the embedding of the (co)reflective subcategory, $\mathbf{H}_{\square} \hookrightarrow \mathbf{H}$, and the (co)reflection $\mathbf{H} \twoheadrightarrow \mathbf{H}_{\Delta}$

As any reflective subcategory can be understood as the localization of the original category by some class W of weak equivalences, for an s-moment $\bigcirc$ there is some corresponding set of morphisms S for which $\mathbf{H}_{\bigcirc}$ is the localization of $\mathbf{H}$, $S = \bigcirc^{-1}[\mathrm{Iso}(\mathbf{H}_{\bigcirc})]$

As we are projecting our objects down to some of their moments, we can define some notion of similarity. For any two objects in $\mathbf{C}$, we say that they are Δ-similar if their modality is isomorphic,

$$X \cong_\Delta Y \leftrightarrow \Delta X \cong \Delta Y \tag{99.17}$$

[Is this the weak equivalence of the localization?]

Likewise, a coreflective subcategory can be understood as the colocalization of the span category $\mathrm{Span}(\mathbf{H}_\Box)$

Interpretation : localization as equivalence by the relation between two objects, colocalization as equivalence between two objects who have a relation with a third object in common?

”we may naturally make sense of ”pure quality” also for (co-)monads that are not idempotent, the pure types should be taken to be the “algebras” over the monad.”

A particular moment that we will see more in details later but that will be of great importance throughout is the one given by the smallest possible subtopos, which is just the terminal topos $\mathbf{1} \cong \mathrm{Sh}(\mathbf{0})$. Being the smallest possible subtopos, it corresponds in some sense to an object with no qualities left beyond being an object, what we will call its quality of *being*. This can be seen in the sense that the category only contains a single object and a single relation (the identity), so that all objects are identical under this modality. This will be defined in more details in 107, but I include it here as it will be used for a few basic constructions of moments.

Example 99.0.1. The adjunction Even $\dashv$ Odd is an opposition of the form $\Box \dashv \bigcirc$

From the form of the monads and comonads, where the mapping of the basic type X to their moment ΔX for some moment Δ is either given by the unit (return operation) for the monad

$$X \to \Delta X \tag{99.18}$$

or the counit (extend operation) for the comonad

$$\Delta X \to X \tag{99.19}$$

we will also call monads *successive moments*, or s-moments, and comonads as *preceding moments*, or p-moments, and we will denote them by default as $\bigcirc$ and $\Box$ respectively, with Δ for a moment of unstated type.

Semantics : any p-moment is given by the embedding of $\mathbf{H}_\Box \hookrightarrow \mathbf{H}$

As monads	As moments	Notation
Comonad	p-moment	$\square$
Monad	s-moment	$\bigcirc$

Table 99.1: Caption

Example 99.0.2. The basic example of a monad that we've seen for the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ of

$$\text{Ceiling} \dashv \text{Floor} \tag{99.20}$$

is a pair of an idempotent monad and comonad. The unit of Ceiling is given by the order relation in $\mathbb{R}$

$$x \in \mathbb{R} \to \text{Ceiling}(x) \tag{99.21}$$

or $x \leq \text{Ceiling}(x)$ for short, so that the integral moment of x is indeed literally *successive* here, ie it is the following element. Likewise for Floor,

$$\text{Floor}(x) \to x \tag{99.22}$$

it is preceding.

Definition 99.0.2. Given a moment $\bigcirc$, for $X \in \mathbf{H}$, $\bigcirc_X$ is a moment on $\mathbf{H}_{/X}$ sending $p : E \to X$ to $\bigcirc_X E \to X$ via the pullback

$$\begin{array}{ccc} \bigcirc_X E & \longrightarrow & \bigcirc E \\ \downarrow & & \downarrow{\scriptstyle \bigcirc p} \\ X & \xrightarrow{\eta_X^{\bigcirc}} & \bigcirc X \end{array}$$

[Are moments Cartesian (co)monads??? Only some? The negations?]

[Not gonna be all since $\circledast$ is not a Cartesian monad : $1 \times_1 Y = Y \neq X$]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X^{\bigcirc}}\downarrow & & \downarrow{\scriptstyle \eta_Y^{\bigcirc}} \\ \bigcirc X & \xrightarrow[\eta^{\bigcirc}(f)]{} & \bigcirc Y \end{array}$$

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Unity of opposites

[164]

As we've seen previously 38.392, we can define a specific notion of adjunction for (co)monads. This can be done in a few different ways, either directly at the level of the pair of monad and comonad, or via a string of adjoint functors.

First, we can define them as traditional adjoint with some additional properties. Let's first look at the case of a monad and comonad

$$(\bigcirc \dashv \square) : \mathbf{H} \underset{-\bigcirc\to}{\overset{\leftarrow\square-}{\rightleftarrows}} \mathbf{H}$$

with the associated unit and counit of this adjunction,

$$\eta : \mathrm{Id}_{\mathbf{H}} \to \square\bigcirc \tag{100.1}$$

$$\epsilon : \bigcirc\square \to \mathrm{Id}_{\mathbf{H}} \tag{100.2}$$

In addition to the (co)monads being adjoint as functors, we ask that their respective units and counits be adjoint in a more specific sense, where we also have the mate bijection between (η, μ) and (ϵ, δ) :

$$\eta^{\bigcirc} : \mathrm{Id}_{\mathbf{H}} \to \square\bigcirc \tag{100.3}$$

$$\epsilon^{\square} : \bigcirc\square \to \mathrm{Id}_{\mathbf{H}} \tag{100.4}$$

$$(\Box \dashv \bigcirc) : \mathbf{H} \underset{\Box}{\overset{\bigcirc}{\leftrightarrows}} \mathbf{H}$$

We can also define them as an adjoint triple. If we have three adjoint functors

$$(F \dashv G \dashv H) : \mathbf{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \\ \xrightarrow{H} \end{matrix} \mathbf{D}$$

where $F, H : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, we can define a pair of monads $\Delta, \bigstar$ on $\mathbf{C}$

$$\Delta = G \circ F : \mathbf{C} \to \mathbf{C} \tag{100.5}$$

$$\bigstar = G \circ H : \mathbf{C} \to \mathbf{C} \tag{100.6}$$

The two adjunctions imply that G preserves all limits and colimits in D

Gives rise to an adjoint pair of a monad GF and a comonad GH on $\mathbf{C}$

$$(GF \dashv GH) : \mathbf{C} \underset{GF}{\overset{GH}{\leftrightarrows}} \mathbf{C}$$

and dually, the adjoint pair of the comonad FG and the monad HG on $\mathbf{D}$

$$(FG \dashv HG) : \mathbf{D} \underset{FG}{\overset{HG}{\leftrightarrows}} \mathbf{D}$$

Theorem 100.0.1. For $F \dashv G \dashv H$, F is fully faithful iff H is.

F being fully faithful is equivalent to $\eta : \mathrm{Id} \to GF$ being a natural isomorphism.

H being fully faithful is equivalent to $\varepsilon : GH \to \mathrm{Id}$ being a natural isomorphism.

GF is isomorphic to the identity if GH is

$F \dashv G \dashv H$ is a fully faithful adjoint triple in this case. "This is often the case when D is a category of "spaces" structured over C, where F and H construct "discrete" and "codiscrete" spaces respectively."

If we denote a pair of idempotent adjoint monad $\bigcirc$ and comonad $\Box$, with the associated reflective and co-reflective categories

$$(\iota_\Box \dashv T_\Box) : \mathbf{H}_\Box \underset{T_\Box}{\overset{\iota_\Box}{\leftrightarrows}} \mathbf{H}$$

$$(FG \dashv HG) : \mathbf{H}_\bigcirc \underset{FG}{\overset{HG}{\leftrightarrows}} \mathbf{H}$$

From the adjunction, we have that $\mathbf{H}_{\square} \cong \mathbf{H}_{\bigcirc}$ (for $\mathbf{H}$ one of the category in the adjoint triple of functor, this is the other category). Given either the adjunction $(\bigcirc \dashv \square)$ or $(\square \dashv \bigcirc)$, the adjoint triple will therefore be either $T_{\square} \ldots$

The opposite of a moment $\bigcirc$ is a moment $\square$ such that they form either a left or right adjunction, ie :

$$(\square \dashv \bigcirc) : \mathbf{H}_{\bigcirc} \cong \mathbf{H}_{\square} \begin{array}{c} \xhookrightarrow{\iota_{\square}} \\ \xleftarrow{T_{\square}\cong T_{\bigcirc}} \\ \xhookrightarrow{\iota_{\bigcirc}} \end{array} \mathbf{H}$$

or

We will denote the unity of opposites $\square \dashv \bigcirc$ as a unity of a preceding to a successive moment, or *ps* unity, and $\bigcirc \dashv \square$ as an *sp* unity.[9]

Theorem 100.0.2. A *ps* unity $(\square \dashv \bigcirc)$ defines an essential subtopos.

$$(\square \dashv \bigcirc) : \mathbf{H}_{\bigcirc} \cong \mathbf{H}_{\square} \begin{array}{c} \xhookrightarrow{\iota_{\square}} \\ \xleftarrow{T_{\square}\cong T_{\bigcirc}} \\ \xhookrightarrow{\iota_{\bigcirc}} \end{array} \mathbf{H}$$

with the monads

$$\square = \iota_{\square} \circ T \tag{100.7}$$
$$\bigcirc = \iota_{\bigcirc} \circ T \tag{100.8}$$

and unit and counit of adjunction

$$\eta^{\bigcirc} : \mathrm{Id}_{\mathbf{H}} \to \bigcirc\square \tag{100.9}$$
$$\epsilon^{\square} : \square\bigcirc \to \mathrm{Id}_{\mathrm{H}} \tag{100.10}$$

Proof. The monad and comonad are adjoint endofunctors, that

□

Interpretation : two different opposite "pure moments", level of a topos

The reflector and coreflector are the same, $T_{\square} = T_{\bigcirc}$. Any object in the category is mapped to the same object in the category of "pure moment" regardless of the moment (this is their "unity"). The inclusion map however will not be the same, and will typically be quite different, although it remains possible that a given object share the same moments for both.

Examples of *ps*-unities : even-odd adjunction, being-nothing, flat-sharp

Theorem 100.0.3. An *sp*-unity $(\bigcirc \dashv \square)$ defines a bireflective subcategory

$$(\bigcirc \dashv \square) : \mathbf{H}_{\bigcirc} \cong \mathbf{H}_{\square} \begin{array}{c} \xleftarrow{T_{\bigcirc}} \\ \xhookrightarrow{\iota_{\square}\cong\iota_{\bigcirc}} \\ \xleftarrow{T_{\square}} \end{array} \mathbf{H}$$

with

$$\Box = \iota \circ T_{\Box} \quad (100.11)$$
$$\bigcirc = \iota \circ T_{\bigcirc} \quad (100.12)$$

Proof. □

The inclusion maps of the reflection/coreflection are such that $\iota_{\Box} \cong \iota_{\bigcirc}$, they are "the same objects"

"one pure moment, but two opposite ways of projecting onto it."

This means that for any object X in $\mathbf{H}$, its projected moment for either adjoint (say $\Box$) will be equivalent to the projection by the other adjoint of another object. If we take project our object, $\Box X = \iota_{\Box} \circ T_{\Box} X$, this maps it to some object in the bireflective subcategory before mapping it back to $\mathbf{H}$. As $\iota_{\Box} \cong \iota_{\bigcirc}$, this bireflection corresponds similarly to some object $Y \in \mathbf{H}$ for which $\Box X \cong \bigcirc Y$, given by $Y = \iota_{\bigcirc} T_{\Box} X$ (this is some pure $\bigcirc$ moment object in $\mathbf{H}$). So we have

$$\forall X \in \mathbf{H},\ \exists Y \in \mathbf{H},\ \Box X \cong \bigcirc Y \quad (100.13)$$

and using a similar argument, we have

$$\forall X \in \mathbf{H},\ \exists Y \in \mathbf{H},\ \bigcirc X \cong \Box Y \quad (100.14)$$

This property gives us the following

Theorem 100.0.4. For an *sp*-unity, the composition of the two modalities is isomorphic to the inner modality :

$$\Box \bigcirc X \cong \bigcirc X \quad (100.15)$$
$$\bigcirc \Box X \cong \Box X \quad (100.16)$$

Proof. Given the map $\eta^{\bigcirc} : \mathrm{Id} \to \Box\bigcirc$, Using the idempotence of the moments,

$$\Box \bigcirc X \cong \Box\Box Y \quad (100.17)$$
$$\cong \Box Y \quad (100.18)$$
$$\cong \bigcirc X \quad (100.19)$$

and dually, $\bigcirc\Box X = \Box X$ □

Corrolary 3. The morphism $\eta^{\bigcirc}(\epsilon^{\Box}_X)$ is an isomorphism.

Proof. □

Theorem 100.0.5. For a *ps*-unity, we have

$$\square \bigcirc X \cong \square X \tag{100.20}$$
$$\bigcirc \square X \cong \bigcirc X \tag{100.21}$$

Proof. by the triangle identity,

$$\square \bigcirc X \cong \iota_\square T \iota_\bigcirc T \tag{100.22}$$
$$\cong \iota_\square T \tag{100.23}$$

□

For a unity of opposites, we have the function defined by the composition of its unit and counit,

$$\eta_X^\bigcirc \circ \epsilon_X^\square : \square X \to X \to \bigcirc X \tag{100.24}$$

which is sometimes called the canonical natural transformation of unity of opposites, and is also given more specific names for some oppositions such as the becoming for 107, or the point-to-piece transform for 109. For brevity we will denote it as Υ in the general case :

$$\Upsilon = \eta \circ \epsilon : \square \to \bigcirc \tag{100.25}$$

Examples of *sp*-unities : ceiling-floor, shape-flat

Ceiling-floor : the similar objects under both modalities are simply given by $\lfloor x \rfloor$ and $\lceil x \rceil$, both mapped to integers and given the pure moment by including them back in $\mathbf{R}$. That is, for some real $x = k + \delta$, $\delta \in [0, 1]$ (and $\neq 0$), we have

$$\square x = \lceil x \rceil = k = \lfloor k \rfloor \tag{100.26}$$

for any $k \in \mathbb{Z}$, we have $\iota(k)$ being the same

$$\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor \tag{100.27}$$
$$\lfloor \lceil x \rceil \rfloor = \lceil x \rceil \tag{100.28}$$

Shape-flat :

$$\int \flat X = \flat X \tag{100.29}$$
$$\flat \int X = \int X \tag{100.30}$$

Pieces of points are the points, points of the pieces are the pieces

spaces among retractive spaces? zero-vector bundles among all vector bundle?

There is an additional case of unity of opposites we can consider, where the adjoint functors form an ambidextrous adjunction, ie both of the outer functors of the adjoint triple are identical (or isomorphic, anyway)

$$(T \dashv \iota \dashv T) \tag{100.31}$$

in which case the monad and comonad are isomorphic,

$$L \cong R \tag{100.32}$$

a unique modality that we will call $\natural$.

Theorem 100.0.6. In an ambidextrous adjunction, $\eta \circ \epsilon \cong \mathrm{Id}$.

Frobenius monads and comonads

Definition 100.0.1. Classical modality (quintessential localizations)

Example 100.0.1. The name of classical modality stems from its application as a map on mixed quantum-classical types in information theory, where the mixed type is given by a bundle of Hilbert spaces over a set, $\prod_{s:S} \mathcal{H}_s \to S$. The classical modality in this case factors through the underlying set via

$$T_\natural(\prod_{s:S} \mathcal{H}_s \to S) \cong S \tag{100.33}$$

and sent back via

$$\iota_\natural S \cong (\prod_{s:S} \mathbb{C}^0 \to S) \cong (\mathbb{C}^0 \times S \to S) \tag{100.34}$$

The T map is both reflective and coreflective.

$$\mathrm{Hom}_{\mathbf{C}}((X \to S), (\{0\} \times S' \to S')) \cong \mathrm{Hom}_{\mathbf{Set}}(S, S') \tag{100.35}$$

which is true as every bundle map $(f, \phi) : (X \to S) \to (\{0\} \times S' \to S')$ has to obey

$$\begin{array}{ccc} X & \xrightarrow{f} & S' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_S} \\ S & \xrightarrow{\phi} & S' \end{array}$$

So that $f = \phi \circ \pi$, the bundle morphisms $(f, \phi) = (\phi \circ \pi, \phi)$ are entirely determined by the underlying functions of the set. And dually,

$$\mathrm{Hom}_{\mathbf{C}}((S' \times \{0\} \to S'), (X \to S)) \cong \mathrm{Hom}_{\mathbf{Set}}(S', S) \tag{100.36}$$

$$\begin{array}{ccc} S' & \xrightarrow{f} & X \\ {\scriptstyle \mathrm{Id}_{S'}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\phi} & S \end{array}$$

giving $\phi = \pi \circ f$, so that the bundle map $(f\phi) = (f, \pi \circ f)$ [?]

Definition 100.0.2. A subtopos is an *essential subtopos* if the inclusion map $\iota : \mathbf{S} \hookrightarrow \mathbf{H}$ is an essential geometric morphism, ie in addition to ι being a full and faithful functor with a left adjoint ι^* that preserves finite limits, ι^* furthermore has a left adjoint $\iota_!$

Theorem 100.0.7. A *ps*-unity $(\Box \dashv \bigcirc)$ defines an essential subtopos.

Proof. As a set of two moments, we have $\Box$

Theorem 100.0.8. For an *sp*-unity $(\bigcirc \dashv \Box)$, we have

$$\Upsilon_X = \eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \cong \eta^{\bigcirc}_{\Box X} \tag{100.37}$$

Proof. Given an object X, if we look at the component $\epsilon^{\Box}_X : \Box X \to X$, its naturality square under $\eta^{\bigcirc}$ is given by

$$\begin{array}{ccc} \Box X & \xrightarrow{\eta^{\bigcirc}_{\Box X}} & \bigcirc\Box X \\ {\scriptstyle \epsilon^{\Box}_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta^{\bigcirc}(\epsilon^{\Box}_X)} \\ X & \xrightarrow{\eta^{\bigcirc}_X} & \bigcirc X \end{array}$$

As $\eta^{\bigcirc}(\epsilon^{\Box}_X) : \bigcirc\Box X \to \bigcirc X$ is an equivalence for an *sp*-unity 100.0.4, we have

$$\eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \cong \eta^{\bigcirc}_{\Box X} \tag{100.38}$$

$\Box$

Interaction of the hom functor adjunction and adjoint modalities?

Adjunction $(L \dashv R)$:

$$\Phi : \mathrm{Hom}_{\mathbf{C}}(L(-), -) \to \mathrm{Hom}_{\mathbf{C}}(-, R(-)) \tag{100.39}$$

$$\Phi^{-1} : \mathrm{Hom}_{\mathbf{C}}(-, R(-)) \to \mathrm{Hom}_{\mathbf{C}}(L(-), -) \tag{100.40}$$

If $\mathbf{C} = \mathbf{D} = \mathbf{H}$:

$$\Phi : \mathrm{Hom}_{\mathbf{H}}(L(-), -) \to \mathrm{Hom}_{\mathbf{H}}(-, R(-)) \tag{100.41}$$

$$\Phi^{-1} : \mathrm{Hom}_{\mathbf{H}}(-, R(-)) \to \mathrm{Hom}_{\mathbf{H}}(L(-), -) \tag{100.42}$$

Using the identity $LR \cong L$, $RL \cong R$: If we apply Φ to a function of type $X \to RY$, in $\mathrm{Hom}(X, RY)$

Theorem 100.0.9. For an adjoint modality $(\bigcirc \dashv \Box)$ $[(\Box \dashv \bigcirc)?]$, the components of the unit $\eta_X^{\bigcirc}$ are surjections if both modalities preserve the terminal object :

$$\Box 1 \cong 1 \cong \bigcirc 1 \tag{100.43}$$

Proof. Given the unit component at an object X,

$$\eta_X^{\bigcirc} : X \to \bigcirc X \tag{100.44}$$

if we define a point $p : 1 \to \bigcirc X$, its adjunct $\tilde{p}$ under the $(\bigcirc \dashv \Box)$ adjunction is the point

$$\tilde{p} = \Phi^{-1}(p) = \Box(p) \circ \epsilon_X^{\Box} : \Box 1 \to \Box \bigcirc X \to X \tag{100.45}$$

As both functors preserve the terminal object, we have

$$\begin{aligned} \eta_1^{\bigcirc} &= \mathrm{Id}_1 && (100.46) \\ &= \epsilon_1^{\Box} && (100.47) \\ &= \Phi(\mathrm{Id}_{\dots}) && (100.48) \end{aligned}$$

, $\Box 1 \cong 1$, giving us a map $\tilde{p} : 1 \to X$. Furthermore, if we compose

$$\begin{aligned} \eta_X^{\bigcirc} \circ \Phi^{-1}(p) &= \Phi(\mathrm{Id}_{\bigcirc X}) \circ \Phi^{-1}(p) && (100.49) \\ &= \circ \tilde{p} && (100.50) \end{aligned}$$

□

Unity of opposites in terms of localization and colocalization? In terms of Quillen negation?

What is the Quillen negations of η_X, ϵ_X and $\eta_X \circ \epsilon_X$

interpretation in terms of the arrow category of the Quillen negation? without isomorphisms?

101

Types

As with monads in general, we define the various types associated with modalities. So that for $\bigcirc$, we define the modal types to be the ones for which $\eta^{\bigcirc}$ is an isomorphism, and submodal types if it is a monomorphism, and likewise for the comodality $\square$, we define comodal types if $\epsilon^{\square}$ is an isomorphism, and the supcomodal types if it is a monomorphism.

Theorem 101.0.1. For an *sp*-unity, the modal types and the comodal types are isomorphic.

Proof. If X is a modal type, we have $X \cong \bigcirc X$. By the properties of the *sp*-unity,

$$\begin{aligned} X &\cong \bigcirc X && (101.1)\\ &\cong \square \bigcirc X && (101.2)\\ &\cong \square X && (101.3) \end{aligned}$$

and likewise for a comodal type,

$$\begin{aligned} X &\cong \square X && (101.4)\\ &\cong \bigcirc \square X && (101.5)\\ &\cong \bigcirc X && (101.6) \end{aligned}$$

$\square$

So that we can simply talk of modal types in general for *sp*-unities.

Every object X of a topos $\mathbf{H}$ can be given an associated submodal object, given by the submodal functor :

Theorem 101.0.2. The submodal type of a given object X is given by the image of the unit of the modality :

$$\mathrm{submod}(X) = \mathrm{Im}(\eta^{\bigcirc}_X) \tag{101.7}$$

Proof. For □

Theorem 101.0.3. The unit of a modality factors via its submodal functor as the epi-mono factorization

$$\eta^{\bigcirc}_X = \iota \circ \eta^{\mathrm{submod}}_X : X \twoheadrightarrow \mathrm{submod}(X) \hookrightarrow \bigcirc X \tag{101.8}$$

Definition 101.0.1. Given two moments, Δ and $\bigstar$, both either monad or comonads, we can define the relation

$$\Delta \prec \bigstar \tag{101.9}$$

signifying that the (co)modal types of Δ are also (co)modal types of $\bigstar$. In the case of monads, this means that for a Δ-modal type $X \cong \Delta Y$, we have that

$$\eta^{\bigstar}_{\Delta Y} : \Delta Y \to \bigstar \Delta Y \tag{101.10}$$

$$\bigstar \Delta \cong \Delta \tag{101.11}$$

That is indeed a partial order as we have $\Delta \prec \Delta$ (by idempotence), $\Delta \prec \bigstar$ and $\bigstar \prec \Delta$ implying $\Delta \cong \bigstar$, as we have [?]

$$\begin{aligned} \bigstar\Delta &\cong \Delta && (101.12)\\ &\cong \Delta\bigstar\Delta && (101.13)\\ &\cong && (101.14) \end{aligned}$$

Theorem 101.0.4. Given two *s*-moments, $\bigcirc_1$ and $\bigcirc_2$,

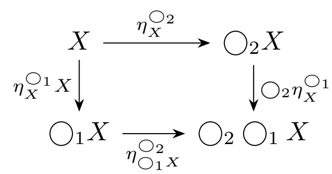
$$\bigcirc_1 \prec \bigcirc_2 \tag{101.15}$$

Any $\bigcirc_1$-modal type is a $\bigcirc_2$-modal type.

Proof. An object X is a $\bigcirc_1$-modal type if we have the isomorphism

$$\eta^{\bigcirc_1}_X : X \xrightarrow{\cong} \bigcirc_1 X \tag{101.16}$$

If we apply $\bigcirc_2$'s unit to it, the naturality square is given by



From the moment inclusion, we have that $\eta_{\bigcirc_1 X}^{\bigcirc_2}$ has an inverse, and from the fact that X is a $\bigcirc_1$-modal type, so does $\eta_X^{\bigcirc_1} X$.

Therefore, $\eta_X^{\bigcirc_2}$ has an inverse given by

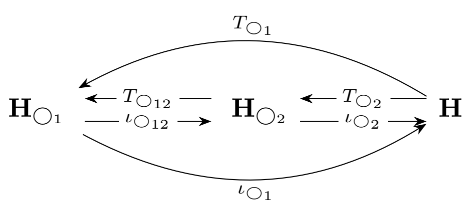
$$(\eta_X^{\bigcirc_1})^{-1} \circ (\eta_{\bigcirc_1 X}^{\bigcirc_2})^{-1} \circ \bigcirc_2 \eta_X^{\bigcirc_1} \tag{101.17}$$

making it an isomorphism. □

Theorem 101.0.5. For two s-moments, $\bigcirc_1$ and $\bigcirc_2$,

$$\bigcirc_1 \prec \bigcirc_2 \tag{101.18}$$

the corresponding reflective subcategories are structured as



such that the compositions of the reflective functors give

$$T_{\bigcirc_1} = T_{\bigcirc_{12}} \circ T_{\bigcirc_2} \tag{101.19}$$

$$\iota_{\bigcirc_1} = \iota_{\bigcirc_2} \circ \iota_{\bigcirc_{12}} \tag{101.20}$$

Proof. From the relation $\bigcirc_1 \prec \bigcirc_2$, we have that □

102

Morphisms

Definition 102.0.1. Given a modality $\bigcirc$ on $\mathbf{C}$, we say that a morphism $f : X \to Y$ is $\bigcirc$-*modal* if

Definition 102.0.2. Given a modality $\bigcirc$ on $\mathbf{C}$, we define the class of morphisms $f : X \to Y$ to be $\bigcirc$-*étale*[165] or $\bigcirc$-*closed*[166] the morphisms for which the naturality square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta^{\bigcirc}_X}\downarrow & & \downarrow{\scriptstyle \eta^{\bigcirc}_Y} \\ \bigcirc X & \xrightarrow[\bigcirc f]{} & \bigcirc Y \end{array}$$

is a pullback :

$$X \cong \bigcirc X \times_{\bigcirc Y} Y \tag{102.1}$$

For comodalities, with the comodality naturality square being a pushout?

$$\begin{array}{ccc} \Box X & \xrightarrow{\Box f} & \Box Y \\ {\scriptstyle \epsilon^{\Box}_X}\downarrow & & \downarrow{\scriptstyle \epsilon^{\Box}_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

$$Y \cong X +_{\Box X} \Box Y \tag{102.2}$$

Interaction with exponential objects

Theorem 102.0.1. In a sheaf topos, for an adjunction $(\Delta \dashv \bigstar)$, we have the identity

$$[X, \bigstar Y]\,(U) = [1, \bigstar\,[\Delta(X), Y]]\,(U) \tag{102.3}$$

Proof. By using the identity of an internal hom in a sheaf topos,

$$\begin{aligned}
[X, \bigstar Y](U) &\cong \mathrm{Hom}_{\mathbf{H}}(\text{よ}(U) \times X, \bigstar Y) && (102.4)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\Delta(\text{よ}(U) \times X), Y) && (102.5)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\Delta(\text{よ}(U)) \times \Delta(X), Y) && (102.6)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\Delta(\text{よ}(U)), [\Delta(X), Y]) && (102.7)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\text{よ}(U), \bigstar[\Delta(X), Y]) && (102.8)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\text{よ}(U) \times 1, \bigstar[\Delta(X), Y]) && (102.9)\\
&\cong [1, \bigstar[\Delta(X), Y]]\,(U) && (102.10)
\end{aligned}$$

□

Theorem 102.0.2. In a Grothendieck topos, for a $\bigcirc$-type in a $(\square \dashv \bigcirc)$ adjunction, we have that the internal hom to that type is also a $\bigcirc$-modal type

$$[X, \iota_{\bigcirc} A] \cong \iota_{\bigcirc} B \tag{102.11}$$

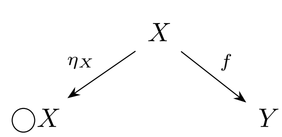
Proof. As a *ps*-unity, we have the adjunction $(\iota_{\square} \dashv T \dashv \iota_{\bigcirc})$, meaning that T is both a left and right adjoint, and thus preserves all limits and colimits.

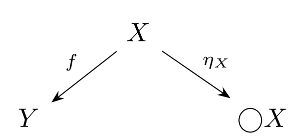
$$\begin{aligned}
[X, \iota_{\bigcirc} A](U) &\cong \mathrm{Hom}_{\mathbf{H}}(\text{よ}(U) \times X, \iota_{\bigcirc} A) && (102.12)\\
&\cong \mathrm{Hom}_{\mathbf{H}_{\bigcirc}}(T(\text{よ}(U) \times X), A) && (102.13)\\
&\cong \mathrm{Hom}_{\mathbf{H}_{\bigcirc}}(T(\text{よ}(U)) \times T(X), A) && (102.14)\\
&\cong \mathrm{Hom}_{\mathbf{H}_{\bigcirc}}(T(\text{よ}(U)), [T(X), A]) && (102.15)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\text{よ}(U), \iota_{\bigcirc}([T(X), A])) && (102.16)\\
&\cong \mathrm{Hom}_{\mathbf{H}}(\text{よ}(U) \times 1, \iota_{\bigcirc}([T(X), A])) && (102.17)\\
&\cong [1, \iota_{\bigcirc}([T(X), A])] && (102.18)
\end{aligned}$$

□

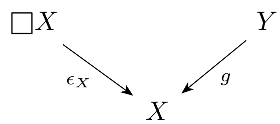
102.0.1 Quillen negations

Given the morphisms of $\epsilon_X : \square X \to X$ and $\eta_X : X \to \bigcirc X$, we can define the following span and cospan. for any morphism $f : X \to Y$ and $g : Y \to X$, we can define two possible spans,





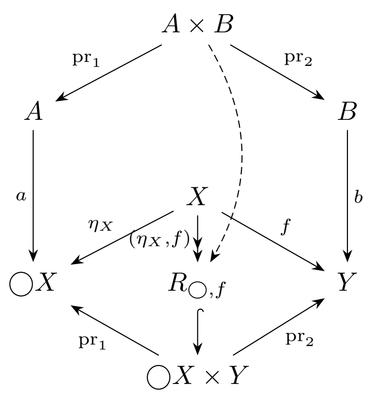
and two possible cospans,



The first span defines a relation between $\bigcirc X$ and Y,

$$R_{\bigcirc} = \mathrm{Im}(\eta_X, f) \hookrightarrow \bigcirc X \times Y \tag{102.19}$$

where given any two elements, $a : A \to \bigcirc X$, $b : B \to Y$, those objects obey the relation if they factor through $R_{\bigcirc, f}$:



A specific case of this is if we have an element $a : A \to X$, and associated element $\bigcirc a : A \to \bigcirc X$. The morphism $a : A \to X$ defines a relation with the span (Id_X, f), ie the relation of the function, while the corresponding $\bigcirc a$ defines a

Adjunction : $A \to \bigcirc X$ has the adjoint $\Box A \to X$?

103

Negation

In addition to oppositions, monads and comonads can also have *negations*. The negation of a moment will be, if it exists, an operator which removes specifically the attributes of a given moment. This is given for a monad $\bigcirc$ and comonad $\Box$ by the fiber and cofiber of the unit and counit, with a specific basepoint $p : 1 \to X$ for the fiber :

$$\overline{\bigcirc}_p X = \mathrm{Fib}_p(X \to \bigcirc X) \tag{103.1}$$

$$\overline{\Box} X = \mathrm{Cofib}(\Box X \to X) \tag{103.2}$$

For the negation of a monad, this corresponds to the following pullback :

$$\begin{array}{ccc} \overline{\bigcirc}_p X & \xrightarrow{!_{\overline{\bigcirc} X}} & 1 \\ {\scriptstyle p^*\eta^{\bigcirc}_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X \end{array}$$

The rough meaning of this is that we are looking for a subobject of X for which the monad $\bigcirc$ will map everything onto a single point, and vice versa, the negation of the modal type $\bigcirc X$ is a single point. One way of this is easy enough to show, as this is the fiber of an isomorphism, by [X], this is simply the terminal object:

$$\begin{aligned}
\overline{\bigcirc}_p \bigcirc X &\cong \mathrm{Fib}_p(\bigcirc X \to \bigcirc \bigcirc X) && (103.3)\\
&\cong \mathrm{Fib}_p(\mathrm{Id}_X) && (103.4)\\
&\cong 1 && (103.5)
\end{aligned}$$

Furthermore, $\epsilon^{\overline{\bigcirc_p}}_{\bigcirc X} \cong p$, since we have $\bigcirc p \cong p$ as both 1 and $\bigcirc X$ are isomorphic under $\bigcirc$, and the left and right morphisms are isomorphic since the top and bottom morphisms are isomorphisms.

$$\begin{array}{ccc}
\overline{\bigcirc}_p \bigcirc X & \xrightarrow{\cong} & 1 \\
{\scriptstyle \epsilon^{\overline{\bigcirc_p}}_{\bigcirc X}}\downarrow & & \downarrow{\scriptstyle p} \\
\bigcirc X & \xrightarrow{\cong} & \bigcirc^2 X
\end{array}$$

p^*

The other way around, $\bigcirc\overline{\bigcirc}_p X$ is not necessarily true however. But if we have the identity $\bigcirc 1 \cong 1$ (for instance if the monad preserves products), and that the monad preserves the fiber (via preserving pullbacks, or products again and the equalizer), we get furthermore

$$\begin{aligned}
\bigcirc\overline{\bigcirc}_p X &\cong \bigcirc\mathrm{Fib}_p(X \to \bigcirc X) && (103.6)\\
&\cong \mathrm{Fib}_{\circ p}(\bigcirc X \to \bigcirc \bigcirc X) && (103.7)\\
&\cong \mathrm{Fib}_{\circ p}(\mathrm{Id}_{\bigcirc X}) && (103.8)\\
&\cong 1 && (103.9)
\end{aligned}$$

$$\begin{array}{ccc}
\bigcirc\overline{\bigcirc} X & \xrightarrow{!_{\overline{\bigcirc} X}} & 1 \\
\downarrow & & \downarrow{\scriptstyle \bigcirc p} \\
\bigcirc X & \xrightarrow{\cong} & \bigcirc X
\end{array}$$

So that $\bigcirc\overline{\bigcirc} X \cong \overline{\bigcirc}_p \bigcirc X \cong 1$, which is the semantics that we would like : each modality preserves mutually exclusive moments of the object.

While the modality depends on a specific point, it can occur that it is independent of a specific choice of point, or that this dependence is shifted to the category by instead using the category of pointed objects $\mathbf{H}^{1/}$ (or as we will see later on, by picking connected objects in the homotopic case). In this case, we will simply denote it by $\overline{\bigcirc}$.

Similarly for the comonad $\Box$, we have this identity easily enough in one case by idempotence,

$$\overline{\Box}\Box X \cong \mathrm{Cofib}(\Box\Box X \to \Box X) \quad (103.10)$$
$$\cong 1 \quad (103.11)$$

$$\begin{array}{ccc} \Box X & \xrightarrow{\epsilon_X} & X \\ \downarrow {\scriptstyle !_{\Box X}} & & \downarrow \\ 1 & \longrightarrow & \overline{\Box} X \end{array}$$

If we have both the comonad and its negation acting on an object,

$$\Box\overline{\Box}X = \Box\mathrm{Cofib}(\Box X \to X) \quad (103.12)$$
$$= \mathrm{Cofib}(\Box\Box X \to \Box X) \quad (103.13)$$
$$= \mathrm{Cofib}(\mathrm{Id}_{\Box X}) \quad (103.14)$$
$$= 1 \quad (103.15)$$

We are left with the terminal object with no specific properties. Equivalently, $\Box\overline{\Box} = \bigcirc_*$, the modality of being that we will see later on.

If we are in the context of a unity of opposites, we have the two cases of an *sp*-unity, $\bigcirc \dashv \Box$, and a *ps*-unity : $\Box \dashv \bigcirc$. As right adjoints preserve limits and left adjoints colimits, we have naturally that for an *sp*-unity,

$$\bigcirc 1 \cong 1 \quad (103.16)$$
$$\bigcirc \mathrm{Fib} \cong \mathrm{Fib}\bigcirc \quad (103.17)$$
$$\Box \mathrm{Cofib} \cong \mathrm{Cofib}\Box \quad (103.18)$$

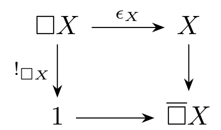
meaning that the property of the negation $\Box\overline{\Box} = \circledast$ is guaranteed, and for a *ps*-unity, $\Box 1 \cong 1$, meaning that additional conditions are required for it to guarantee it [CHECK IT], such as the preservation of the fiber and cofiber.

To do this, we need to find a map from the category to the subcategory containing only objects that the moment map to the terminal object. Given the counit $\Box X \to X$, this is the cofiber :

Definition 103.0.1. The negation of a comonadic moment $\Box$ is given objectwise by the cofiber of its counit :

$$\overline{\Box}X = \mathrm{Cofib}(\Box X \to X) \quad (103.19)$$

Or as a limit, it is the pullback of the cospan $1 \leftarrow \Box X \to X$:



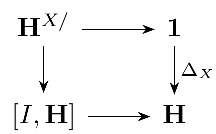
To define : composition of a pullback, fiber, cofiber with a natural transformation, (co)monad

Limit as a natural transformation in the category of functors $[I, C]$

$$\eta : \Delta_{\lim F} \to F \tag{103.20}$$

For fiber : arrow category, slice category C_*, coslice category?

Diagram for coslice categories :



Theorem 103.0.1. The negation of a monad defines a comonad, and the negation of a comonad defines a monad.

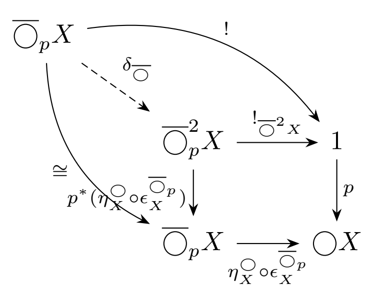
Proof. The fiber naturally defines a counit

$$p^* \eta^{\bigcirc}_X = \epsilon^{\overline{\bigcirc}_p}_X : \overline{\bigcirc}_p X \to X \tag{103.21}$$

and we can construct a comultiplication map via a second pullback, using the composite map

$$\eta^{\bigcirc}_X \circ \epsilon^{\overline{\bigcirc}_p}_X : \overline{\bigcirc}_p X \to \bigcirc X \tag{103.22}$$

where, using the universal property of the pullback, we can define a unique map



$$\delta_{\overline{\bigcirc}} : \overline{\bigcirc}_p X \to \overline{\bigcirc}_p^2 X \tag{103.23}$$

obeying the property

$$\delta_{\overline{\bigcirc}} \circ p^*(\eta_X^{\bigcirc} \circ \epsilon_X^{\overline{\bigcirc}_p}) \cong \mathrm{Id}_{\overline{\bigcirc}_p X} \tag{103.24}$$

To define a multiplication map, it needs to obey

$$\overline{\bigcirc}_p(\epsilon_X^{\overline{\bigcirc}_p}) \circ \delta_X^{\overline{\bigcirc}} \cong \mathrm{Id} \cong \delta_X^{\overline{\bigcirc}} \circ \epsilon_{\overline{\bigcirc}_p X}^{\overline{\bigcirc}_p} \tag{103.25}$$

Alt : if the negation commutes with the fiber,

$$\overline{\bigcirc}_q \overline{\bigcirc}_p X \cong \mathrm{Fib}_p(\overline{\bigcirc}_q X \to \overline{\bigcirc}_q \bigcirc X) \tag{103.26}$$

Since $\overline{\bigcirc}_q \bigcirc X \cong 1$,

$$\overline{\bigcirc}_q \overline{\bigcirc}_p X \cong \mathrm{Fib}_p(\overline{\bigcirc}_q X \to 1) \tag{103.27}$$

[...] For a comonad $\square$, the cofiber naturally defines a unit,

$$!^*_{\square X} \epsilon_X^{\square} \eta_X^{\overline{\square}} : X \to \square X \tag{103.28}$$

$\square$

As in the case of an *sp*-unity, the modal types are in some sense "the same", ($\bigcirc \dashv \square$), we also have that $\square \bigcirc X = \bigcirc X$, ie there is

Determinate negation : if ($\bigcirc \dashv \square$) is such that $\bigcirc 1 \cong 1$ and $\square \to \bigcirc$ is epi, there is determinate negation

For determinate negation, we have

$$\bigcirc \overline{\square} \cong 1 \tag{103.29}$$

Dually we can also define the determinate negation of a monadic moment, but while a cofibration only involves the pushout $1 \leftarrow \square X \to X$, where the morphism $\square X \to 1$ is unique, the fibration is the pullback $1 \to X \leftarrow \bigcirc X$, which depends on a specific choice of a point $1 \to X$ in the object. This means that this negation will either be defined if the object contains a single point, if we are given a specific choice of a point, if the result is independent from the choice of basepoint, or if we allow more flexible negations such as the homotopy fiber of a connected object.

Definition 103.0.2. The determinate negation of a monadic moment

$$\overline{\bigcirc} X = \mathrm{Fib}(X \to \bigcirc X) \tag{103.30}$$

$$\Box\overline{\Box} = \circledast \tag{103.31}$$

Show that the intersection of subcategories is something

Example 103.0.1.

Definition 103.0.3. *Determinate negation* of a unity of opposite moments $\bigcirc \dashv \Box$ if $\Box, \bigcirc$ restrict to 0-types and

- $\bigcirc * \cong *$
- $\Box \to \bigcirc$ is an epimorphism.

"For an opposition with determinate negation, def. 1.14, then on 0-types there is no $\bigcirc$-moment left in the negative of $\Box$-moment"

$$\bigcirc\overline{\Box} \cong * \tag{103.32}$$

Proof. By left adjoints preserving colimits,

$$\bigcirc\overline{\Box}X = \bigcirc\mathrm{cofib}(\Box X \to X) \cong \mathrm{cofib}(\Box X \to \bigcirc X) \tag{103.33}$$

Since $\Box X \to \bigcirc X$ is epi, which is preserved by pushout, this is an epimorphism from the terminal object, therefore the terminal object itself. $\Box$

As we have that there is no $\bigcirc$ moment left in the negative of $\Box$, we also have dually that the negation of $\bigcirc$ is just the identity on such a type :

Theorem 103.0.2. For a determinate negation on an *sp*-unity ($\bigcirc \dashv \Box$), we have

$$\overline{\bigcirc}_p\overline{\Box}X \cong \overline{\Box}X \tag{103.34}$$

Proof. The modal negation diagram on $\overline{\Box}X$ is

$$\begin{array}{ccc} \overline{\bigcirc}_p\overline{\Box}X & \xrightarrow{!_{\overline{\bigcirc}\overline{\Box}X}} & 1 \\ {\scriptstyle p^*\eta^{\bigcirc}_{\overline{\Box}X}}\downarrow & & \downarrow{\scriptstyle p} \\ \overline{\Box}X & \xrightarrow[\eta^{\bigcirc}_{\overline{\Box}X}]{} & \bigcirc\overline{\Box}X \end{array}$$

as we have furthermore the identity $\bigcirc_p\overline{\Box}X \cong 1$, p is an isomorphism here so that as pullbacks preserves isomorphisms, $p^*\eta^{\bigcirc}_{\overline{\Box}X}$ is as well, giving us

$$\overline{\bigcirc}_p\overline{\Box}X \cong \overline{\Box}X \tag{103.35}$$

$\Box$

[...]

From all these identities, we can draw the following tables :

	$\bigcirc$	$\Box$	$\overline{\bigcirc}$	$\overline{\Box}$
$\bigcirc$	$\bigcirc^2 \cong \bigcirc$	$\bigcirc\Box \cong \Box$	$\bigcirc\overline{\bigcirc}$	$\bigcirc\overline{\Box}$
$\Box$	$\Box\bigcirc \cong \bigcirc$	$\Box^2 \cong \Box$	$\Box\overline{\bigcirc}$	$\Box\overline{\Box}$
$\overline{\bigcirc}$	$\overline{\bigcirc}\bigcirc \cong 1$	$\overline{\bigcirc}\Box$	$\overline{\bigcirc}^2$	$\overline{\bigcirc}\overline{\Box}$
$\overline{\Box}$	$\overline{\Box}\bigcirc$	$\overline{\Box}\Box \cong 1$	$\overline{\Box}\overline{\bigcirc}$	$\overline{\Box}^2$

Table 103.1: For an *sp*-unity

	$\bigcirc$	$\Box$	$\overline{\bigcirc}$	$\overline{\Box}$
$\bigcirc$	$\bigcirc^2 \cong \bigcirc$	$\bigcirc\Box \cong \bigcirc$	$\bigcirc\overline{\bigcirc}$	$\bigcirc\overline{\Box}$
$\Box$	$\Box\bigcirc \cong \Box$	$\Box^2 \cong \Box$	$\Box\overline{\bigcirc}$	$\Box\overline{\Box}$
$\overline{\bigcirc}$	$\overline{\bigcirc}\bigcirc \cong 1$	$\overline{\bigcirc}\Box$	$\overline{\bigcirc}^2$	$\overline{\bigcirc}\overline{\Box}$
$\overline{\Box}$	$\overline{\Box}\bigcirc$	$\overline{\Box}\Box \cong 1$	$\overline{\Box}\overline{\bigcirc}$	$\overline{\Box}^2$

Table 103.2: For a *ps*-unity

Theorem 103.0.3. The negation of a $\bigcirc$-submodal object is the terminal object [subterminal?].

Proof. If X is a $\bigcirc$-submodal object, we have that $\eta^{\bigcirc}_X$ is a monomorphism, so that by the preservation of monomorphisms of the pullback, we have that $!_{\overline{\bigcirc}_p}$ is also a monomorphism :

$$\begin{array}{ccc} \overline{\bigcirc}_p X & \xhookrightarrow{!_{\overline{\bigcirc} X}} & 1 \\ {\scriptstyle p^*\eta^{\bigcirc}_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xhookrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X \end{array}$$

so that $\overline{\bigcirc}_p$ is subterminal. Furthermore, $\Box$

Theorem 103.0.4. For an *sp*-unity ($\bigcirc \dashv \Box$) where the unit $\eta^{\bigcirc}_X : X \to \bigcirc X$ is an epimorphism [on X?], we have the identity

$$\overline{\Box} \bigcirc X \cong 1 \tag{103.36}$$

Proof. If $\eta^{\bigcirc}_X$ is epi, we have

$$g_1 \circ \eta^{\bigcirc}_X = g_2 \circ \eta^{\bigcirc}_X \to g_1 = g_2 \tag{103.37}$$

$$g_1 \circ \eta_X^{\bigcirc'} = g_2 \circ \eta_X^{\bigcirc'} \tag{103.38}$$

□

relation between the lifting property of a uop and its corresponding negations?

103.1 de Rham modalities

In addition to the negations of modalities, we will also consider their duals, the *de Rham* modalities, which are simply given by the fiber of the counit for a comonad, and cofiber of the unit for a monad.

$$\widetilde{\bigcirc} X = \mathrm{Cofib}(X \to \bigcirc X) \tag{103.39}$$
$$\widetilde{\square} X = \mathrm{Fib}_p(\square X \to X) \tag{103.40}$$

Those types of modalities will mostly be of use in the ∞-categorical case in VII, where some specific properties of such categories make them more relevant, but they will however still function as intended in the 1-categorical case. To see their relationship to negation, we will have to look at the 1-categorical case of the loop and suspension functors, using the fact that 1-categories can be considered as model categories with interval object 1 :

$$\Omega X \cong 1 \times_X 1 = \mathrm{eq}(1 \rightrightarrows X) \tag{103.41}$$

$$\Sigma X \cong 1 +_X 1 \tag{103.42}$$

$\Omega X \cong X$?

103.2 Pointed types

Another way to interpret the negations and de Rham modalities of a given moment is to consider them as functors between the category of pointed objects on the topos, $\mathbf{H}^{1/}$, and the topos itself, as the cofiber and fiber define respectively a pointed object from an object and vice versa.

$$(\overline{\bigcirc} \dashv \overline{\square}) : \mathbf{H}^{1/} \underset{\overline{\square}}{\overset{\overline{\bigcirc}}{\rightleftarrows}} \mathbf{H}$$

$$(\tilde{\bigcirc} \dashv \tilde{\square}) : \mathbf{H}^{1/} \underset{\tilde{\bigcirc}}{\overset{\tilde{\square}}{\rightleftarrows}} \mathbf{H}$$

As the pointed objects form a monoidal category, we have a natural structure of a tensor product given by the smash product and a zero object given by the identity $1 \to 1$.

104
Adjoint string

It can happen that modalities form longer adjunctions than two, usually up to three. If we consider the case of three such modalities, this can give us two cases, either the case of a *psp*-unity,

$$\square \dashv \bigcirc \dashv \square' \tag{104.1}$$

or an *sps*-unity

$$\bigcirc \dashv \bigcirc \dashv \bigcirc' \tag{104.2}$$

For the case of a *psp*-unity, this represents the following quadruple adjunction of functors to the Eilenberg-Moore category

$$(f_! \dashv f^* \dashv f_* \dashv f^!) : \mathbf{H}_{\bigcirc} \begin{array}{c} \xhookrightarrow{\quad f_! \quad} \\ \xleftarrow{\quad f_! \quad} \\ \xhookrightarrow{\quad f_* \quad} \\ \xleftarrow{\quad f^! \quad} \end{array} \mathbf{H}$$

which, in terms of the (co)reflective subcategories of $(\square \dashv \bigcirc)$ and $(\bigcirc \dashv \square)$, is

$$(\iota_\square \dashv T \dashv \iota \dashv T_{\square'}) : \mathbf{H}_{\bigcirc} \begin{array}{c} \xhookrightarrow{\quad \iota_\square \quad} \\ \xleftarrow{T_\square \cong T_\bigcirc} \\ \xhookrightarrow{\iota_\bigcirc \cong \iota_{\square'}} \\ \xleftarrow{\quad T_{\square'} \quad} \end{array} \mathbf{H}$$

with the identities

$$\square = f^* f_! \tag{104.3}$$

$$\bigcirc = f^* f_* \tag{104.4}$$

$$\square' = f^! f_* \tag{104.5}$$

$$\square' \square = \tag{104.6}$$

No identities for basic $\square\square'$, $\bigcirc\bigcirc'$? Check with EM category

For *sps*-unity,

$$(f_! \dashv f^* \dashv f_* \dashv f^!) : \mathbf{H}_{\bigcirc} \begin{array}{c} \xleftarrow{\quad f_! \quad} \\ \xhookrightarrow{\quad f^* \quad} \\ \xleftarrow{\quad f^* \quad} \\ \xhookrightarrow{\quad f^! \quad} \end{array} \mathbf{H}$$

In terms of reflective subcategories :

$$(T_{\bigcirc} \dashv \iota \dashv T \dashv \iota_{\bigcirc'}) : \mathbf{H}_{\bigcirc} \begin{array}{c} \xleftarrow{\quad T_{\bigcirc} \quad} \\ \xhookrightarrow{\iota_{\bigcap} \cong \iota_{\sqcap}} \\ \xleftarrow{T_{\square} \cong T_{\bigcirc'}} \\ \xhookrightarrow{\quad \iota_{\bigcirc'} \quad} \end{array} \mathbf{H}$$

$\bigcirc\bigcirc'$ would correspond to something like $\int\sharp$ in this case, which is not guaranteed to be 1 (only for a codiscretely connected topos).

$$\bigcirc \bigcirc' X \cong T_{\bigcirc} \iota_{\bigcirc} T_{\bigcirc'} \iota_{\bigcirc'} \tag{104.7}$$

$$\cong T_{\bigcirc} \iota_{\square} T_{\square} \iota_{\bigcirc'} \tag{104.8}$$

$\bigcirc'\bigcirc$: similar to $\sharp\int$, which would be $\cong \int$?

$$\bigcirc' \bigcirc X \cong T_{\bigcirc'} \iota_{\bigcirc'} T_{\bigcirc} \iota_{\bigcirc} \tag{104.9}$$

$$\cong T_{\square} \iota_{\bigcirc'} T_{\bigcirc} \iota_{\square} \tag{104.10}$$

Is there perhaps a point to invoke the converse monads, ie $\overleftarrow{\bigcirc} = \iota_{\bigcirc} T_{\bigcirc}$, $\overleftarrow{\square} = T_{\square} \iota_{\square}$

105
Sublation

Sublation (or *Aufhebung* in the original German), levels of a topos

§180 The resultant equilibrium of coming-to-be and ceasing-to-be is in the first place becoming itself. But this equally settles into a stable unity. Being and nothing are in this unity only as vanishing moments; yet becoming as such is only through their distinguishedness. Their vanishing, therefore, is the vanishing of becoming or the vanishing of the vanishing itself. Becoming is an unstable unrest which settles into a stable result.

§181 This could also be expressed thus: becoming is the vanishing of being in nothing and of nothing in being and the vanishing of being and nothing generally; but at the same time it rests on the distinction between them. It is therefore inherently self-contradictory, because the determinations it unites within itself are opposed to each other; but such a union destroys itself.

§182 This result is the vanishedness of becoming, but it is not nothing; as such it would only be a relapse into one of the already sublated determinations, not the resultant of nothing and being. It is the unity of being and nothing which has settled into a stable oneness. But this stable oneness is being, yet no longer as a determination on its own but as a determination of the whole.

§183 Becoming, as this transition into the unity of being and nothing, a unity which is in the form of being or has the form of the onesided immediate unity of these moments, is determinate being.

Unity of opposites, like idempotent monads themselves, can be ordered. As idempotent monads and comonads can be classified by inclusion, being described by (co)reflective subcategories, we can also do so for the adjunctions.

[Differentiate between higher level, resolution and sublation, talk about j-sheaves and j-skeleta]

[Maybe replace ♠♡♣♢ by $\bigcirc\Box\dot{\bigcirc}\dot{\Box}$, or $\Box_i\bigcirc_j$?]

In the case of a *ps*-unity, if we have some original adjunction $(\Box_1 \dashv \bigcirc_1)$, we say that another adjunction $(\Box_2 \dashv \bigcirc_2)$ is of a *higher level* if we have both $\Box_1 \prec \Box_2$ and $\bigcirc_1 \prec \bigcirc_2$, which we denote diagrammatically by

$$\begin{array}{ccc} \Box_2 & \dashv & \bigcirc_2 \\ \curlyvee & & \curlyvee \\ \Box_1 & \dashv & \bigcirc_1 \end{array}$$

In other words, we have

$$\Box_2\Box_1 \cong \Box_1 \tag{105.1}$$

$$\bigcirc_2\bigcirc_1 \cong \bigcirc_1 \tag{105.2}$$

Likewise for an *sp*-unity, the adjunction $(\bigcirc_1 \dashv \Box_1)$ has a higher level adjunction $(\bigcirc_2 \dashv \Box_2)$ if we have $\bigcirc_1 \prec \bigcirc_2$ and $\Box_1 \prec \Box_2$

$$\begin{array}{ccc} \bigcirc_2 & \dashv & \Box_2 \\ \curlyvee & & \curlyvee \\ \bigcirc_1 & \dashv & \Box_1 \end{array}$$

$$\bigcirc_2\bigcirc_1 \cong \bigcirc_1 \tag{105.3}$$

$$\Box_2\Box_1 \cong \Box_1 \tag{105.4}$$

Furthermore, for a *ps*-unity, if we have that one of the lower modality has its image in the opposite of its upper modality, ie $\Box_1 \prec \bigcirc_2$ or $\bigcirc_1 \prec \Box_2$, we say that the higher modality is a *resolution* of the lower, denoted by

$$\begin{array}{ccc} \Box_2 & \dashv & \bigcirc_2 \\ \curlyvee\!\!\!\curlyvee & & \curlyvee\!\!\!\curlyvee \\ \Box_1 & \dashv & \bigcirc_1 \end{array}$$

The specific cases of $\square_1 \prec \bigcirc_2$ and $\bigcirc_1 \prec \square_2$ are called respectively a *left resolution* and a *right resolution* of the lower opposition, corresponding to, for the left resolution

$$\bigcirc_2\square_1 \cong \square_1 \tag{105.5}$$

Denoted by the diagram

$$\begin{array}{ccccc} & \square_2 & \dashv & \bigcirc_2 & \\ & \curlyvee & & \curlyvee & \\ \square_1 & \dashv & & \bigcirc_1 & \end{array}$$

and for the right resolution

$$\square_2\bigcirc_1 \cong \bigcirc_1 \tag{105.6}$$

Denoted by the diagram

$$\begin{array}{ccccc} \square_2 & \dashv & \bigcirc_2 & & \\ \curlyvee & & \curlyvee & & \\ \square_1 & & \dashv & & \bigcirc_1 \end{array}$$

For an *sp*-unity, by 100.0.4, we have that for a higher level $(\bigcirc_1 \dashv \square_1) \ll (\bigcirc_2 \dashv \square_2)$

$$\begin{aligned} \bigcirc_2\square_1 &\cong \bigcirc_2\,\bigcirc_1\,\square_1 && (105.7)\\ &\cong \bigcirc_1\square_1 && (105.8)\\ &\cong \square_1 && (105.9) \end{aligned}$$

$$\begin{aligned} \square_2\bigcirc_1 &\cong \square_2\square_1\,\bigcirc_1 && (105.10)\\ &\cong \square_1\,\bigcirc_1 && (105.11)\\ &\cong \bigcirc_1 && (105.12) \end{aligned}$$

meaning that any higher level of an *sp*-unity is always both a left and right resolution.

Finally, if a resolution of an opposition is the minimal resolution, ie if for any other resolution of $(\bigcirc_1 \dashv \square_1) \ll (\bigcirc_2 \dashv \square_2)$

Definition 105.0.1. Given some adjoint moments ($\boxed{\spadesuit} \dashv \clubsuit$), we say that a second pair of adjoints moments ($\boxed{\heartsuit} \dashv \diamondsuit$) is a *sublation* of the first adjunction if we have the equalities

$$\boxed{\heartsuit}\boxed{\spadesuit} \cong \boxed{\spadesuit} \tag{105.13}$$
$$\diamondsuit\clubsuit \cong \clubsuit \tag{105.14}$$

In other words, the objects of pure $\boxed{\heartsuit}$ moments of the category are also of pure $\boxed{\spadesuit}$ moment.

Theorem 105.0.1. For an *sp*-opposition, the sublation ($\bigcirc_1 \dashv \square_1$) to ($\bigcirc_2 \dashv \square_2$), with unit and counit ($\eta^{\bigcirc_1}, \epsilon^{\square_1}$) and ($\eta^{\bigcirc_2}, \epsilon^{\square_2}$),

$$\eta_X^{\bigcirc_1} : X \to \bigcirc_1 X \tag{105.15}$$
$$\epsilon_X^{\square_1} : \square_1 X \to X \tag{105.16}$$
$$\eta_X^{\bigcirc_2} : X \to \bigcirc_2 X \tag{105.17}$$
$$\epsilon_X^{\square_2} : \square_2 X \to X \tag{105.18}$$

then the unit and counit of the original opposition factor through their sublated moments :

$$\eta_X^{\bigcirc_1} : X \to \bigcirc_2 X \to \bigcirc_1 X \tag{105.19}$$
$$\epsilon_X^{\square_1} : \square_1 X \to \square_2 X \to X \tag{105.20}$$

Proof. Looking at the commutative square of $\eta^{\bigcirc_2} : \mathrm{Id} \to \bigcirc_2$,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^{\bigcirc_2}} & \bigcirc_2 X \\ {\scriptstyle \eta_X^{\bigcirc_1}} \downarrow & & \downarrow {\scriptstyle \bigcirc_2 \eta_X^{\bigcirc_1}} \\ \bigcirc_1 X & \xrightarrow[\eta_{\bigcirc_1 X}^{\bigcirc_2}]{} & \bigcirc_2 \bigcirc_1 X \end{array}$$

From $\bigcirc_2\bigcirc_1 \cong \bigcirc_1$, we have the existence of an inverse natural transformation [inverse specifically to $\eta^{\bigcirc_2}$?]

$$\alpha : \bigcirc_2\bigcirc_1 \to \bigcirc_1 \tag{105.21}$$

So that, applying it to our commutative diagram,

$$\alpha \bigcirc_2 \eta_X^{\bigcirc_1} \eta_X^{\bigstar} = \alpha \eta_{\bigcirc_1 X}^{\bigstar} \eta_X^{\bigcirc_1} \tag{105.22}$$

$$\alpha \bigcirc_2 \eta_X^{\bigcirc_1} \eta_X^{\bigstar} = \eta_X^{\bigcirc_1} \tag{105.23}$$

Furthermore, we can apply the monad $\bigcirc_2$ to the unit as [whiskering?]

$$(\bigcirc_2 \eta^{\bigcirc_1})_X : \bigcirc_2 X \to \bigcirc_2 \bigcirc_1 X \tag{105.24}$$

Then given the composition

$$(\bigcirc_2 \eta^{\bigcirc_1} \eta^{\bigcirc_2}) : X \xrightarrow{\eta_X^{\bigcirc_2}} \bigcirc_2 X \xrightarrow{\bigcirc_2 \eta_X^{\bigcirc_1}} \bigcirc_2 \bigcirc_1 X \cong \bigcirc_1 X \tag{105.25}$$

$$\bigcirc_2 \bigcirc_1 \tag{105.26}$$

□

$$\begin{array}{ccc} X & \xrightarrow{\eta^{\square_1}} & \square_1 X \\ {\scriptstyle \eta_X^{\bigcirc_1}} \downarrow & & \downarrow {\scriptstyle \square_1 \eta_X^{\bigcirc_1}} \\ \bigcirc_1 X & \xrightarrow[\eta^{\square_1}_{\bigcirc_1 X}]{} & \square_1 \bigcirc_1 X \end{array}$$

If the unit/counit is epi/mono, does that generalize to the sublation?

Action of sublation on negations?

Theorem 105.0.2. Given a *ps*-unity with some higher level

$$\begin{array}{ccc} \boxed{\heartsuit} & \dashv & \circledcirc{\diamondsuit} \\ \vee & & \vee \\ \blacksquare{\spadesuit} & \dashv & \bullet{\clubsuit} \end{array}$$

The $\blacksquare{\spadesuit}$-negation of a $\boxed{\heartsuit}$-type is

$$\overline{\blacksquare{\spadesuit}}\, \boxed{\heartsuit}\, X \cong 1 \tag{105.27}$$

Proof.

$$\boxed{\heartsuit}\blacksquare{\spadesuit} \cong \blacksquare{\spadesuit} \tag{105.28}$$
$$\circledcirc{\diamondsuit}\bullet{\clubsuit} \cong \bullet{\clubsuit} \tag{105.29}$$

□

[167]

106 Slice topos

As a lot of topoi of interest are constructed as the slice of another topos, we should look at the action of modalities on such categories.

Given a topos $\mathbf{H}$, a modality $\bigcirc$ which preserves fibers over $\bigcirc$-modal types, and an object $X \in \mathbf{H}$, then the slice category $\mathbf{H}_{/X}$ admits a moment $\bigcirc_X$ given by the pullback of the modality of the fibration $\bigcirc(p : E \to X) = \bigcirc p : \bigcirc E \to \bigcirc X$ [14]:

$$\begin{array}{ccc} \bigcirc_X E & \longrightarrow & \bigcirc E \\ \downarrow & & \downarrow {\scriptstyle \bigcirc p} \\ X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X \end{array}$$

Universal factorization of $p : E \to X$ by $\bigcirc_X E \to X$

$$\begin{array}{ccccc} E & \longrightarrow & \bigcirc_X E & \longrightarrow & \bigcirc E \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow {\scriptstyle \bigcirc p} \\ X & = & X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X \end{array}$$

via a $\bigcirc$-equivalence $E \to \bigcirc_X E$

They form an orthogonal factorization system ($\bigcirc$-equivalences, $\bigcirc_X$-modal morphisms) in $\mathbf{H}$.

Part IX

Objective logic

"These many different things stand in essential reciprocal action via their properties; the property is this reciprocal relation itself and apart from it the thing is nothing"

As there will be many notations for very similar concepts of different types, we will require the following convention :

- Unless a more specific unambiguous symbol exists, monads will be denoted by $\bigcirc$, with possibly some overlaid symbol to differentiate it
- Similarly, comonads will be denoted by $\Box$, with possibly some overlaid symbol to differentiate it
- A generic topos $\mathbf{H}$
- The terminal object is 1
- The initial object is 0

This to avoid circumstances such as $*$ to represent both an object, functor, category and monad.

A trivial opposition we have in the objective logic is

$$\mathrm{Id} \dashv \mathrm{Id} \tag{106.1}$$

This is an opposition defined by the triple of endofunctors

$$(\mathrm{Id} \dashv \mathrm{Id}) : \mathbf{H} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathbf{H}$$

Representing three identity functors, composing into the two identity monad and comonad, with the subtopi being $\mathbf{H}$ itself. (Moment of identity?)

107

Being and nothingness

“Being, pure being, [...] it has no diversity within itself nor any with a reference outwards”

“Nothing, pure nothing: it is simply equality with itself, complete emptiness”

The most basic type of moments are the monads of *being* (*Sein*) and *nothingness* (*Nichts*). This can be seen easily enough on a topos level by considering that the smallest subtopos is the initial topos $\mathrm{Sh}(\varnothing) \cong \mathbf{1}$. Let's consider the unique functor from our topos to it, the constant functor on this subtopos,

$$\Delta_* : \mathbf{C} \to \mathbf{1} \tag{107.1}$$

which maps every object of $\mathbf{C}$ to the unique object $*$ of the terminal category $\mathbf{1}$. This corresponds to the *unit type* in type theory term, and this is what is referred to as *The One* (*Das Eins*) in objective logic.

There are only two possibilities here : either the underlying opposition is a *ps*-unity, with Δ_* its (co)reflector, or it is an *sp*-unity, in which case we have some adjoint string $(\Delta_* \dashv F \dashv \Delta_*)$ where Δ_* is the same on both sides (since there is no other functor between those categories), making it an ambidextrous adjunction.

If we try to look at the *sp*-unity case,

$$\mathrm{Hom}_{\mathbf{1}}(\Delta_* X, *) = \mathrm{Hom}_{\mathbf{H}}(X, F(*)) \tag{107.2}$$
$$\mathrm{Hom}_{\mathbf{H}}(F(*), Y) = \mathrm{Hom}_{\mathbf{1}}(*, \Delta_*(Y)) \tag{107.3}$$

meaning that the hom-set of functions to and from $F(*)$ only have one element. As any topos has at least an initial and terminal object, any object outside of the initial and terminal object should have at least two such morphisms (the identity morphism and the morphism from the initial object/to the terminal object), and if it is either the initial object or the terminal object, it should have at least one morphism to or from any other object. Therefore there is no such adjunction, unless we are dealing with the initial topos itself, in which case this adjunction is simply the identity.

For a *ps*-adjunction, we give this functor a left and right adjoint, which as we will see are the constant functors of the initial and terminal object, so that we will denote them as Δ_0 and Δ_1, forming the adjoint cylinder

$$(\Delta_0 \dashv \Delta_* \dashv \Delta_1) : \mathbf{1} \underset{\underset{\Delta_1}{\hookrightarrow}}{\overset{\overset{\Delta_0}{\hookrightarrow}}{\xleftarrow{\Delta_*}}} \mathbf{H}$$

An easy way to see this is via the adjunction of hom-sets :

$$\mathrm{Hom}_{\mathbf{C}}(\Delta_0(*), X) \cong \mathrm{Hom}_{\mathbf{1}}(*, \Delta_*(X)) \tag{107.4}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between $\Delta_0(*)$ and any object X, making it the initial object of the topos. Similarly,

$$\mathrm{Hom}_{\mathbf{1}}(\Delta_*(X), *) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Delta_1(*)) \tag{107.5}$$

There is only one element in the hom-set for $* \to *$, and therefore only one in the hom-set between any object X and $\Delta_1(*)$, making it the terminal object of the topos, confirming our choice of those adjoints as constant functors.

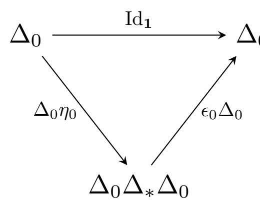
We can also look at this in terms of the unit and counit of the adjunction. First if we look at the adjunction $\Delta_0 \dashv \Delta_*$, as adjoint functors, they are equipped with the unit and counit natural transformations,

$$\eta_0 \quad : \quad \mathrm{Id}_{\mathbf{1}} \Rightarrow \Delta_* \circ \Delta_0 \tag{107.6}$$

$$\epsilon_0 \quad : \quad \Delta_0 \circ \Delta_* \Rightarrow \mathrm{Id}_{\mathbf{C}} \tag{107.7}$$

As there is only one endofunctor on $\mathbf{1}$, η_0 is simply the identity natural transformation.

they have to obey the triangle identities



In terms of components, this means that for any object $X \in \mathbf{C}$ (and the only object $*$ in $\mathbf{1}$),

$$\begin{aligned} \text{Id}_{\Delta_0(*)} &= \epsilon_{\Delta_0(*)} \circ \Delta_0(\eta_*) && (107.8) \\ \text{Id}_{\Delta_*(X)} &= \Delta_*(\epsilon_X) \circ \eta_{\Delta_*(X)} && (107.9) \end{aligned}$$

We have the identities $\Delta_*(X) = *$, and any component of the counit can only be the identity morphism on $*$, so that

$$\begin{aligned} \text{Id}_{\Delta_0(*)} &= \epsilon_{\Delta_0(*)} \circ \text{Id}_{\Delta_0(*)} && (107.10) \\ \text{Id}_* &= \text{Id}_* && (107.11) \end{aligned}$$

The second line is trivial, but the first line tells us that ϵ is the identity on $\Delta_0(*)$

for any object $X \in \mathbf{C}$, there exists an object $\Delta_*(X) \in \mathbf{1}$ and a morphism $\epsilon_X : \Delta_0 \circ \Delta_*(X) \to X$ such that for every object in $\mathbf{1}$ (so only for $*$), and every morphism $f : \Delta_0(*) \to X$, there exists a unique morphism $g : * \to \Delta_*(X) = *$ with $\epsilon_X \circ \Delta_0(g) = f$.

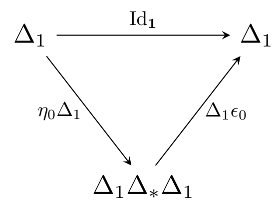
The unique morphism g is simply the identity function Id_1, and our natural transformation at c is simply $\epsilon_X : \Delta_0(*) \to X$. We therefore need the constraint

$$\epsilon_X \circ \Delta_0(\text{Id}_1) = f \qquad (107.12)$$

However, as g can only be one function, we cannot have more than one such possible morphism f, and we need exactly one to map to Id_1. This means that the empty functor Δ_0 maps the single object of 1 to the initial object 0 in $\mathbf{C}$ (if the category contains one), as its name indicates, justifying our notation of the constant functor Δ_0.

[Do the other triangle?]

Conversely, the right adjoint Δ_1 has to obey



Δ_1 maps the unique object of the terminal category to the terminal object 0 of $\mathbf{C}$, if this object exists.

Therefore, we have an adjoint triple $(\Delta_0 \dashv \Delta_* \dashv \Delta_1)$ between our topos and the initial topos, forming a *ps* unity of opposites, giving rise to an adjoint modality that we will write as $(\boxed{\varnothing} \dashv \circledast)$, with $\boxed{\varnothing}$ the modality of nothingness (or empty comonad) and $\circledast$ the modality of being (or unit monad), defined by

$$\boxed{\varnothing} = \Delta_0 \circ \Delta_* \tag{107.13}$$
$$\circledast = \Delta_1 \circ \Delta_* \tag{107.14}$$

where the unit and counit of the adjunction are given by those that we have seen, the unit of the monad $\circledast$ and the counit of the comonad $\boxed{\varnothing}$

$$\eta^{\circledast} : \quad \mathrm{Id}_{\mathbf{H}} \to \Delta_1 \circ \Delta_* \tag{107.15}$$
$$\epsilon^{\boxed{\varnothing}} : \Delta_0\Delta_* \to \mathrm{Id}_{\mathbf{1}} \tag{107.16}$$

Forming the adjunction

$$(\boxed{\varnothing} \dashv \circledast) : \mathbf{1} \begin{array}{c} \xhookrightarrow{\ \Delta_0\ } \\ \xleftarrow{\ \Delta_*\ } \\ \xhookrightarrow{\ \Delta_1\ } \end{array} \mathbf{H}$$

As a *ps*-unity, we find what we expect : by necessity, only one projection, and two different inclusions of this in the topos.

In terms of their modal action, the empty monad maps any object of the category to its initial element,

$$\boxed{\varnothing}(X) = 0 \tag{107.17}$$

and any morphism to the identity on the initial object

$$\boxed{\varnothing}(f : X \to Y) = \mathrm{Id}_0 \tag{107.18}$$

while the unit monad maps any object of the category to its terminal element,

$$\circledast(X) = 1 \tag{107.19}$$

and any morphism to the identity on the terminal object

$$\circledast(f : X \to Y) = \mathrm{Id}_1 \tag{107.20}$$

Relation to $X \to 1$, where does Ω factor in? Is it the "truth" diagram $X \to 1 \to \Omega$, with the semantics that this is the map of the subobject $X \hookrightarrow X$?

Something like $\vdash \forall X,\ X = X$?

[Stack semantics?]

The unit and empty modality also have a variety of alternative interpretations in terms of other adjunctions. For instance, given the dependent adjunction of the reader monad and coreader comonad, from the base change $f : X \to Y$

$$(\Diamond_f \dashv \Box_f) : \mathbf{H}_{/X} \underset{\prod_f}{\overset{\sum_f}{\rightleftarrows}} \mathbf{H}_{/Y} \quad (\xleftarrow{f^*})$$

the adjunction of those two monads will correspond to the base change $0_1 : 0 \to 1$, as we have $\mathbf{H}_{/1} \cong \mathbf{H}$ and $\mathbf{H}_{/0} \cong \mathbf{1}$. [corresponds to Ex falso quodlibet?]

As a duality of hom and product :

$$((-) \times 0 \dashv (-)^0) \tag{107.21}$$

conjunction with falsehood is true v. implication of falsity is anything

In terms of morphisms?

In terms of localization, the unit modality is the case of the localization by every morphism of the topos, so that

$$\circledast \cong \mathrm{loc}_{\mathrm{Mor}(\mathbf{H})} \tag{107.22}$$

ie every morphism becomes an equivalence so that every two objects are isomorphic.

Likewise, the empty modality is the colocalization [...]

This notion can be formalized internally by the terminal object, as this is the only object for which any other object has a single relation to.

Now if we would like to look at the properties of this being, we can

In the Hegelian sense : $\circledast$ maps every object of $\mathbf{H}$ to a single object (they all share the same characteristic of existence, "pure being", and there is nothing differentiating them in that respect, no further qualities). The image of any two objects under this modality are identical, as there are no other characteristics to differentiate them, and the only relation they can have is that of the identity. This is also true of the empty object []

“In its indeterminate immediacy it is equal only to itself. It is also not unequal relatively to an other; it has no diversity within itself nor any with a reference outwards”

Conversely, $\boxed{\varnothing}$ maps every object to nothing, the opposition of being.

The unit of the monad and counit of the comonad are given in terms of components by the typical morphisms of the terminal and initial object,

$$\epsilon_X : \quad X \to \circledast X \cong 1 \tag{107.23}$$
$$\eta_X : 0 \cong \boxed{\varnothing} X \to X \tag{107.24}$$

Both of those adjoint functors roughly reflect the fact that each has to map elements to a single element and morphisms between that element and every other element to a single morphism.

The composition of the unit and counit give us the *unity of opposites* for being and nothingness

$$0 \to X \to 1 \tag{107.25}$$

“there is nothing which is not an intermediate state between being and nothing.”

An alternative interpretations of this modality is given by the opposition of the dependent sum and dependent product on the empty context

$$\sum_{\varnothing}(-) \ \vdash \ \prod_{\varnothing}(-) \tag{107.26}$$

Cartesian product v. internal home adjunction of the unit type

$$((-) \times \varnothing) \ \dashv \ (\varnothing \to (-)) \tag{107.27}$$

Negation in categories : internal hom to the initial object : $\neg = [-, \varnothing]$

Examples of those two modalities on a topos will not give us very different results overall, as they all roughly have the same behaviour. For **Set** for instance,

$$\forall A \in \mathrm{Obj}(\mathbf{Set}), \ \boxed{\varnothing}(A) = \varnothing \tag{107.28}$$
$$\forall f \in \mathrm{Mor}(\mathbf{Set}), \ \boxed{\varnothing}(f) = \mathrm{Id}_{\varnothing} \tag{107.29}$$

$$\forall A \in \mathrm{Obj}(\mathbf{Set}),\ \circledast A \ = \{\bullet\} \tag{107.30}$$
$$\forall f \in \mathrm{Mor}(\mathbf{Set}),\ \circledast (f) = \mathrm{Id}_{\{\bullet\}} \tag{107.31}$$

[...]

As the examples we have given thus far do not have particularly varied definitions for the initial and terminal object (being mostly concrete categories where $I = \varnothing$ and $T = \{\bullet\}$, the modality of being and nothingness do not offer particularly more insight in those topos. Smooth sets simply get mapped to the empty space and the one point space, etc

"In geometric language these are categories equipped with a notion of discrete objects and codiscrete objects."

As any further adjoint would have to be a new functor from $\mathbf{H}$ to $\mathbf{1}$, they would simply be just the functor Δ_* again, so that the further adjoints would simply be the same monads again. There is therefore no further moments at this level.

$\circledast$ is monoidal :

$$\circledast(X) \times \circledast(Y) \cong 1 \times 1 \cong 1 \cong \circledast(X \times Y) \tag{107.32}$$

[...]

The types involved in the initial opposition are fairly easy to work through : For $\circledast$, we only have a single modal type, which is the terminal object 1, and for $\boxslash$, the only modal type is the terminal object, 0. The submodal types of $\circledast$ are the subterminal objects, the objects for which the unique morphism $X \to 1$ is a monomorphism. By topoi having strict initial objects, there is only one submodal type for $\boxslash$, which is the initial object itself.

Subterminal objects : serve to detect open sets?

$\circledast_X$ as a map to pointed objects?

Adjoint functions :

$$f : \boxslash X \to Y \tag{107.33}$$

Adjunct :

$$\overline{f} = \circledast(f) \circ \circledast_X = \mathrm{Id}_1 \tag{107.34}$$

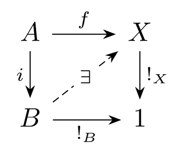
$$\tilde{f} : X \to \circledast Y \tag{107.35}$$

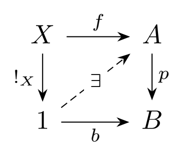
Adjunct :

$$\overline{f} = \boxslash_Y \circ \boxslash(f) = \mathrm{Id}_0 \tag{107.36}$$

Every morphism is $\circledast$-closed?

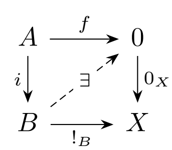
Quillen negation of the unit $\eta_X^{\circledast} : X \to 1$: left Quillen negation is the set of $i : A \to B$ for which given any morphism $f : A \to X$, there is an extension $\tilde{f} : B \to X$.





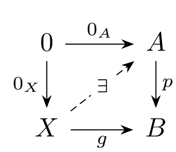
Injective object ?

Quillen negation of the counit $\epsilon_X^{\boxslash} : 0 \to X$:



As we are in a topos, there is no morphism $A \to 0$ outside of the identity morphism Id_0, and the same applies for the diagonal filler, meaning that the only possible morphism i is just the identity map $\mathrm{Id}_0 : 0 \to 0$, which does make the diagram commute and is unique.

$$\{\eta_X^{\circledast}\}^{\boxslash} = \{\mathrm{Id}_0\} \tag{107.37}$$



Surjective functions for generalized elements X?

107.1 Negations

As the unit monad $\circledast$ is a right adjoint, it preserves limits, and therefore admits a negation with all the appropriate properties.

Theorem 107.1.1. The negation of the monad of being is the identity monad :

$$\overline{\circledast} = \mathrm{Id} \tag{107.38}$$

Proof. The computation is simple enough, as

$$\begin{aligned}
\overline{\circledast}(X) &= \mathrm{Fib}(X \to \bigcirc_* X) && (107.39)\\
&= \mathrm{Fib}(X \to 1) && (107.40)\\
&= \mathrm{Fib}(!_X) && (107.41)\\
&= X \times_1 1 && (107.42)\\
&= X && (107.43)
\end{aligned}$$

This is independent of the choice of basepoint, so that the negation of being is therefore the identity. □

This indeed obeys $\overline{\circledast}\circledast = \circledast\overline{\circledast} = \circledast$.

This relates well enough to the interpretation of "the determinate negation containing the part of the structure that is trivialized by the unit $X \to \bigcirc X$", as the unit monad removes all structure from the object, the trivialized part is all of it. Being the identity functor, its further adjoints are all itself, so that nothing further of interest can be gotten.

In terms of pointed objects, this simply transforms any pointed object into its equivalent non-pointed object, ie it is the forgetful functor between the pointed category $\mathbf{H}^{1/}$ and the topos $\mathbf{H}$.

$$\overline{\circledast} : \mathbf{H}^{1/} \to \mathbf{H} \tag{107.44}$$

On the other hand, the modality of nothingness's cofibration does not give us exactly a determinate negation :

Theorem 107.1.2. The negation of the comonad of nothingness is the maybe monad :

$$\overline{\boxed{\varnothing}} = \mathrm{Maybe} \tag{107.45}$$

Proof.

$$\begin{aligned}
\overline{\boxed{\varnothing}} &= \mathrm{Cofib}(\boxed{\varnothing} X \to X) && (107.46)\\
&= X +_{\boxed{\varnothing} X} 1 && (107.47)\\
&= X +_{\varnothing} 1 && (107.48)\\
&= X + 1 && (107.49)
\end{aligned}$$

□

While the determinate [?] negation is well-defined, it is not an idempotent monad (since the components of its multiplication map is not an isomorphism except in the degenerate case where we have the initial topos, as $\mu_0 : 2 \to 1$ is clearly not), as it simply adds a new element to any object.

$$\text{Maybe}^2 X = (\text{Maybe}X) \sqcup \{\bullet\} = X \sqcup \{\bullet_1, \bullet_2\} \tag{107.50}$$

$$\overline{\boxed{\varnothing}}\,\boxed{\varnothing}\, X = 1 \tag{107.51}$$

$\boxed{\varnothing}\overline{\boxed{\varnothing}} = 0$

One way it can be understood is that there cannot be any property that we remove from an object to make it into the unit type, as there is no object to simplify here, therefore the only way to get us to a unit type is to add it.

As the maybe monad adds a new copy of the terminal object, it will clearly not preserve either the initial or terminal object,

$$\overline{\boxed{\varnothing}}0 \cong 1 \tag{107.52}$$

$$\overline{\boxed{\varnothing}}1 \cong 2 \tag{107.53}$$

Unless we are in the initial topos, so that it does not preserve either limits or colimits, meaning that it has no further adjoints.

There is little point to the de Rham modalities here [?], but they are as follow :

$$\tilde{\circledast}X = \text{Cofib}(X \to 1) \tag{107.54}$$

$$= X +_1 1 \tag{107.55}$$

$$= \text{coeq}(X + 1 \rightrightarrows 1) \tag{107.56}$$

[$0 \times X \cong X$ because strict initial object]

$$\tilde{\boxed{\varnothing}}X = \text{Fib}_p(0 \to X) \tag{107.57}$$

$$= 0 \times_X 1 \tag{107.58}$$

$$= \text{eq}(0 \times 1 \rightrightarrows X) \tag{107.59}$$

$$= \text{eq}(0 \rightrightarrows X) \tag{107.60}$$

As this is a strict initial object, only itself can map into it, so that the equalizer must be 0, so that the de Rham modality for the empty modality is simply itself.

Decomposition using *ps*-hexagon?

107.2 Algebra

As a monad, we can try to give the unit monad an associated algebra. For a given element X, and a morphism $x : \circledast X \cong 1 \to X$, ie a point of X, we need to have the commutation

$$\eta_X \circ x = \mathrm{Id}_X \tag{107.61}$$

Unfortunately, as η_X is the unique map to the terminal object, it cannot be true unless X is itself the terminal object [proof]. So our only possible algebra will be $(1, \mathrm{Id}_1)$, as we'd expect since our Eilenberg-Moore category is the terminal category.

Free algebra : algebra on 1 with morphism $\mathrm{Id}_1 : 1 \to 1$, trivial algebra

Coalgebra of the empty comonad $\boxslash$: given X and a morphism $f : X \to \boxslash X \cong 0$

If there is no such map : empty coalgebra? Except on $\varnothing$, the cofree coalgebra, which is the coalgebra with

107.3 Logic

In terms of logic, those two monads will correspond to modalities sending each object (as types) to either the unit type or the empty type. As a modal type theory, we are therefore simply sending the logic of our ambient topos to that of the initial category. So first let's look at its internal logic.

The initial category $\mathrm{Sh}(\varnothing) \cong \mathbf{1}$ has a rather barebone structure as a logical system. Its subobject classifier, initial and terminal object are all the same object, $*$, meaning that both the truth and falsehood morphisms are also the unique morphism Id_*, and the only subobject category is $\mathrm{Sub}(*)$, which is simply $*$ itself (*both* as the object itself and the terminal object). Trivially, any map factors through the terminal object and its truth map, meaning that any proposition is true in there. The unique proposition is simply the one defined by the subobject $* \hookrightarrow *$, which due to the rather collapsed logic could be either the truth or falsehood morphism.

$$\vdash \top \tag{107.62}$$

Likewise, any limit and colimit in the poset of subobjects will still be $*$, making any operation on propositions still ”true” (or false, depending on the viewpoint). This is the trivial logic, in which any proposition is true.

In terms of the ambient topos $\mathbf{H}$, any proposition $p : X \to \Omega$ is therefore mapped to this proposition,

$$\Delta_*(p : X \to \Omega) = \top/\bot : * \to * \tag{107.63}$$

And likewise, any negation is given by the same formula

$$\Delta_*(\neg p : X \to \Omega) = \top/\bot : * \to * \tag{107.64}$$

Going back to the ambient category, this gives us, depending on the modality,

$$\boxed{\varnothing}(p : X \to \Omega) \quad = \quad 0 \hookrightarrow 0 \tag{107.65}$$
$$\circledast(p : X \to \Omega) \quad = \quad 1 \hookrightarrow 1 \tag{107.66}$$

The first modality maps any proposition to the $0 \hookrightarrow 0$ proposition, which is the ”trivially true” proposition, both false and true but with no subobject that could actually be used to test it. The other is simply the true proposition $\top$.

Interpretation of the unit $X \to 1$ as a proposition

The modality of being $X \to 1$ defines a relation for $f : X \to Y$

$$R_{\circledast_X, f} \hookrightarrow 1 \times Y \cong Y \tag{107.67}$$

which is simply a property of the object Y itself, the property of being in the image of f.

Likewise, for the comodality of nothingness, we have a partition

$$0 + Y \cong Y \twoheadrightarrow \mathrm{Im}([\boxed{\varnothing}, g]) \tag{107.68}$$

The canonical unity of opposite $\Upsilon : 0 \hookrightarrow 1$ defines the relation with some arbitrary $f : 0 \to Y$

$$R_{\Upsilon, f} \hookrightarrow 1 \times Y \cong Y \tag{107.69}$$

which is simply the empty relation on Y.

as a partition with $g : Y \to 1$:

$$0 + Y \cong Y \twoheadrightarrow P_{\Upsilon,g} \cong \mathrm{Im}([\Upsilon_X, g]) \tag{107.70}$$

which is a partition on Y to the image of $[\Upsilon_X, g]$, or $1+1$, a partition into two elements, one for the elements which are in 0 (so none), and the other for the elements which are in Y.

[...]

107.3.1 Quillen negations

If we consider our three canonical morphisms, $\eta_X : X \to 1$, $\epsilon : 0 \to X$ and $\Upsilon : 0 \to 1$, the Quillen negations are the following

Theorem 107.3.1. The Quillen negations for $X \to 1$

Proof. For the left Quillen negation, we consider some function $X \to A$ and a point $b : 1 \to B$

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle !_X}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[b]{} & B \end{array}$$

Diagonal filler is a point of A which is the preimage of b by p. The relation $b \circ !_X = p \circ f$ means that all points of X are sent to a single value of B via $p \circ f$. As this is true for any morphism f, this means that this is true for p as well. As this must be true for any choice of point b, this means that B is a single point? [try to do a better proof] Is it self-dual?

$$\{\eta_X^{\circledast}\}^{\boxslash l}_{X \in \mathbf{H}} = \{\eta_X^{\circledast}\}_{X \in \mathbf{H}} \tag{107.71}$$

Right negation :

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle !_X} \\ B & \xrightarrow[!_B]{} & 1 \end{array}$$

This is simply the lifting property for objects, where X

fibrant resolution??? □

Right lifting for η is related to injective objects, and $0 \to 1$ defines injective functions, relation?

Determination of an object by using epimorphism/monomorphism to differentiate from isomorphism? Using of lifting properties and Quillen negations? Relation with the quillen negation of $0 \to 1$

Quillen negation of the unity of opposites $0 \to 1$, its opposite being the functions which are not surjective, relation with the existence of subobjects, ”different” objects?

Injective functions from $2 \to 1$, related to $\overline{\boxed{\varnothing}}$ and $\circledast$?

$\overline{\circledast} \to \circledast$

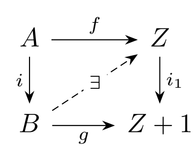
If we now consider our negations, $\eta_X^{\overline{\boxed{\varnothing}}}$ and $\epsilon^{\overline{\circledast}}$,

Theorem 107.3.2. Quillen negations of the identity

Proof. Any function has the right lifting property wrt the identity, as g is itself the diagonal morphism, and likewise for left lifting (f) □

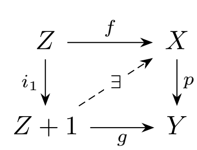
Theorem 107.3.3. The right Quillen negation of the maybe monad is the class of surjective function.

Proof. For the left Quillen negation of $\overline{\boxed{\varnothing}}$,



for the left lifting property : use the universal property of the coproduct? For any function $f : Z \to A$ and a morphism $a : 1 \to A$, there is a unique morphism $[f, a] : Z + 1 \to A$ that commutes with both. try here mb

The right Quillen negation of $\overline{\boxed{\varnothing}}$



As g is a morphism from a coproduct, it is equivalent to a pair of morphisms 20.12.1

$$g \circ i_1 : Z \quad \to \quad Y \tag{107.72}$$
$$g \circ i_2 : 1 \quad \to \quad Y \tag{107.73}$$

where one is simply the diagonal of the diagram and the other is the basepoint of $Z+1$ which is mapped to Y via g. Likewise for the filler $h : Z+1 \to X$, this corresponds to the pair of morphisms

$$h \circ i_1 : Z \quad \to \quad X \tag{107.74}$$
$$h \circ i_2 : 1 \quad \to \quad X \tag{107.75}$$

The commutativity of the external diagram means that for any function $Z \to X$, $p \circ f$ is in fact

□

What is the Quillen negations associated with $\boxslash X \to \overline{\boxslash} X$, which is $1 \to X$, ie the set of

List of functions to consider :

$$\eta_X^{\circledast} : X \to 1 \tag{107.76}$$
$$\epsilon_X^{\boxslash} : 0 \to X \tag{107.77}$$
$$\Upsilon_X^{\boxslash \circledast} : 0 \to 1 \tag{107.78}$$
$$\eta_X^{\overline{\boxslash}} : X \to X+1 \tag{107.79}$$
$$\epsilon_X^{\overline{\circledast}} : X \to X \tag{107.80}$$

107.4 Interpretation

If we try to apply Hegel's notions onto all this, this is the property of every object of our topos simply being an object, a property shared by all objects. Similarly, in the equivalence in type theory, this is just given by the type judgement : any object X is a type.

$$\vdash X : \text{Type} \tag{107.81}$$

[subobject classifier as a type of type?]

[Logical equivalence?]

From the notion that the being is somewhat related to the belonging to the type universe, logically this corresponds to its relation with the subobject classifier?

[...]

If we start the construction of the system with Being, we have merely some unknown topos $\mathbf{H}$, our domain of ideas, along with a subtopos $\mathbf{1}$, which is the moment of being for those ideas. All objects of our unknown topos will only manifest at one there, and therefore cannot be differentiated. If we were to assume that we were done, our topos would therefore be **1** itself. Due to the collapse of the logic to the trivial logic however, there is no difference between an object as

If we try to consider **1**

What is the ”nothingness” of this idea of being?

Stack semantics? If we pick our first undifferentiated universe of discourse **1**, the sheaf over this category is $\mathrm{Sh}(\mathbf{1}) \cong \mathbf{Set}$, and its fundamental fibration is

$$x \tag{107.82}$$

Kripke-Joyal semantics : ”Object has property P” : predicate P, ”Object does not have property P”

The subobject classifier of the initial topos is $*$ itself, with $\mathrm{Sub}(*) \cong \mathbf{1} \cong \mathrm{Hom}(*, *)$. Its negation is given by the terminal object, also $*$, into the subobject classifier, therefore the negation of any property is also the property.

[Intuition of the properties that an object doesn’t have via the use of Quillen negation?]

108

Necessity and possibility

Before looking further into the "standard" sublation of the ground opposition, let's briefly look at another direction to generalize.

The interpretation of being and nothingness as the duality between the dependent product and sum on the empty context gives us a possibility of generalization in this direction, in which we simply generalize to an arbitrary context.

As we've seen before, the context Γ of the internal logic corresponds to the slice category $\mathbf{C}_\Gamma$. If we wish to change our context, this is done via a display morphism $f : X \to Y$ which induces the functor

$$f^* : \mathbf{C}_{/Y} \to \mathbf{C}_{/X} \tag{108.1}$$

The ground that we've seen is done on the empty context, which is given by the terminal object 0. The corresponding context is that of the display morphism $!_0 : 0 \to 1$, changing the context from 0, falsity, to 1, truth. The corresponding contexts are $\mathbf{C}_1 \cong \mathbf{C}$ and $\mathbf{C}_0 = \mathbf{1}$, giving the base change functor

$$f^* : \mathbf{C} \to \mathbf{1} \tag{108.2}$$

which is exactly the functor that we used as its basis.

The interpretation in this sense is therefore that $\circledast$ is adding the context

For a morphism $f : X \to 1$ (what context is that?) :

$$f^* : \mathbf{C} \to \mathbf{C}_{/X} \tag{108.3}$$

Adjoints : $(f_! \dashv f^* \dashv f_*)$

$$(\textstyle\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}_{/X} \; \underset{\sum_f}{\overset{\prod_f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{f^*}}}}} \; \mathbf{H}_{/Y}$$

for $f : X \to 1$, we have $\mathbf{H}_{/Y} \cong \mathbf{H}$

[...]

Theorem 108.0.1. $\prod_f$ is a right adjoint and therefore preserves limits, in particular the terminal object :

$$\prod_f 1 = 1 \tag{108.4}$$

Theorem 108.0.2. $\sum_f$ is a left adjoint and therefore preserves colimits, in particular the initial object :

$$\sum_f 0 = 0 \tag{108.5}$$

From this, we can see that each of those adjunctions is a sublation of the ground.

Base change f^* preserves limits and colimits because toposes are stable under pullbacks?

so $f_* f^* = \prod_f f^* = \Box_f$ is such that

$$\Box_f 0 = \prod_f 0 \tag{108.6}$$

Adjoint modality : $(f_! f^* \dashv f_* f^*)$

Writer comonad and reader monad

Possibility comonad and necessity monad $(\Box \dashv \Diamond)$

[...]

The interpretation of this modality in terms of standard modal logic (the modality of necessity and possibility) can be understood using the Kripke semantics of modal logic.

interpretation of the ground in this context : $\circledast$ sends every proposition to true and $\boxslash$ to false. Every proposition is *possibly* true and none are *necessarily* true.

Show that they are not idempotent? Is that specifically if f fails to be an epimorphism?

109
Cohesion

[168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183]

geometry

cohesion in rome

Differential Cohesive Type Theory (Extended Abstract)

Precohesive Toposes over Arbitrary Base Toposes "Quantity is the unity of these moments of continuity and discreteness"

Qualität/Quality, Etwas/Something, Die Endlichket/Finitude,

The next step in the objective logic, on a more philosophical ground, is to allow for the notion of a difference, where there exists an object which is different from another object. Our underlying terminal topos for the opposition of being and nothingness has precisely that problem, where all objects are the same, which is related to the localization by all morphisms. Every object within our realm of discourse is isomorphic to every other object. If we want to be able to talk about more than one thing, and in this sense as more than mere existence, we will need to have some morphisms which are not localized. That is also why the underlying logic of this topos is trivial, as there is only a single truth value.

In topos terms, this is caused among other things by the conflation of all important objects of the topos into one, ie the initial and terminal object, and the subobject classifier. Not only is there only one object, but there is also a single morphism (otherwise the exponential object $[*, *]$ would be different from the terminal object).

This basic issue is at the root of a few different axioms. The law of non-contradiction requires the existence of two truth values, and also the existence of negation as fundamentally different from the identity.

The typical way to do this in the context of set theory is via the axiom of power sets, where for any set X, we have the existence of a power set Y which contains every subset of X :

$$\forall X,\ \exists Y,\ \forall Z,\ [Z \in Y \leftrightarrow \forall W,\ (W \in Z \to W \in X)] \tag{109.1}$$

while not evidently advocating the existence of another set, this can be shown via Cantor's theorem, as a set X and its power set $\mathcal{P}(X)$ can never be isomorphic, since there is no surjection $X \twoheadrightarrow \mathcal{P}(X)$. Therefore, along with the basic guaranteed set (the empty set via the axiom of empty set), we can generate another object,

$$\mathcal{P}(\varnothing) \cong \{\bullet\} \tag{109.2}$$

and so on. The axiom of infinity can also be used for the definition of the existence of at least infinitely many objects, but this will be looked into later on with respect to ideality.

The "something and another" thing in terms of the basic axiom "there's more than one thing", related to the axiom of choice / law of non-contradiction (Diaconescu's theorem), axiom of infinity, etc

To express this notion internally is difficult, as many definitions in category theory assume the existence of *something and other*.

This is generally implicit in the use of *processes*

[inductive types?]

We have $\exists$ to

Externally, if a topos has more than one object, it in fact has infinitely many. This can be shown since if the set of objects of a topos is of cardinality at least 2, there must be at least the initial object and terminal object as different objects. Given this, we can construct objects

$$n = \coprod_{s \in S} 1 \tag{109.3}$$

for an n-fold and m-fold coproduct, $n \leq m$, we have that there exists a canonical injection (this can be shown inductively)

$$n \hookrightarrow m \tag{109.4}$$

n is a complemented subobject of m, whose complement is

$$m - n = \coprod_{i=1}^{m-n} 1 \tag{109.5}$$

$$\begin{aligned} \mathrm{Hom}_{\mathbf{H}}(n, \Omega) &= \mathrm{Hom}_{\mathbf{H}}(\coprod_{i=1}^{n} 1, \Omega) & (109.6) \\ &= \prod_{i=1}^{n} \mathrm{Hom}_{\mathbf{H}}(1, \Omega) & (109.7) \\ &= \prod_{i=1}^{n} \mathrm{Hom}_{\mathbf{H}}(1, \Omega) & (109.8) \end{aligned}$$

From the equivalence $\mathrm{Sub}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \Omega)$, we have therefore that the category of subobjects of m is given by the set Ω^m, and each coproduct injection $\iota_i : 1 \to m$ corresponds to the characteristic morphism χ_i

The inclusion of n in m is therefore given by

"extensivity makes this true: if f+g is an iso then f,g are both isos; apply this to e.g. 1+1 0+1"

which we can show inductively to be different from each other. If the terminal object is different from the initial object, consider the following induction : the 0-fold coproduct of 1 is just 0. The 1-fold coproduct is simply the identity, so that the 1-fold coproduct of 1 is just 1. We now have to prove that if n is different from $n + 1$, then $n + 1$ is different from $n + 1 + 1$.

From the definition of the n-fold coproduct, we have some

Then we have at least as many objects as we do integers.

As existence of an object beyond one allows us to have at the very least an initial object and a terminal object, we have the existence of the falsity arrow $\bot : 1 \to \Omega$, and the negation $\neg : \Omega \to \Omega$.

Relation to span and partition?

This relates back to the negation of the empty comonad. As we've seen, this negation is the maybe monad, $X \to X + 1$, which, if in a non-degenerate topos, would imply a different object, but with this lack of differentiated objects, is just the identity. In terms of the right Quillen negation of this monad, we have the existence of surjective functions, but in a degenerate topos, this is all isomorphisms.

The connection of this notion to Hegel's objective logic is given by his notion of *something and another* (Etwas und ein Anderes), where to resolve the opposition of nothing and being (the fact that $0 \cong 1$ in the relevant subtopos), we

109.1 Continuity

From the existence of more than one object, we get the existence of the negation. This allows us further directions in which to sublate the initial opposition ($\boxslash \dashv \circledast$) to find higher ones. To find some resolution of ($\boxslash \dashv \circledast$) to some higher adjunction ($\boxslash' \dashv \circledast'$), we need some adjunction obeying the property

$$\begin{aligned} \boxslash' \boxslash &= \boxslash && (109.9) \\ \circledast' \circledast &= \circledast && (109.10) \end{aligned}$$

In other words, each of these preserve the initial and terminal object

$$\begin{aligned} \boxslash' 0 &= 0 && (109.11) \\ \circledast' 1 &= 1 && (109.12) \end{aligned}$$

We will look here more specifically at a *right sublation*, with the additional property

$$\circledast' \boxslash \cong \boxslash \qquad (109.13)$$

From the properties of the empty comonad, this simply means that the sublated monad preserves the terminal product :

$$\circledast' 0 \cong 0 \qquad (109.14)$$

which is the exact property of $\mathbf{H}_{\circledast'}$ being a dense subtopos. Fortunately there is a natural choice for this, given by this theorem

Theorem 109.1.1. The smallest dense subtopos of a topos is that of local types with respect to double negation $\mathrm{loc}_{\neg\neg}$. (Johnstone 02, corollary A4.5.20)

Proof. □

From this, we have that the natural sublation of the ground opposition can be constructed from the localization by the double negation. This is called the *sharp modality*, denoted by $\sharp$.

$$\circledast' = \sharp = \mathrm{loc}_{\neg\neg} \tag{109.15}$$

As a side note, if we had looked for a left sublation, the condition would have been

$$\boxslash' \circledast \cong \circledast \tag{109.16}$$

meaning that it must preserve the terminal object,

$$\boxslash' 1 \cong 1 \tag{109.17}$$

As we will see, this is also true of $\sharp$, meaning that $\sharp$ is also a left resolution of $(\boxslash \dashv \circledast)$, although this does not show that it would also be a left sublation, where we do not have any intermediary adjunction in between.

109.1.1 Intuition

To understand the role of the sharp modality here, let's consider why a double negation is not typically already localized. This is true in any boolean topos (by definition), including **Set**, but if we look at any topos with a more topological aspect to it, such as **Smooth** or **sSet**, we will see some issues with it.

Consider some topological space X (or its closest equivalent in some topological topos) and a subspace $\iota : U \hookrightarrow X$. We can try to consider its negation in X by using the naive complement in X, simply picking its complement as a set, with the appropriate subspace topology on this complement. In this case, we do have the law of excluded middle we'd expect from a boolean topos :

$$U + \neg U \cong X \tag{109.18}$$

Implying that there is some isomorphism between the two, but this is not the case. There is generally no continuous map

$$U + \neg U \to X \tag{109.19}$$

as can be seen by looking at this example. Let's take U some open subset of X and U^c its (point-wise) complement, in the case of U^c containing a non-empty boundary ∂U. For instance, $X = (0,1)^2$ and U some open disk

Since those two objects are meant to be homeomorphic, the image of any open set in one is an open set in the other. For any neighborhood V of X containing

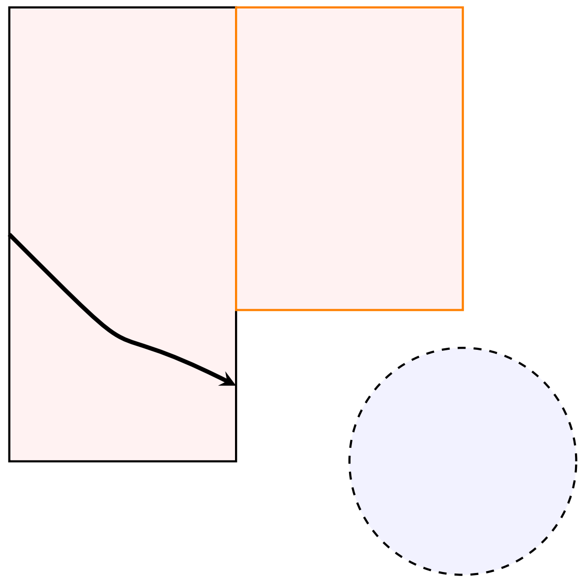


Figure 109.1: Caption

a point of the boundary ∂U, by the properties of a boundary point, this neighborhood will overlap both subspaces. For continuity, we should have an open preimage, but the preimage in U^c will contain the boundary ∂U which overlaps with the preimage of V, and therefore fails to be open in the subspace topology.

So if we look at a topological space in logical terms, its "negation" cannot merely be the point-wise complement, but only the pseudo-complement given by

$$\neg U = \mathrm{Int}(X \setminus U) \tag{109.20}$$

While the preimage of an open set does end up in an open set [prove], this means that the law of excluded middle does not hold, since the disjoint union of an open subspace and its complement will not even contain every point of X, and on a more spatial level, it will also lack every open set straddling both subspaces, making it disconnected along that missing boundary.

Likewise, the double negation (whose equivalence to the identity implies excluded middle, see [X]) will not generally be an equivalence :

$$\neg\neg U = \mathrm{Int}(X \setminus \mathrm{Int}(X \setminus U)) \tag{109.21}$$

If we take a fairly simple case like a closed disk in the plane, its pseudo-complement will be the exterior of the disk minus the circle itself, and its double negation is the interior of this disk. Typically, any double negation will get rid of any boundary in this context.

As we can see, both from the law of excluded middle and the double negation, the issue is always due to the "spatial structure". If we try to consider a space X as merely the sum of its parts (via the coproduct), we are simply missing out on the spatial cohesion along the division of those part. This is understandable simply enough due to the fact that if we take some originally connected space and split it into multiple subspaces, the coproduct of any number of those subspaces will not be connected, but connectedness should be preserved by homeomorphisms. The open sets meant to connect those regions have gone missing in the process.

[Relate to reconstruction of topological spaces? cf here]

[184]

From the laws of Heyting algebras however, we do always have the inclusion

$$U \hookrightarrow \neg\neg U \tag{109.22}$$

In logical terms, the proposition $P : U \hookrightarrow X$ implies its double negation. A property that is true "continuously" in U will be true in $\neg\neg U$.

[In terms of lifting property? Codiscrete space as maximally non-boolean space

"a space D is discrete iff $\varnothing \to D$ is in $(\varnothing \to \{\bullet\})^{\boxslash rl}$"

Left Quillen negation of $(\varnothing \to \{\bullet\})^{\boxslash r}$, the set of surjective function, which is the right Quillen negation of the unity ($\boxslash \dashv \circledast$)

a space D is codiscrete iff $D \to \{\bullet\}$ is in $(\{a,b\} \to \{a=b\})^{\boxslash rr} = (\{a \leftrightarrow b\} \to \{a=b\})^{\boxslash lr}$"

109.1.2 Sharp modality

The localization of the double negation should tell us that in some sense we may be trying to get back to the naive negation where we consider the actual pointwise complement. As usual, this localization defines a reflective subcategory

$$(T \dashv \iota) : \mathbf{H}_{\neg\neg} \underset{\iota}{\overset{T}{\rightleftarrows}} \mathbf{H}$$

where the idempotent monad is the composition $\sharp = \iota \circ T$. To look a bit more at the behavior of the sharp modality, let's look at the Eilenberg-Moore category of our modality.

Theorem 109.1.2. The subtopos by the localization of the double negation is boolean.

Proof. As the double negation is a local operator $\neg\neg : \Omega \to \Omega$, the sheaf topos $\mathrm{Sh}_{\neg\neg}(\mathbf{H})$ is boolean if $\neg\neg = q(U)$ for a subterminal object. [j sheaves, q is the quasi-closed local operator]

$$\Omega \xrightarrow{\cong} \Omega \times 1 \xrightarrow{\mathrm{Id}_\Omega \times u \times u} \Omega \times \Omega \times \Omega \xrightarrow{\to \times \mathrm{Id}_\Omega} \Omega \times \Omega \xrightarrow{\to} \Omega \tag{109.23}$$

$u : U \to 1$, $q(0)$ is double negation □

The subtopos $\mathbf{H}_\sharp$ is therefore a *Boolean topos*. It can be understood in terms of the existence of a complement for any subobject, where every object can be split exactly in two parts by a subobject, one part which contains every point of A, and another part which contains every other point. [etc etc]

As a boolean topos, $\mathbf{H}_\sharp$ is typically either the topos of sets **Set** or some variant thereof, such as some ETCS variant of it. We will generally assume that we are using **Set** here, but most of the properties used here should generalize to a wider variety of boolean topos.

There are some obvious counter examples, such as taking the initial topos **1**, which is its own double negation subtopos, [This? ref]

If we do actually pick **Set** to be our underlying Eilenberg-Moore category, the most natural choice for the adjoint functors of our monad $\sharp$ is given by the global section functor, the common notion of a forgetful functor to **Set**.

$$\Gamma : \mathbf{H} \to \mathbf{Set} \tag{109.24}$$

$$X \mapsto \mathrm{Hom}_{\mathbf{H}}(1, X) \tag{109.25}$$

The global section functor is part of the terminal geometric morphism,

$$(\mathrm{LConst} \dashv \Gamma) : \mathbf{H} \underset{\mathrm{LConst}}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set} \tag{109.26}$$

which is an embedding of **Set** in **H** [proof],

$$\mathrm{Sh}_j(\mathbf{H}) \hookrightarrow \mathbf{H} \tag{109.27}$$

From this, we have that

Theorem 109.1.3. The global section functor is a localization by the double negation.

Proof. We need to show that the geometric morphism $(\mathrm{LConst} \dashv \Gamma)$ is given equivalently by the adjoint pair of the embedding of j-sheaves and the sheafification functor

$$(\mathrm{Sh}_j \dashv \iota_j) : \mathrm{Sh}_j(\mathbf{H}) \underset{\iota_j}{\overset{\mathrm{Sh}_j}{\leftrightarrows}} \mathbf{H}$$

with $j = \neg\neg$.

Natural closure from a geometric morphism : [66, p. 373]

□

If this geometric morphism admits a further right adjoint, making it a locally local geometric morphism, we will call it the codiscrete functor, CoDisc.

Definition 109.1.1. A *locally local topos* is a topos for which the global section functor admits a right adjoint.

$$(\Gamma \dashv \mathrm{Codisc}) : \mathbf{Set} \underset{\mathrm{Codisc}}{\overset{\Gamma}{\leftrightarrows}} \mathbf{H}$$

The adjunction tells us that

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(X), S) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{CoDisc}(S)) \tag{109.28}$$

The codiscrete functor will therefore transform any set S into a space for which the hom-set from any other object will be isomorphic to that of the set of functions between X and S. This is a property that is similar to the trivial topology : any function on the points of the underlying set of that space lifts to a "continuous" function in the topos.

On a more "topological" description, we can look at the coverage of $\sharp X$. As an adjunction of geometric morphisms, $(\Gamma \dashv \mathrm{CoDisc}) \cong (f^* \dashv f_*)$, with respect to the terminal topos $\mathbf{Set} \cong \mathrm{Sh}(\mathbf{1})$,

Embedding geometric morphism

For any subobject $[U] \hookrightarrow \mathrm{CoDisc}(X)$, or in subobject classifier term, $\chi_U : X \to \Omega$, the closure map is given by

$$\chi_{\overline{U}} = j \circ \chi_U \tag{109.29}$$

Show that $\mathrm{CoDisc}(X)$ is a $\neg\neg$-sheaf of $\mathbf{H}$,

Monomorphism $U \hookrightarrow X$ is dense if $\overline{U} = X$, ie $\chi_{\overline{U}} \cong \top \circ !_X$.

$$\chi_{\overline{U}} \quad = \quad \neg \circ \neg \circ \chi_U \tag{109.30}$$

[As a j sheaf all images are dense in X and therefore]

[Compute the Grothendieck topology]

In other words, every non-empty subobject of CoDisc(X) is dense, which is another characteristic of the codiscrete topology : $\overline{U} = X$. In particular, if $\mathbf{H}$ is a Grothendieck topos, its Grothendieck topology for $\sharp X$ is the trivial one, with simply $\{0 \to \sharp X, \sharp X \to \sharp X\}$(?)

Theorem 109.1.4. A codiscrete space has the same point content as the set it is generated from

$$\Gamma \circ \mathrm{Codisc} \circ \Gamma \cong \Gamma \tag{109.31}$$

Proof. Triangle identity □

Our sharp modality therefore simply maps a space X to its set of points $\Gamma(X)$, before mapping it back to the codiscrete space composed of these points. As the type of space with the strongest topology, $\sharp X$ does not obey the law of excluded middle or have a double negation isomorphic to the identity, and in fact in some sense is maximally in violation of these. In terms of our motivating example with trivial topologies, we have (for $U \neq \varnothing$)

$$\begin{aligned} \neg\neg U &= \mathrm{Int}(X \setminus \mathrm{Int}(X \setminus U)) && (109.32)\\ &= \mathrm{Int}(X \setminus \varnothing) && (109.33)\\ &= X && (109.34)\\ & && (109.35) \end{aligned}$$

and

$$\begin{aligned} U \vee \neg U &= U \cup \mathrm{Int}(X \setminus U) && (109.36)\\ &= U && (109.37) \end{aligned}$$

Given some (strict) subobject $U \hookrightarrow \sharp X$

$$\neg\neg U = 0 \tag{109.38}$$

[$\neg\neg X = X$ otoh]

and $U \vee \neg U = U$. It is in some sense the worst at reconstructing the initial space by negation of subobjects, as any split of the original space will lose the entire cohesive structure.

Measuring the violation of the excluded middle : for any X with the canonical embedding $X \hookrightarrow \neg_X \neg_X X$, how

Therefore, under the sharp modality, if we split our object into subobjects, there will be no loss of "spatial" information, as would be the typical case from what we expect.

Property under the local operator?

From this subtopos, we can determine an important property of the sharp modality : if we have take any two spaces $X, Y \in \mathbf{H}$ and look at their underlying sets, then any morphism between those sets

$$f : \Gamma(X) \to \Gamma(Y) \tag{109.39}$$

can be lifted to a continuous function $X \to \sharp Y$. Via adjointness,

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(X), \Gamma(Y)) \cong \mathrm{Hom}_{\mathbf{C}}(X, \mathrm{CoDisc} \circ \Gamma(Y)) \tag{109.40}$$

This property marks it as the sharp modality being similar to the trivial topology. In particular, any morphism $S \to \sharp X$ that does not have a corresponding factoring through X via $S \to X \to \sharp X$ corresponds to a discontinuous function to X.

Sublation by $\sharp$ always exists for any topos?

Theorem 109.1.5. A topos $\mathbf{H}$ is boolean if and only if it only has a single dense subtopos, $\mathbf{H}_{\neg\neg}$, for which $\mathbf{H}_{\neg\neg} \cong \mathbf{H}$.

Properties of $\sharp$:

As a reflector preserving all limits (by being right adjoint to $\flat$), $\sharp$ defines a Lawvere-Tierney topology. That is, for any $X \in \mathbf{H}$, we have some closure operator

$$j_\sharp : \Omega \xrightarrow{\chi_{\sharp\top} \circ \sharp} \Omega \tag{109.41}$$

For the truth map $\top : 1 \to \Omega$, the sharp truth map $\sharp\top : 1 \to \sharp\Omega$

This is the Lawvere-Tierney topology defined by the constant truth value morphism, ie

$$x \tag{109.42}$$

Proof that this is the codiscrete one

[...]

Relation with the internal hom :

Theorem 109.1.6. The internal hom of two objects has the global section of the hom-set :

$$\Gamma([X, Y]) = \mathrm{Hom}_{\mathbf{H}}(X, Y) \tag{109.43}$$

Proof.

$$\begin{aligned}\Gamma([X,Y]) &= \mathrm{Hom}_{\mathbf{H}}(1,[X,Y]) && (109.44)\\ &= \mathrm{Hom}_{\mathbf{H}}(1\times X,Y) && (109.45)\\ &= \mathrm{Hom}_{\mathbf{H}}(X,Y) && (109.46)\end{aligned}$$

□

Locally local topos has a NNO?

Sharp modality on bundles : For a bundle $p : E \to X$, take the sharp modality, $\sharp p : \sharp E \to \sharp X$.

$$\begin{array}{ccccc} E & \longrightarrow & \sharp_X E & \longrightarrow & \sharp E \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow{\scriptstyle \sharp p} \\ X & = & X & \xrightarrow[\eta^\sharp_X]{} & \sharp X \end{array}$$

$\sharp_X E$ the sharp closure of p?

What are sharp closed objects, ie $\sharp_X E \cong E$?

Property of the pullback for codiscrete objects?

109.2 Discreteness

As we've seen, the natural (right) sublation of the ground is given by the localization by the double negation, $\mathrm{loc}_{\neg\neg}$, the sharp modality $\sharp$. To get the full sublation of the ground adjunction, we will need also the existence of an adjoint modality, called the *flat modality* $\flat$.

We have already seen that Γ admits a natural left adjoint, the locally constant sheaf functor LConst

[recall the properties]

In our context, we will call it the *discrete* functor Disc, which give us the adjoint cylinder

$$(\mathrm{Disc} \dashv \Gamma) : \mathbf{H}_\sharp \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \mathbf{H}$$

As a geometric morphism, Disc preserves all finite limits.

As $\sharp$ can be understood as the space generated from the set of its points with a trivial topology, $\flat$ is the space generated from the set of its points with a discrete topology, ie as a topos it is entirely determined by its points, and is simply the coproduct of every point.

$$\flat X \cong \coprod_{s:\Gamma(X)} 1 \tag{109.47}$$

This is the other extreme of the properties of our topos with respect to double negation, where the given space obeys the law of excluded middle rather than entirely violating it.

$$\forall S \in \mathbf{H}_\sharp,\ S \cong \coprod_{x\in \mathrm{Hom}(1,S)} 1 \tag{109.48}$$

Since for any point $p \in \mathrm{Hom}(1, S)$, we can decompose its object with the law of excluded middle as

$$p + \neg_S p = S \tag{109.49}$$

which leads by induction[?] to the identity[What if the remaining object $\neg^n S$ is pointless but not initial?]. We can therefore get

Theorem 109.2.1. If the base topos' objects are coproducts indexed by their own hom-sets,

$$S \cong \coprod_{x\in \mathrm{Hom}(1,S)} 1 \tag{109.50}$$

Then the left adjoint of the global section functor is the functor of locally constant sheaves LConst

$$\mathrm{LConst}(S) = \coprod_{s\in S} 1 \tag{109.51}$$

Proof. As left adjoints preserve limits,

$$\begin{aligned} \mathrm{LConst}(S) &= \mathrm{LConst}(\coprod_{s\in S} 1) && (109.52)\\ &= \coprod_{s\in S} \mathrm{LConst}(1) && (109.53) \end{aligned}$$

And since we assumed that finite limits are preserved,

$$\mathrm{LConst}(S) = \coprod_{s\in S} 1 \tag{109.54}$$

□

As we are in a topos, the category is extensive, so that this coproduct of terminal objects is just a collection of disjoint points, giving us the intuition of this modality sending an object to its equivalent discrete topology.

Dually to the codiscrete case, we also have that discrete objects obey the universal property that any function from a discrete space is continuous, ie that

$$\mathrm{Hom}_{\mathbf{H}}(\mathrm{Disc}(S), X) \cong \mathrm{Hom}_{\mathbf{H}_\sharp}(S, \Gamma(X)) \tag{109.55}$$

Topos localized by $\neg\neg$, the new topos is $H_\sharp$. The opposition $\sharp \dashv \flat$ is the ground topos of $H_\sharp$

the law of excluded middle for the flat case?

$$\flat(U + \neg_X U) \tag{109.56}$$

Definition 109.2.1. A topos that admits a geometric morphism $(\flat \dashv \sharp)$ is called a *local topos*

[185, 186]

Adjuncts :

$$f : \flat X \to Y \tag{109.57}$$

$$\overline{f} = \sharp(f) \circ \eta^\sharp_X : X \to \sharp X \to \sharp Y \tag{109.58}$$

$$f : X \to \sharp Y \tag{109.59}$$

$$\overline{f} = \epsilon^\flat_Y \circ \flat(f) : \flat X \to \flat Y \to Y \tag{109.60}$$

Theorem 109.2.2. Given the adjunction $(\flat \dashv \sharp)$, The components of the unit of the sharp modality $\eta^\sharp_X$ are surjections. [?]

Proof. Given a point of the codiscrete space $p : 1 \to \sharp X$, we can describe it as

$$\eta^\sharp_X \circ p : 1 \to \sharp\flat X \cong \sharp X \tag{109.61}$$

So that its adjunct

$$\tilde{p} = \Phi(p) = \flat(p) \circ \epsilon^\flat_X : \flat(1) \cong 1 \to \flat(\sharp(X)) \cong \flat X \to X \tag{109.62}$$

is isomorphic to

$$\Phi(\eta^\sharp_X \circ p) \cong \mathrm{Id}_X \circ \tilde{p} \tag{109.63}$$

[CHECK] □

109.3 Quality

If the flat modality admits a further adjunction, we will call its left adjoint the *shape modality*, $\int$. This will allow us to define the full notion of cohesion properly by adding some appropriate requirements to it. The shape modality roughly corresponds to the topological notion of connected objects, so that first we need to define what it means for an object to be connected in a topos. If we take a connected object to be the lack of decomposition into the disjoint sum of other objects, as they are for topological spaces, we will show that this definition gives out that result :

Definition 109.3.1. An object X in a topos $\mathbf{H}$ is *connected* if $\mathrm{Hom}_{\mathbf{H}}(X, -)$ preserves finite coproducts.

A simple enough way to convince ourselves of this is to consider that as a topos is an extensive category, its coproduct is disjoint, so that a finite coproduct of objects always has empty intersection :

$$\bigcap_i \dots \tag{109.64}$$

and we can now show that our definition is equivalent to the notion that an object cannot be expressed as a coproduct of subobjects, which is the categorical equivalent of the topological notion of connectedness :

Theorem 109.3.1. If an object is connected, it cannot be expressed as the coproduct of more than one non-empty subobject.

Proof. □

Definition 109.3.2. A topos is *locally connected* if every object is the coproduct of connected objects $(X_i)_{i\in I}$:

$$X \cong \coprod_{i\in I} X_i \tag{109.65}$$

Theorem 109.3.2. The index set I is unique up to isomorphism

Proof. □

Definition 109.3.3. The connected component functor maps objects of a locally connected topos to its index set of connected objects :

$$\Pi_0(X) \cong \Pi_0(\coprod_{i\in I} X_i) \cong I \tag{109.66}$$

Theorem 109.3.3. The connected component functor is indeed a functor

Proof. □

Theorem 109.3.4. The connected component functor is left adjoint to the discrete object functor.

Proof. As a discrete object is given by

$$\mathrm{Disc}(S) \cong \coprod_{s \in S} 1 \tag{109.67}$$

with $\Pi_0(1) \cong 1$ (as from its universal property, it cannot have more than one coprojection), we have therefore that

$$\Pi_0(\mathrm{Disc}(S)) \cong S \tag{109.68}$$

□

As an adjoint triple of modalities, we therefore have two pairs of units and counits. First is the pair for $(\flat \dashv \sharp)$, a *ps*-opposition, which are components-wise

$$\epsilon^{\flat}_X : \flat X \quad \to \quad X \tag{109.69}$$
$$\eta^{\sharp}_X : X \quad \to \quad \sharp X \tag{109.70}$$

which give us relations of objects and their discrete and codiscrete equivalents, and the pair for $(\textstyle\int \dashv \flat)$

$$\epsilon^{\flat}_X : \flat X \quad \to \quad X \tag{109.71}$$
$$\eta^{\int}_X : X \quad \to \quad \textstyle\int X \tag{109.72}$$

which deal with the relations of objects, their points and their connected components, which is an *sp*-opposition.

As an *sp*-opposition, both $\int$ and $\flat$ have the same modal type, which are the *discrete objects*, similarly to spaces of discrete topologies, with opposite ways to project on it.

From the canonical natural transformation of the opposition 100.24, which is called the *point-to-piece transform*,

$$\mathrm{ptp}_X = \eta^{\int}_X \circ \epsilon^{\flat}_X : \flat X \to X \to \textstyle\int X \tag{109.73}$$

which is meant to signify the association of points of a space to its connected components

[Diagram]

For most of our cases here, the base topos will be consider will be the topos of sets, **Set**, so that it is best to understand cohesion in terms of functors to sets. The archetypical cohesion is done using the global section functor Γ :

Theorem 109.3.5. The initial object is the only object with no connected components :

$$\int X \cong 0 \leftrightarrow X \cong 0 \tag{109.74}$$

Proof. As $\int$ preserves colimits, this is trivially true in one direction. In the other direction, for any object X with

$$\int X \cong 0 \tag{109.75}$$

we have the unit

$$\eta^{\int}_X : X \to \int X \cong 0 \tag{109.76}$$

As 0 is a strict initial object, this means that $X \cong 0$. □

Definition 109.3.4. If an object X has its exponential objects X^Y for any other object Y with the shape of the terminal object, we say that X is *contractible.*

This definition stems from the case of topological spaces, where a space X is contractible if every continuous map into it $Y \to X$ is null homotopic, ie if the mapping space $C(Y, X)$ is path connected (to the identity map, in particular).

109.4 Cohesiveness

The mere adjoint triple $(\int \dashv \flat \dashv \sharp)$ does not fully define what we wish to have for the notion of cohesiveness, as many properties that would seem intuitively important for a spatial object may still be lacking. For instance, we would typically imagine that the connected components of a space obey basic algebraic rules. Take some space $(I + I)$, two intervals, and take its product space $(I + I) \times (I + I)$. From the preservation of the coproduct, if we have $\int I \cong 1$, we have $\int(I + I) \cong 1 + 1$, a space with two components, and we would expect the product of spaces to translate to a product of those components,

$$\int((I + I) \times (I + I)) = 4 \tag{109.77}$$

In particular, we want the conservation of the terminal object, the empty product, as if we are meant to interpret $\int$ as the connected components of an object, we would also like that $\int 1 \cong 1$. While guaranteed by

Definition 109.4.1. For a topos with the geometric morphism [...] we say that it is *strongly connected* if

$$\Pi_0(X \times Y) = (\Pi_0 X) \times (\Pi_0 Y) \tag{109.78}$$

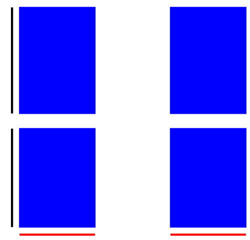


Figure 109.2: Example of a product of two spaces with two connected components

Counterexample 109.4.1. The topos of G-sets is connected but not strongly connected.

Proof. From the definition of G-sets 71 for a given group G, let's look at its cohesive structure. The point content of a G-set as a set differs from that of its points in a cohesive manner. Its standard forgetful functor :

$$U(X_\rho) = X \tag{109.79}$$

gives us the points of the underlying set X, while the global section functor Γ gives us the set of equivariant functions

$$f(g \cdot \bullet) = f(\bullet) = g \cdot f(\bullet) \tag{109.80}$$

In other words, every element of X that is invariant under G-action. Generally, there is no guarantee that a G-set will have any points at all, for instance if ρ is a free group action. Its right adjoint

$$(\Gamma \dashv \mathrm{Codisc}) : \mathbf{Set} \underset{\text{Codisc}}{\overset{\Gamma}{\rightleftarrows}} G\mathbf{Set}$$

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(X_\rho), Y) = \mathrm{Hom}_{G\mathbf{Set}}(X_\rho, \mathrm{Codisc}(Y)) \tag{109.81}$$

$$f(\rho_{X,g} \cdot x) \quad = \quad \rho_{\mathrm{Codisc},g} \cdot f(x) \tag{109.82}$$

Left adjoint :

$$\mathrm{Hom}_{\mathbf{Set}}(X, \Gamma(Y_\rho)) = \mathrm{Hom}_{G\mathbf{Set}}(\mathrm{Disc}(X), Y_\rho) \tag{109.83}$$

Connected modality : quotient set $\Pi_0(X_\rho) = X/\rho$?

$$\Pi_0(G) = 1 \tag{109.84}$$

$$\Pi_0(G \times G) = \tag{109.85}$$

[...]

Therefore, for G**Set**, the shape modality does not preserve products. □

Also while we typically try to admit spaces more broadly than what is defined by point set topology, we will ask that there is a minimum of cohesion : every connected component of a space should have at least one point (another property not fulfilled by the topos of G-sets). This is given by the condition that the point to piece transform is an epimorphism :

$$\flat X \to X \to \textstyle\int X \tag{109.86}$$

Definition 109.4.2. A topos $\mathbf{H}$ is *cohesive* over a base topos $\mathbf{B}$ if it is equipped with the geometric morphisms

$$(f^* \dashv f_*) : \mathbf{H} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{B}$$

and obeys the following properties :

- It is a locally connected topos : there is a further left adjoint $(f_! \dashv f^*)$ and every object is a coproduct of connected objects.
- It is connected : $f_!$ preserves the terminal object
- It is strongly connected : $f_!$ preserves finite products.
- It is local : there is a further right adjoint $(f_* \dashv f^!)$

Together these form the adjoint string that we have seen, which in the specific notation of cohesion gives us (for $\mathbf{B} \cong \mathbf{Set}$)

$$(\Pi_0 \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc}) : \mathbf{H} \begin{array}{c} \xrightarrow{\Pi_0} \\ \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\mathrm{Codisc}} \end{array} \mathbf{Set}$$

giving us the corresponding adjoint triple of cohesive modalities,

$$(\textstyle\int \dashv \flat \dashv \sharp) : \mathbf{H} \to \mathbf{H} \tag{109.87}$$

with the modalities

$$\textstyle\int = \mathrm{Disc} \circ \Pi_0 \tag{109.88}$$

$$\flat = \mathrm{Disc} \circ \Gamma \tag{109.89}$$

$$\sharp = \mathrm{Codisc} \circ \Gamma \tag{109.90}$$

Counterexample 109.4.2. The category of topological G-sets admits the adjunction $(\int \dashv \flat \dashv \sharp)$ but is not a cohesive topos.

Proof. For a given group G, $G - \mathbf{Set}$ admits the terminal object given by the terminal topological space with a trivial G-action on it. Its global section functor Γ is the set of equivariant functions

$$\Gamma \tag{109.91}$$

□

In addition to this basic definition, it is common to ask for a few additional conditions.

Axiom Punctual local connectedness. The points-to-pieces transform is an epimorphism :

$$\mathrm{ptp}_X : \flat X \twoheadrightarrow \textstyle\int X \tag{109.92}$$

This is meant to signify that for any object with at least one connected component ($\int X \neq 0$), each of those connected component has at least one component, avoiding the case of a space that is entirely pointless.

Nullstellensatz condition? [187]

How does this relate to the dual space

Axiom . pieces of powers are powers of pieces

Axiom Sufficient cohesion. the subobject classifier is connected

Equivalent to

Theorem 109.4.1. A cohesive topos is sufficiently cohesive iff for every object X, there exists a contractible object Y, $\int Y^A = 1$ for all A, such that X is a subobject of Y :

$$\iota : X \hookrightarrow Y \tag{109.93}$$

Proof. □

Theorem 109.4.2. If the terminal object 1 is connected, $\mathrm{Hom}_{\mathbf{H}}(1, -)$ preserves coproducts, then the point content of an object is given by any given subobject and its negation.

$$\Gamma(A + \neg_X A) \cong \Gamma(X) \tag{109.94}$$

Proof. From the preservation of coproducts, we have that

$$\Gamma(A + \neg_X A) \cong \Gamma(A) + \Gamma(\neg_X A) \tag{109.95}$$

□

Preservation of pushout requires the geometric morphism to be surjective idk

Similarly $\flat(A + \neg_X A) \cong \flat(X)$?

Vague : $\flat$ maps to the points of X, and every point of $\int X$ is an image of one of those points. (point to pieces transform)

To give it its proper meaning of being about the connected components of

$$(\Pi_0 \dashv \mathrm{Disc}) : \mathbf{H} \xrightarrow{\Pi_0} \mathbf{Set} \xrightarrow{\mathrm{Disc}} \mathbf{H} \tag{109.96}$$

$$\mathrm{Hom}_{\mathbf{Set}}(\Pi_0(X), A) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{Disc}(A)) \tag{109.97}$$

The hom-set of functions from our space to the discrete space from a set A is isomorphic to the set of functions from $\Pi_0(X)$ to that set.

Interpretation : Π_0 send each element - subobject in the same "connected component" to a different point.

Connected object : X is a connected object if the hom-set functor preserves coproducts

Shape preserves connected objects?

Shape modality : $\int = \mathrm{Disc} \circ \Pi_0$. *sp*-unity, so $\int$ and $\flat$ share the same space : $\int X$ is a discrete space : $\flat \int = \int$

Commutating diagram for adjoint quadruples :

for $(\Pi_0 \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc})$, $X \in \mathbf{H}$ and $S \in \mathbf{Set}$,

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\epsilon^{-1}_{\Gamma X}} & \Pi_0 \mathrm{Disc}\Gamma(X) \\ {\scriptstyle \Gamma(\eta_X)}\downarrow & \searrow^{\theta_X} & \downarrow{\scriptstyle \Pi_0(\epsilon_X)} \\ \Gamma\mathrm{Disc}\Pi_0(X) & \xrightarrow[\eta^{-1}_{\Pi_0(X)}]{} & \Pi_0(X) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Disc}S & \xrightarrow{\eta \mathrm{Disc}S} & \mathrm{CoDisc}\Gamma\mathrm{Disc}(S) \\ {\scriptstyle \mathrm{Disc}(\epsilon^{-1}_S)}\downarrow & \searrow^{\phi_S} & \downarrow{\scriptstyle \mathrm{CoDisc}(\eta^{-1}_S)} \\ \mathrm{Disc}\Gamma\mathrm{CoDisc}(S) & \xrightarrow[???]{} & \mathrm{CoDisc}(S) \end{array}$$

Another intuitive property that we could ask of a topos is that its codiscrete objects are connected, that is

$$\int \sharp X \cong 1 \tag{109.98}$$

This is not generally true (**Set** is cohesive and does not obey it), not even for a sufficiently cohesive topos, but it is true in a category called codiscretely connected[174]

Definition 109.4.3. A cohesive topos is *codiscretely connected* if for some set S, the unique map $\Pi_0 \mathrm{CoDisc} S \to \{\bullet\}$ is a monomorphism.

This property is equivalent to it, as we can show that

Theorem 109.4.3. A cohesive topos is codiscretely connected if and only if $\int \mathrm{CoDisc}(2) \cong 1$

property

Theorem 109.4.4. For any codiscrete object $\sharp X$, this object is connected and contractible :

$$\int \sharp X \cong 1,\ \forall Y,\ \int (\sharp X)^Y \cong 1 \tag{109.99}$$

Proof. Given a set S and its associated codiscrete object $\overline{S} = \mathrm{CoDisc}(S)$, □

[...]

Hierarchy :

Topos (every topos has a terminal geometric morphism with adjoint?), sheaf topos (geometric morphism?)

splitting : existence of a further left / right adjoint :

local topos (codisc adjoint), locally connected topos ()

essential topos? (???)

Theorem 109.4.5. If a topos has a site with an initial object

Theorem 109.4.6. If a topos has a site with an initial and terminal object, it is both locally local and locally connected.

Proof. □

Cohesive site

Definition 109.4.4. Given a cospan composed of a point $x : 1 \to X$ and the counit of the $(\flat \dashv \sharp)$ adjunction, $\epsilon^\flat_X : \flat X \to X$, the *de Rham flat modality* $\flat_{\mathrm{dR}}$ is given by the pullback

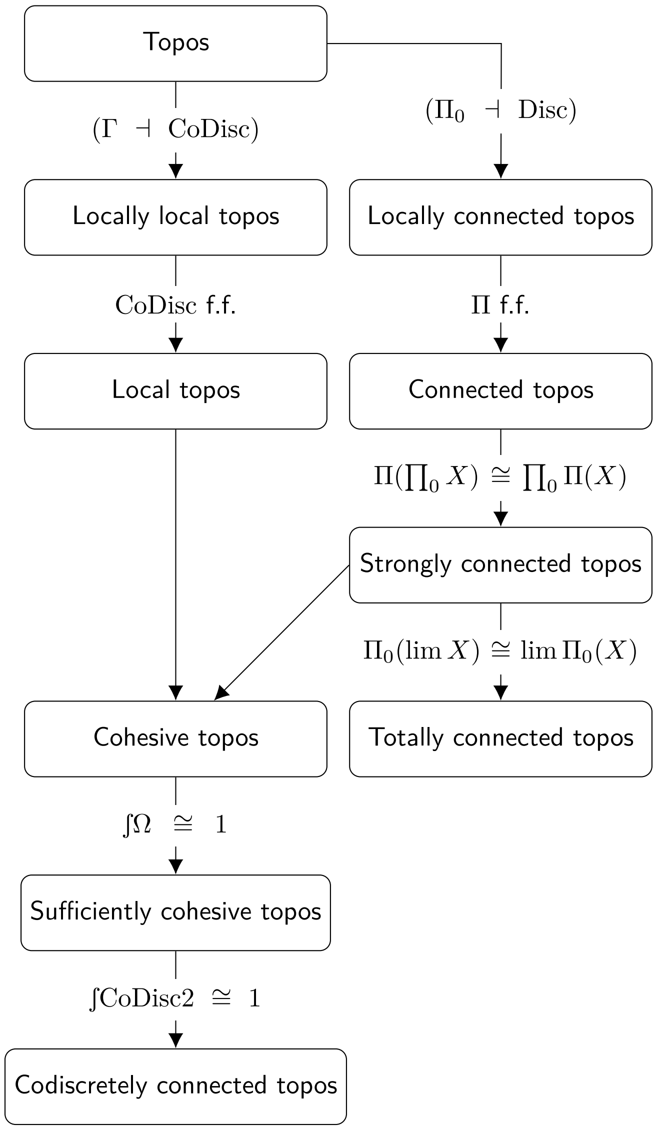


Figure 109.3: Hierarchy of cohesive topoi

Interpretation : $\bar{\flat}\flat = \square_*$, ie the de Rham flat modality negates the flat modality. The de Rham flat modality of a discrete space is the terminal object.

For co-concrete objects, $\flat X \to X$ is an epimorphism

Nullstellensztz, the point to piece map is epi, equivalent to mono map from discrete to codiscrete spaces

Example 109.4.1. As we have that for the real line, $\flat\mathbb{R} \cong |\mathbb{R}|$, the discrete space with the cardinality of $\mathbb{R}$, given some point $x : 1 \to \mathbb{R}$ and the inclusion map $|\mathbb{R}| \to \mathbb{R}$, its de Rham flat modality is given by the dependent product

$$\bar{\flat}\mathbb{R} = |\mathbb{R}| \times_{\mathbb{R}} 1 = 1 \tag{109.100}$$

which does give us $\flat\bar{\flat}\mathbb{R} = 1$

The de Rham flat modality does not seem to be of much use, however it will reveal itself to be much more important later on for the case of higher category theory, and will earn its name of "de Rham" there.

Definition 109.4.5. Given a morphism $f : X \to Y$, its $\int$-closure $c_\int f$ is given by the pullback

$$c_\int f = Y \times_{\int Y} \int X \tag{109.101}$$

"Morphisms that are $\int$-closed may be identified with the total space projections of locally constant inf-stacks over Y"

Decomposition of morphisms into $\int$-equivalences and $\int$-closed

$$\tilde{X} = D^{\int_1}(X, 1) = \sum_{x:X} \eta(x) = \eta(1) \tag{109.102}$$

Theorem 109.4.7.

$$\flat(\int X \to Y) \cong \flat(X \to \flat Y) \tag{109.103}$$

Proof. □

Adjuncts :

$$f : \int(X) \to Y \tag{109.104}$$

$$\tilde{f} : X \to \flat(Y) \tag{109.105}$$

$$f : \flat(X) \to Y \tag{109.106}$$

$$\tilde{f} : X \to \sharp(Y) \tag{109.107}$$

109.5 Interpretation

As a sublation of the initial opposition, we are trying to find a larger domain of thinking for which we have a resolution of the opposition. Rather than the initial topos **1**, where the only object $*$ is both terminal and initial (in terms of internal logic, truth and falsehood are the same), we want some additional concepts which embody both of those notions as intermediaries. We need some *determinate* objects, whose properties differentiate them. In other words, given any property of the domain, we want to be able to define it as either *being* those properties or *not being* those properties, rather than the case we previously saw, where there is only one property

The sublation of the initial opposition corresponds here to the notion of *determinate being* :

The two basic notions involved with the initial adjoint ($\flat \dashv \sharp$) is to consider our objects as either collections of objects, or as a single whole. The unity of opposites involved therefore being the behaviour of objects with subobjects with respect to both those opposite notions.

[...]

When we take our initial opposition of being and nothingness, $0 \to X \to 1$, this is an opposition that has no information on our object. This is identically true

First, we need to look at what the notion of negation corresponds to here, which is that of *something and other*.

To consider the notion of objects having some kind of parts, we have first to assume the existence of more than one object.

“Something and other are, in the first place, both determinate beings or somethings.

Secondly, each is equally an other. It is immaterial which is first named and solely for that reason called something; (in Latin, when they both occur in a sentence, both are called aliud, or 'the one, the other', alius alium; when there is reciprocity the expression alter alterum is analogous). If of two things we call one A, and the other B, then in the first instance B is determined as the other. But A is just as much the other of B. Both are, in the same way, others. The word 'this' serves to fix the distinction and the something which is to be taken affirmatively. But 'this' clearly expresses that this distinguishing and signalising of the one something is a subjective designating falling outside the-something itself. The entire determinateness falls into this external pointing out; even the expression 'this' contains no distinction; each and every something is just as well a 'this' as it is also an other. By 'this' we mean to express something completely determined; it is overlooked that speech, as a work of the understanding, gives expression only to universals, except in the name of a single object; but the individual name is meaningless, in the sense that it does not express a universal, and for the same reason appears as something merely posited and arbitrary; just as proper names, too, can be arbitrarily assumed, given or also altered."

Those correspond to the notion of continuity and discreteness, as seen in many different discussions on the topic through history. This is the sort of notion we saw in the introduction 1, which we can also find in Iamblichus'

"The nature of the continuous and the discrete for all that is, that is to say for the whole structure of the cosmos, may be conceived in two ways: there is the discrete through juxtaposition and through piling up, and the continuous through unification and through conjunction.

In accordance with the essence of magnitude the cosmos would be conceived as one and would be called solid, spherical, and fused together, extended and conjoined; but, again, according to the form and concept of plurality the ordering, disposition, and joining together of the whole would be thought of as, we may say, being constructed of so many oppositions and similarities of elements, spheres, stars, kinds, animals, and plants. But in the case of the unified, division from the totality is without limit, while its increase is to a limited point; while conversely, in the case of plurality, increase is unlimited, but division limited."

Or Grassmann's Ausdehnungslehrer :

""Each particular existent brought to be by thought can come about in one of two ways, either through a simple act of generation or through a twofold act of placement and conjunction. That arising in the first way is the continuous form, or magnitude in the narrow sense, while that arising in the second way is the discrete or conjunctive form.

The simple act of becoming yields the continuous form. For the discrete form, that posited for conjunction is of course also produced by thought, but for the act of conjunction it appears as given; and the structure produced from the givens as the discrete form is a mere correlative thought. The concept of continuous becoming is more easily grasped if one first treats it by analogy with the more familiar discrete mode of emergence. Thus since in continuous generation what has already become is always retained in that correlative thought together with the newly emerging at the moment of its emergence, so by analogy one discerns in the concept of the continuous form a twofold act of placement and conjunction, but in this case the two are united in a single act, and thus proceed together as an indivisible unit. Thus, of the two parts of the conjunction (temporarily retaining this expression for the sake of the analogy), the one has already become, but the other newly emerges at the moment of conjunction itself, and thus is not already complete prior to conjunction. Both acts, placement and conjunction, are thus merged together so that conjunction cannot precede placement, nor is placement possible before conjunction. Or again, speaking in the sense appropriate for the continuous, that which newly emerges does so precisely upon that which has already become, and thus, in that moment of becoming itself, appears in its further course as growing there.

The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can also be regarded as continuous, and the continuous as discrete. The discrete may be regarded as continuous if that conjoined is itself again regarded as given, and the act of conjunction as a moment of becoming. And the continuous can be regarded as discrete if every moment of becoming is regarded as a mere conjunctive act, and that so conjoined as a given for the conjunction." "

and Hegel [...]

Any adjoint modality $\Box \dashv \bigcirc$ that includes the modalities $\varnothing \dashv *$, ie $\varnothing \subset \Box$, $* \subset \bigcirc$, formalizes a more determinate being (Dasein)

109.6 Types

From the adjoint modalities, we can define the various (co)modal types involved.

Definition 109.6.1. An object in a cohesive topos is *concrete* if the unit of the adjunction ($\Gamma \dashv$ CoDisc) is a monomorphism, ie is a $\sharp$-*submodal type*, so that

$$\eta^\sharp_X : X \hookrightarrow \sharp X \tag{109.108}$$

Definition 109.6.2. An object in a cohesive topos is *codiscrete* if the unit of the adjunction ($\Gamma \dashv$ CoDisc) is an isomorphism, ie is a $\sharp$-*modal type*, so that

$$X \cong \sharp X \tag{109.109}$$

This notion of concreteness can be understood in the sense that for an actual topological space, (X, τ), given a coarser topology (X, τ'), we have that the identity function (as sets) lifts to a continuous function on topological spaces, and is therefore an injective map.

Counterexample 109.6.1. For the moduli space of 1-forms, as it has a single point, we have

$$\sharp\Omega \cong 1 \tag{109.110}$$

If it were concrete, for any two morphisms $\omega_1, \omega_2 : \mathbb{R} \to \Omega$, corresponding to two 1-forms on $\mathbb{R}$, we should have

$$! \circ \omega_1 =! \circ \omega_2 \to \omega_1 = \omega_2 \tag{109.111}$$

which is always true for the terminal morphism, but as we have more than one 1-form on $\mathbb{R}$, that cannot be true.

As the name implies, a concrete object in a cohesive topos corresponds to a concrete sheaf :

Theorem 109.6.1. The sharp object $\sharp X$ is a concrete sheaf.

Proof. Its underlying set is simply $|\sharp X| = \Gamma(X)$, and we have [...] Define as V-separated objects?

$$V = \Gamma^{-1}(\mathrm{iso}(\mathbf{Set})) \tag{109.112}$$

□

In terms of what we saw about concrete categories31, the largest possible subcategory of $\mathbf{H}$ that is concrete is the category of all concrete objects. This is because for any functor $L : \mathbf{C} \to \mathbf{D}$ with a right adjoint R, the unit $X \to RLX$ is a monomorphism iff U is faithful on morphisms with target X, so Γ (forgetful functor) is faithful, therefore concrete

Proof. By composing with the unit,

$$\mathrm{Hom}_{\mathbf{C}}(X', X) \to \mathrm{Hom}_{\mathbf{D}}(LX', LX) \cong \mathrm{Hom}_{\mathbf{C}}(X', LRX) \tag{109.113}$$

function is injective etc □

The interpretation of a concrete space is that, if we consider our spaces from both the lens of their points $\Gamma(X)$ and their algebra of subobjects Ω^X, the locale given by Ω^X can be entirely generated by $\Gamma(X)$

$X \hookrightarrow \sharp X \cong \mathrm{CoDisc}(\Gamma(X))$

$X \to X$ is mono obviously, and so on a coarser topology, remains continuous (conserve the mono idk). This is true for every concrete space

$\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$, $\Gamma(X) \cong \mathrm{Hom}(1, X)$

For every subobject $S \hookrightarrow X$, we have a characteristic morphism $\chi_S : X \to \Omega$, and we have the mapping

$$\Gamma(\chi_S) : \Gamma(X) \to \Gamma(\Omega) \tag{109.114}$$

Only makes sense if $\Gamma(\Omega)_{\mathbf{Set}}$?

The submodal functor associated to $\sharp$ is the concretification functor :

Definition 109.6.3. For a cohesive topos, the concretification functor

$$\mathrm{conc} : X \mapsto \mathrm{im}(\eta^{\sharp}_X) \tag{109.115}$$

As a submodal type, we can factor our modality as

$$\mathbf{H} \underset{\iota_{\mathrm{conc}}}{\overset{\mathrm{conc}}{\rightleftarrows}} \mathbf{H}_{\mathrm{conc}} \rightleftarrows \mathbf{Set}$$

Theorem 109.6.2. The subcategory of concrete objects is a quasitopos.

Proof. □

Dually to the concrete and discrete objects, we also have the types associated with the flat modality, the coconcrete and codiscrete types :

Definition 109.6.4. The supcomodal type of the comodality $\flat$ is that of the coconcrete objects, for which the counit is an epimorphism :

$$\flat X \twoheadrightarrow X \tag{109.116}$$

Theorem 109.6.3. For a coconcrete object X, the points are defined entirely by the locale[?] So they can have "not enough points" ???

Proof. □

Definition 109.6.5. If a morphism has a concrete codomain, ie $f : X \to Y$ is such that X is a concrete object, it is called *intensive*.

This is the generalization of the notion used in thermodynamics, where the function is determined entirely by its value at points. We could see this as considering our function as associating values to points

For any subobject $S \hookrightarrow X$, the

Concrete objects and separated presheaves

Copresheaves of algebras?

Definition 109.6.6. An object in a cohesive topos is *discrete* if the counit

$$X \cong \flat X \tag{109.117}$$

From this definition, we have that any discrete object is the image of the set $\Gamma(X)$ via the discrete functor Disc. As a property, this gives us

$$\mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(X)) \cong \mathrm{Hom}_{\mathbf{H}}(\mathrm{Disc}(S), X) \tag{109.118}$$

So that the hom-set of any discrete object $\mathrm{Disc}(S)$ to any object X is the hom-set of all functions from their underlying sets, which is a property similar to that of the discrete topology.

Discrete objects are roughly speaking an inclusion of sets in the topos, in that they have no "cohesion"

Theorem 109.6.4. The closure of any subobject of a discrete object is itself.

Theorem 109.6.5. A discrete object is concrete.

Proof. From the epimorphism of the points to pieces transformation $\flat \to \int$, □

Theorem 109.6.6. Any subobject of a discrete object is discrete[?]

Proof. For it to be discrete, we need to have the property that for any object Z,

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(U), \Gamma(Z)) \cong \mathrm{Hom}_{\mathbf{H}}(U, Z) \tag{109.119}$$

If we have some monomorphism $U \hookrightarrow X$, as we have $X \cong \flat X$, we also have by the monomorphic morphism,

$$\mathrm{Hom}_{\mathbf{H}}(Z, \flat X) \to \mathrm{Hom}_{\mathbf{H}}(Z, U) \tag{109.120}$$

is injective

$$\flat(U \hookrightarrow X) \cong \flat U \hookrightarrow X \tag{109.121}$$

counit : $\flat U \to U$, two out of three property? Composition $\epsilon^{\flat}_U \circ \iota_U$:

$$\flat U \to U \hookrightarrow X \tag{109.122}$$

modality :

$$\flat(\flat U \to U \hookrightarrow X) \cong (\flat U \xrightarrow{\cong} \flat U \hookrightarrow X) \tag{109.123}$$

global section functor is fully faithful on concrete objects, therefore preserves monomorphisms :

$$\Gamma(U) \hookrightarrow \Gamma(X) \tag{109.124}$$

$$\mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(X)) \hookrightarrow \mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(U)) \tag{109.125}$$

□

Definition 109.6.7. For an intensive morphism, we have [some isomorphism idk]

Extensive objects : maximally non-concrete codomain, ie X is

From Lawvere : example of extensive quantity as $M(X) = L_R(R^X, R)$, smooth linear functionals on the ring object R

the integration/end/eval of extensive and intensive quantity :

$$\int_X : R^X \times M(X) \to R \tag{109.126}$$

Does it have to be wrt a ring object?

[102]

Theorem 109.6.7. For a cohesive sheaf topos on a concrete site $\mathbf{C}$, every locally

As $(\int \dashv \flat)$ is an *sp*-unity, the modal type of $\int$ is the same as that of $\flat$, that of the discrete objects. [submodal type?]

109.7 Morphisms

$\sharp$-closed morphism? For some function $f : X \to Y$, with modal map $\sharp f : \sharp X \to \sharp Y$,

$\int$-closed

From the identity 102.4, we have that

$$[X, \sharp Y] = \sharp\,[\flat X, Y] \tag{109.127}$$

$$[X, \flat Y] = \flat\,[\textstyle\int X, Y] \tag{109.128}$$

109.8 Infinitesimal cohesive topos

One particular special case regarding cohesion is the case where the points to pieces transform is an equivalence,

$$\flat \cong \textstyle\int \tag{109.129}$$

In other words, for every connected component of a topos, there is a unique point in that piece. As the left adjoint of a functor being an equivalence implies that its right adjoint also is, this leads to the collapse of the cohesion adjoint triple ($\int \dashv \flat \dashv \sharp$) into a single modality $\natural$

$$\textstyle\int = \mathrm{Disc} \circ \Pi_0 = \mathrm{Disc} \circ \Gamma = \flat \tag{109.130}$$

This ambidextrous modality manifests as a bireflective subcategory,

$$(T_\natural \dashv \iota_\natural \dashv T_\natural) : \mathbf{H}_\natural \cong \mathbf{1} \underset{\xleftarrow{T_\natural}}{\overset{\xleftarrow{T_\natural}}{\hookrightarrow_{\iota_\natural}}} \mathbf{H}_{\mathrm{inf}}$$

As both the inclusion and (co)reflector are ambidextrous, they preserve all limits and colimits, and so does $\natural$. This means in particular that

$$\natural 1 = 1 \tag{109.131}$$

The notion of such spaces being infinitesimal comes from the general idea that infinitesimal spaces in the context of a topos are really just single points with a "halo" of pointless regions around them. This is however not necessarily the general structure of all infinitesimal spaces, as

Beware of overextending this idea however as this is not always an appropriate interpretation. **Set** is such an "infinitesimal cohesive topos", but has no real extension around its points (although it could be understood as the topos over the infinitesimally thickened point of order 0, or the dual to the trivial Weil algebra that is just $\mathbb{R}$, ie just a point). They also include such objects as the dual of non-commutative algebras or some moduli spaces.

This is why another way to interpret them is via the notion of *quality types*.

Definition 109.8.1. A fully faithful functor f^*

"Hence from a more geometrical point of view, an object in a quality type is a particular simple kind of space with 'degenerate' components, or, if you prefer, a space with 'thick' or 'coarse' points which in turn can be viewed as a minimal vestige of cohesion: when a set is a space with no cohesion, an object in a quality type is a space with almost no cohesion."

"The claim that quality types intend to model the philosophical concept of quality reoccurs at several places in Lawvere's writings though the concrete connection still needs to be spelled out. Some motivation is provided in Lawvere (1992) where it is linked to the negation of quantity as a logical category of being that is indifferent to non-being (cf. Hegel's Science of Logic) e.g. whereas the temperature varies continuously or "indifferently" below and above zero degree, the same transition makes a crucial difference for the phase or "qualitative being" of water."

"Intuitively, an intensive quality is compatible with the points of its domain spaces and an extensive quality with the connected components."

Nullstellensatz implies that infinitesimal objects as a category is closed under arbitrary subobjects and is thus epi reflective

If co-reflective as well, the geometric map(?) p (Γ?) factorizes as

$$p = qs \tag{109.132}$$

q has an adjoint string of length two because the left and right adjoints of q^* are isomorphic (see the "classical modality" $\natural$)

s has adjoint length 3 due to a lack of codiscreteness, left adjoint of q doesn't preserve product.

q is the *quality type*, s is the *intensive quality* because it is compatible with right adjoint points functor, while an extensive quality is a map to a quality type compatible with the pieces aspect of p. Example : s is an extensive quality in the category of infinitesimal spaces

[188]

We have seen previously some examples of quality types, such as the universal moduli space of k-forms, Ω^k, the moduli space of Riemannian metrics Met, or the moduli space of symplectic structures ω.

Example of quality type :

109.9 Negation

[165]

109.9.1 Anti-sharp

Being a monad which is a right adjoint, the sharp modality preserves limits, and therefore admits a negation. To find the negation of the sharp modality, we need to find the fiber of its unit.

$$\overline{\sharp}_p X = \mathrm{Fib}_p(X \to \sharp X) \tag{109.133}$$

$$\begin{array}{ccc} \overline{\sharp}_p X & \xrightarrow{!_{\overline{\sharp}X}} & 1 \\ \downarrow & & \downarrow p \\ X & \xrightarrow{\eta^{\sharp}_X} & \sharp X \end{array}$$

$$\overline{\sharp}_p X \cong X +_{\sharp X} 1 \tag{109.134}$$

In terms of products : the equalizer of $X \times 1 \cong X$ by the morphisms $p : 1 \to \sharp X$ (which is equivalent to some point in X by etc), and $\eta^{\sharp}_X$. For a given point p, we are looking for an object which, set-wise, is the

$$\Gamma(p, \eta^{\sharp}_X) = \{(x, \bullet) \in X \times 1 \mid p(\bullet) = \eta^{\sharp}_X(x)\} \tag{109.135}$$

So that we are given an object whose only element is p. As far as the point content of an object goes, this means that every negation of continuity has the same content (a unique point). This is about what we would expect since continuity and its negation would reduce any object to a bare object, $\sharp(\overline{\sharp}(X)) = 1$.

However, there are additional informations left after the negation of the continuity. If we look for instance at an infinitesimal object X, $\sharp X \cong 1$, as the negation will merely be $X \times_1 1 = X$, any infinitesimal object (or "quality type") will be preserved. This means that such objects as infinitesimally thickened points, coarse moduli spaces, etc will remain invariant under this.

Theorem 109.9.1. Any anti-sharp modality is infinitesimal.

Proof. By the definition of the negation, we have

$$\sharp\overline{\sharp}_p X = 1 \tag{109.136}$$

meaning that it only has a single point, $\Gamma(\overline{\sharp}_p X) \cong \{\bullet\}$. □

For a broader example we can look at the case of a space with infinitesimal extension, like the Cahier topos.

Theorem 109.9.2. The negation of the continuity modality on a Cahier topos at a point $p : 1 \to X$ gives its infinitesimally thickened point.

This is the general behavior of the anti-sharp modality, which is to give the "quality" of the space at that point, ie the maximally non-concrete part of that space around that point.

Behaviour on a Q-category?

Theorem 109.9.3. Given a Q-category

$$\mathbf{H} \;\; \begin{array}{c} \xleftarrow{\quad \Pi_0 \quad} \\ \xhookrightarrow{\quad \mathrm{Disc} \quad} \\ \xleftarrow{\quad \Gamma \quad} \end{array} \;\; \mathbf{H}_{\mathrm{th}}$$

where the subcategory $\mathbf{H}$ is cohesive but with trivial infinitesimals, $\mathbf{H}_{\mathrm{inf}} \cong \mathbf{Set}$, and the thickened topos has some nilpotent ideal I, the anti-sharp modality gives us [...]

Proof. □

What about some moduli space, ie some space M for which the internal hom $[X, M]$ is a moduli space of some sort, such that for the point, $[1, M] \cong 1$?

Moduli space defined by some family of objects such that for any object $B \in \mathbf{H}$, there is a surjective morphism $\pi : M \to B$, such that every fiber $M_b = \pi^{-1}(b)$ for $b \in B$ is also in the family of objects.

If M has a single point, it is infinitesimal

[Does the quality type being a quality (at a point) relate to the fact that a fine moduli space cannot have non-trivial automorphisms]

Universal property with this object : for any object Y such that there is a morphism $f : Y \to X$, there exists a unique function $\beta : Y \to \overline{\sharp} X$ such that

Since the exponential functor $(-)^X$ is a right adjoint, we have the preservation of pullbacks (and therefore fibers), so that $(\overline{\sharp}_p Y)^X$ defines the diagram

$$\begin{array}{ccc} (\overline{\sharp}_p Y)^X & \xrightarrow{\quad ! \quad} & 1 \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y^X & \xrightarrow{\quad \eta^{\sharp}_{Y^X} \quad} & \sharp(Y^X) \end{array}$$

Use adjunction with the product?

anti-sharp of a concrete object?

Theorem 109.9.4. The anti-sharp modality of a concrete object is the terminal object[or subterminal?].

Proof. As a concrete object X has a monomorphism to its sharp modality, we have that since pullbacks preserve monomorphisms,

$$\begin{array}{ccc} \overline{\sharp}_p X & \xhookrightarrow{!_{\overline{\sharp}X}} & 1 \\ {\scriptstyle p^*\eta^\sharp_X}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xhookrightarrow[\eta^\sharp_X]{} & \sharp X \end{array}$$

$\overline{\sharp}X$ is therefore subterminal. If furthermore we take a second anti-sharp modality for this, at the adjunct point $\tilde{p}$, we get

$$\begin{array}{ccccc} \overline{\sharp}_{\tilde{p}}\overline{\sharp}_p X & \longrightarrow & \overline{\sharp}_p X & \xhookrightarrow{!_{\overline{\sharp}X}} & 1 \\ {\scriptstyle \tilde{p}^* p^*\eta^\sharp_X}\downarrow & & {\scriptstyle p^*\eta^\sharp_X}\downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\tilde{p}]{} & X & \xhookrightarrow[\eta^\sharp_X]{} & \sharp X \end{array}$$

As a pasting of two pullbacks, this is also a pullback, and the pullback by two terminal objects at the same point [since $\eta^\sharp_X \circ \tilde{p} = p$] is the terminal object, meaning that we have

$$1 \hookrightarrow \overline{\sharp}_p X \hookrightarrow 1 \tag{109.137}$$

so that

$$\overline{\sharp}_p X \cong 1 \tag{109.138}$$

□

If X is not concrete, ie the concretification functor η_{conc} is not an isomorphism, is it guaranteed to not be terminal?

Theorem 109.9.5. For a non-concrete object, the anti-sharp modality is not isomorphic to the terminal object

Proof. If we take the factorization of the sharp unit, as we have $\sharp \mathrm{conc} X \cong \sharp X$, we obtain the diagram

$$\begin{array}{ccccc} \overline{\sharp}_p X & \longrightarrow & \overline{\sharp}_p \mathrm{conc}(X) \cong 1 & \xhookrightarrow{!_{\overline{\sharp}X}} & 1 \\ {\scriptstyle p^*\eta^\sharp_X}\downarrow & & {\scriptstyle p^*\eta^\sharp_{\mathrm{conc}(X)}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \overset{}{\underset{\eta^{\mathrm{conc}}_X}{\twoheadrightarrow}} & \mathrm{conc}(X) & \xhookrightarrow[\eta^\sharp_{\mathrm{conc}(X)}]{} & \sharp X \end{array}$$

where the outer and right square are pullbacks, and therefore so is the left square. As monomorphisms are preserved by pullbacks, if $\overline{\sharp}_p X$ was a subobject of 1, so would be $X \to \mathrm{conc}(X)$, which is not the case as the object is not concrete, so that $\overline{\sharp}_p X$ is not subterminal. □

This hints at the fact that the anti-sharp modality gives us the non-concrete structure of a space at a given point.

Decomposition of a space if $\overline{\sharp}_p X$ is isomorphic for all p?

109.9.2 Anti-flat

The flat modality, $\flat$, being a right adjoint to $\int$, also admits a negation.

$$\overline{\flat} = \mathrm{Cofib}(\flat X \to X) \tag{109.139}$$

with the pullback diagram

$$\begin{array}{ccc} \flat X & \xrightarrow{\epsilon^\flat_X} & X \\ {\scriptstyle !_X}\downarrow & & \downarrow{\scriptstyle !^*_X \epsilon^\flat_X} \\ 1 & \xrightarrow[p]{} & \overline{\flat} X \end{array}$$

Giving us some pointed object $(\epsilon^\flat_X)^* !_X = p : 1 \to \overline{\flat} X$

Furthermore, also being in an *sp*-unity with $\int$, and the map $\flat \to \int$ being an epimorphism by definition of a cohesive topos, it is furthermore a determinate negation, so that we have

$$\int \overline{\flat} X \cong 1 \tag{109.140}$$

So that the anti-flat modality of an object is always connected.

Theorem 109.9.6. The anti-flat modality of a co-concrete object is isomorphic to the terminal object.

As a quotient :

$$\overline{\flat} \cong \mathrm{coeq}(1 \times X \rightrightarrows \flat X) \tag{109.141}$$

$$\overline{\flat} = X / \flat X \tag{109.142}$$

In the case of a topological space, this would be something akin to the quotient of a topological space

$$\bar{\flat} X = \flat X +_X 1 \tag{109.143}$$

As $\flat X$ is already a coproduct, we are considering the space constituted by

$$\coprod_{x \in \Gamma(X) + \{\bullet\}} 1 \tag{109.144}$$

where we quotient out the relation given by $\flat X \to X$. As there is always an element of this coproduct which is such that $!_{\flat X}()$

$$\bar{\flat} X = \mathrm{coeq}(!_{\sharp X}, \eta^{\flat}_X) \tag{109.145}$$

Is it just $\bar{\flat} X \cong 1$??? It should probably include the "lost" cohesion, so only 1 for discrete spaces

Topological example : if $X = \mathbb{R}$ or something, with $\flat\mathbb{R} \to \mathbb{R}$ the surjection of a finer topology to the original, we identify every point of $\flat\mathbb{R}$ with $\mathbb{R}$, singleton topology, only one

If it must preserve anything, it must be non-concrete.

Example 109.9.1. Given the smooth space of k-forms Ω^k, its anti-flat modality is

$$\bar{\flat}\Omega^k = 1 +_{\Omega^k} 1 \tag{109.146}$$

Sum of two infinitesimal spaces?

$$\begin{aligned} \bar{\flat}(X+Y) &= \mathrm{Cofib}(\flat(X+Y) \to (X+Y)) && (109.147)\\ &= \mathrm{Cofib}((\flat X + \flat Y) \to (X+Y)) && (109.148)\\ &= \mathrm{Cofib}(2 \to (X+Y)) && (109.149) \end{aligned}$$

For a concrete object, $X \hookrightarrow \sharp X$, we have the adjunct

$$\flat X \xrightarrow{\flat\eta^{\sharp}_X} \flat\sharp \xrightarrow{\epsilon^{\flat}_X} X \tag{109.150}$$

$$\flat(\textstyle\int X \to Y) \cong \flat(X \to \flat Y) \tag{109.151}$$

109.9.3 Anti-shape

The last negation is that of the shape modality, given by

$$\bar{\textstyle\int}_p X = \mathrm{Fib}_p(\textstyle\int X \to X) \tag{109.152}$$

represented by the pullback diagram

$$\begin{array}{ccc} \bar{\textstyle\int} X & \xrightarrow{!_{\bar{\int} X}} & 1 \\ \downarrow & & \downarrow p \\ X & \xrightarrow{\eta^{\int}_X} & \textstyle\int_p X \end{array}$$

As a left adjoint, this modality is not guaranteed to have a proper negation, in the sense that it is not guaranteed to preserve limits. This will in fact be an issue in the higher categorical case, where the higher categorical version of the shape modality will not preserve homotopy pullbacks.

Theorem 109.9.7. The shape modality admits a negation?

This is the common usage of the fiber of a morphism, so that the anti-shape modality at a point p will give the connected component at that point.

$$\bar{\textstyle\int} X = X \times_{\int X} 1 \tag{109.153}$$

This simply gives us the space back if it is connected, as we would expect. If we have multiple connected components, $X \to \int X$ is essentially a bundle over the discrete space of its points [prove it? Is it epi?], and the choice of point in $\int X$ is simply the choice of the connected component we will consider. In other words, if X is expressed as a coproduct of connected components,

$$X \cong \coprod_{i \in \int X} X_i \tag{109.154}$$

Then for a choice of point $p \in X_i$, we have

$$\bar{\textstyle\int} X \cong X_i \tag{109.155}$$

109.9.4 de Rham modalities

In addition to the negations of our various modalities, cohesion also comes with the notion of *de Rham* flat and shape modalities. Unlike the case of the negation, which is the cofiber of flat and the fiber of shape, it is the fiber of flat and the cofiber of shape :

$$\flat_{\mathrm{dR},p} X = \mathrm{Fib}_p(\flat X \to X) = \flat A / A \tag{109.156}$$

$$\int_{\mathrm{dR}} X = \mathrm{Cofib}(X \to \int X) \tag{109.157}$$

de Rham modality of concrete object?

Properties wrt flat and shape?

$$\flat \flat_{\mathrm{dR},p} X = \tag{109.158}$$

109.9.5 Modal hexagon

$$\begin{aligned} X &\cong \int X \times_{\int \bar{\flat} X} \bar{\flat} X && (109.159) \\ &\cong \int X \times_1 \bar{\flat} X && (109.160) \\ &\cong \int X \times \bar{\flat} X && (109.161) \end{aligned}$$

109.9.6 de Rham modalities

Definition 109.9.1. The de Rham flat modality $\tilde{\flat}_p$

$$\tilde{\flat}_p X = \mathrm{Fib}_p(\epsilon^\flat : \flat X \to X) \tag{109.162}$$

Tangent space???

Definition 109.9.2. The de Rham shape modality $\tilde{\int}$

$$\tilde{\int} X = \mathrm{Cofib}(\eta^{\int} : X \to \int X) \tag{109.163}$$

109.10 Algebra

If the topos is cohesive, the modalities for cohesion are all monoidal.

Theorem 109.10.1. The sharp, flat and shape modalities are monoidal.

Proof. As Γ is both a left and right adjoint, it preserves limits and colimits, and CoDisc as a right adjoint preserves limits.

$$\begin{aligned}
\sharp(X \times Y) &= \mathrm{CoDisc} \circ \Gamma(X \times Y) && (109.164)\\
&= \mathrm{CoDisc}(\Gamma(X) \times \Gamma(Y)) && (109.165)\\
&= \mathrm{CoDisc}(\Gamma(X)) \times \mathrm{CoDisc}(\Gamma(Y))) && (109.166)\\
&= \sharp X \times \sharp Y && (109.167)
\end{aligned}$$

And since Γ and Disc are both left and right adjoints, this follows naturally with a similar proof.

$$\flat(X \times Y) = \flat X \times \flat Y \qquad (109.168)$$

The case for the shape modality follows from the same property of Disc and the preservation of products for Π_0 in a cohesive topos.

$$\int(X \times Y) = \int X \times \int Y \qquad (109.169)$$

□

From 1 :

- $\flat$: possibly discontinuous
- $\int$: Constant on each connected component

109.11 Logic

Every inhabited object of a cohesive topos has at least a point :

$$[X] \cong 1 \leftrightarrow \{\bullet\} \subseteq \mathrm{Hom}(1, X) \qquad (109.170)$$

Nullenst etc

As we saw early on, one basic property of cohesion is that the law of excluded middle applies to varying degrees depending on the modalities involves.

Theorem 109.11.1. For a discrete object, the law of excluded middle holds :

$$U + \neg_X U \cong X \qquad (109.171)$$

Proof. As we have $X \cong \flat X$, where $\flat$ preserves colimits, we have

$$\Gamma(U + \neg_X U) \cong \Gamma(X) \qquad (109.172)$$

□

Definition 109.11.1. A proposition $p : A \hookrightarrow X$ is *discretely true* if in the pullback

$$\begin{array}{ccc} \sharp A\big|_X & \longrightarrow & \sharp A \\ \downarrow & & \downarrow \\ X & \xrightarrow{\eta_X} & \sharp X \end{array}$$

$\sharp A\big|_X \to X$ is an isomorphism

Proposition that is true over discrete spaces.

Theorem 109.11.2. If p

Is there a way to classify truths by their modalities, and is there a semantics of flat "collections are ensembles of parts" v. sharp "collections are wholes"?

What is the appropriate method? If we have some predicate $p[X]$ for some space X, ie a dependent type(?), what is the condition for the predicate being "as a whole" v. "as a collection"?

Predicates of type Y are related to morphisms $p : X \to Y$, its valuation is given by specifying

Is the modality to be applied to the free term of the predicate or the predicate as a whole?

Ex. of part properties : "U is a subobject of X", "the point x has value $f(x)$", "Has a number of elements"

Ex. of whole properties : "X has total volume V", "two magnitudes are congruent"

Relation to intensive v. extensive properties? (this is for $\sharp$ and $\bar{\sharp}$ though $\flat$ isn't involved)

Let's check :

"X has n points" :

$$\Gamma(X) = n \tag{109.173}$$

Replace by :

$$\Gamma(\flat X) = n \tag{109.174}$$

Type :

$$\vdash \tag{109.175}$$

"X has n connected components" :

"X is constant over connected components"

Moduli space???

"X has volume V" : first pick a volume structure vol : $X \to$ Vol, then take the coend idk with the constant function to $1 : 1 \to R$,

$$\int_{x:X} \mathrm{vol} \times 1 \tag{109.176}$$

Careful : example of X having a volume V, this is a proposition which involves another space $\mathbb{R}$

Is it something like $p(\sharp X) \dashv p(X)$

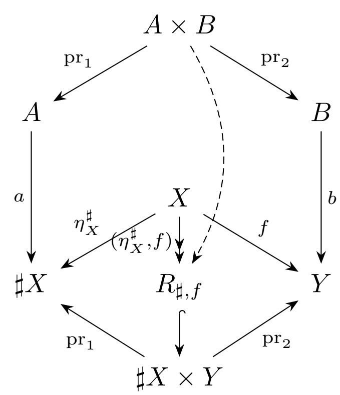
For vol : a volume of X is given by some value $1 \to \mathbb{R}$

Integration map :

$$\int_X : [X, \mathbb{R}] \times [X, \mathbb{R}]^* \quad \to \quad \mathbb{R} \tag{109.177}$$

"the general sheaves for this base level are commonly called "codiscrete" or "chaotic" objects within the big category of Being, and the subtopos of them may be called "pure Becoming". The negative objects for this level are commonly called "discrete" and the subcategory of them deserves to be styled "non Becoming"."

Relations : for a function $f : X \to Y$, the span given by $\sharp$



The relation of f ("a has the image b via f") has an associated relation $R_{\sharp,f}$

This relation is always true for global elements [?]

What does it mean otherwise?

Quillen negation of

109.12 Dimension

In the context of a cohesive topos, we can

Definition 109.12.1. A discrete object $X \cong \flat X$ has dimension zero.

Lebesgue covering dimension? Poincaré's definition?

"Having described a basic framework, we can now return to the question of the intrinsic meaning of "one-dimensionality" of an object within such a framework. The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level. Thus a zero-dimensional space is just a discrete one (there are several answers, not gone into here, to the objection which general topologists may raise to that) and dimension one is the Aufhebung of dimension zero. Because of the special feature of dimension zero of having a components functor to it (usually there is no analogue of that functor in higher dimensions), the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of of components of the skeleton at that dimension of the space, or more pictorially: if two points of any space can be connected by anything, then they can be connected by a curve. Here of course by "curve" we mean any figure in (i.e. map to) the given space whose domain is one-dimensional."

"Continuing, two-dimensional spaces should be those negating a subtopos which itself contains both the one-dimensional spaces and the identical-but-opposite sheaves which the one-dimensional spaces negate."

109.13 Cohesion on sets

Set is trivially a cohesive topos, but it fails to have the stronger property of being *sufficiently cohesive*. If we consider the case where its base topos is itself, then we need to investigate the adjoint cylinder from its functor of global sections, ie

$$(\mathrm{Disc} \dashv \Gamma \dashv \mathrm{CoDisc}) : \mathbf{Set} \underset{\hookleftarrow \ \mathrm{Codisc} \ \to}{\overset{\hookleftarrow \ \mathrm{Disc} \ \to}{\longleftarrow \Gamma \longrightarrow}} \mathbf{Set}$$

The global section functor, as the hom functor from the terminal object, is simply the identity on sets, as sets are entirely defined by their points. As the adjoints of the identity are always the identity, any adjunction string is simply going to be more identity functor. Our adjunction for cohesion is therefore

$$(\mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}} \dashv \mathrm{Id}_{\mathbf{Set}}) \qquad (109.178)$$

And its corresponding monads are just the identity monad.

$$(\mathrm{Id} \dashv \mathrm{Id} \dashv \mathrm{Id}) \tag{109.179}$$

From there, every property of cohesion is fulfilled simply by every limit and colimit being preserved by the identity functor and every unit and counit being appropriately mono or epi, as they are always isomorphisms.

The behaviour of Disc is about what we would expect for discrete objects, sending an object to the coproduct of terminal objects over its point content, in other words a set with the same cardinality. However, the codiscrete objects, being the same, does not seem to follow what we would expect of a codiscrete object, of being "one whole" in some sense, although they do obey the more decisive property of every function on its points lifting to a continuous function. Connected components are likewise about what we would expect since every point is its own component in the "topology" of a set. We have that every piece (a single point) has a point, and every point has its piece, and that this is a sublation of the ground by $\sharp 0 = 0$. That pieces of powers are powers of pieces is obviously true again due to the identity :

$$\Pi_0(X^{\mathrm{Disc}(S)}) = (\Pi_0 X)^S = X^S \tag{109.180}$$

The issue that prevents it from being a more intuitive cohesive topos is that of the connectedness of the subobject classifier, 2, as trivially, $\Pi_0 2 = 2$, and not 1 as we would need to. This stems from an obstruction found in [189], saying that a localic topos cannot both have a shape modality over sets that preserves the product and have a connected subobject classifier. [proof?]

This means that we cannot embed any object into some connected or contractible space, and in fact, there is no connected space outside of 1, meaning that its topology is fairly uninteresting as we would guess. In other words, there's no equivalence of our own "standard" Euclidian space of a dimension superior to 0, some simply connected space that can contain multiple disjoint objects.

As we've seen in the case of **Set**, there are two possible Lawvere-Tierney closure operators we could try over sets, and the $\mathrm{loc}_{\neg\neg}$ closure is the discrete topology, in the sense that every subset $S \subseteq X$ is its own closure, including singletons $\{\bullet\}$. No point is "in contact" with another, we can entirely separate a given element from the whole. The other closure operator is the trivial one $j(x) = 1$, giving us the *chaotic* topology on 1.

Equivalence between the lawvere-tierney topology and Grothendieck topology, chaotic grothendieck topology, collapse of sets into triviality?

As $\neg\neg = \mathrm{Id}_\Omega$, the localization does not do anything and the smallest dense subtopos is simply itself, there are no levels in between Set and the ground.

The global section functor $\Gamma(-) \cong \mathrm{Hom}_{\mathbf{Set}}(1, -)$ is simply the identity, as

$$\Gamma(S) \cong \mathrm{Hom}_{\mathbf{Set}}(1, S) \cong S \tag{109.181}$$

Since the hom-set of the point to a set is of the same cardinality as the set itself, and

$$\Gamma(f) \tag{109.182}$$

The left adjoint functor Disc here will work out as

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Disc}(-), -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, -) \tag{109.183}$$

while the right adjoint is

$$\mathrm{Hom}_{\mathbf{Set}}(-, -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{109.184}$$

As far as the objects go, these are identities, so that the discrete and codiscrete objects of sets are the same objects (they have the same point content). However, the actions on morphisms does change, and most importantly, on the Lawvere-Tierney topology of the topos. For the morphism $j : \Omega \to \Omega$, $j \in \mathrm{Hom}_{\mathbf{Set}}(2, 2)$, we have

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(-), -) \cong \mathrm{Hom}_{\mathbf{Set}}(-, \mathrm{CoDisc}(-)) \tag{109.185}$$

The left adjoint modality $\flat \dashv \sharp$

"Note that in this example, the “global sections” functor $S \to Set$ is not the forgetful functor $Set/U \to Set$ (which doesn't even preserve the terminal object), but the exponential functor $\Pi U = \mathrm{Hom}(U, -)$. This is the direct image functor in the geometric morphism $Set/U \to Set$, whereas the obvious forgetful functor is the left adjoint to the inverse image functor that exhibits S as a locally connected topos."

Negation of sharp modality :

$$\begin{aligned} \bar{\sharp} X &= X \sqcup_{\sharp X} * & (109.186) \\ &= X \sqcup_{\sharp X} * & (109.187) \end{aligned}$$

Is it just $*$? The negation of the moment of continuity (maximally non-concrete object)

From the definition of the cohesion, we have that every sheaf of **Set** is in fact infinitesimal, but only in the trivial sense. While they do obey $\flat \cong \int$, this is only due to the pieces being actual points without any "infinitesimal extension" to them.

109.14 Cohesion of simplicial sets

Simplicial sets can be seen as the simplest model of a ”space” that has non-trivial cohesion, as there is a notion of ”cohesion” between points as instantiated by the higher simplices between them.

The global section functor for simplicial sets is simply given by the 0-simplices.

$$\Gamma(X) = \mathrm{Hom}_{\mathbf{sSet}}(1, X) = X_0 \tag{109.188}$$

Theorem 109.14.1. The discrete simplicial sets $\mathrm{Disc}(S)$ are the simplicial sets whose only non-degenerate simplices are those of dimension zero.

Proof. The left adjoint of Γ is as usual the set-tensoring functor, so that

$$\mathrm{Disc}(X) = \coprod_{i\in\Gamma(X)} 1 \tag{109.189}$$

$$= \coprod_{i\in X_0} 1 \tag{109.190}$$

Disjoint sum of simplices etc

□

In other words, the discrete simplicial sets are simply a collection of disjoint points, as we would expect from a discrete space.

Theorem 109.14.2. The codiscrete simplicial sets CoDisc are the simplicial sets which have as degree k-faces the set of $(k+1)$-tuples

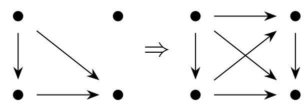
Proof. For a simplicial set X and a set S,

$$\mathrm{Hom}_{\mathbf{H}}(X, \mathrm{CoDisc}(S)) = \mathrm{Hom}_{\mathbf{Set}}(\Gamma(X), S) \tag{109.191}$$

which means that for every function between the vertices of X and S, there is a corresponding morphism. In particular, for simplices Δ_n, we have that there is a face map For the face map of n, we have a corresponding hom-set $\Gamma(X) \times S$, ie every point is connected to every other point (including itself) Likewise for degeneracy map? □

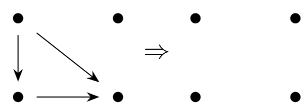
In other words, the codiscrete simplicial sets are the *maximally connected simplicial set* of that set. Every pair of point is connected by a line, every trio by a face, etc. This is also the appropriate intuition we would have for a ”codiscrete” combinatorial space, where every point is ”connected to” every other point.

The sharp modality is therefore the monad that will ”complete” a simplicial set to its maximally connected version.



(109.192)

while the flat modality will forget all cohesion and simply return a set of disconnected points.



(109.193)

Theorem 109.14.3. The connected component functor is given by the set of vertices X_0 quotiented by the relation that if any two vertices are connected by an edge, they belong to the same component, given as the coequalizer of X_0 by the degeneracy maps of each point of an edge

$$X_1 \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} X_0 \longrightarrow \Pi_0(X)$$

This simply means that the connected components are the components of a graph as far as the vertices and edges are concerned. The shape modality is then simply the discrete space generated from those components.

$$\int X = \tag{109.194}$$

As the codiscrete space of a simplicial set is its maximally connected version, any simplicial set with at most one edge between two vertices will be concrete

[prove equivalence]

Any non-concrete simplicial sets will therefore correspond to pairs of vertices connected by more than one edge.

The submodal types are given, for the sharp modality, by the concrete simplicial sets, which are

Theorem 109.14.4. A concrete simplicial set is a simplicial set for which every pair of points has at most one edge.

Co-concrete? $\flat X \to X$ is an epimorphism, ie a surjection on every component $\eta^\flat_{X,n}$:.

Theorem 109.14.5. The infinitesimal objects of the simplicial topos are the degenerate simplices of a single point and more than one edge between the two.

Proof. If we take some simplicial set with $X_0 = \{\bullet\}$ and $|X_1| \geq$, then we have

$$\flat X = \int X = \Delta_0 \tag{109.195}$$

□

The simplest non-trivial case of this is to consider some single vertex with n edges to itself. This is the simplicial circle for $n = 1$, and additional such edges will make it the simplicial version of a bouquet of circles.

Theorem 109.14.6. The bouquet of circles is not a concrete object.

Proof. If we take this bouquet of circles, we have

$$\begin{aligned} \sharp X &= \mathrm{Codisc} \circ \Gamma(X) && (109.196) \\ &= \mathrm{Codisc}(\{\bullet\}) && (109.197) \\ &= 1 && (109.198) \end{aligned}$$

As we have $|X_1| = n$, it cannot be a subsheaf of the terminal object. □

Morphisms into the bouquet object?

Is it an interesting moduli object? $[X, S_n]$:

$$[X, S_n]_n = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta[\vec{\mathbf{n}}], S_n) \tag{109.199}$$

X_0 is the hom set X, S_n

109.15 Cohesion of a spatial topos

Spatial topos always has disconnected truth values hence not a sufficiently cohesive topos cf Lawvere

109.16 Cohesion of smooth spaces

As a Grothendieck topos, the cohesiveness of smooth spaces relies on the cohesiveness of its site, the category of Cartesian spaces.

First, **CartSp** has the terminal object ($\mathbb{R}^0$), and every object $\mathbb{R}^n$ admits global sections (at least $0 : \mathbb{R}^0 \to \mathbb{R}^n$). **CartSp** is also cosifted in that it has finite products, as we have

$$\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2} \tag{109.200}$$

We therefore just have to prove that it is a locally connected site with its topology of differentiably good open covers, ie for any object $X \in \mathbf{CartSp}$, any covering sieve on X is a connected subcategory of $\mathbf{CartSp}_{/X}$.

Theorem 109.16.1. **Smooth** is sufficiently connected.

Proof. □

[...]

As a cohesive site, **CartSp** has the global section functor Γ which simply assigns to each open set of $\mathbb{R}^n$ its set of points

$$\Gamma(U_{\mathbf{CartSp}}) = U_{\mathbf{Set}} \tag{109.201}$$

corresponding to the $\mathbb{R}^0$ plots of diffeological spaces, and as its left adjoint the functor $\Gamma^* \cong \mathrm{LConst} \cong \mathrm{Disc}$ which associate to any set the smooth space composed by an equinumerous number of disconnected copies of $\mathbb{R}^0$,

$$\mathrm{Disc}(X) = \coprod_{x \in X} \mathbb{R}^0 \tag{109.202}$$

This is equivalently the fine diffeology on a set, given by a diffeology where the only plots are the locally constant maps. This does respect the adjunction as every function from a discrete space to a smooth space is a smooth map, since [...]

Its right adjoint is given by the coarse diffeology on a set

Right adjunction :

$$\mathrm{Hom}_{\mathbf{Set}}(\Gamma(U), X) = \mathrm{Hom}_{\mathbf{CartSp}}(U, \mathrm{CoDisc}(X)) \tag{109.203}$$

Every possible function (as a set) between some Cartesian space U and our codiscrete space X correspond to a valid plot.

In terms of intuition, this means that every point is "next to" every other point in some sense. For instance, given the probe $[0, 1]$, there is a smooth curve for every possible combination of points, ie the set of smooth curves is just $X^{[0,1]}$.

There is therefore no meaningful way to separate points (as we would expect from the smooth equivalent of the trivial topology).

Given these two functors, we can construct our two modalities. Our flat modality is

$$\flat = \mathrm{Disc} \circ \Gamma \tag{109.204}$$

which first maps a smooth space to its points, and then to the coproduct of $\mathbb{R}^0$ over those points, giving us the discrete space $\flat X$, switching the space X to its finest diffeology.

The sharp modality $\sharp$ is

$$\flat = \text{Codisc} \circ \Gamma \tag{109.205}$$

which maps a smooth space to its points and then to the

Are all smooth spaces in the Eilenberg category diffeological spaces?

Theorem 109.16.2. Given a concrete smooth space[necessary?], $X \hookrightarrow \sharp X$, its sharp modality is the coarse diffeology on the underlying set.

Proof. Given a smooth space X, its sharp modality $\sharp X$

$$\sharp X(U) = \tag{109.206}$$

□

Theorem 109.16.3. Given a concrete smooth space[necessary?], $X \hookrightarrow \sharp X$, its flat modality is the fine diffeology.

"Every topological space X is equipped with the continuous diffeology for which the plots are the continuous maps."

Law of excluded middle? For a subobject $S \hookrightarrow X$

$$\neg_X () \tag{109.207}$$

109.16.1 Extensive and intensive quantities

109.17 Cohesion of classical mechanics

As a slice topos of the topos of smooth sets, the topos of classical mechanics $\mathbf{H}_{/\omega}$ is trivially cohesive.

109.18 Cohesion of quantum mechanics

There will not be any obvious cohesive structure on the usual category of quantum mechanics, as the category of Hilbert spaces is not a topos, but we can try to define such a structure on the Bohr topos of a theory.

The terminal object in the Bohr topos is given by the constant sheaf which maps all commutative operators in a context to a spectrum of a single element, the singleton $\{\bullet\}$ in **Set**. This is the spectral presheaf which has a spectrum of a single value for all

Global section functor : hom-set of

$$\Gamma : \mathbb{C}^0 \to \tag{109.208}$$

110
Elastic substance

A further sublation we can get is that of $(\int \dashv \flat)$, the opposition given by

$$(\int \dashv \flat) : \mathbf{H}_\sharp \underset{\Gamma}{\overset{\Pi_0}{\leftarrow}} \hookrightarrow_{\text{Disc}} \mathbf{H}$$

As $(\int \dashv \flat)$ is an *sp*-unity, any sublation of it will be automatically a left and right sublation. We will call this sublation $(\Im \dashv \&)$, and from the properties described, we will have

$$\Im\int \cong \int \quad (110.1)$$
$$\&\flat \cong \flat \quad (110.2)$$
$$\&\int \cong \int \quad (110.3)$$
$$\Im\flat \cong \flat \quad (110.4)$$

Those modalities will therefore not act on any space contracted to its connected components or that has lost its cohesion. In other words, it does not change discrete types. They are therefore in some sense modalities about the continuous structure of a space.

[190, 166]

From the factorization of unities of opposites, we have the

$$X \to \Im X \to \int X \quad (110.5)$$

There is some notion of a *partial* contraction of the entire space's components. If we are considering in particular the A^1-cohesion that we saw, this is a partial contraction of paths,

$$R \to \Im R \to \int R \cong 1 \tag{110.6}$$

None of the topos we have seen thus far are differentially cohesive, except in a trivial sense (all their objects are reduced, $\Re(X) = X$), but it is simple enough to extend them to be. This is generally done concretely by changing the site to include an infinitesimal structure. The basic example for this is the site of *formal Cartesian spaces*, **FormalCartSp**, which is the site with objects being open sets of $\mathbb{R}^n$ composed with an *infinitesimally thickened point*, D. As we will see, D is a point if we forget the elastic structure, $\Re(D) \cong *$, but in basic mathematical terms, this relates to Weil algebras, the algebras of infinitesimal objects, as seen in 67.

Definition 110.0.1. The category of formal Cartesian spaces **FormalCartSp** is the full subcategory of smooth locus $\mathbb{L}$ of the form

$$\ell A \cong \mathbb{R}^n \rtimes D \tag{110.7}$$

[Open subset $D(f) = \mathrm{Spec}(A) \setminus V(f)$, $V(f)$ all prime ideals containing f, then $D(f)$ is empty if f is nilpotent?]

Definition 110.0.2. The *Cahier topos* **Cahier** is the sheaf topos over the site of formal Cartesian spaces :

$$\mathbf{Cahier} = \mathrm{Sh}(\mathbf{FormalCartSp}) \tag{110.8}$$

with the coverage

Theorem 110.0.1. The Cahier topos is cohesive

Proof. □

[...]

As an *sp*-unity, we can construct the essential subtopos $\mathbf{H}_{\Re}$, the "reduced" topos

$$(\Im \dashv \&) : \mathbf{H}_{\Re} \underset{\xleftarrow{T_{\&}}}{\overset{\xleftarrow{T_{\Im}}}{\xrightarrow{\iota_{\&} \cong \iota_{\Im}}}} \mathbf{H}$$

In terms of essential geometric morphisms, this means that

$$i^! := T_{\&} \tag{110.9}$$
$$i_* := \iota \tag{110.10}$$
$$i^* = T_{\Im} \tag{110.11}$$

It is common to write out the reduced topos as the basic one, ie $\mathbf{H}_{\mathfrak{R}} := \mathbf{H}$, and the topmost topos here as an "augmented" topos, the infinitesimally thickened topos $\mathbf{H}_{\text{th}}$. Furthermore, by analogy with the cohesive case, we will call the inclusion ι the infinitesimal discrete functor Disc_{inf}, and the reflectors $T_{\mathfrak{I}}$ and $T_{\&}$ as the infinitesimal connected component functor Π_{inf} and the infinitesimal global section functor Γ_{inf}

$$\begin{aligned} \mathfrak{I} &= \text{Disc}_{\text{inf}} \circ \Pi_{\text{inf}} && (110.12) \\ \& &= \text{Disc}_{\text{inf}} \circ \Gamma_{\text{inf}} && (110.13) \end{aligned}$$

Subtopos	Geometric	Inf.
$T_{\mathfrak{I}}$	i^*	Π_{inf}
ι	i_*	Disc_{inf}
$T_{\&}$	$i^!$	Γ_{inf}

Table 110.1: Notations

The reduced subtopos, unlike the boolean subtopos, does not seem to have any easy definition in terms of its own properties, so that we will have to look at some of its models. But first we can certainly deduce a few properties from its raw definition.

First, this reduced subtopos is the middle step in between the elastic topos and the cohesive subtopos, ie we have the following hierarchy

$$\mathbf{H}_{\sharp} \;\underset{\xleftarrow{\;T_{\&}\;}}{\overset{\xleftarrow{\;T_{\mathfrak{I}}\;}}{\xhookrightarrow{\;\iota_{\&} \cong \iota_{\mathfrak{I}}\;}}}\; \mathbf{H}_{\mathfrak{R}} \;\underset{\xleftarrow{\;T_{\&}\;}}{\overset{\xleftarrow{\;T_{\mathfrak{I}}\;}}{\xhookrightarrow{\;\iota_{\&} \cong \iota_{\mathfrak{I}}\;}}}\; \mathbf{H}$$

Chain the functors?

Theorem 110.0.2. If $\Pi_{\text{inf}} X \cong 1$, then X is infinitesimal.

Proof. As we have that $\Gamma_{\text{inf}} 1 \cong 1$ [proof? Due to $\&1 \cong \&\flat 1 \cong \flat 1 \cong 1$]

$$\begin{aligned} \Gamma(X) &= \text{Hom}(1, X) && (110.14) \\ &= \text{Hom}(T_{\&} 1, X) && (110.15) \\ &= \text{Hom}(1, \Pi_{\text{inf}} X) && (110.16) \\ &= \text{Hom}(1, 1) && (110.17) \\ &= 1 && (110.18) \end{aligned}$$

So that X only has a single point, and likewise, a single piece. □

This corroborates the interpretation of Π_{inf} as the contraction of infinitesimal paths.

Jet comonad

Definition 110.0.3. For an object X, given the unit of the infinitesimal shape modality

$$\eta^{\Im} : X \to \Im X \tag{110.19}$$

inducing the base change functors

$$\mathbf{H}_{/X} \underset{\eta^{\Im *}}{\overset{\eta^{\Im}_*}{\rightleftarrows}} \mathbf{H}_{\Im X} \tag{110.20}$$

Then the *jet comonad* at X Jet_X is the map given by their composition

$$\mathrm{Jet}_X = \eta^{\Im^*} \eta^{\Im}_* : \mathbf{H}_{/X} \to \mathbf{H}_{/X} \tag{110.21}$$

Left adjoint of jet comonad : forming infinitesimal disk bundle monad

Example 110.0.1. The main interpretation of the jet comonad in our context is that of its action on bundles for smooth spaces. Given some bundle $\pi : E \to X$ in $\mathbf{H}_{/X}$, the action of the jet comonad is to send this bundle to its ∞ jet bundle.

$$\mathrm{Jet}_X(\pi) = J^\infty \pi \tag{110.22}$$

Theorem 110.0.3.

The topos from that site is the *Cahier topos*,

$$\mathbf{Cahier} = \mathrm{Sh}(\mathbf{FormalCartSp}) \tag{110.23}$$

[191, 192, 193]

Example 110.0.2. As **Set** was already only trivially cohesive, so too is it only trivially differentially cohesive. We can understand the differential cohesion of sets as simply being from the trivial Weil algebra, so that our underlying site is just $\mathbf{1} \rtimes \{0\}$. Any set is therefore its own reduced element

Theorem 110.0.4.

$$\int(\Re X \to X) \tag{110.24}$$

is an equivalence

Proof. □

Definition 110.0.4. In a Cartesian closed category, we say that an object Δ is *infinitesimal atomic* if the exponential object functor

$$(-)^\Delta : \mathbf{H} \to \mathbf{H} \tag{110.25}$$

has a right adjoint $(-)^{1/\Delta}$:

$$\mathrm{Hom}_{\mathbf{H}}(X^\Delta, Y) = \mathrm{Hom}_{\mathbf{H}}(X, Y^{1/\Delta}) \tag{110.26}$$

Definition 110.0.5. An object is a *tiny object*

Theorem 110.0.5. Any infinitesimal atomic object is tiny.

Proof. As the exponential $(-)^\Delta$ is a left adjoint, it preserves all colimits. □

Therefore $(-)^\Delta$ preserves colimits.

[194]

[Is there a nuance on the difference between set-wise (exponential) infinitesimal and synthetic infinitesimal as seen in the difference between Cahier topos infinitesimal v. hyperreal infinitesimal topos, where the infinitesimal region has (maybe) more than one point? Also infinitesimal Sierpinski topos?]

110.1 Synthetic infinitesimal geometry

[195] [196, 197, 198, 199, 200, 196, 201, 202, 203]

Probably the most common model of differential cohesion in topos theory is the one given by synthetic differential geometry.

Definition 110.1.1. On a topos T with a ring object R, if there exists an object D such that

$$D = \{x \in R \mid x^2 = 0\} \tag{110.27}$$

for which the canonical map

$$\begin{aligned} R \times R &\to R^D && (110.28)\\ (x, d) &\mapsto (\varepsilon \mapsto x + \varepsilon d) && (110.29) \end{aligned}$$

is an isomorphism of objects.

Koch-Lawvere axioms

This axiom cannot work properly in the context of a boolean internal logic. Otherwise, if we picked two different "points" in D, given a function $g : D \to R$, we would have

$$x \tag{110.30}$$

110.2 Relative cohesion

Relative shape modality :

$$\int^{\mathrm{rel}} = \int X +_{\Re X} X \tag{110.31}$$

Relative flat modality :

$$\flat^{\mathrm{rel}} X = \flat X \times_{\Im X} X \tag{110.32}$$

$$(\int^{\mathrm{rel}} \dashv \flat^{\mathrm{rel}}) \tag{110.33}$$

(co)modal types : infinitesimal types (such that $\flat \to \Im$) is an equivalence

Relative shape preserves the terminal object

Relative sharp modality $\sharp^{\mathrm{rel}}$

110.3 Non-standard analysis

While a perfectly good model for differential structures in a topos, there are other alternative models to synthetic differential geometry outside of formal spaces.

As a space, infinitesimally thickened points are entirely structureless. Even beyond the single point, it also has a completely trivial mereology, so that its only differentiating aspect from a normal point is given by its sheaf structure. Unlike actual points, there are multiple possible maps $X \to D$, including the core $\mathrm{Aut}(D)$

[Due to the nilpotence of the algebra?]

An alternative to this is to have invertible infinitesimals in the given algebra. There are a variety of models of such infinitesimals, such as the hyperreals, the generalized numbers or the asymptotic numbers

110.4 Differential cohesion of the Cahier topos

[204]

The various topos we have seen previously will not typically be models of differential cohesion, except in a trivial way, that is, where every object is its own reduced object

$$\Re X \cong X \tag{110.34}$$

The most common model used to define a differential cohesion in a ”geometric” way is to use the Weil algebras we saw earlier67. To generalize this infinitesimally thickened point to a full on infinitesimal geometry, we will use, in addition to the Weil algebra, the smooth R-algebra associated with smooth spaces.

[...]

This gives us a sheaf topos on the site of formal Cartesian space **FormalCartSp**, where every object is the product of a Cartesian space with a Weil algebra :

$$\mathbb{R}^n \times \ell W \tag{110.35}$$

Theorem 110.4.1. Isomorphism

$$C^\infty(M, W) \cong C^\infty(M) \otimes W \tag{110.36}$$

via

$$(fw)(p) = f(p)x \tag{110.37}$$

[Is the ”infinitesimally thickened point”, ie an infinitesimal neighbourhood, guaranteed to be infinitesimal in the sense that $\flat \cong \int$, or is that specific to the Cahier topos, diff. with NSA]

110.5 The elastic Sierpinski topos

The simplest non-trivial case of an elastic topos is given by the infinitesimally thickened Sierpienski site, which is the presheaf on the total order $\Delta[\vec{\mathbf{2}}] = 0 \to 1 \to 2$. This is the non-spatial frame we saw in 45.0.1.

$$0 \longrightarrow 1 \longrightarrow 2$$

Theorem 110.5.1. The presheaves on $\Delta[\vec{\mathbf{2}}]$ correspond to sequences of sets $A_2 \to A_1 \to A_0$.

As a differential cohesion, we need to look at its Q-category, which is simply its pairing with the Sierpienski topos as a cohesive category.

$$(i_! \dashv i^* \dashv i_* \dashv i^!) : \mathrm{PSh}(\mathbf{\Delta}[\vec{\mathbf{3}}]) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \\ \xrightarrow{H} \end{array} \mathrm{PSh}(\mathbf{\Delta}[\vec{\mathbf{2}}])$$

In this context, the semantics we can give it is that 0 corresponds to a point, 1 correspond to an infinitesimally close point, and 2 is another point.

As an infinitesimal thickening, it requires

Theorem 110.5.2. The infinitesimally thickened Sierpinski topos forms a Q-category with the Sierpinski topos.

Proof. □

110.6 Crystalline cohomology

110.7 The standard limit

The derivative on a differential cohesive topos can be defined with the use of the jet comonad, but does this match up with the usual definition of derivatives with limits?

Function : map from the space to a ring object of the topos

Value at a point : stalk, net over that point

Standard definition of a limit : using the limit of the undercategory of monomorphisms over a point? Connected to the differential way by doing the same thing over a thickened point?

110.8 Negation

Interaction of elastic modalities with negations of cohesion?

$$\Im\overline{\sharp} \tag{110.38}$$

Colimit commutes with left adjoint :

$$\begin{aligned}
\Im\overline{\flat}X &= \Im\mathrm{Cofib}(\flat X \to X) && (110.39)\\
&= \mathrm{Cofib}(\Im\flat X \to \Im X) && (110.40)\\
&= \mathrm{Cofib}(\flat X \to \Im X) && (110.41)\\
&= \mathrm{Cofib}(\flat\Im X \to \Im X) && (110.42)\\
& && (110.43)
\end{aligned}$$

110.9 Interpretation

Does the notion of finite set get involved here

111

Solid substance

Given the opposition $\mathfrak{R} \dashv \mathfrak{I}$, a *ps*-unity, we seek another sublation, ie we want a further opposition $(\mathfrak{R}' \dashv \mathfrak{I}')$ obeying

$$\mathfrak{R}'\mathfrak{R} = \mathfrak{R} \quad (111.1)$$
$$\mathfrak{I}'\mathfrak{I} = \mathfrak{I} \quad (111.2)$$

We will look here specifically at a left sublation, so that furthermore

$$\mathfrak{I}'\mathfrak{R} = \mathfrak{R} \quad (111.3)$$

We will call the opposition of this sublation the opposition of *solidity*, given by the bosonic modality $\rightsquigarrow$ and the rheonomic modality Rh,

$$(\rightsquigarrow \dashv \mathrm{Rh}) \quad (111.4)$$

where the sublation gives us the identities

$$\rightsquigarrow \mathfrak{R} = \mathfrak{R} \quad (111.5)$$
$$\mathrm{Rh}\mathfrak{I} = \mathfrak{I} \quad (111.6)$$
$$\mathrm{Rh}\mathfrak{R} = \mathfrak{R} \quad (111.7)$$

Meaning that those modalities have no effect on a reduced space and that the rheonomic modality has no effect on an infinitesimal space.

$$\Re X \to \overset{\rightsquigarrow}{X} \to X \tag{111.8}$$

Partial reduction of the quality type[?], from entirely reduced to only reduced as far as the infinitesimal component goes

$$x \tag{111.9}$$

[205]

Bosonic v. fermionic spaces

Definition 111.0.1. A k Grassmann algebra on a k-vector space is an algebra defined by the anticommuting product

$$x \wedge y = -y \wedge x \tag{111.10}$$

As this is a non-commuting algebra, the interpretation of its dual as a space is much more complicated. There is no obvious way to define a sober space from a non-commutative algebra[206], nor a locale[207], nor a sheaf[208], so that the geometric interpretation of that dual should be taken cautiously.

This dual space is the superpoint :

Definition 111.0.2. The *superpoint* is the formal dual to the Grassmann algebra [...]

What is its lattice structure

The general naming scheme of the opposition of solidity is meant to reflect Hegel's talking points on light and matter, not from the Science of Logic but Hegel's Encyclopaedia of the Philosophical Sciences [209].

"Light behaves as a general identity, initially in this determination of diversity, or the determination by the understanding of the moment of totality, then to concrete matter as an external and other entity, as to darkening. This contact and external darkening of the one by the other is colour."

meant to mirror the parallel that light relates to the bosonic moment as it is a bosonic particle while matter is of a fermionic nature.

This is probably the least convincing parallel in the hierarchy of nlab's objective logic here, as while the rigidity of matter can be connected to its fermionic nature, this is not something that can be deduced simply from the raw properties of fermionic spaces and is very much dependent on the model of physics that we use, which is a much more empirical fact than purely metaphysical. It is probably just included as solidity for purely aesthetic reasons, much like the I Ching notation for the classification of cohesive toposx

112 Other stuff

complex analytic cohesion

$$\mathbb{C}\mathrm{Analytic}\infty\mathbf{Grpd} = \mathrm{Sh}_\infty(\mathbb{C}\mathrm{Man}) \tag{112.1}$$

Pointed arithmetic homotopy types

113

Interpretation

The construction of the objective logic is sort of the reverse process presented here. Rather than providing opposites to consider for existing categories, the goal is more to start with very simple ones and construct more complex ones to resolve their oppositions.

As the first step of this construction, the notion of Being (initial topos)

what is the "negation" of being, looking at the "diversity" (collection of morphisms minus the isomorphisms), creating also the initial topos for nothingness?

For any topos, being as the localization by all morphisms? Being simply means having any possible relation with any other object?

Interpretation as arrow category?

Nothingness :

Diversity in $\mathbf{1}$: functor category with $*$?

One and many for sets : opposition involving the terminal object? Would this process work for any other object

Related to the families of open sets on the topology in the resolution? isomorphic to $[1, \Omega]$ and $[X, \Omega]$?

Does it relate to the lower topos? The "one" is $\circledast X$, the "many" is X. Relationship to $\boxslash$ for the opposition? See also determinate being, being for self, being for other stuff

One v. many [something and other?] using the notion of cospan and partition? Partition using the subobject classifier?

Part X

Higher order objective logic

[210]

We have done everything thus far in the context of category theory, but a generalization of this can be done in the context of ∞-categories. This will be useful as the most interesting parts of physics work best in the context of category theory where we use the full ∞-category context to account for gauge groups in a more natural way.

Most concepts are substituted easily enough. The domain of discourse is given by some ∞-topos, and the moments upon it are given by ∞-monads.

One of those construction is the notion of *accidence* : we say that a moment $\bigcirc$ is exhibited by a type J if $\bigcirc$ is a J-homotopy localization :

$$\bigcirc \cong \mathrm{loc}_J \tag{113.1}$$

which implies trivially that the object J has none of the qualities of $\bigcirc$:

$$\bigcirc J \cong * \tag{113.2}$$

In essence, a moment is exhibited by J if it contains none of the qualities of J [?]

Is there some property such that if $\Delta J = J$, then $\bigcirc \Delta X = 1$ or something

Homotopy localization : for an object $A \in \mathbf{C}$, take the class of morphisms W_A

$$X \times (A \overset{\exists !}{\to} *) : X \times A \xrightarrow{p_1} X \tag{113.3}$$

what means

"The idea is that if A is, or is regarded as, an interval object, then "geometric" left homotopies between morphisms $X \to Y$ are, or would be, given by morphisms out of $X \times A$, and hence forcing the projections $X \times A \to X$ to be equivalences means forcing all morphisms to be homotopy invariant with respect to A."

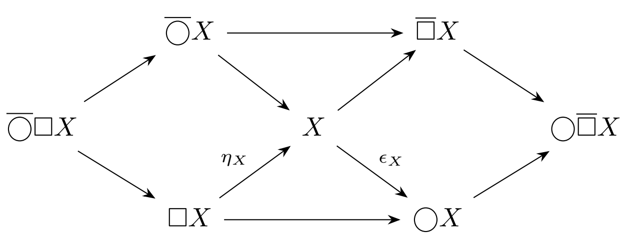
Example using the real line for homotopy localization?

114

Negation

Determinate negation : also has to restrict to 0-types?

An aspect of note for higher order objective logic is the relations given by its adjunction and their determinate negations, $(\Box, \bigcirc, \overline{\Box}, \overline{\bigcirc})$. As we've seen in the 1-categorical case, various applications of this quadruple of modalities gives us the following commuting diagram in the case of an *sp*-unity 114.1



where both squares are pullbacks. In the ∞-categorical case however, we naturally replace the notion of determinate negation from the (co)fiber of the (co)modality to the homotopy (co)fiber of the ∞-(co)modality.

$$\overline{\bigcirc}X \cong \mathrm{HoFib}_p(X \to \bigcirc X) \quad (114.1)$$
$$\overline{\Box}X \cong \mathrm{HoCofib}(\Box X \to X) \quad (114.2)$$

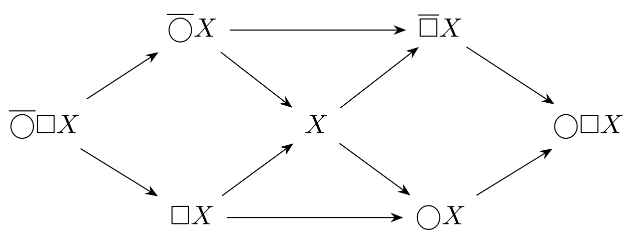
Fracturing for determinate negation, see the hexagonal diagram[165]

A special case of the application of negation in the ∞-categorical case is that for a stable object,

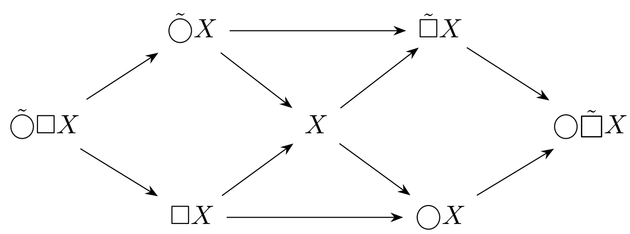
Theorem 114.0.1. For a stable object X, any pushout square is also a pullback square.

From this property, we get

Theorem 114.0.2. For an *sp*-unity $(\bigcirc \dashv \Box)$ on a stable ∞-category $\mathrm{Stab}(\mathbf{H})$, the modal hexagon is equivalently



or



and has the property that

- both squares are homotopy pullbacks and pushouts
- the boundary sequences are long homotopy fiber sequences

Proof. □

This allows us to do the decomposition of a stable type along one of those way :

$$X \cong (\bigcirc X) \underset{\bigcirc\overline{\Box}X}{\oplus} (\overline{\Box}X) \tag{114.3}$$

$$X \cong (\overline{\bigcirc}X) \underset{\overline{\bigcirc}\Box X}{\oplus} (\Box X) \tag{114.4}$$

$$X \cong (\bigcirc X) \underset{\bigcirc\tilde{\Box}X}{\oplus} (\tilde{\Box}X) \tag{114.5}$$

$$X \cong (\tilde{\bigcirc}X) \underset{\tilde{\bigcirc}\Box X}{\oplus} (\Box X) \tag{114.6}$$

114.1 Modal hexagons

In the case of an ∞-category, given a pointed object $p : 1 \to X$, we have the following long fiber sequence for some morphism $f : X \to Y$:

$$\ldots \to \Omega \mathrm{Fib}_p(f) \to \Omega X \to \Omega Y \to \mathrm{Fib}_p(f) \to X \to Y \tag{114.7}$$

This means that for a modality $\bigcirc$, we have the sequence given by its unit,

$$\ldots \to \Omega \overline{\bigcirc}_p X \to \Omega X \to \Omega \bigcirc X \to \overline{\bigcirc}_p X \to X \to \bigcirc X \tag{114.8}$$

If furthermore, X admits a delooping $\mathbf{B}X$, we can look at its sequence, using the identity $X \cong \Omega(\mathbf{B}X)$ [and $\mathbf{B}$ commutes with the fiber??? Limits commute with limits, $\mathbf{B}X \cong 1 \times^h_X 1$]

$$\ldots \to \overline{\bigcirc}_p X \to X \to \bigcirc X \to \overline{\bigcirc}_p(\mathbf{B}X) \to \mathbf{B}X \to \bigcirc \mathbf{B}X \tag{114.9}$$

Likewise, for cofibers, we have the long fiber sequence

$$X \to Y \to \mathrm{Cofib}(f) \to \Omega X \to \Omega Y \to \Omega \mathrm{Cofib}(f) \to \ldots \tag{114.10}$$

$$\Box X \to X \to \overline{\Box} X \to \Omega \Box X \to \Omega X \to \Omega \overline{\Box} X \to \ldots \tag{114.11}$$

$$\Box \mathbf{B}X \to \mathbf{B}X \to \overline{\Box}\mathbf{B}X \to \Box X \to X \to \overline{\Box} X \to \ldots \tag{114.12}$$

From what we have seen, given an *sp*-unity ($\bigcirc \dashv \Box$), we have the following diagrams, for the modality

$$\overline{\bigcirc}_p X \xrightarrow{p^* \eta^{\bigcirc}_X} X \xrightarrow{\eta^{\bigcirc}_X} \bigcirc X$$

and the comodality

$$\Box X \xrightarrow{\epsilon^{\Box}_X} X \xrightarrow{!^*_X \epsilon^{\Box}_X} \overline{\Box} X$$

We can furthermore apply the modality and its negation to the comodality, and using the property of *sp*-unities, $\bigcirc\Box \cong \Box$

[REDO ALL]

[Pullback square : apply the negation of modality to the comodality fibration?]

Given the ptp transform, its naturality square with respect to $\epsilon^{\overline{\bigcirc}}$

$$\begin{array}{ccccc}
\overline{\bigcirc}_p \Box X & \xrightarrow{\epsilon^{\overline{\bigcirc}}(\epsilon^{\Box}_X)} & \overline{\bigcirc}_p X & \xrightarrow{\epsilon^{\overline{\bigcirc}}(\eta^{\bigcirc}_X)} & \overline{\bigcirc}_p \bigcirc X \cong 1 \\
{\scriptstyle \epsilon^{\overline{\bigcirc}_p}_{\Box X}}\downarrow & & \downarrow{\scriptstyle \epsilon^{\overline{\bigcirc}_p}_X} & & \downarrow{\scriptstyle \epsilon^{\overline{\bigcirc}_p}_{\bigcirc X}} \\
\Box X & \xrightarrow[\epsilon^{\Box}_X]{} & X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X
\end{array}$$

or, replacing some of those morphisms,

$$\begin{array}{ccccc}
\overline{\bigcirc}_p \Box X & \xrightarrow{\epsilon^{\overline{\bigcirc}}(\epsilon^{\Box}_X)} & \overline{\bigcirc}_p X & \xrightarrow{!} & 1 \\
{\scriptstyle p^* \eta^{\bigcirc}_{\Box X}}\downarrow & & \downarrow{\scriptstyle p^* \eta^{\bigcirc}_X} & & \downarrow{\scriptstyle p} \\
\Box X & \xrightarrow[\epsilon^{\Box}_X]{} & X & \xrightarrow[\eta^{\bigcirc}_X]{} & \bigcirc X
\end{array}$$

From 100.0.8, we have that

$$\eta^{\bigcirc}_{\Box X} \cong \eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \tag{114.13}$$

so that we can see that both the right square and the whole square of this diagram are pullbacks, and therefore so is the left square by the pullback pasting law.

The naturality of the ptp transform with respect to $\eta^{\overline{\Box}}$ is given by

$$\begin{array}{ccccc}
\Box X & \xrightarrow{\epsilon^{\Box}_X} & X & \xrightarrow{\eta^{\bigcirc}_X} & \bigcirc X \\
{\scriptstyle \eta^{\overline{\Box}}_{\Box X}}\downarrow & & \downarrow{\scriptstyle \eta^{\overline{\Box}}_X} & & \downarrow{\scriptstyle \eta^{\overline{\Box}}_{\bigcirc X}} \\
1 \cong \overline{\Box}\Box X & \xrightarrow[\eta^{\overline{\Box}}(\epsilon^{\Box}_X)]{} & \overline{\Box} X & \xrightarrow[\eta^{\overline{\Box}}(\eta^{\bigcirc}_X)]{} & \overline{\Box} \bigcirc X
\end{array}$$

$$\begin{array}{ccccc}
\Box X & \xrightarrow{\epsilon^{\Box}_X} & X & \xrightarrow{\eta^{\bigcirc}_X} & \bigcirc X \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \eta^{\overline{\Box}}_X} & & \downarrow{\scriptstyle \eta^{\overline{\Box}}_{\bigcirc X}} \\
1 & \xrightarrow[\eta^{\overline{\Box}}(\epsilon^{\Box}_X)]{} & \overline{\Box} X & \xrightarrow[\eta^{\overline{\Box}}(\eta^{\bigcirc}_X)]{} & \overline{\Box} \bigcirc X
\end{array}$$

Naturality for comodality fibration

$$\begin{array}{ccccc}
\Box X & \xrightarrow{\epsilon^{\Box}_X} & X & \xrightarrow{!^*_X \epsilon^{\Box}_X} & \overline{\Box} X \\
{\scriptstyle \eta^{\bigcirc}_{\Box X}}\downarrow & & \downarrow{\scriptstyle \eta^{\bigcirc}_X} & & \downarrow{\scriptstyle \eta^{\bigcirc}_{\overline{\Box} X}} \\
\bigcirc \Box X & \xrightarrow[\eta^{\bigcirc}(\epsilon^{\Box}_X)]{} & \bigcirc X & \longrightarrow & \bigcirc \overline{\Box} X
\end{array}$$

Do $\bigcirc$ and $\overline{\Box}$ commute?

Right whiskering of $\eta^{\bigcirc}_X$?

So that this sequence can be connected to the triangle given by X, $\Box X$ and $\bigcirc X$ commute.

If we apply the negation of the modality to the cofibration $\Box X \to X \to \overline{\Box} X$, we get

$$\overline{\bigcirc}_p \Box X \to \overline{\bigcirc}_p X \to \overline{\bigcirc}_p \overline{\Box} X \cong \overline{\Box} X \tag{114.14}$$

Prove that

$$p^* \eta^{\bigcirc}_X \circ \overline{\bigcirc}_p (\epsilon^{\Box}_X) \cong p^* \eta^{\bigcirc}_{\Box X} \circ \epsilon^{\Box}_X \tag{114.15}$$

so that we have the square $\overline{\bigcirc}_p \Box X, \overline{\bigcirc}_p X, \Box X, X$

Furthermore, this square is a pullback. If we consider the pullback squares

$$\begin{array}{ccccc}
\overline{\bigcirc}_p \Box X & \xrightarrow{\overline{\bigcirc}_p(\epsilon^{\Box}_X)} & \overline{\bigcirc}_p X & \xrightarrow{!_{\overline{\bigcirc}_p X}} & 1 \\
{\scriptstyle p^* \eta^{\bigcirc}_{\Box X}} \downarrow & & \downarrow {\scriptstyle p^* \eta^{\bigcirc}_X} & & \downarrow {\scriptstyle p} \\
\Box X & \xrightarrow{\epsilon^{\Box}_X} & X & \xrightarrow{\eta^{\bigcirc}_X} & \bigcirc \Box X \cong \Box X
\end{array}$$

[show that it's commutative : top arrow is just !, bottom arrow is

$$\eta^{\bigcirc}_X \circ \epsilon^{\Box}_X \cong \eta^{\bigcirc}_{\Box X} \tag{114.16}$$

this is just the fiber diagram of $\Box \to \bigcirc \Box$]

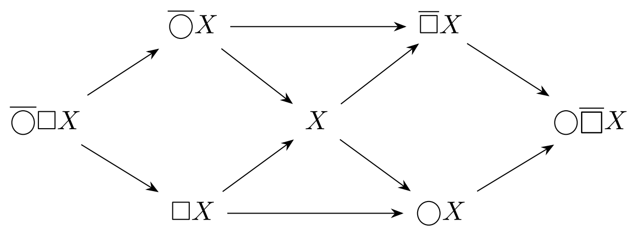
By the pasting law for pullbacks, this means that the left square is a pullback.

Other pullback :

$$\begin{array}{ccccc}
X & \xrightarrow{\overline{\bigcirc}_p(\epsilon^{\Box}_X)} & \overline{\Box} X & \xrightarrow{!_{\overline{\bigcirc}_p X}} & 1 \\
{\scriptstyle p^* \eta^{\bigcirc}_{\Box X}} \downarrow & & \downarrow {\scriptstyle p^* \eta^{\bigcirc}_X} & & \downarrow {\scriptstyle p} \\
\bigcirc X & \xrightarrow{\epsilon^{\Box}_X} & \bigcirc \overline{\Box} X & \xrightarrow{\eta^{\bigcirc}_X} & \Box X \cong \bigcirc \Box X
\end{array}$$

[...]

This means that given an *sp*-unity with determinate negation, we have the following commutative diagram :



where both squares are pullbacks. This is the *modal hexagon.* This means that we have

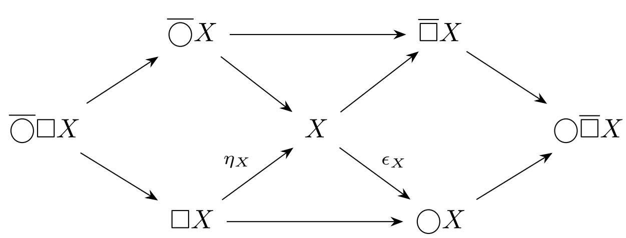
$$X \cong \square X \times_{\overline{\bigcirc}\square X} \overline{\bigcirc} X \tag{114.17}$$

In particular, if $\square$ is entirely a $\bigcirc$ moment, so that

"However, this fiber depends on a chosen basepoint, so it only makes sense on types which have only one constituent (but possibly this constituent has (higher) equalities), or, thought of homotopically, have only one connected component. In this case,$\bigcirc T$ contains that part of the structure of T that is trivialized by $T \to \bigcirc T$. Note that $\square$ and $\overline{\square}$ (or the dual notions) do not form a unity of oppositions. However, for each object T , a sequence $\square T \to T \to \overline{\square} T$ exists and this sequence decomposes each T , in the sense that T could be reconstructed from its aspects under a moment and its negative, as well as their relation. This is not generally true for unities of oppositions."

Is there a relation between $\overline{\square}$ and $\bigcirc$, and $\overline{\bigcirc}$ and $\square$, for both *ps* and *sp* unities?

A useful identity that we will look at later on is given for *sp*-unities ($\bigcirc \dashv \square$) by the following diagram



While the full utility of this diagram comes from the use of higher categories, it remains however valid in the 1-categorical case

Proof : $\bigcirc\overline{\square}X$ is simply 1

Is $\overline{\bigcirc}\square X \cong 1$?

$$\begin{aligned} \overline{\bigcirc}\Box X &= \mathrm{Fib}(\epsilon_{\Box X} : \Box X \to \bigcirc\Box X) && (114.18)\\ &= \mathrm{Fib}(\epsilon_{\Box X} : \Box X \to \Box X) && (114.19)\\ & && (114.20) \end{aligned}$$

Determinate negation as completion/residue v. modalities as localizations/torsions [152]?

For the case of a *ps*-unity $(\Box \dashv \bigcirc)$:

Like the case of the *sp*-unity, we need to look at the modalities of the various fibrations. As usual, we have the two basic diagrams

$$\overline{\bigcirc}_p X \xrightarrow{p^*\eta_X^{\bigcirc}} X \xrightarrow{\eta_X^{\bigcirc}} \bigcirc X$$

$$\Box X \xrightarrow{\epsilon_X^{\Box}} X \xrightarrow{!_X^*\epsilon_X^{\Box}} \overline{\Box} X$$

Example 114.1.1.

Modal hexagon as completion-torsion-localization-residual x

[211]

115

Being and nothingness

In the case of being and nothingness, the main change is simply to replace the terminal topos with the terminal ∞-topos, which is the topos defined by a single k-morphism for every level k connecting level $k-1$ morphisms to themselves. There is not a lot to say on this level, as most everything related to this unity is roughly similar to the 1-categorical case.

Theorem 115.0.1. The ground opposition is given by the adjoint pair of ∞ modalities $\Box_\varnothing \dashv \bigcirc_*$, which map objects of the ∞-topos to the initial and terminal object.

Proof. $\square$

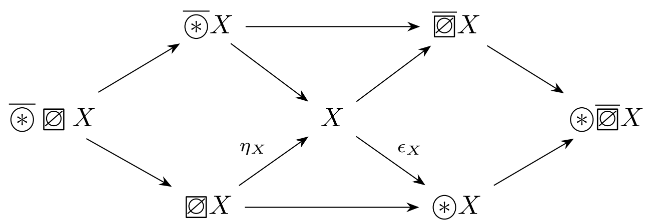
Exhibited by the terminal object?

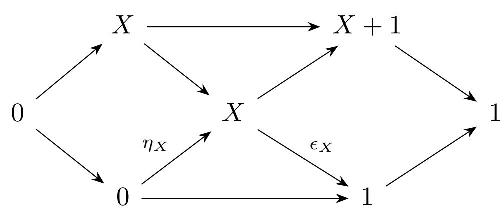
[Negation of the nothing modality?]

Theorem 115.0.2. The negation of the unit modality is the identity.

Proof. Identical to other case? $\square$

Modal hexagon





The next level is given by the localization by the double negation.

116
Cohesion

[212]

Cohesion is the first level of the hierarchy for which the higher morphisms start to be of importance.

Is the double negation subtopos of an ∞-topos also boolean?

As in the 1-categorical case, we consider the category of (∞-)sheaves over the terminal object as our base object, the ∞-category of ∞-groupoids,

$$\mathrm{Sh}_\infty(\mathbf{1}) \cong \infty\mathbf{Grpd} \tag{116.1}$$

where we pick as our forgetful functor the global section functor mapping between every two objects their groupoid, the hom-groupoid of the terminal groupoid.

$$\Gamma = \mathrm{Hom}_{\mathbf{H}}(1, -) \tag{116.2}$$

Example 116.0.1. Consider a topological space (in some topological topos like **Smooth**), with higher morphisms the homotopy transformations between lower morphisms given by the interval object of the real line (the A^1 homotopy).

The global section functor of such a space is the set of its points, with the endomorphisms on those points being the homotopy equivalence classes between those points, and all higher morphisms likewise being higher homotopy equivalences.

For instance, given the circle S^1, we have

$$\tau_0\Gamma(S^1) = |S^1| \tag{116.3}$$

For a point $p \in |S^1|$, its core is the equivalence class of paths from the interval, which for a circle is classified by the winding number :

$$\mathrm{core}(p) = \mathbb{Z} \tag{116.4}$$

As the higher homotopies of the circle are all trivial, any higher k-morphism is simply given by the identity.

As with before, we take the left adjoint functor of locally constant ∞-stacks LConst, which given a groupoid $\mathcal{G} \in \infty\mathbf{Grpd}$ associates the ∞-stack of

$$\mathrm{LConst}(X) \cong \mathrm{Hom}_{\mathbf{H}}(X, \mathrm{LConst}(?)) \tag{116.5}$$

This adjunction always exists for any ∞-topos (?), giving us the flat ∞-modality $\flat$:

$$\flat = \mathrm{LConst} \circ \Gamma \tag{116.6}$$

which first maps the object to its bare ∞-groupoid and then back to our ∞-topos, but importantly, unlike the 1-categorical case, while the (local) spatial information is lost, its *homotopical information*, global spatial information, is preserved, at least for each point. While we would not be able to reconstruct the underlying space(?), we still have all the equivalence classes of loop spaces up to homotopy preserved.

Example 116.0.2. For our circle in the topos of smooth ∞-groupoids, its flat modality will be given by some object which is the coproduct of a single point with the core $\mathbb{Z}$ on each point.

Definition 116.0.1. If Γ admits a right adjoint functor, we will call it the codiscrete functor CoDisc. A topos with this right adjoint is ∞-locally local.

Definition 116.0.2. If Disc admits a left adjoint functor, we will call it the path ∞-groupoid functor, Π. A topos with this left adjoint is called ∞-locally connected.

Definition 116.0.3. If it exists, the left adjoint to Disc $\cong$ LConst is the *path ∞-groupoid* functor,

$$\Pi : \mathbf{H} \to \infty\mathbf{Grpd} \tag{116.7}$$

Properties :

$$\mathrm{Hom}_{\mathbf{H}}(X, \mathrm{Disc}(G)) \cong \mathrm{Hom}_{\infty\mathbf{Grpd}}(\Pi(X), G) \tag{116.8}$$

The groupoid morphisms between the path ∞-groupoid $\Pi(X)$ and any other groupoid G is isomorphic to the hom-groupoid of that space with the discrete groupoid G

The modality Π maps the space given to its homotopy ∞-group. This is an expression of the homotopy groups at a base point being all isomorphic in a path-connected component, ie if $\int X = 1$ [?].

If we look at the 0-truncation, this is simply the same as the connected component functor of the 1-categorical case.

1-morphisms on a connected component : Given a morphism on the stack, ie on some U we have some non-trivial morphisms on an element of $X(U)$,

For cohesion, our subtopos will be some boolean ∞-category, typically the ∞**Grpd** category. As in the 1-categorical case, any ∞-stack ∞-topos admits a canonical geometric morphism and a left adjoint for it,

$$(\mathrm{LConst} \dashv \Gamma) : \mathbf{H} \underset{\mathrm{LConst}}{\overset{\Gamma}{\rightleftarrows}} \infty\mathbf{Grpd} \tag{116.9}$$

As Π_0 refers specifically to the fundamental group, ie the first homotopy group of the space, the higher order case will be Π, the full fundamental groupoid of the space.

Example 116.0.3. On the circle S^1, defined by some smooth 0-type sheaf $S^1(\mathbb{R}^1) = 3$ with the usual good cover on a circle,

[213]

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc}) : \mathbf{H} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\mathrm{Codisc}} \end{array} \mathbf{Grp}_\infty$$

Theorem 116.0.1. Shape and flat preserve quotients by some G-action :

$$\int(X /\!/ G) \cong (\int X) /\!/ (\int G) \tag{116.10}$$

$$\flat(X /\!/ G) \cong (\flat X) /\!/ (\flat G) \tag{116.11}$$

116.1 Truncated cohesion

In addition to the cohesion itself, we can investigate what the various truncations of the topos give us.

Inclusion :

$$\tau_{\leq n}\mathbf{H} \hookrightarrow \mathbf{H} \tag{116.12}$$

Reflective subcategory :

$$(\tau_{\leq n} \dashv \iota_n) : \tau_{\leq n}\mathbf{H} \underset{\iota_n}{\overset{\tau_{\leq n}}{\rightleftarrows}} \mathbf{H}$$

116.2 Real cohesion

An extra condition to emulate the most common of spaces is that of real cohesion, saying that the shape modality is exhibited by the localization of the affine line object. This is a property which is specific to the homotopical case, as the notion of localization by an object is mostly relevant there.

In many cases, we will ask that the shape modality correspond to a retraction to a point of the (path)-connected components of the space. Using the usual notion of path connectedness, two points are path connected if there is a path $[0,1] \to X$ between them, in which case the space is contractible if for any two points, there is a path between them.

The categorical equivalent is to consider the path space object I, and localization by I, typically $\mathbb{R}$.

$$\int \cong \mathrm{loc}_I \tag{116.13}$$

[210]

116.3 Negations

In the ∞-categorical case, the negation conveys much more information than before, as the

Theorem 116.3.1. The negation of the shape modality $\bar{\int}_p$ is the ∞-universal cover of the connected component at p.

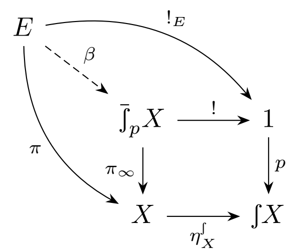
Proof. From the negation of the shape modality, the object obtained by this modality gives a fibration sequence

$$\bar{\int}_p X \longrightarrow X \longrightarrow \int X \tag{116.14}$$

such that $\bar{\int}_p X$ is an ∞-connected object (since $\int\bar{\int}_p X \cong 1$ by the properties of the negation, meaning all fundamental groups are trivia). By the universal property of pullbacks, if there is another ∞-covering of X,

$$\pi : E \to X \tag{116.15}$$

with the additional map $!_E : E \to 1$, we have a unique map $\beta : E \to \bar{\int}_p X$



Case of a covering space? ie with product fiber $\mathrm{Fib}_p(\pi) = F$ for all p □

We define the fiber map $p^* \eta^{\int}_X$ by the ∞-*covering map* of X, π :

$$\pi_p : \bar{\int}_p X \to X \tag{116.16}$$

and we say that $\bar{\int}_p X$ is the universal ∞-cover of X.

If X is a connected object, the homotopy pullback will be independent of the chosen point, so that we can write it as $\bar{\int}$:

$$\pi : \bar{\int} X \to X \tag{116.17}$$

Example 116.3.1. The universal ∞-cover of S^1 is the real line :

$$\bar{\int} S^1 \cong R \tag{116.18}$$

Proof. As we can define the circle as the homotopy quotient of the real line and the integers

$$S^1 = R /\!/ \mathbb{Z} \tag{116.19}$$

$$\bar{\int} S^1 = \mathrm{HoFib}(\eta^{\int} : S^1 \to \mathbf{B}\mathbb{Z}) \tag{116.20}$$

□

Definition 116.3.1. $\bar{\flat}$ is the infinitesimal remainder of the object[?]

$$\bar{\flat} X \cong X /\!/ \flat X \tag{116.21}$$

Theorem 116.3.2. Lie algebroid???

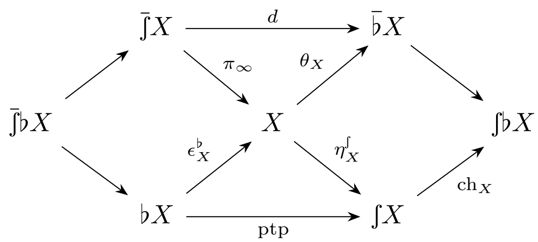
Example 116.3.2. The infinitesimal remainder of the real line R

[dR-shape : universal cover?]

[dR-flat : infinitesimal structure around all points?]

116.4 Cohesive hexagon

As the unity of opposites ($\int \dashv \flat$) is an *sp*-unity, for any object X admitting a delooping $\mathbf{B}X$, we can look at its modal hexagon



In addition to our usual maps for the lower triangle,

ch_X is the *Chern character* of the object X

$$\mathrm{ch}_X = \Pi(\theta_X) \tag{116.22}$$

de Rham modalities

Poincaré lemma : $\flat\mathbb{R} \cong \Omega^\bullet$??? Apply to fracture square

Example 116.4.1. For the line object R, being a group object, its delooping $\mathbf{B}R$ is the classifying space

Example 116.4.2. For the circle S^1, also known as the group object $\mathrm{U}(1)$, its loop space is homotopy equivalent to the group $\mathbb{Z}$, and its delooping is $\mathbb{Z}$.

117
Singular cohesion

In addition to the usual poset of modalities we saw previously, the ∞-categorical case also has the case of singular cohesion, meant to model the notion of some geometric singularities.

Given the cohesive structure on an ∞-topos,

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc}) : \mathbf{H} \underset{\overset{\longleftarrow \mathrm{Codisc} \hookleftarrow}{\longrightarrow \Gamma \longrightarrow}}{\overset{\overset{\longrightarrow \Pi \longrightarrow}{\longleftarrow \mathrm{Disc} \hookrightarrow}}{}} \mathbf{Grp}_\infty$$

We can consider additionally the

[214, 215]

118
Elasticity

119 Solidity

Exhibited by the odd line

Part XI

Nature

120

Mechanics topos

The general topos typically used by nlab to respond to the requirements of all those modalities is the *super formal smooth* ∞*-groupoid*. This is the ∞-sheaf on a special site composed from the category of Cartesian spaces (to give the cohesion), the category of infinitesimally thickened points (to give the elasticity), and the category of superpoints (to give the solidity).

We have already seen the category of Cartesian spaces in detail, so let's now look into infinitesimally thickened points.

Infinitesimally thickened points are a geometrical realization of the formal notion of infinitesimals as provided by Weil algebras.

$$(x, \epsilon) \mapsto f(x, \epsilon) = f(x) \tag{120.1}$$

$$\mathbf{SuperFormalSmooth} = \mathrm{Sh}() \tag{120.2}$$

[...]

Lawvere's topos : nlab

[216]

Using the shape modality for defining constant values, ie Noether's theorem? Lagrangian as object of a cohomology

Zero variation : constant on the infinitesimal neighbourhood of the configuration space? Using $\Im$ for "constant on infinitesimal neighbourhood" ?

121
Gauge theory

In a real cohesive topos, $\int$ will correspond to the path ∞-groupoid of an object. As we've seen, the path groupoid of an object lacks too much informations to give us the full informations of a connection, but as it identifies any two homotopic paths, it is however good enough for the space of flat connections.

$$\mathrm{Hom}(\int X, A) \tag{121.1}$$

Assignment to every point of X a fiber in A, to every path an equivalence of those fibers, ...

[217]

122

Geometry

[218]

The confluence of infinitesimal geometry, higher order morphisms and cohesion makes this system a good setting in which to work in the general notion of a geometrical theory of physics, such as general relativity or other such theories.

Infinitesimal disk :

$$D_p^X = 1 \times_{\Im(X)} X \tag{122.1}$$

Order k

$$D_{p,k}^X = 1 \times_{\Im(X)} \Im_{k)} X \tag{122.2}$$

[219]

Theorem 122.0.1. In $\mathbf{FormalSmooth}_\infty$, The automorphism group of the formal k-dimensional disk D_k is isomorphic to the jet group $\mathrm{GL}_k(n)$

Proof. Given the Weil algebra of the formal disk,

$$D_k^n = \{x \in \mathbb{R}^n \mid x^k = 0\} \tag{122.3}$$

Any automorphism $\phi : D_k^n \to D_k^n$ will have the action, by microlinearity

$$\phi(x) = \phi(0) + cx \tag{122.4}$$

Automorphism in a category of rings : maps 0 to 0? Linear? idk Inverse operation :

$$\phi^{-1}(cx) = cc^{-1}x = x \tag{122.5}$$

Is the automorphism in the dual category isomorphic to the automorphism in the original category [190] □

In particular, the first order jet group is simply the general linear group,

$$\mathrm{GL}(n) = \mathrm{Aut}(D^n) \tag{122.6}$$

We will define the jet group $\mathrm{GL}_k(V)$ in general to be the automorphism group

Definition 122.0.1. The *jet group* of the typical infinitesimal disk of an object X is

$$\mathrm{GL}(X) = \mathrm{Aut}(D_k^X) \tag{122.7}$$

Definition 122.0.2. The ∞-jet bundle of a smooth groupoid is given by the action of the jet comonad

$$J^\infty X = \mathrm{Jet}(X) \tag{122.8}$$

Definition 122.0.3. The k-th jet bundle is the object given by the action of the k-th order jet comonad, where

In particular, given the line object $\mathbb{R} \in \mathbf{FormalSmooth}_\infty$, the tangent bundle is the first jet bundle of the bundle given by the product projection. ie for the product $\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$, the tangent bundle of M is given by

$$T\mathbb{R} \times TM = \mathrm{Jet}(\mathrm{pr}_1) \tag{122.9}$$

Alternative viewpoint : the line $\mathbb{R}$ is reduced to the linelet D^1_1, $\pi : M \to$

Frame bundle : For

123 Quantization

Quantization via linearization of a classical theory

Definition 123.0.1. On a Cartesian monoidal ∞-category $\mathbf{H}$

$$\mathrm{Mod}(X) \tag{123.1}$$

124
Formal ontology

While the examples from physics can be useful, perhaps a more basic description of ontology would be more in line with the original spirit of the objective logic.

there exists a few attempts at formal descriptions of general ontologies

[...]

Topos for objects simply being assemblies of properties : products of discrete sites, or at least finite sites?

Consider a finite site S of elements $\{P_A, P_B, P_C, \ldots\}$. The only covering families on S are simply either the empty coverage or $\{A \hookrightarrow A\}$, giving a total of $2^{|S|}$ possible coverages.

A presheaf $\mathrm{PSh}(S)$

$$F(P_i) = \{\bullet, \bullet, \ldots\} \tag{124.1}$$

In other words, the sheaf associates to each property P_i some cardinality.

As the category is discrete, no need to concern about

Part XII

Other stuff

125
The Ausdehnungslehre

One early attempt at mathematization of similar philosophical notions was the Ausdehnungslehre of Grassmann[6]

[220, 221, 222]

"Each particular existent brought to be by thought can come about in one of two ways, either through a simple act of generation or through a twofold act of placement and conjunction. That arising in the first way is the continuous form, or magnitude in the narrow sense, while that arising in the second way is the discrete or conjunctive form.

The simple act of becoming yields the continuous form. For the discrete form, that posited for conjunction is of course also produced by thought, but for the act of conjunction it appears as given; and the structure produced from the givens as the discrete form is a mere correlative thought. The concept of continuous becoming is more easily grasped if one first treats it by analogy with the more familiar discrete mode of emergence. Thus since in continuous generation what has already become is always retained in that correlative thought together with the newly emerging at the moment of its emergence, so by analogy one discerns in the concept of the continuous form a twofold act of placement and conjunction, but in this case the two are united in a single act, and thus proceed together as an indivisible unit. Thus, of the two parts of the conjunction (temporarily retaining this expression for the sake of the analogy), the one has already become, but the other newly emerges at the moment of conjunction itself, and thus is not already complete prior to conjunction. Both acts, placement and conjunction, are thus merged together so that conjunction cannot precede placement, nor is placement possible before conjunction. Or again, speaking in the sense appropriate for the continuous, that which newly emerges does so precisely upon that which has already become, and thus, in that moment of becoming itself, appears in its further course as growing there.

The opposition between the discrete and the continuous is (as with all true oppositions) fluid, since the discrete can also be regarded as continuous, and the continuous as discrete. The discrete may be regarded as continuous if that conjoined is itself again regarded as given, and the act of conjunction as a moment of becoming. And the continuous can be regarded as discrete if every moment of becoming is regarded as a mere conjunctive act, and that so conjoined as a given for the conjunction."

126
Kant's categories

[223]

Quantity : unity, plurality, totality

Quality : reality, negation, limitation

Relation : inherence and subsistence, causality and dependence, community

Modality : possibility, actuality, necessity

127

Lauter einsen

Cantor's original attempt at set theory[224] involved the notion of *aggregates* (*Mengen*), which is what we would call a sequence today, some ordered association of various objects. If we have objects $a, b, c, \ldots$, their aggregate M is denoted by

$$M = \{a, b, c, \ldots\} \tag{127.1}$$

Despite the notation reminiscent of sets, the order matters here. The notion closer to that of a set is given by an abstraction process $\overline{M}$, in which the order of elements is abstracted away (this would be something akin to an equivalence class under permutation nowadays).

The *cardinality* of the aggregate is given by a further abstraction, $\overline{\overline{M}}$, given by removing the nature of all of its element, leaving only "units" behind :

$$\overline{\overline{M}} = \{\bullet, \bullet, \bullet, \ldots\} \tag{127.2}$$

$\bullet$ here is an object for which all characteristics have been removed, and all instances of $\bullet$ are identical. In some sense this is the application of the being modality on its objects : we only have as its property that the object exists, similarly to *das eins*.

Comment from Zermelo [225, p. 351]:

"The attempt to explain the abstraction process leading to the "cardinal number" by conceiving the cardinal number as a "set made up of nothing but ones" was not a successful one. For if the ”ones” are all different from one another, as they must be, then they are nothing more than the elements of a newly introduced set that is equivalent to the first one, and we have not made any progress in the abstraction that is now required.”

Relation to the discrete/continuous modality

From Lawvere : The maps between the *Menge* and the *Kardinal* is the adjunction

$$(\text{discrete} \dashv \text{points}) : M \underset{\text{discrete}}{\overset{\text{points}}{\rightleftarrows}} K \tag{127.3}$$

The functor points maps all elements of an object in M to the ”bag of points” of the cardinal in K, the functor discrete sends back

[226]

From Iamblichus :

these, we say, are three fingers, the smallest, the next, and the middle one – consider me as intending them as seen from nearby. But please consider this about them: each of them appears to be equally a finger, I take it, and they are in that way in no way different, whether seen as being in the middle or at the end, whether pale or dark, fat or thin, and so on, for all similar categories.

128

Spaces and quantities

From Lawvere[227]

Distributive v. other categories

Intensive / extensive

"The role of space as an arena for quantitative "becoming" underlies the qualitative transformation of a spatial category into a homotopy category, on which extensive and intensive quantities reappear as homology and cohomology."

Definition 128.0.1. A *distributive category* **C** is a category with finite products and coproducts such that the canonical distributive morphism

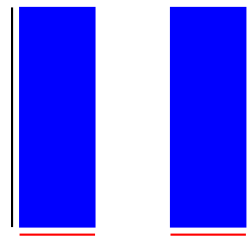
$$(X \times Y) + (X \times Z) \to X \times (Y + Z) \tag{128.1}$$

is an isomorphism, ie there exists a morphism

$$X \times (Y + Z) \to (X \times Y) + (X \times Z) \tag{128.2}$$

that is its inverse.

Distributive categories are typically categories that are "like a space" in some sense, in the context that concerns us. In terms of physical space, we can visualize it like this :



Whether we compose this figure by first spanning each red line along the black line and then summing them, or first summing the two red lines and spanning them along the black line does not matter and will give the same figure.

Example 128.0.1. If a category is Cartesian closed and has finite coproducts, it is distributive.

Proof. In a Cartesian category, the Cartesian product functor $X \times -$ is a left adjoint to the internal hom functor $[X, -]$. As colimits are preserved by left adjoint tmp, we have

$$(X + Y) \tag{128.3}$$

□

This means in particular that any topos is distributive, such as **Set** and **Smooth**.

Example 128.0.2. The category of topological spaces **Top** is distributive.

Proof. In **Top**, the product is given by spaces with the product topology,

$$(X_1, \tau_1) \prod (X_2, \tau_2) = (X_1 \times X_2, \tau_1 \prod \tau_2) \tag{128.4}$$

where the product of two topologies is the topology generated by products of open sets in X_1, X_2 :

$$\tau_1 \prod \tau_2 = \{U \subset\} \tag{128.5}$$

[...] and the coproduct is the disjoint union topology :

□

[Category of frames?]

As can be seen, those are some of the most archetypal categories of spaces.

Definition 128.0.2. A *linear category* is a category for which the product and coproduct coincide, called the *biproduct*. For any two objects $X, Y \in \mathbf{C}$, the biproduct is

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y \tag{128.6}$$

Proposition 128.0.1. A linear category has a zero object 0, which is both the initial and terminal object.

Proof. This stems from the equivalence of initial and terminal objects as the product and coproduct over the empty diagram. □

Proposition 128.0.2. In a linear category, there exists a zero morphism $0 : X \to Y$ between any two objects X, Y, which is the morphism given by the terminal object map $0 \to Y$ and the initial object map $X \to 0$:

$$0_{X,Y} : X \to 0 \to Y \tag{128.7}$$

A linear category is so called due to its natural enriched structure over commutative monoids.

Proposition 128.0.3. For any two morphisms $f, g : X \to Y$ in a linear category, there exists a morphism $f \oplus g$ defined by the sequence

$$X \to X \times X \cong X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \cong Y + Y \to Y \tag{128.8}$$

Proposition 128.0.4. The sum of two morphisms is associative and commutative

Proposition 128.0.5. The zero morphism is the unit element of the sum.

"in any linear category there is a unique commutative and associative addition operation on the maps with given domain and given codomain, and the composition operation distributes over this addition; thus linear categories are the general contexts in which the basic formalism of linear algebra can be interpreted."

Definition 128.0.3. If in a linear category every morphism $f : X \to Y$ has an inverse denoted $-f : X \to Y$, such that $f \oplus -f = 0$, then it is enriched over the category of Abelian groups *Ab*, and is called an *additive category*.

https://philarchive.org/archive/NOTTPO-4

https://philarchive.org/archive/MEIGIT

129
Parmenides

Arguments from Zeno & Parmenides :

"All objects are similar to each other and all objects are different from each other"

Parmenides proceeded: If one is, he said, the one cannot be many? Impossible. Then the one cannot have parts, and cannot be a whole? Why not? Because every part is part of a whole; is it not? Yes. And what is a whole? would not that of which no part is wanting be a whole?

Certainly. Then, in either case, the one would be made up of parts; both as being a whole, and also as having parts?

To be sure. And in either case, the one would be many, and not one? True. But, surely, it ought to be one and not many? It ought. Then, if the one is to remain one, it will not be a whole, and will not have parts?

No. But if it has no parts, it will have neither beginning, middle, nor end; for these would of course be parts of it.

Right. But then, again, a beginning and an end are the limits of everything?

Certainly. Then the one, having neither beginning nor end, is unlimited? Yes, unlimited. And therefore formless; for it cannot partake either of round or straight.

But why? Why, because the round is that of which all the extreme points are equidistant from the centre?

Yes. And the straight is that of which the centre intercepts the view of the extremes?

True. Then the one would have parts and would be many, if it partook either of a straight or of a circular form?

Being is indivisible, since it is equal as a whole; nor is it at any place more, which could keep it from being kept together, nor is it less, but as a whole it is full of Being. Therefore it is as a whole continuous; for Being borders on Being.

[...]

In categorical terms, this argument would obviously make the whole structure collapse immediately, as the ousting of non-being [negation?] would simply make the external logic incapable of producing an argument, but let's see what happens if we merely look at the internal logic.

If we try to look at this in terms of subobjects, the existence of a division, ie some subobject $S \hookrightarrow X$, relates to the existence of a negation, and therefore fundamentally to the existence of the initial object, ie "nothingness".

Given our poset of subobjects $\mathrm{Sub}(X)$, the existence of a division is that of some subobject $(S \hookrightarrow X) \in \mathrm{Sub}(X)$, and if we are considering it as fundamentally divisible, we have to consider at least another subobject S' which is not itself part of S. In other words, there exists no further subobject S'' of both S and S' which is part of both.

We could phrase this more elegantly with the use of the coproduct, as this is a more adapted notion to handle the existence of disjoint objects, but the coproduct will already imply the existence of an initial object, making it a non-starter.

Given this, if we assume that this poset of subobjects is a join semilattice, corresponding to the category being finitely complete (in other words, we just have the notion of logical conjunction in our category), this leads to a contradiction, as this subobject always exists, and this is the initial object. Our two subsets S, S' have the join

$$S \wedge S' = 0 \tag{129.1}$$

[equivalent method with negation from the Heyting implication]

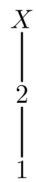
In terms of logic, the subobject relation $S \hookrightarrow X$ defines a proposition p, as well as $S' \hookrightarrow X$ defines p', and their lack of overlap is defined by their

$$p \wedge p' \leftrightarrow \bot \tag{129.2}$$

Theorem 129.0.1. In a finitely complete category, the existence of disjoint objects requires the existence of an initial object.

Any notion of division therefore requires some notion of nothingness.

We could of course conceive of posets of subobjects which can circumvent this, such as some entirely linear mereology



but this is not a particularly model of space. In mereological terms, if our space has at least two regions which are not related by parthood, $X \not\subseteq Y$, $Y \not\subseteq X$, and they do not overlap, $\neg(X \cap Y)$, then the

The only object that would avoid such an issue being of course the terminal object, simply having itself as a subobject and none other

130
Bridgman and identity

Another related philosophical issue to this is the notion of *identity*, ie what we can consider to be an individual object, both as a division (what specific parts of the world do we include in the object) as well as equivalence (what can we define to be the same object despite being differing from what we previously defined it as).

cf. Bridgman :

“We must, for example, be able to look continuously at the object, and state that while we look at it, it remains the same. This involves the possession by the object of certain characteristics — it must be a discrete thing, separated from its surroundings by physical discontinuities which persist.”

For instance, if we look at an object in physics, like a ball, to study its motion, this is somewhat arbitrary in a number of ways. If we look at it from an atomic perspective, it is nothing but some assemblage of particles, which we have chosen to consider together as it appears to form an individual object to the human mind. If we were to take the quantum field theory perspective, this would be even worse as due to the non-local nature of all relativistic quantum fields, there would be no obvious way to split

[13, 228]

Definition of a system? Definition via partition, negation?

131
Ludwig Gunther

Relation of moments to Ludwig's structuralism?

Axiomatization from Ludwig [229]

[230]

132
Dialectics of nature

When we look at dialectical logic in practice, ie [15], the examples given are much more concrete. We are considering the identity of some *entity*, such as an object, a group, etc, and considering what it means for at entity to be identical to itself. All concrete entities are never identical to themselves, either in time, context, etc.

From Engels :

The law of the transformation of quantity into quality and vice versa; The law of the interpenetration of opposites; The law of the negation of the negation.

Example : an object moving in space, an organization changing, ship of Theseus, etc

To keep things concrete, let's try to consider a simple example of both category theory and dialectical logic, which is an object in motion. We will simply consider the kinematics here and not the dynamics as this is unnecessary.

The simplest case is the $(1 + 1)$-dimensional case, of a point particle moving along a line $x : L_t \to L_s$.

First notion : We are considering the "identity" of an object under a certain *lens* (ie with respect to its relations with a number of other objects). Its identity is only assured under the full spectrum of those relationships, ie we say that two objects A, B are identical if

$$\forall X \in \mathbf{H},\ \mathrm{Hom}_{\mathbf{H}}(A, X) \cong \mathrm{Hom}_{\mathbf{H}}(B, X),\ \mathrm{Hom}_{\mathbf{H}}(X, A) \cong \mathrm{Hom}_{\mathbf{H}}(X, B) \tag{132.1}$$

Relation to Yoneda? Relation to Hegel's "only the whole is true" thing?

This corresponds to the hom covariant and contravariant functor

$$h^A \cong h^B,\ h_A \cong h_B \tag{132.2}$$

Two objects are identical if their hom functors are naturally isomorphic, ie if there exists a natural transformation

$$\eta : h^A \Rightarrow h^B,\ \epsilon : h_A \Rightarrow h_B \tag{132.3}$$

with two-sided inverses each.

Yoneda embedding :

$$\mathrm{Nat}(h_A, h_B) \cong \mathrm{Hom}_{\mathrm{H}}(B, A) \tag{132.4}$$
$$\mathrm{Nat}(h^A, h^B) \cong \mathrm{Hom}_{\mathrm{H}}(A, B) \tag{132.5}$$
$$\tag{132.6}$$

those isomorphisms are elements of this, therefore isomorphic to morphisms from A to B. Therefore A and B are identical in term of their relationships to every other object if they are identical in the more typical sense of the existence of an isomorphism.

As those are functors, this analysis also applies to elements of the objects X, or subobjects. For any two monomorphisms

$$x, y : S \to X \tag{132.7}$$

We say that those elements are identical if the hom functors applied to them [...]

The breakage of the law of identity comes by considering different perspectives. If given an object X [or element $x : \bullet \to X$?], we attempt to use different relations with other objects to "probe" it, its moments and the assessment of its identity will change.

[Abstract example ?]

Expression via the subobject classifier of the topos

Example : case of motion. What does it means for a moving object to *be* in different positions.

Ex : consider the position of an object wrt two different time intervals $[t_a, t_b]$, $[t'_a, t'_b]$, rather than all possible intervals.

Characterization by observables at those different times

Observables : $O_i : \mathrm{Conf} \to \mathbb{R}$ [Isbell duality thing idk]

133
misc

[231]

A
The Hegel dictionary

As a rather esoteric writer, it is not uncommon for commentaries related to Hegel to include a little lexicon explaining the terms involved, and this work will not detract from it.

As different translations of Hegel can show up, the original German word is also included.

Glossary

concept (*Begriff*) Test. 683

determinate (*bestimmt*) Test. 683

essence Test. 683

immanence Test. 683

immediacy Test. 683

—Birth of Isis, the heavens rained

Rhind papyrus

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